Delegated Learning in Asset Management*

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ABSTRACT

We develop a tractable framework of delegated asset management with flexible information acquisition in a multi-asset economy in which fund managers face moral hazard in portfolio allocation decisions. We explore the features of the optimal affine compensation contract for fund managers, in which benchmarking arises endogenously as part of their optimal compensation. In the equilibrium with delegated learning, asset prices reflect both the acquired private information of fund managers and their desire to hedge their exposure to the benchmark. We show a potential gap between our model-implied measure and several widely-adopted empirical statistics intended to capture managerial ability. In a multi-period extension, we propose a new performance measure of fund manager skill.

Keywords: Fund Manager Compensation, Endogenous Benchmark, Moral Hazard, Information Acquisition

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1 Introduction

There is growing concern that active fund managers lack the superior ability in garnering higher returns to justify their higher fees compared to their passive counterparts.[1] The existing literature has focused on either improving empirical measures to evaluate the unobservable skill of fund managers, or on developing theories to justify the lack of empirical support for their superior ability.[2] Despite the progress of this fast growing literature, the relationship between fund manager ability and the incentives that they face, in equilibrium, is still not well-understood. In this paper, we ask to what extent such unobservable ability is an outcome of the incentives provided to active managers in the equilibrium and shed light on measuring skills in the asset management industry.

To investigate this issue, we cast the information acquisition and portfolio allocation decisions of a delegated asset manager as a principal-agent problem between the active fund manager and its investors. We refer to this as the delegated learning channel. We study an economy in which asset managers can trade on behalf of investors in a multi-asset financial market, similar to that in Admati (1985). In the spirit of Kacperczyk et al. (2016), active fund managers are able to exert costly effort to learn about the aggregate and asset-specific components of the payoffs of the assets in which they can invest. The inability of investors to observe the effort and portfolio decisions of their delegated asset managers, however, forces investors to offer a contract that is incentive compatible to managers, who seek only to maximize their compensation. In equilibrium, these incentives feed into asset prices as fund managers trade on their private information in financial markets, which then feed back

[1] Consistent with this view, in recent years there has been an accelerating shift in fund flows from active to passive strategies. Since 2005, actively managed equity and fixed income funds have lost fund flows to passive strategies globally. According to MorningStar, over last decade, actively run U.S. stock funds saw net outflows every year, totaling about $600 billion, while their indexed counterparts saw net inflows of approximately $700 billion. See http://www.marquetteassociates.com/research/a-continued-shift-from-active-to-passive-in-u-s-equities.

[2] The existing literature has developed several theories to help explain the lack of empirical support that active managers have superior ability, including that fund performance exhibits decreasing-returns-to-scale (Berk and Green, 2004), that managers choose investments based on their benchmark and flow-performance sensitivity (Brennan (1993), Admati and Pfleiderer (1997), Buffa et al. (2014)), and that skill reflects a choice to acquire information over the business cycle (Kacperczyk et al. (2014), Kacperczyk et al. (2016)).

into the determinants of the optimal contract in the principal-agent problem between the manager and investors.

The optimal affine contract for active fund managers that we derive features three components: a fixed fee, a performance-based reward that evaluates a fund manager for its performance, and benchmarking relative to the ex-ante mean-variance efficient portfolio. In contrast to such frameworks as those of Basak and Pavlova (2013) and Buffa et al. (2014), the optimal benchmark we derive in our economy is endogenous and arises as an outcome of the compensation contract. Since the performance-based reward influences the aggressiveness with which active fund managers trade on their private information, it feeds into the informativeness of asset prices in equilibrium. Through this channel, the performance-based piece impacts the overall uncertainty that active fund managers face when choosing their portfolio, and consequently their incentives to exert effort to acquire private information. By benchmarking, the investor effectively endows the active fund manager with a tilted short position in the benchmark portfolio, which leads it to hedge its benchmark risk with passive investing in financial markets. This tilt, consequently, impacts the level of risk-sharing between the investor and the active manager. Our analysis therefore highlights a separation between information acquisition and risk-sharing incentives in delegated asset management.

To illustrate how the optimal affine contract varies with the asset environment, we perform comparative statics when active managers are more risk-averse by altering the overall risk in the economy and the cross-sectional correlation of asset payoffs. As the overall level of uncertainty about asset payoffs increases, the optimal contract places less emphasis on the performance-based component, and more weight on benchmark-based incentives. This is optimal because the marginal benefit of exerting effort to learn is higher for active fund managers, even in the absence of incentives. The shift toward benchmarking reflects the increased value fund managers are expected to add over direct investment by the fund's investors. When the correlation of payoffs increases, in contrast, the optimal contract instead puts more weight on the performance-based component, and less on benchmarking. This occurs because the increased correlation reduces the cross-section of risk in the economy about which active fund managers can learn, and makes prices more revealing about the aggregate sources of risk. As such, active managers reduce the overall effort that they exert to acquire private information, which motivates the need for stronger performance-based incentives and lessens the benefit of benchmarking for sharing risk.

Our model has novel implications for identifying skill among fund managers. We highlight a gap between our model-implied measure of fund manager skill, the reduction in uncertainty about asset payoffs, and empirical statistics meant to capture asset management ability, such as the active share proposed by Cremers and Petajisto (2009) and the return gap of Kacperczyk et al. (2008). When the overall level of payoff uncertainty increases, active fund managers devote more effort to acquiring private information and take more active positions, when compared to the benchmark portfolio, and this is correctly reflected in our theoretical analogues of the two empirical measures. When asset payoffs become more correlated, however, active fund managers exert less overall effort to learn, but may appear more active because the optimal benchmark, the ex-ante mean-variance efficient portfolio, takes smaller positions in the risky assets because of the diminished benefits from diversification. Consequently, our analysis cautions in the interpretation of these empirical measures as proxies for managerial ability, and also highlights the importance of endogenizing the benchmark for theoretical predictions.

We then propose a incentive-based measure of manager skills by investigating the dynamic extension of our baseline model. Assume now that trading by managers occurs over multiple periods. This dynamic extension illustrates that having multiple periods introduces intertemporal incentives for fund managers to acquire private information and, more importantly, can provide investors with a time-series of past fund behavior to improve monitoring. We show that the historical variance of a fund's return gap, downweighted by the dispersion of asset payoffs, provides a consistent measure of average portfolio selection skill, and argue how investors learning about a fund manager's skill through this channel could help explain the nonlinear relationship between performance and fund flows observed empirically.

Our work is related to the literature on delegated asset management under asymmetric information. García and Vanden (2009) and Gárleanu and Pedersen (2015) explore the implications for market efficiency of the formation of mutual funds in the presence of costly information acquisition in a single asset setting. García and Vanden (2009) also consider a model of delegated asset management with information acquisition, yet their focus is on market efficiency, and they assume that managers sell funds to households and pay a fixed fee to become informed in the spirit of Grossman and Stiglitz (1980). Our work focuses on an affine contract between investors and fund managers in a multi-asset principal-agent setting. Kapur and Timmermann (2005) investigate the impact of relative performance contracts on the equity premium and on portfolio herding. Dybvig et al. (2010) and He and Xiong (2013) consider the market-timing benefits of benchmarking in a partial equilibrium setting. Kyle et al. (2011) investigates the incentives to acquire information under delegated asset management for a large informed fund, in the spirit of Kyle (1985), while Glode (2011) and Savov (2014) microfound delegated asset management as a vehicle for investors to hedge their background risk. Huang (2015) studies the market for information brokers in an equilibrium setting with optimal contracting to explain home bias, comovement in asset idiosyncratic volatility, and the possibility of herding and equilibria multiplicity.

This paper is connected to the growing literature on equilibrium asset pricing with flexible information acquisition. Van Nieuwerburgh and Veldkamp (2009, 2010) and Kacperczyk et al. (2016) study the flexible information acquisition problem faced by investors who have limited attention that they can allocate to learning about risky asset payoffs, the latter of which focuses on business cycle implications. Maćkowiak and Wiederholt (2012) investigate the information acquisition decisions of investors who have limited liability, while Huang et al. (2016) models information acquisition as part of a dynamic reputation game between the fund and its investors. In contrast to these studies, we model the information acquisition of managers as being subject to agency issues within an equilibrium framework.

In addition, our work is also related to the literature on manager incentives and benchmarking in the asset management industry. Basak and Pavlova (2013) and Buffa et al. (2014) investigate the asset pricing implications of benchmarking against an exogenous index in a multi-asset setting, with Buffa et al. (2014) embedding benchmarking in a principal-agent framework. Buffa and Hodor (2017) explore the asset pricing implications of heterogeneous benchmarking. Starks (1987) studies the role of symmetric versus bonus performance-based contracts in incentivizing asset managers. Brennan (1993) examines the CAPM implications of delegated management with both exogenous and optimal benchmarking. Admati and Pfleiderer (1997) analyzes benchmarking and manager incentives in a partial equilibrium framework in which managers have superior information to investors, while Ou-Yang (2003) investigates the optimal affine contract in a finite horizon setting in which managers bear a time-varying cost of investing in a portfolio. van Binsbergen et al. (2008) explores how benchmarking can overcome moral hazard issues that arise with decentralization. Cuoco and Kaniel (2011) study the implications for asset pricing when manager compensation is linked to a benchmark, and Li and Tiwari (2009) study nonlinear performance-based contracts in the presence of benchmarking. In our work, we derive the optimal benchmark jointly with the optimal affine contract and equilibrium prices, and study their empirical implications for intermediary holdings and asset returns.

2 A Model of Delegated Asset Management

In this section, we present a model of delegated asset management with flexible information acquisition in a multi-asset framework. In this economy, there are investors who can allocate their wealth between an active fund, which is subject to agency issues in its portfolio allocation decisions, and a passive fund. We first introduce the asset environment, and then discuss the problem faced by each type of agent. Finally, we define the asset market equilibrium.

2.1 The Environment

Asset Fundamentals There are three dates $t = \{0, 1, 2\}$. Suppose that there are N assets with risky payoffs f_i, $i \in \{1, 2, ..., N\}$, which realize at date 2 that satisfy the following decomposition:

$$f_i = \begin{cases} b_1\theta_1 \\ a_i\theta_i + b_i\theta_1, \ i \in \{2, ..., N\} \end{cases}$$

The common component θ_1 can be viewed as aggregate payoff risk, with b_i being the loading on this aggregate payoff risk of the asset, while the $a_i\theta_i, i \in \{2, ..., N\}$ are the asset-specific components of the risky asset payoffs. This payoff structure we employ is similar to that in Buffa et al. (2014) and Kacperczyk et al. (2016). For interpretation of θ_1 as aggregate payoff risk, we assume that $a_1 = 0, b_1 = 1$ and that the first asset is a composite asset of the remaining assets in the economy with a payoff that loads only on this aggregate payoff risk.[3] In addition to the N assets, there is a risk-free asset, which can be viewed as asset 0, in perfectly elastic supply with gross return $R^f > 1$. Asset i has price P_i at $t = 1$, and we stack the N prices into the $N \times 1$ vector $\mathbf{P}$. In what follows, bold symbols represent vectors. For convenience, we define the vector $\Theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_N \end{bmatrix}'$ such that:

$$\mathbf{f} = F\Theta,$$

for the $N \times N$ matrix F, which is invertible since F is lower triangular provided that $b_i > 0$ $\forall i$. In our setting, aggregate risk arises through the correlation structure of asset payoffs, and is represented by the common fundamental θ_1.[4]

We assume that all agents in our model have a normal prior over Θ, and initially believe

[3]Kacperczyk et al. (2016) employ a similar assumption for the asset payoff structure. While not essential for our analysis, it helps with exposition by ensuring that the map from risk factors $\left\{\theta_1, \{\theta_i\}_{i\in\{2,\ldots,N\}}\right\}$ to asset payoffs $\{f_i\}_{i\in\{1,\ldots,N\}}$ is invertible.

[4]This is in contrast to Kacperczyk et al. (2016), where aggregate risk takes the form of the asset fundamental with a higher supply variance. Our derivations will, in fact, be valid more generally for any arbitrary invertible matrix F.

that $\Theta \sim \mathcal{N}\left(\bar{\Theta}, \tau_\theta^{-1} Id_N\right)$, where τ_θ is the common precision of the prior over the hidden factors driving asset payoffs. One can view the prior as reflecting all publicly available information about the asset payoffs, such as financial disclosures, earnings announcement, and macroeconomic news that agents have before contracting at date 0.

Agents There is a unit mass of investors, each with initial wealth W_0, that can allocate this wealth at date 0 between an active and a passive fund. Investor i invests a fraction y_i of its wealth in an active fund, and the remaining fraction $1 - y_i$ in the passive fund. Although the funds in which investors delegate their wealth can borrow and lend freely at the risk-free rate, R^f, we assume for simplicity that investors cannot. As a consequence, $y_i \in [0, 1]$.[5] The portfolio allocation decisions of all investors are publicly observable.

There is an active and a passive advisory firm or management company available to investors. Both types of firms employs a mass of ex-ante identical fund managers that each manages its own fund. As is standard, we assume these managers have no initial wealth. While the passive firm employs managers to implement a passive strategy that is known to investors, the active firm pays managers to take active positions by exerting costly effort to acquire private information. Their information acquisition activities and portfolio allocation decisions, however, are unobservable to investors and the advisory firm, and this gives rise to an agency conflict between investors and active fund managers.[6] This is what we refer to as *delegated learning* channel.

The active fund offers one compensation contract that is incentive compatible to all managers of its funds that maximizes investor utility subject to their participation. We abstract from issues of decentralization, as in van Binsbergen et al. (2008). We also abstract from tournament incentives featured in, for instance, Kapur and Timmermann (2005), which

[5] A previous version considered allocation by investors on the extensive margin, in which an investor either delegated all of their wealth to an active manager or invested directly as a passive fund. In practice, however, investors can freely allocate capital across investment opportunities.

[6] Brown and Davies (2016) also studied the moral hazard in the active asset management industry in a partial equilibrium framework. They assume the effort exerted by managers are directly linked to returns, while we focus on the incentives to acquire costly information.

can lead to herding when manager participation constraints bind.[7] Our approach is different from García and Vanden (2009) and Gárleanu and Pedersen (2015), who investigate the fee setting by asset management companies. Given the relatively stable mutual fund fee structure, we focus on studying the direct incentive provision from the compensation contract to managers.

Since our model features a static setting, we are unable to address implicit incentives that arise from career concerns and fund flows. In addition, as there is only one observation of a fund manager's performance, there is little that can be inferred about the manager's skill in acquiring information, as noted in Admati and Pfleiderer (1997). We return to these issues when we consider the dynamic extension of the model.

In what follows, we assume that there is a mass χ of managers in the active fund available, and $1-\chi$ in the the passive fund. Each manager manages one fund, so that the mass of active funds is the same of the mass of the active fund managers. Clearing in the market for funds requires that investors, on aggregate, allocate their capital to respect these fixed proportions. The fraction χ is observable public information.

We now discuss the problem faced by each of these agents in turn.

2.2 Passive Fund

For simplicity, and since it is not the focus of our analysis, we keep our specification of the passive fund very simple. The passive fund employs managers who at date 1 allocate all their capital to holding the mean-variance efficient portfolio, ω_1^D, after observing market prices, $\mathbf{P}$.

The passive fund has a final AUM at date 2, W_2^D given by:

$$W_2^D = R^f W_0 + \omega_1^{D\prime} \left(\mathbf{f} - R^f \mathbf{P}\right).$$

[7] An earlier version allowed the compensation contract to condition on the realized performance of the passive fund. This mainly modified the optimal benchmark. Since it did not add much additional insight for the additional notational complexity, we omit it to simplify the exposition.

We assume that the passive fund managers trade as if they have the same coefficient of absolute risk aversion, $\gamma > 0$, as investors. Since the mean-variance efficient portfolio is publicly observable, there is no agency conflict between passive fund managers and investors, and they will charge a fixed fee for their services.[8] For parsimony, we normalize this fee to zero.

This simplification avoids the issue that passive fund managers may want to tilt the portfolio to take into account the background risk of investors who also invest in active funds. Since the portfolios of active funds are unobservable, passive managers would have to form expectations about this unknown component, and this would render the problem intractable. Given that passive managers only have access to public information, this alternative portfolio would still be observable to investors, and would not introduce any new agency issues. The mean-variance efficient portfolio is also a portfolio of independent interest, since it is the portfolio an investor would choose if invested directly in financial markets at date 1.

If the unit mass of investors, on aggregate, allocate a fraction $1 - y_i$ of their capital W_0 to passive funds, then they will receive an aggregate dollar payoff $(1 - y_i)W_2^D$ that can be distributed among them, and dollar return from each fund W_2^D/W_0.

2.3 Active Fund

We assume that investors are randomly allocated to a mass χ of fund managers by the active advisory firm, and are therefore exposed to fund-specific risk. There is a many-to-one matching protocol because there is a unit mass of investors who are matched with a smaller set χ of fund managers. Though investors could achieve full diversification by investing in all active funds, this is, in part, an artifact of the information structure, and also at variance with what is observed in practice.[9]

[8]Since there is no agency conflict, investors can extract the surplus from the relationship to make the participation constraint of the passive fund manager bind. If the passive fund manager is risk-averse, then the cheapest form of compensation is a fixed fee, which we normalize to zero.

[9]Empirically, investor fund flows are correlated with past performance, which we would not expect to observe empirically if investors diversified their holdings across active funds. Furthermore, the premise of

Fund managers in the active fund each face a portfolio choice problem at date 1. Given their information and initial AUM W_0, fund managers choose a portfolio allocation strategy ω_1^S at date 1 across the N assets *after* observing market prices $\mathbf{P}$ and their private information, so that the final AUM W_2^S is given by:

$$W_2^S = R^f W_0 + \omega_1^{S\prime} \left(\mathbf{f} - R^f \mathbf{P} \right).$$

As described above, in addition to a portfolio choice problem, fund managers must exert costly effort at date 0 to acquire their private information about asset payoffs at date 1. While asset prices are publicly observable, active managers also acquire a vector of noisy private signals $\mathbf{s}_j$ about θ_1 and the asset-specific component of asset payoffs θ_i, $i \in \{2, ..., N\}$. They can exert a vector of efforts $e = \mathbf{e}'\mathbf{1}_{N\times 1} \geq 0$, with $\mathbf{e} \geq \mathbf{0}$ element-by-element, to reduce the variance of these signals $\Sigma(\mathbf{e})$.

Specifically, active manager j receives a vector of noisy signals $\mathbf{s}_j$ about Θ given the effort level $\mathbf{e}_j$:

$$\mathbf{s}_j = \Theta + \Sigma_j (\mathbf{e}_j)^{1/2} \varepsilon_j,$$

where $\varepsilon_j \sim N(\mathbf{0}_{N\times 1}, Id_N)$ is independent across j and satisfies the Strong LLN $\int_{-\infty}^{\infty} \varepsilon_j d\Phi(\varepsilon_j) = \mathbf{0}_{N\times 1}$ for $\Phi(\cdot)$, the CDF of the standard normal distribution. Following Kacperczyk et al. (2014), we assume that $\Sigma_j(\mathbf{e}_j)$ is a diagonal matrix with entry $K_{ii}^{-1}(e_{ij})$ that satisfies a monotonicity condition.[10] We assume that $\Sigma_j(\mathbf{e}_j)$ is diagonal so that there is a direct link between the effort manager j exerts to learn about the i^{th} component of Θ, $\mathbf{e}_{ij}$, and the precision of the signal manager j receives about that component, s_{ij}.[11] The monotonicity condition we impose ensures that a higher level of effort (weakly) implies the manager receives more

active funds is to be exposed to "active alpha" strategies, and much of the literature, in the spirit of (Berk and Green, 2004), has focused on the problem of investors identifying managers that can deliver superior performance. García and Vanden (2009) make a similar assumption in that, despite holding a portfolio of funds, households are still exposed to fund idiosyncratic risk.

[10] The monotonicity condition we require is that : $\Sigma_j(\mathbf{e}_j'') - \Sigma_j(\mathbf{e}_j')$ is positive-semi definite (PSD) whenever $\mathbf{e}_j' \geq \mathbf{e}_j''$.

[11] Our results are robust to the more general specification of $\Sigma_j(\mathbf{e}_j)$.

informative signals. To ensure prices are always informative, we regulate $\Sigma(\mathbf{e}_j)$ by assuming that $\sup_i \Sigma(\mathbf{0}_{N\times 1}) \leq M^{-1} < \infty$.[12] In what follows, we parameterize $K_{ii}(e_{ij}) = M + e_{ij}$. One can view this observation of private information by a fund manager as their security selection or "stock picking ability".

Active fund managers have CARA preferences over their compensation from investors C_0^S and the monetary cost of exerting effort to acquire private information:

$$u\left(C_0^S; \omega_1^S, \mathbf{e}\right) = -\exp\left(H(\mathbf{e}) - \gamma_M C_0^S\right),$$

where γ_M is the coefficient of absolute risk aversion and $H(\cdot)$ is the dollar cost for effort $\mathbf{e}$, an increasing and (strictly) convex function in each of its arguments, such that $\partial_i H(\mathbf{e}) > 0$ and $\partial_{ii} H(\mathbf{e}) \geq 0$, and $H(\mathbf{0}_{N\times 1}) = 0$ as a normalization. We specialize $H(\mathbf{e})$ to the case that $H(\mathbf{e}) = \frac{1}{2} h(\mathbf{e}'\mathbf{1}_{N\times 1})$, where $h'(\cdot) > 0$, $h''(\cdot) \geq 0$, and $h(0) = 0$. This functional form induces potential substitutability in manager learning decisions, and therefore a tradeoff to learning too much about one source of asset-specific risk.

An agency conflict arises because the effort that active managers exert, and their portfolio choice, are not observable. An active manager must therefore find it optimal to follow the recommendation of investors, which gives rise to the incentive compatibility (IC) constraint:

$$\mathbf{e} \in \operatorname{argsup}_{\mathbf{e}\in\mathbb{R}_+^N} E\left[\sup_{\omega\in\mathbb{R}^N} E\left[u\left(C_0^S; \omega_1^S, \mathbf{e}'\right) \mid \mathcal{F}_j\right]\right] \quad (IC), \tag{1}$$

where $\mathcal{F}_j$ is the fund manager's information set, and the optimization implies a natural timing to their decisions. The fund manager first determines the effort to exert based on the compensation contract C_0^S with investors at date 0. At date 1, the fund manager observes prices and private signals, and makes portfolio allocation choice. The fund manager's information set is then the sigma algebra generated from observing the vector of prices $\mathbf{P}$ and its private signals $\mathbf{s}_j$, $\mathcal{F}_j = \sigma\left(\mathbf{P}, \mathbf{s}_j(\mathbf{e}_j)\right)$.

[12]Our results will be valid in the limit that $M \searrow 0$.

In addition to the IC constraint, active fund managers are also subject to a participation constraint:

$$E\left[u\left(C_0^S;\omega_1^S,\mathbf{e}\right)\ \mid \mathcal{F}_j\right] \geq u_0$$

In what follows, the IC constraint will always bind, and we primarily will consider parameter restrictions to focus on the case in which the participation constraint does not bind. Since investors will have to be indifferent to investing the marginal dollar for the market for intermediaries to clear, they implicitly face a participation constraint that will always bind when splitting the surplus of active management with the active fund managers. If the participation constraint also binds for managers, then this imposes a limit on the effort that active managers will exert to acquire private information.

Finally, we assume that the signal noise of all active fund managers is uncorrelated, so that a Weak Law of Large Numbers, in the spirit of Uhlig (1996), holds. Once returns are realized, the aggregate return of the active fund is then independent of the fund-specific risk of any individual active fund. We then assume that the advisory firm pays out to each investor this aggregate return adjusted by the individual performance of the fund manager with whom the investor has been matched.

If the unit mass of investors, on aggregate, allocate a fraction y_i of their capital W_0 to passive funds, then with a mass of y_i active funds, they will receive, as a group, this aggregate return. Consequently, wealth is neither created nor destroyed by this return protocol. Each active fund offers a dollar return W_2^S/W_0.

We assume the active management company chooses the compensation contract, C_0^S, that it offers to all fund managers, since they are ex-ante identical, to maximize the return to investors subject to the market for intermediaries clearing.[13]

[13]With this protocol, it is not important whether the management company, the fund managers, or the investors offer the contract. Maximizing the surplus of active management to investors allows managers to extract the most rent from the relationship, as their fixed fee will capture the marginal benefit of a dollar allocated to active management instead of the passive fund.

2.4 Investors

Investors have CARA preferences over the final AUM at date 2, W_2^S. They choose their asset allocation policy y_i across the active and passive fund of funds to maximize their utility subject to incurring the cost of compensating the active fund manager:

$$U\left(W_2^I, y_i; C_0^S\right) = -\exp\left(-\gamma\left(x_i\left(W_2^S - C_0^S\right) + (1-y_i)W_2^D\right)\right),$$

where $\gamma > 0$ is their coefficient of absolute risk aversion.

The investors solve the optimization problem when allocating their capital:

$$V_0 = \sup_{y_i} E^{\mathbf{e}\left(C_0^S\right)}\left[U\left(W_2^I, y_i; C_0^S\right)\right], \tag{2}$$

subject to the IR and IC constraints, where $E^{\mathbf{e}(\cdot)}[\cdot]$ is understood as the expectation under the probability distribution induced by the recommended effort level $\mathbf{e}\left(C_0^S\right)$. Consequently, V_0 is the indirect utility of investors.

Similar to Ou-Yang (2003), Kapur and Timmermann (2005), Buffa et al. (2014) and Sotes-Paladino and Zapatero (2017), we restrict our attention to the space of affine contracts between investors and active fund managers through the active advisory firm for several reasons. First, is to advance our understanding of how the incentives faced by active fund managers extend to an equilibrium setting. Previous partial equilibrium studies have found negative results on affine incentive contracts for fund managers. Stoughton (1993) and Admati and Pfleiderer (1997), for instance, both show that affine contracts fail to provide incentives for managers to acquire private information with CARA preferences. We analyze the optimal contract in this linear paradigm and show that affine contracts can indirectly provide managerial incentives when asset prices that aggregate private information feed back into managers' compensation. Second, investors in our setting contract indirectly with fund managers through advisory firms, and regulations restrict the form of this compensation to fulcrum fees that are the symmetric around the return of a benchmark.[14] Third, option-like incentives hamper risk-sharing between investors and managers. In addition, they are likely to worsen the effort that managers exert to acquire private information, since the down-side protection of the option-like component gives managers incentive to take risk by remaining less informed.[15] Finally, since we are solving for noisy rational expectations within the linear paradigm of Grossman and Stiglitz (1980) and Hellwig (1980), such a restriction may be seen as a natural extension of the focus on linear equilibria.

2.5 Intermediary and Asset Markets Clearing

Intermediary Market Clearing We assume that investors can freely choose to allocate capital to the active and passive investment management companies. Since there is a fixed fraction of funds of each type available to investors in the two management companies, in equilibrium they must be indifferent between these two options at the margin.

This free-entry assumption is similar to that in Berk and Green (2004), where the "net fees" of active funds versus passive funds offers similar returns, while "gross of fees" reflects manager skill. Furthermore, Berk and van Binsbergen (2015) find that active managers capture the surplus in the advisory relationship, and this will show up as a fixed fee in our affine contract. Given that the investment decisions of fund managers will be independent of initial wealth in this CARA-normal setting, we are abstracting from the decreasing returns to scale at the fund level that are observed empirically, the consequences of which are explored, for instance, in Berk and Green (2004) and Pástor and Stambaugh (2012). Since investors in our model are indifferent to with whom they invest, the market for intermediation between investors and managers then trivially clears. To be internally consistent, we assume that

[14] The 1970 SEC amendment to the Investment Company Act of 1940 requires that performance-based contracts should not contain the "bonus" performance-based fee and should only be symmetric around the benchmark returns.

[15] Starks (1987) shows that linear contracts lead to an optimal portfolio risk exposure by managers, but an under-provision of effort compared to the first-best, while asymmetric contracts with bonus incentives lead to both a suboptimal risk exposure and an even lower level of effort than the symmetric case. Sotes-Paladino and Zapatero (2017) also finds that option-like payoffs can be suboptimal to linear contracts with asymmetric information between investors and managers, though in the absence of moral hazard.

$y_i = \chi$ to pin down the level of active compensation in the equilibrium.

Asset Market Clearing Let $\omega_1^S(i)$ be the portfolio allocation of active fund manager $i \in [0,1]$, and similarly with ω_1^D for passive fund manager i. We assume the supply of the asset is given by the vector $\mathbf{x}$ for the N assets. Since there are a fraction χ of active fund managers, and a fraction $1-\chi$ of passive fund managers, market-clearing requires that:

$$\chi \int_0^1 \omega_1^S(i)\, di + (1-\chi)\, \omega_1^D = \mathbf{x}. \tag{3}$$

As is common in the literature, we assume that asset supply $\mathbf{x}$ is noisy to prevent beliefs from being degenerate. We assume that, from the perspective of all agents, $\mathbf{x} \sim \mathcal{N}(\bar{\mathbf{x}}, \tau_x^{-1} Id_N)$ has a multivariate normal distribution, and $\bar{\mathbf{x}} > \mathbf{0}$ (element-by-element). Since all fund managers are atomistic, they take prices as given and each has negligible impact on the price formation process.

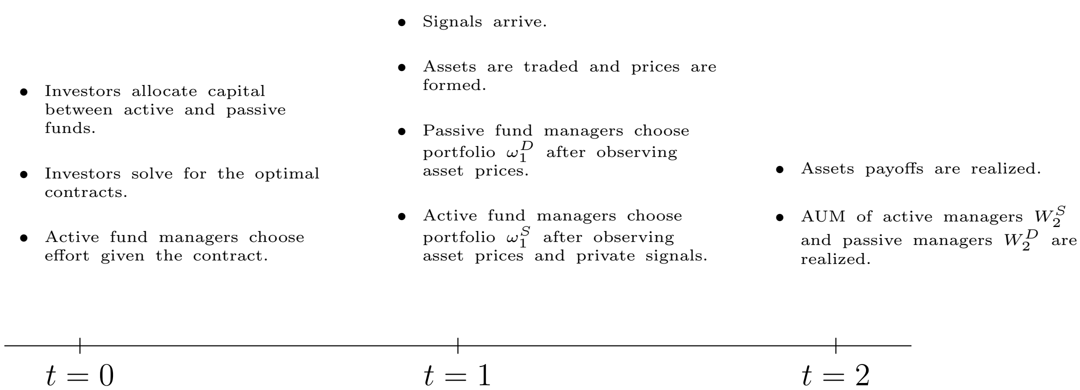


Figure 1: Timeline

Figure 1 illustrates the time line. We solve for a perfect Bayesian noisy rational expectations equilibrium defined as follows:

Equilibrium A perfect Bayesian noisy rational expectations equilibrium in this economy is a list of policy functions y_i, $\mathbf{e}\left(C_0^S\right)$, $\omega_1^S\left(\mathbf{s}_j, \mathbf{P}\right)$, and $\omega_1^D\left(\mathbf{P}\right)$, compensation contract C_0^S for fund managers, and prices $\mathbf{P}$ such that:

- Active Fund Manager Optimization: Given contract C_0^S, prices $\mathbf{P}$, and information set $\mathcal{F}_j$, $\mathbf{e}\left(C_0^S\right)$, and $\omega_1^S\left(\mathbf{s}_j, \mathbf{P}\right)$ solve each fund manager's IC constraint.

- Active Advisory Firm Optimization: Given y_i, contract C_0^S solves the investor's optimization problem (2) and delivers expected utility V_0.

- Investor Optimization: Given contract C_0^S, allocation y_i solves the investor's optimization problem (2) and delivers expected utility V_0.

- Market Clearing: The intermediary market clears through $y_i = \chi$ and the asset markets clear through equation (3).

- Consistency: Investors and passive fund managers form their expectations about Θ based on their information set $\mathcal{F}^c$, while active fund managers form their expectations based on their information set $\mathcal{F}_j$, according to Bayes' rule.

- Sequential Rationality: For each realization of prices $\mathbf{P}$ and private signals $\mathbf{s}_j$, passive and active fund managers find it optimal at date 1 to follow investment policy $\omega_1^S\left(\mathbf{s}_j, \mathbf{P}\right)$ and $\omega_1^D\left(\mathbf{P}\right)$, respectively.

3 The Equilibrium

We search for a symmetric linear equilibrium in which we conjecture that asset prices $\mathbf{P}\left(\Theta, \mathbf{x}\right)$ take the linear form:

$$\mathbf{P}\left(\Theta, \mathbf{x}\right) = \Pi_0 + \Pi_\theta \Theta + \Pi_x \mathbf{x}, \tag{4}$$

where $Rank\left(\Pi_\theta\right),\ Rank\left(\Pi_x\right) = N$. As discussed above, we also focus on linear contracts.

We first derive the conditional beliefs of investors and fund managers. We then derive the optimal investment policy for passive and active fund managers, the latter whom face both effort and portfolio choice decisions that must be incentive compatible. Imposing market clearing allows us to solve for equilibrium asset prices. Finally, we solve for the optimal contracts offered by the active fund of funds, and the allocation decision of investors.

3.1 Learning

We begin by deriving the learning process for investors and passive fund managers. Since they have a normal prior, after observing the linear Gaussian signals $\mathbf{P}(\Theta)$, they update to a posterior for Θ that is also Gaussian $\Theta \mid \mathbf{P}(\Theta) \sim \mathcal{N}\left(\hat{\Theta}, \Omega\right)$ with conditional mean $\hat{\Theta}$ and conditional variance Ω, given by:

$$\hat{\Theta} = \Omega\tau_\theta\bar{\Theta} + \Omega\tau_x\Pi_\theta'\left(\Pi_x\Pi_x'\right)^{-1}\left(\mathbf{P} - \Pi_0 - \Pi_x\bar{\mathbf{x}}\right), \tag{5}$$

$$\Omega^{-1} = \tau_\theta Id_N + \tau_x\Pi_\theta'\left(\Pi_x\Pi_x'\right)^{-1}\Pi_\theta. \tag{6}$$

We now describe the learning process of active fund managers. To get to the posterior of active fund manager j, we recognize that we can first have the manager update his beliefs based on the publicly observed prices, and then treat these beliefs as an updated prior for when the manager then observes its vector of private signals $\mathbf{s}_j$. After observing the public signals $\mathbf{P}(\Theta)$, the new prior of fund manager j from above is $\Theta \mid \mathbf{P}(\Theta) \sim \mathcal{N}\left(\hat{\Theta}, \Omega\right)$, with $\hat{\Theta}$ and Ω given by equations (5) and (6), respectively.

After observing its vector of private signals, the posterior of active fund manager j is also Gaussian $\Theta \mid \{\mathbf{P}(\Theta), \mathbf{s}_j\} \sim \mathcal{N}\left(\hat{\Theta}(j), \Omega(j)\right)$ with conditional mean $\hat{\Theta}(j)$ and the conditional variance $\Omega(j)$ summarized by the following two expressions:

$$\hat{\Theta}(j) = \Omega(j)\Omega^{-1}\hat{\Theta} + \Omega(j)\Sigma_j(\mathbf{e}_j)^{-1}\mathbf{s}_j, \tag{7}$$

$$\Omega(j)^{-1} = \Omega^{-1} + \Sigma_j(\mathbf{e}_j)^{-1}. \tag{8}$$

This completes our characterization of learning by investors and fund managers. Having solved for the conditional beliefs of all agents, we next analyze the optimal portfolio and effort policies of fund managers.

3.2 Passive Fund Managers

As discussed in the previous section, we assume that passive fund managers trade to hold the mean-variance efficient portfolio:

$$\omega_{\mathbf{1}}{}^{D} = \frac{1}{\gamma}\left(F\Omega F'\right)^{-1}\left(F\hat{\Theta} - R^{f}\mathbf{P}\right). \tag{9}$$

This portfolio is consistent, for instance, with an optimization over mean-variance preferences in this normal setting. The superscript D indicates that this is the investment portfolio for passive fund managers given information set $\mathcal{F}^{c}$.

3.3 Active Fund Managers

Fund managers must be incentivized since they add value to the investor's portfolio through their hidden, costly acquisition of private information. As such, they can no longer be perfectly monitored since they are free to choose $\mathcal{F}_j-$measurable portfolio strategies, $\omega_1^S = \omega_1^S\left(\mathbf{P}, \mathbf{s}_j\right)$, and $\mathcal{F}^c \subseteq \mathcal{F}_j$. Consequently, it is not generically possible for the investor to invert the private signals $\mathbf{s}_j$ the manager received from the realized portfolio excess payoff $W_2^S - R^f W_0$ to ensure that the manager followed the investor's recommendation contingent on observing signals $\mathbf{s}_j$. What is worse is that, even if the investor could observe $\mathbf{s}_j$ directly, the investor could not ex-post verify that the fund manager exerted the recommended effort level $\mathbf{e}$ to obtain the desired precision of the signals.[16]

These considerations motivate us to consider compensation schedules C_0^S that are contingent on outcomes observable to investors at date 2 and, as such, we consider contracts

[16] In part, the assumption that the variance of private signals is regulated from above in the sup norm by $\frac{1}{M}$ ensures there is asymmetric information even with zero effort, and avoids uninformative prices.

that condition on the realized portfolio return per share of the fund $W_2^S - R^f W_0$ and the realized excess payoffs of the risky assets $\mathbf{f} - R^f \mathbf{P}$, $C_0^S = C_0^S \left(W_2^S - R^f W_0, \mathbf{f} - R^f \mathbf{P} \right)$.[17] We focus on linear contracts and specify the optimal contract C_0^S of the form:

$$C_0^S = \rho_0 + \rho_S \left(W_2^S - R^f W_0 \right) + \rho_R' \left(\mathbf{f} - R^f \mathbf{P} \right). \tag{10}$$

Conditioning compensation on the realized excess payoff of the portfolio potentially helps to align the incentives of the fund manager and investor by giving the manager an equity stake in the portfolio. In our static model, this feature is similar to the fixed fraction of assets under management fee that mutual funds charge in practice, consistent with the recent finding by Ibert et al. (2017) using a unique data set of compensation on Swedish mutual fund managers. In addition, allowing the compensation schedule to vary with observed excess payoffs $\mathbf{f} - R^f \mathbf{P}$ can also improve incentives by providing flexibility for the contract to take into account realized market conditions through $\mathbf{f} - R^f \mathbf{P}$.

Since their effort and portfolio choice are unobservable, active fund managers choose incentive compatible portfolios that solve the inner optimization program (1). Conditional on this portfolio choice, which has both a mean-variance component and a hedge against the excess payoff portion of their contract, they choose their optimal effort to minimize the conditional variance of their excess payoff. This is summarized in Proposition 1.

Proposition 1 *The optimal portfolio of an active manager ω^S is given by:*

$$\omega_1^S(j) = \frac{1}{\gamma_M \rho_S} \left(F \Omega(j) F' \right)^{-1} \left(F \hat{\Theta}(j) - R^f \mathbf{P} \right) - \frac{1}{\rho_S} \rho_R,$$

and the optimal level of effort $\mathbf{e}$ *when the manager's participation constraint does not bind*

[17] We also considered a version in which the compensation contract conditions on the realized return of passive funds W_2^D. Since W_2^D is based on public information, and is exogenous to the choices of any active fund manager, it has no substantive impact on their information acquisition decisions. It does, however, affect the hedging incentives in their portfolio choice. See Kapur and Timmermann (2005) for this type of incentive contract.

satisfies:

$$Diag\left[\left(\Omega^{-1}+M\cdot Id_N+diag\left(\mathbf{e}\right)\right)^{-1}\right]\leq h'\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right)\mathbf{1}_{N\times 1}, \tag{11}$$

where Diag is the diagonal operator. If F is diagonal, then this condition further reduces to

$$\frac{1}{\Omega_{ii}^{-1}+M+\mathbf{e}_i}\leq h'\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right)\ \forall\ i\in\{1,...,N\}\,. \tag{12}$$

When the participation constraint binds, the optimal effort instead satisfies:

$$\begin{aligned}&\frac{1}{2}h\left(\left(\mathbf{e}'\right)'\mathbf{1}_{N\times 1}\right)-\frac{1}{2}\log\left|Id_N+\Sigma_j\left(\mathbf{e}'\right)^{-1}\Omega\right| \\ &=\log\left(-u_0\right)+\gamma_M\rho_0-\log E\left[\exp\left(-\frac{1}{2}\left(\hat{\Theta}-R^fF^{-1}\mathbf{P}\right)'\Omega^{-1}\left(\hat{\Theta}-R^fF^{-1}\mathbf{P}\right)\right)\right]\end{aligned} \tag{13}$$

From Proposition 1, the linear contract induces the active fund manager to take the optimal mean-variance portfolio given its beliefs, with effective risk aversion $\gamma_M\rho_S$, corrected by a hedging position $-\frac{1}{\rho_S}\rho_{\mathbf{R}}$ that takes into account that the manager is exposed to payoff risks $\mathbf{f}-R^f\mathbf{P}$ independent of the return on the portfolio it manages. The optimal level of effort $\mathbf{e}$ from equation (11), when the participation constraint does not bind. is determined only by the second moments of the conditional excess payoff $F\hat{\Theta}\left(j\right)-R^f\mathbf{P}$, and seeks to minimize $\Omega\left(j\right)$, since $\Omega^{-1}+\Sigma_j\left(\mathbf{e}_j\right)^{-1}=\Omega\left(j\right)^{-1}$ is the expression within the trace operator in the condition for optimality. When instead the participation constraint binds, then effort is constrained by the reservation utility u_0, and targets the impact of effort on the level of the active manager's utility rather than its marginal utility. Interestingly, the manager's fixed fee ρ_0 induces more effort in this constrained case.

The correlation structure of asset payoffs F induces substitutability in learning across asset fundamentals Θ for active fund managers, in addition to the ex post correlation in beliefs captured in Ω. Active fund managers choose their effort recognizing that learning about asset-specific fundamental $\theta_i, i\in\{2,3,...,N\}$ also reveals information about the aggregate fundamental θ_1 through prices, which further reveals information about the other asset-specific fundamentals θ_j for $j \neq i$. In the special case that F is diagonal, the FOC for the optimal effort from Proposition 1 reduces to equation (12). As one can see from equation (12), the benefit to the active fund manager for increasing effort becomes separable across assets $\frac{1}{\Omega_{ii}^{-1}+M+\mathbf{e}_i}$. With (weakly) convex costs to exerting effort, it then makes sense for the active fund manager to allocate all its attention to the asset that reduces the conditional variance of its excess payoff the most. Consequently, one would expect corner solutions to the active fund manager's optimal effort problem when F is diagonal, and for the active manager to allocate its attention to the fundamentals for which he is able to equate its marginal benefit of learning with the marginal cost.

Having characterized the optimal policies of passive and active fund managers, we solve for equilibrium asset prices by imposing market clearing. Appendix A.1 contains the solution of equilibrium asset prices.

We can then examine how different components of the linear contract ρ_0, ρ_S, and ρ_R impact the information acquisition choice of active fund managers. Substituting equation of equilibrium asset prices (A1) into equation (11) from Proposition 1, we can find the equilibrium level of effort exerted by active fund managers in a symmetric equilibrium:

$$Diag\left[\left(\begin{array}{c} (\tau_\theta + M)\cdot Id_N + \tau_x\left(\frac{\chi}{\gamma_M\rho_S}\right)^2 (M\cdot Id_N + diag(\mathbf{e}))\cdot \\ (F'F)^{-1}(M\cdot Id_N + diag(\mathbf{e})) + diag(\mathbf{e}) \end{array}\right)^{-1}\right] \leq \; h'(\mathbf{e}'\mathbf{1}_{N\times 1}). \qquad (14)$$

With equation (14) characterizing the optimal effort of the active fund manager in equilibrium, we can perform several comparative statics to understand how optimal effort changes with different features of the economic environment, taking into account that changes in effort at the industry level impact the informational content of prices. These comparative statics are summarized in Proposition 2.

Proposition 2 *The optimal choice of active fund manager effort* $\mathbf{e}$*, in equilibrium, is increasing (element-by-element) in the coefficient of active manager risk aversion,* γ_M*, and the industry sensitivity of active manager compensation to the realized portfolio return,* ρ_S*. It is decreasing in the precision of the prior on* Θ, τ_θ*, the precision of the prior on the liquidity trading* $\mathbf{x}, \tau_x$*, and the fraction of active fund managers,* χ*.*

From Proposition 2, the industry sensitivity of the active manager's compensation, in equilibrium, increases the effort that each active fund manager exerts to learn about the payoffs of risky assets. The more that each active manager's compensation depends on the excess payoff of its fund, the less private information is incorporated into asset prices, and the more incentive each active manager has to acquire information to improve the fund's performance. Similarly, the more risk-averse are active managers (higher γ_M), the less aggressively they trade on their private information, and the less informative are asset prices, which increases the benefit of learning. Consistent with this substitution between public and private information, the more uncertain the economic environment (lower τ_θ, τ_x, and χ), the more effort each active manager exerts.

Importantly, it is the industry sensitivity of the active fund manager's compensation to the fund's return ρ_S that determines how each active manager's contract impacts the managerial incentives to acquire private signals, along with the active manager's risk aversion γ_M and parameters that characterize the conditional uncertainty of asset payoffs given prices. It is through equilibrium price formation, consequently, that the contract incentives indirectly impact the information acquisition decisions of active managers. This is one dimension of the contract externality that we wish to highlight.

3.4 The Optimal Affine Contract

We now focus on the optimal linear contract C_0^S that the active fund advisory company offers to investors. Having solved for the determinants of optimal effort of active fund managers, in equilibrium, we provide a characterization of the optimal linear contract, which is summarized in Proposition 3.

Proposition 3 *The optimal affine contract for an active fund manager is a* $N+2\times 1$ *vector* (ρ_0, ρ_S, ρ_R) *with the following properties: 1) The optimal choice of* ρ_R *is given by:*

$$\rho_R = -\left(\rho_S - \frac{\gamma}{\gamma_M}(1-\rho_S)\right)\omega^0,$$

where

$$\omega^0 = \frac{1}{\gamma}F'^{-1}Var\left(\Theta - R^f F^{-1}\mathbf{P}\right)^{-1}E\left[\Theta - R^f F^{-1}\mathbf{P}\right] = \frac{1}{\gamma}F'^{-1}\left(\Omega_Z + \Omega\right)^{-1}\mu,$$

is the ex ante mean-variance efficient portfolio, 2) the optimal sensitivity to the realized excess payoff of the active manager's fund $\rho_S <= \frac{\gamma}{(\gamma+\gamma_M)}$ *satisfies the FONC (A.4), with equality when* $y_i = 1$*, and 3)* ρ_0 *is given by equation (A.4) and is set such that* $y_i = \chi$*, with all equations given in the Appendix.*

Proposition 3 reveals that the optimal performance-based incentive ρ_S is (weakly) below that of perfect risk-sharing $\frac{\gamma}{(\gamma+\gamma_M)}$. When investors allocate only part of their wealth to active management, $y_i < 1$, then the investor and active manager are asymmetrically exposed to the fund-specific risk of noise in the manager's private information. The manager is exposed to every dollar of the fund's return, while the investor is exposed to only y_i of this dollar. Interestingly, it is this departure from perfect risk-sharing that introduces a role for benchmarking to help align preferences, since when $\rho_S = \frac{\gamma}{(\gamma+\gamma_M)}$, then $\rho_R = \rho_0$.

In our setting, ρ_S depends on equilibrium prices, and this introduces a contracting externality because contract incentives determine prices. Substituting for Ω with equation (A1), $\mathbf{e}_j$ with equation (14), and ρ_R from Proposition 3 into the FONC (A.4), we can solve for the fixed point to find the equilibrium value of ρ_S.

It is instructive to consider the case, in which we fix ρ_S at its value under perfect-risk sharing, $\rho_S = \frac{\gamma}{(\gamma+\gamma_M)}$. Then, $\rho_R = \mathbf{0}$, and:

$$\rho_0 = \frac{(1-y_i)\, Tr\left[A\Sigma_j(\mathbf{e_j})^{-1}\right]}{\gamma} > 0$$

Active managers earn a positive fee to ensure that investors are willing to invest a fraction y_i of their wealth in active management. As y_i approaches 1, investors and active managers become symmetrically exposed to the fund's return, and the fixed fee vanishes. We are then back in the standard framework in which the optimal affine contract features only performance-based incentives with perfect-risk sharing. Given that the effective risk-aversion of the investor is $y_i\gamma$, one may instead expect a risk-sharing rule of $\frac{y_i\gamma}{(y_i\gamma+\gamma_M)}$, and our equilibrium ρ_S is indeed closer to this value.

To help further explore the implications of Proposition 3, we rewrite the contract for an active manager as:

$$\begin{aligned} C_0^S &= \rho_0 + \rho_S \omega_1^S(i)' \left(\mathbf{f} - R^f \mathbf{P}\right) - \left(\rho_S - \frac{\gamma}{\gamma_M}(1-\rho_S)\right)\omega^{0\prime}\left(\mathbf{f} - R^f \mathbf{P}\right) \\ &= \rho_0 + \rho_S\left(\omega_1^S(i)' - \omega^{0\prime}\right)\left(\mathbf{f} - R^f \mathbf{P}\right) + \frac{\gamma}{\gamma_M}(1-\rho_S)\,\omega^{0\prime}\left(\mathbf{f} - R^f \mathbf{P}\right) \end{aligned}$$

The first piece of the contract is a constant fee that ensures that investors are, at the margin, indifferent between investing with active and passive fund managers. The second piece is the manager's compensation based on the fund's performance relative to the ex ante mean-variance portfolio ω^0. The third adjusts compensation by the performance of an index that tracks this passive portfolio. Consequently, compensation beyond a fixed fee is offered for the value added by the active manager over the investment strategy that investors could achieve through direct investment without acquiring any public or private information.

Proposition 3 reveals that the optimal benchmark in our setting is the ex-ante mean-variance portfolio formed at date 0, the date that the contract is signed.[18] Intuitively, this

[18] The ex ante mean-variance portfolio will also be the market portfolio if trading is allowed to occur at $t = 0$, since all investors and managers are initially identical. Consequently, one can view the benchmark as the market portfolio in a CAPM world.

benchmark represents the outside option to investors at the time of contracting. This benchmark is weighted by a factor $-\left(\rho_S - \frac{\gamma}{\gamma_M}\left(1-\rho_S\right)\right)$ that aids in risk-sharing by adjusting for the relative risk aversions of the active manager and investor, and is decreasing in ρ_S. This is similar to the optimal benchmark in van Binsbergen et al. (2008), which features a tilt from the minimum variance portfolio that corrects for differences in risk attitudes between the active fund manager and the delegating CIO. Under certain conditions, this passive portfolio is also featured as an optimal benchmark in Admati and Pfleiderer (1997).[19] Ou-Yang (2003) finds a similar optimal benchmark in the special case when the cost of investing for the manager in their framework is zero, and Dybvig et al. (2010) shows the optimality of the efficient portfolio given the available public information on the market return.

To further understand the impact of incentives on an active fund manager's actions, we rewrite the optimal portfolio choice of the fund manager by substituting for ρ_R:

$$\omega_1^S(j) = \frac{1}{\gamma_M \rho_S}\left(F\Omega(j)F'\right)^{-1}\left(F\hat{\Theta}(j) - R^f \mathbf{P}\right) + \left(1 - \frac{\gamma}{\gamma_M}\left(\frac{1}{\rho_S} - 1\right)\right)\omega^0. \tag{15}$$

The portfolio of active fund managers essentially has two components: a mean-variance efficient portfolio with respect to the manager's information set, and a hedging position in the benchmark portfolio. Since the compensation of active fund managers is tied to a benchmark, they are effectively endowed with an exposure to it, and take an opposite position to hedge this risk. This benchmark-driven demand causes managers to potentially over-invest in assets that are representative in their benchmark, and we refer to this demand as *hedging* demand. This hedging channel is also featured in Cuoco and Kaniel (2011), Basak and Pavlova (2013), Leippold and Rohner (2012), and Buffa et al. (2014). In contrast to settings in which benchmarking is assumed in the preferences of investors, such as in Basak and Pavlova (2013) and Duarte et al. (2015), in our model, the benchmark enters into

[19] Admati and Pfleiderer (1997) identify the global minimum variance portfolio, tilted by the assets held by investors in separate accounts, as the optimal benchmark in a partial equilibrium setting. Since prices are determined by market-clearing, the benchmark portfolio is, itself, an equilibrium object that depends on the optimal contract.

security selection through the hedging demand of active fund managers.

Our analysis of the optimal effort and portfolio decisions of active managers illustrates that only the performance-based component of compensation rho_S plays a dual role in our equilibrium setting. In addition to facilitating risk-sharing, it also indirectly influences the information acquisition decision of active managers by impacting the informativeness of prices, since prices act as a substitute source of learning to private information. This feeds back into the optimal choice of ρ_S through the FONC (A.4, similar to García and Vanden (2009), yet our setting also features feedback through the impact of the industry ρ_S on each manager's effort. Our analysis highlights that such a fee-setting externality also arises in a principal-agent setting, and influences performance-based incentives in compensation contracts. In contrast, active managers care about benchmarking insofar as it affects their compensation, and this leads to the sterilization of the benchmark in the manager's optimal portfolio. This aids in risk-sharing but, because the benchmark is based on public information, it does not determine the informativeness of prices.

4 Implications for Performance Evaluation

In this section, we investigate the model's implications for performance evaluation. Since both the choice of benchmark portfolios and the skill of active managers are endogenous and vary with respect to the fundamentals, it allows us to offer empirical predictions without conditioning on actual compensation contracts and observing managerial effort. By relating characteristics of the asset fundamentals to potentially observable fund outcomes such as their holdings and performance through the incentive contracts, our model also provides the theoretical link between the unobservable effort (skill) of the fund manager and the cross sections of fund behavior.

We start by considering a numerical example with two assets to illustrate our predictions.

We choose as our baseline specification:

$$F = \begin{bmatrix} 1 & 0 \\ b & \sqrt{1-b^2} \end{bmatrix}, \ \bar{\Theta} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \bar{\mathbf{x}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

In our discussion in this section, we refer to the asset whose payoff depends only on the aggregate fundamental θ_1 as Asset 1, and the asset that also has an asset-specific fundamental θ_2, with loading b on θ_1, as Asset 2. The F matrix is set to ensure that the comparative static of b is implemented keeping the level of uncertainty constant. Finally, we choose the effort function $h(\cdot)$ to be linear in effort $\mathbf{e}'\mathbf{1}_{2\times1}$, $h(\mathbf{e}'\mathbf{1}_{2\times1}) = \mathbf{e}'\mathbf{1}_{2\times1}$, so that the marginal cost of learning is constant. As a result, any substitutability in learning arises from the co-variance structure of asset prices. Although we consider a two-asset example for ease of exposition, we find that our results hold more generically.[20]

4.1 Optimal Effort and Portfolios

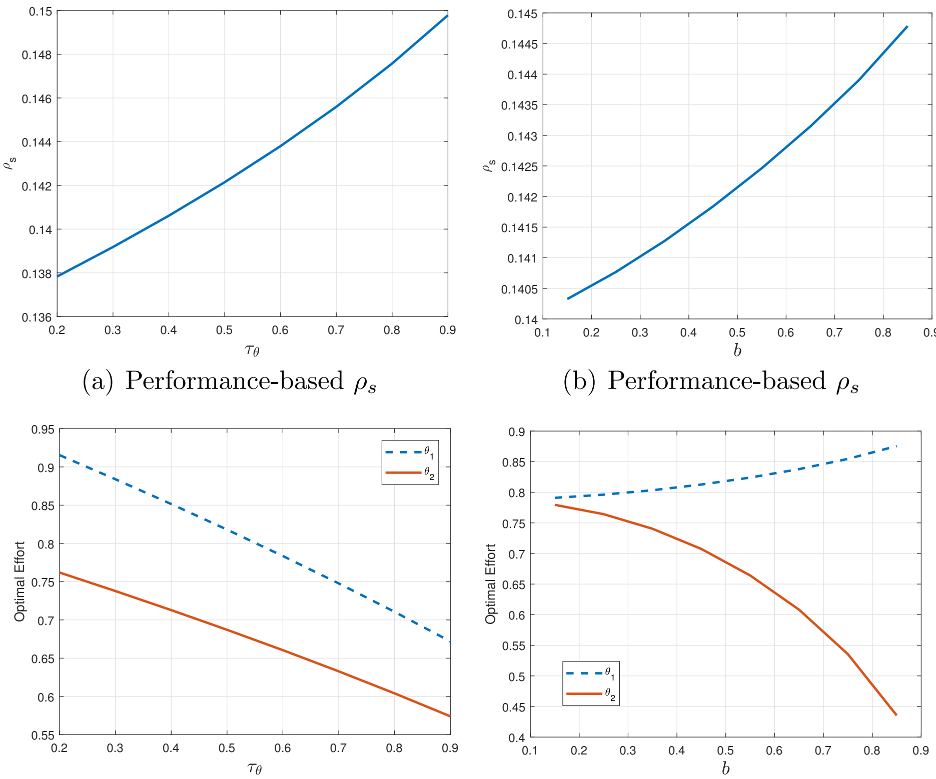
The affine incentive contract has two key features: performance-based incentives ρ_s and benchmarking. Through the equilibrium price formation process, ρ_s determines how the contract impacts each active manager's incentives to acquire private information. Panels (a) and (c) in Figure 2 show that as the ex ante uncertainty of asset payoffs τ_θ^{-1} declines, there are less benefits for the fund manager to acquire information. The optimal contract then puts more weight on the performance-based component ρ_s to provide incentives, and ρ_s increases with respect to τ_θ. Since we allow a general structure for asset payoffs, varying the correlation between asset payoffs also indicates a shift of incentives, as shown in Panels (b) and (d) of Figure 2. As b increases, the individual asset payoffs are more correlated with the aggregate component. Hence, the marginal benefit of learning about asset-specific information is low, while learning about the aggregate component has higher marginal benefit. Since information about aggregate component becomes more important as two assets are more correlated, the

[20]We find qualitatively similar results for a 30-asset case available in Internet Appendix B.

optimal contract provides more incentives to acquire private information by increasing ρ_s.

Figure 2: **Performance-based Compensation and Optimal Effort**

Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1, W_0 = 1, R^f = 1.02$



(a) Performance-based ρ_s

(b) Performance-based ρ_s

(c) Optimal Effort **e**

(d) Optimal Effort **e**

Benchmarking rises endogenously in the optimal contract. To understand the active fund holdings through the hedging channel, we conduct a comparative statics analysis with respect to τ_θ and b. Figure 3 and Figure 4 show the holdings of both active and passive managers, and the composition of the benchmark portfolio. As τ_θ increases, the payoffs of risky assets are less uncertain, and the representation of risky assets in the benchmark portfolio increases. This is because the benchmark is determined based on the risk-return trade off of the assets ex ante. Since active managers' compensation is benchmarked to ω_0, they tend to hedge the benchmark risk by longing[21] the assets that are representative in the benchmark portfolio. This hedging behavior drives up the equilibrium prices of the assets that are representative

[21] ρ_R is negative since the managers are compensated relative to the benchmark.

in the benchmark. The change of correlation structure b impacts the benchmark portfolio as well. As two assets become more correlated, there is less diversification benefit in the ex ante mean-variance portfolio. Hence, as b increases in Figure 4 Panel (b), the holdings of both risky Assets 1 and 2 decline in the benchmark portfolio.

Hedging demand through general equilibrium effects is the key to understanding the implications of benchmarking on the active fund holdings. The expected excess payoffs to the fundamentals $E\left[\Theta - R^f F^{-1}\mathbf{P}\right]$ contain an additional component:

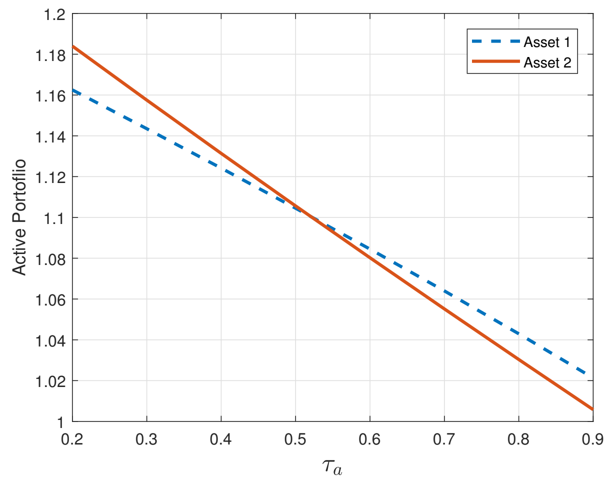
$$-\frac{1}{R^f}\chi\left(1-\frac{\gamma}{\gamma_M}\left(\frac{1}{\rho_S}-1\right)\right)\left(\left(\frac{1-\chi}{\gamma}+\frac{\chi}{\gamma_M\rho_S}\right)\Omega^{-1}+\frac{\chi}{\gamma_M\rho_S}\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)^{-1}F'\omega^0,$$

that reflects the risk of the fund manager's benchmark. If the manager pushes up prices because of hedging demand for assets that are more representative in the benchmark portfolio, this lowers the expected payoff. This effect helps active managers, however, because they are evaluated relative to the performance of the benchmark, which now has a lower expected payoff. This is a general equilibrium effect through which the asset prices that are most inflated by benchmarking are those that are the largest constituents of the benchmark portfolio. As a consequence, active managers both hedge themselves in their portfolios against the benchmark, and benefit from the benchmark's lower expected payoff from their aggregate hedging demand, as shown in Panels (a) and (c) in Figure 3.

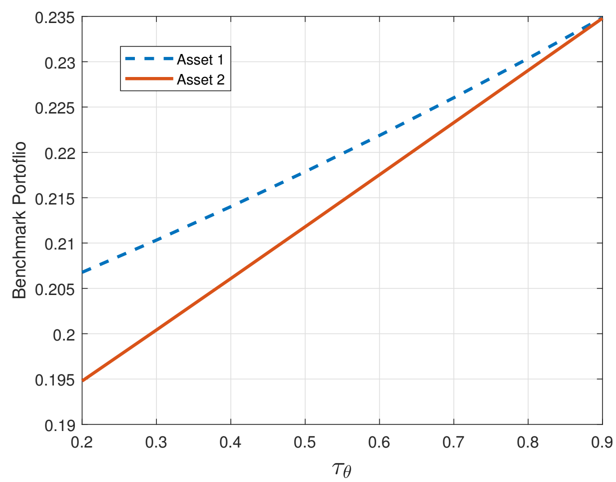
One may wonder what impact benchmarking has on the investment performance of passive managers. Since active fund managers all have a hedging demand for their exposure to the benchmark in their compensation, the liquidity providers for this demand are the passive managers. These passive fund managers are compensated by tilting their portfolios away from the benchmark, and toward assets that offer relatively higher excess payoffs. Consequently, passive fund managers benefit from benchmarking in active manager compensation, since they earn risk premia for insuring active fund managers against their benchmark exposure.

Figure 3: **Portfolios and Benchmark:** τ_θ

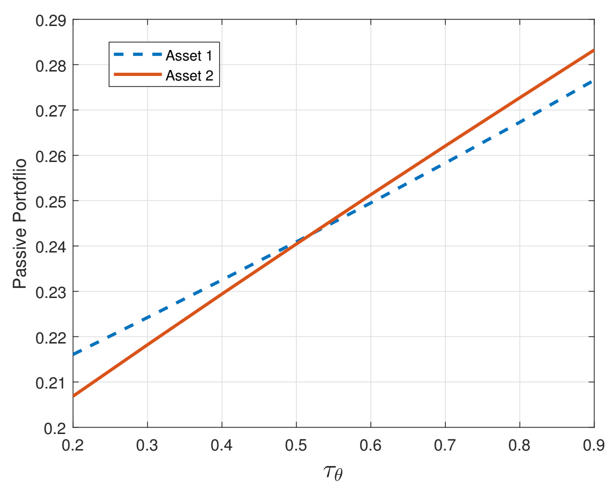
Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1, W_0 = 1, R^f = 1.02$



(a) Active Fund Manager's Portfolio ω_S



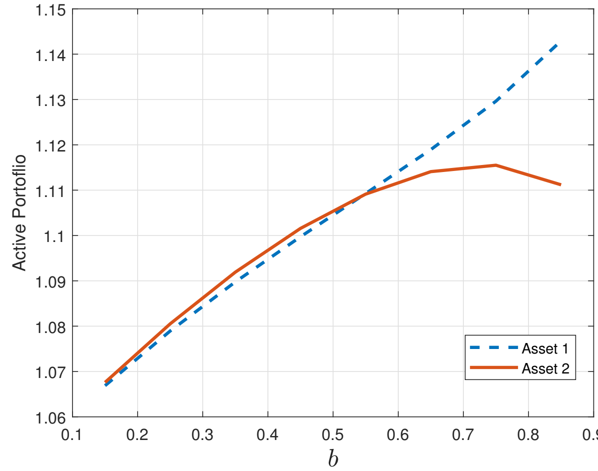
(b) Benchmark Portfolio ω^0



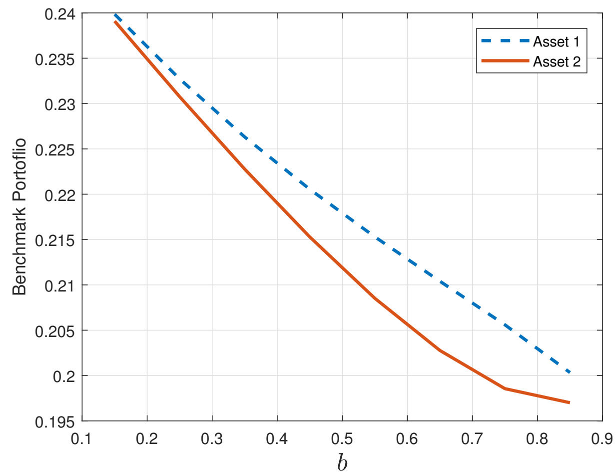
(c) Passive Fund Manager's Portfolio ω_D

Figure 4: **Portfolios and Benchmark:** b

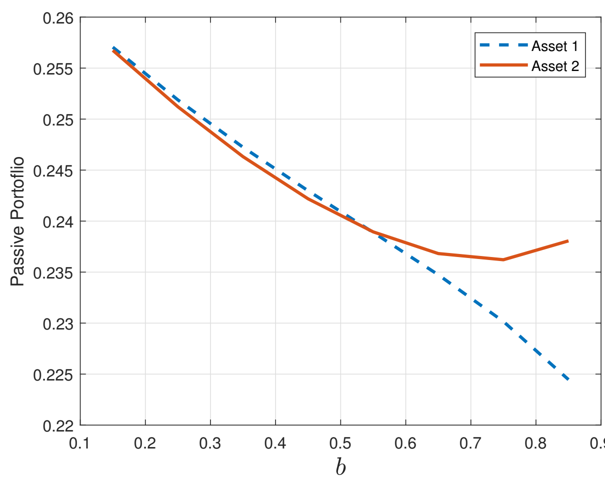
Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1, W_0 = 1, R^f = 1.02$



(a) Active Fund Manager's Portfolio ω_S



(b) Benchmark Portfolio ω^0



(c) Passive Fund Manager's Portfolio ω_D

4.2 Performance Evaluation

We next show our central results of examining the link between our model-implied measure of skills and empirical statistics meant to capture asset manager's ability. In our framework, active manager skill is the optimal effort level given the manager's incentive contract, and we define our model-implied measure as the decrease in uncertainty about asset payoffs $|\Omega| - |\Omega(j)|$.[22] Given the optimal decision, active fund managers choose how much their portfolios deviate from the endogenous benchmark portfolio. To relate the effort exerted by the fund manager to the active share that is first introduced by Cremers and Petajisto (2009), we define our active share as the deviation of an active fund manager's portfolio holdings from the benchmark portfolio. The change in fundamental risk impacts the active share not just from shifting the learning incentives, but also from rebalancing the benchmark portfolio ex ante.

We derive an analogous expression for the average active share of an active fund manager in our economy AS:

$$AS = \frac{1}{2} E\left[\mathbf{1}' \left|\omega_1^S(j) - \omega^0\right|\right],$$

where ω^0 is the benchmark for the active fund manager.[23] Substituting for $\omega_1^S(j)$ with equation (15), ω^0 with Proposition 3, $\hat{\Theta}(j)$ with equation (7), and Π_θ and Π_x with equations (A2) and (A3), respectively, we can employ results for the expectation of a folded normal distribution to arrive at:

$$AS = \frac{1}{2\gamma_M \rho_S} \sum_{i=1}^{N} \left(\sqrt{\frac{2}{\pi}} \sigma_i e^{-\mu_i^2/2\sigma_i^2} + \mu_i \left(1 - 2\Phi\left(-\frac{\mu_i}{\sigma_i}\right)\right)\right),$$

[22] Our results are quantitatively similar if we use other model-implied measures of skill, such as $\mathbf{e_j}$ or $|\Omega - \Omega(j)|$.

[23] One may notice that the definition of active share includes fund leverage. Since the benchmark can also take leveraged positions, this ensures an equitable comparison of portfolios when measuring manager activeness.

where:

$$\mu_i = f_i' \left(\Omega^{-1} + \Sigma_j (\mathbf{e}_j)^{-1} - (1-\rho_S)(\Omega_Z + \Omega)^{-1}\right) \mu,$$
$$\sigma_i^2 = f_i' \left(\Gamma_\theta \tau_\theta^{-1} \Gamma_\theta' + \Gamma_x \tau_x^{-1} \Gamma_x + \Sigma_j (\mathbf{e}_j)^{-1}\right) f_i,$$

where f_i is the i^{th} column of F^{-1}, and

$$\Gamma_\theta = \tau_x \left(\frac{\chi}{\gamma_M \rho_S}\right)^2 \Sigma_j (\mathbf{e}_j)^{-1} (F'F)^{-1} \Sigma_j (\mathbf{e}_j)^{-1} - R^f \left(\Omega^{-1} + \Sigma_j (\mathbf{e}_j)^{-1}\right) F^{-1} \Pi_\theta + \Sigma_j (\mathbf{e}_j)^{-1},$$
$$\Gamma_x = \left(\tau_x \Sigma_j (\mathbf{e}_j)^{-1} (F'F)^{-1} \Sigma_j (\mathbf{e}_j)^{-1} - R^f \left(\frac{\gamma_M \rho_S}{\chi}\right)^2 \left(\Omega^{-1} + \Sigma_j (\mathbf{e}_j)^{-1}\right) F^{-1} \Pi_\theta\right) \Sigma_j (\mathbf{e}_j) F'.$$

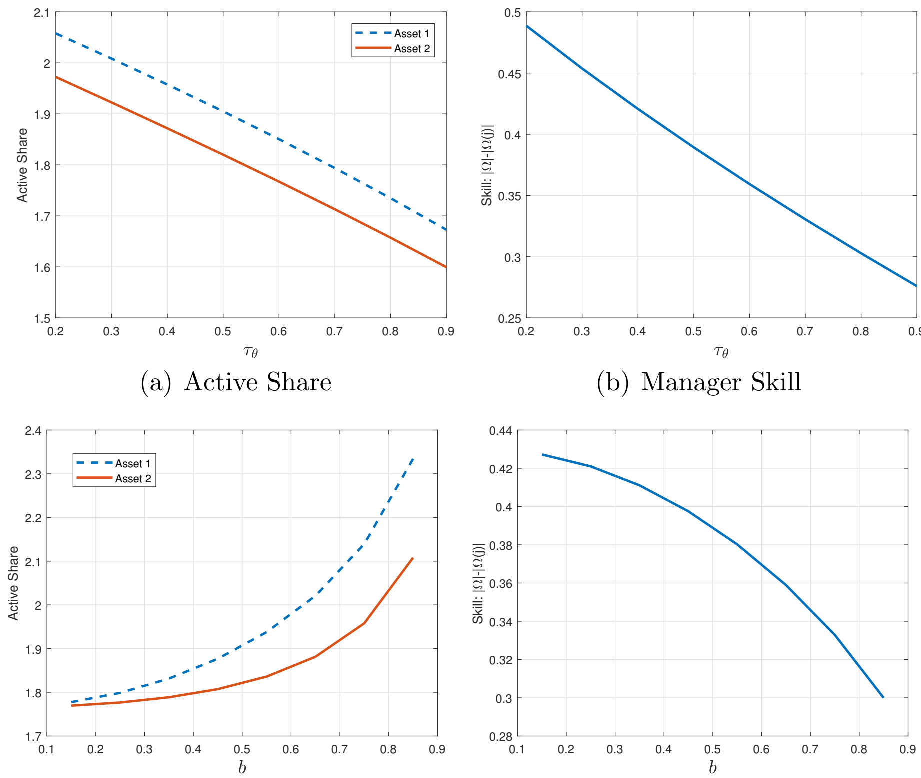
Figure 5 shows the comparative statics of the theoretical active share and the model-implied measure of skill $|\Omega| - |\Omega(j)|$ with respect to the change of τ_θ and b. As the uncertainty of asset payoffs goes up (i.e., τ_θ decreases), the active fund manager exerts more effort to learn and acquires more private information. Hence, both the benefit of learning and the measure of active manager skills increases as τ_θ decreases, as shown in Figure 5 Panel (b).

By acquiring additional private information about asset fundamentals, the active manager's portfolio further deviates from their benchmark portfolio, ω^0. Similarly, when asset fundamentals are less uncertain (higher τ_θ), the manager acquires less private information and takes a more passive position in financial markets.

We then obtain comparative statics for active manager skill as the correlation b between the asset payoff increases (Figure 5 Panel (d)) through the incentive channel. The active shares of the two assets, however, go up. This occurs because the change in the correlation between two assets impacts the ex ante choice of the mean-variance portfolio (i.e., the benchmark portfolio). The benchmark portfolio reduces its weight in both Asset 1 and Asset 2 as there is less benefit from diversification when risky assets are more correlated. Although active managers invest less in Asset 2, which is the asset with both aggregate and asset-specific components in its payoff, the decline in the weight of Asset 2 in the portfolio is not as much

Figure 5: **Active Share and Active Fund Manager Skill: τ_θ & b**

Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1, W_0 = 1, R^f = 1.02$



(a) Active Share (b) Manager Skill

(c) Active Share (d) Manager Skill

as the reduction in the benchmark portfolio.

Our prediction on the relation between a fund's active share and its benchmark also aligns with the critique of Frazzini et al. (2016). The active share measure can deviate from the underlying level of active manager skill, since the incentives for the active manager to learn are not strong enough to dominate the changes in the ex ante choices for benchmark portfolio. The incentives for the active fund manager to acquire private information are shaped by the asset environment, so the effort exerted by the active manager is correlated with its benchmark. While funds that invest in more volatile stocks are and appear more active, funds that invest in more correlated stocks may only appear more active because of their benchmark. Our prediction is also consistent with Jiang and Sun (2014), who studied dispersion in fund managers' beliefs about future stock returns based on their active holdings. The degree of information asymmetry is positively correlated with the dispersion of active mutual funds holdings under our delegated learning channel, since the incentives for active managers to learn rises as the degree of uncertainty of asset payoffs increases, which is proxied by the idiosyncratic volatility of stock returns in Jiang and Sun (2014).

Our setting also allows us to explore another empirical measure for unobservable mutual fund actions, return gap (RG), employed in Kacperczyk et al. (2008). We rewrite the fund manager's portfolio as:

$$\omega_1^S(j) = \frac{\gamma}{\gamma_M \rho_S}\omega_1^D + \left(1 - \frac{\gamma}{\gamma_M}\left(\frac{1}{\rho_S} - 1\right)\right)\omega^0 + \frac{1}{\gamma_M \rho_S} F'^{-1}\Sigma_j \left(\mathbf{e}_j\right)^{-1}\left(\mathbf{s}_j - R^f F^{-1}\mathbf{P}\right).$$

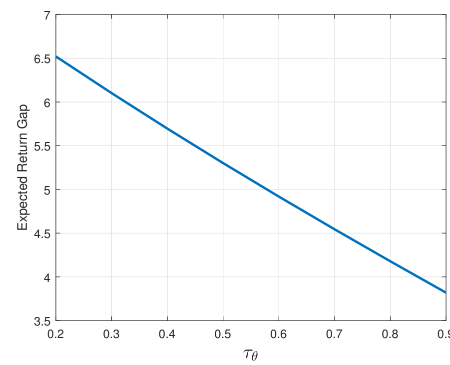
The first two elements reflect the position an active fund manager without private information would take based on public information and the benchmark portfolio, while the last element captures the speculative bet the active fund manager makes based on his informational advantage after observing its private signals. Consequently, we view the first two elements as the "holdings" portfolio that is publicly observable to investors, and measure the expected return gap between the gross return a fund manager garners and that of this "holdings" portfolio, RG. We can then construct the expected return gap:

$$E\left[RG\right] = \frac{1}{\gamma_M \rho_S} \mu' \Sigma_j \left(\mathbf{e}_j\right)^{-1} \mu + \frac{1}{\gamma_M \rho_S} Tr\left[\Sigma_j \left(\mathbf{e}_j\right)^{-1} \left(\Omega_Z + \Omega\right)\right].$$

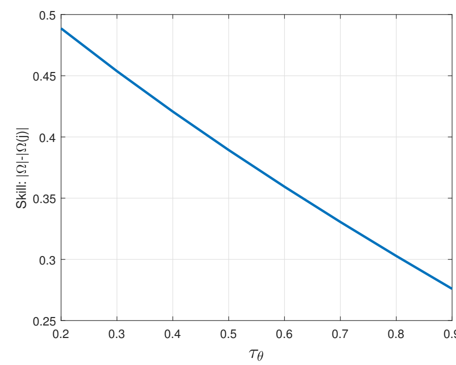
The expected return gap is driven by active managers trading more aggressively to collect the risk premia on assets since they face less risk because of their private information, and from the reduction in overall uncertainty they have when speculating.

Figure 6: **Return Gap and Fund Manager Skill:** τ_θ **&** b

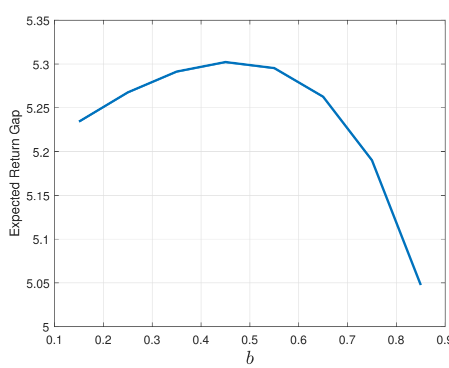
Parameters: $\tau_\theta = 0.5$, $\tau_x = 1$, $b = 0.5$, $\gamma_M = 2$, $M = 0$, $\chi = 0.3$, $\gamma = 1, W_0 = 1, R^f = 1.02$



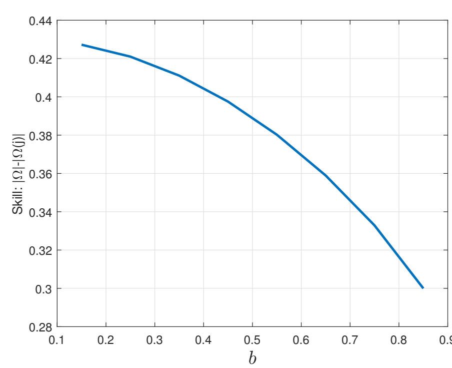
(a) Return Gap



(b) Manager Skill



(c) Return Gap



(d) Manager Skill

Figure 6 shows the comparative statics of the expected return gap and the model-implied measure of fund manager skill. The return gap is moving in the same direction as the skill measure of our model when the uncertainty level of the fundamental τ_θ^{-1} declines. The return gap, however, does not monotonically decrease in the correlation of asset fundamentals. As the correlation between asset fundamentals increases, there is less overall benefit to learning since the asset-specific fundamental is less relevant to returns, and the aggregate fundamental is more revealed by prices to both active and passive fund managers. The expected return gap, in contrast, trades off two competing forces. On the one hand, there is increased risk in asset markets because a higher correlation among asset returns reduces the diversification benefit to holding both assets. Since they have access to private information, active fund managers take larger exposures to the risky assets than passive managers. Hence, the risk premia earned by fund managers who bear more risk is higher through learning. On the other hand, the increased correlation also reveals more information to the passive managers about the aggregate asset fundamental, reducing the information asymmetry between active and passive fund managers. These two forces contribute to the humped-shaped return gap in Panel (c) in Figure 6.

We can also evaluate fund manager performance by computing expected excess returns in our setting, and compare them across the benchmark portfolio and both active and passive fund managers. Given the benchmark portfolio ω^0 in Proposition 3, it is straightforward that an active fund manager with portfolio that has an initial wealth W_0 who invests in the benchmark portfolio with final wealth W_2^0 will have an expected excess return:

$$E\left[W_2^0 - R^f W_0\right] = \frac{1}{\gamma}\mu' \left(\Omega_Z + \Omega\right)^{-1} \mu.$$

Making use of properties of chi-squared random variables, and that the trace operator is linear and satisfies $Tr\left[ABC\right] = Tr\left[BCA\right]$, we arrive at the expected excess return for passive managers:

$$E\left[W_2^D - R^f W_0\right] = \frac{1}{\gamma}\mu'\Omega^{-1}\mu + \frac{1}{\gamma}Tr\left[\Omega^{-1}\Omega_Z\right],$$

and similarly for an active fund manager:

$$
\begin{aligned}
E\left[W_2^S - C_0^S - R^f W_0\right] &= E\left[W_2^0 - R^f W_0\right] - \frac{1}{\gamma_M}\left(\frac{1}{\rho_S} - 1\right)\mu'\left(\left(\Omega_Z + \Omega\right)^{-1} + \Omega^{-1} + \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)\mu \\
&\quad - \frac{1}{\gamma_M}\left(\frac{1}{\rho_S} - 1\right) Tr\left[\left(\Omega^{-1} + \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)\Omega_Z + \Sigma_j\left(\mathbf{e}_j\right)^{-1}\Omega\right] - \rho_0.
\end{aligned}
$$

Kacperczyk et al. (2016) highlights that the additional return that an active manager earns arises from their information acquisition decisions. As volatility falls in our setting, the expected excess return of both passive managers and the benchmark portfolio increases, reflecting the decreased uncertainty in investing and the more liquidity that their portfolios provide. In contrast, the expected excess return of active funds falls as their superior information degrades. As one may expect, the first piece of fund returns is the benchmark portfolio's return.

4.3 Decline in Fund Flow and Performance in Active Management

Since 2006, over 90% of U.S. actively managed equity funds failed to beat their benchmark net of fees. During this time, they have also lost fund flows to passive strategies, both domestically and globally. A potential explanation for these phenomena is that there has been a downward trend in the level of skill among active asset managers. As can be seen in (Figure 5 (d)), our model predicts that a higher (pairwise) correlation between assets reduces the level of effort that active managers exert to acquire private information. Cotter et al. (2016), among others, document a pronounced increase in the level of integration within and among assets classes, and across countries, since the 2008 financial crisis. Such an increase in asset correlations can, consequently, explain why fund managers have under-performed in recent years if their compensation contracts did not adjust to the new asset environment. Consistent with this view, much of the recent debate has been about reforming the incentive structure for the asset management industry.

5 Measuring Skill: The Dynamic Extension

In this section, we discuss a dynamic extension of our model in which fund managers trade over multiple periods. Further details of the model, its derivation, and a more thorough discussion of the results are in the Internet Appendix A. In what follows, we describe the salient features of the analysis.

We use the multi-period model to generalize several of our insights to a dynamic setting. As is endemic to dynamic portfolio choice problems, the portfolio allocation decisions of managers and direct investors not only reflect a speculative component based on the mean and variances of asset payoffs, but also an inter-temporal hedging motive to insure against future fluctuations in the payoff environment. Novel to our setting is that the effort choices of fund managers are now also forward-looking. Whereas in the static setting, fund managers seek to minimize the conditional variance of their portfolio excess payoff through their information acquisition decision, in the dynamic setting managers take into account the benefits of learning in early versus later periods for the same level of dis-utility from exerting effort. Fund investors take into account these inter-temporal incentives when choosing the optimal affine contract to offer to fund managers.

An important point of departure of the multi-period model from its static counterpart is the possibility that investors can observe a time series of fund manager performance during the intermediate trading periods. In practice, the SEC N-Q and N-CSR filings, which are publicly available, require large mutual funds to report all long positions held at the end of a quarter. This potentially enables investors to improve their monitoring of fund managers' behavior. Based on the availability of the SEC N-Q and N-CSR filings for mutual funds, we allow investors to observe an unbiased but noisy measure of their fund's return gap at each date t. This noise, which we assume is i.i.d. across assets and dates, can be thought of as portfolio rebalancing driven by non-fundamental, non-informational reasons or measurement error. Let R_t^i be this noisy observation. Given their observations of past asset and fund-specific returns, investors can form their posterior beliefs about their fund manager's private information conditional on a given path of effort $\{\mathbf{e}_t(i)\}_{s=1}^{T}$. They can then derive a log-likelihood ratio under the null hypothesis that their manager exerts no effort $H_0 : \{\mathbf{e}_t(i)\}_{s=1}^{T} = \{\mathbf{0}_{N\times 1}\}_{s=1}^{T}$, which corresponds to no exhibition of ability. We show that the weighted historical variance of this return gap S_T is related to the log-likelihood ratio of the null hypothesis of no effort to an alternative hypothesis of some schedule of effort:

$$S_T = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\left(R_t^i\right)^2}{Var_U\left[R_t^i \mid \mathcal{F}_t^c\right]} - \frac{\left(R_t^i - E\left[R_t^i \mid \mathcal{F}_t^I(i)\right]\right)^2}{Var\left[R_t^i \mid \mathcal{F}_t^I(i)\right]}\right),$$

and that it is a consistent estimator of whether or not the fund manager exhibits skills. While Admati and Pfleiderer (1997) demonstrate that the first moment of benchmark-adjusted returns is not sufficient to identify skill, our analysis suggests that, instead, investors should focus on second moments when evaluating the skill of fund managers. In the above statistic, $Var_U\left[R_t^i \mid \mathcal{F}_t^c\right]$ is the conditional variance of the return gap if the manager exerts no effort to learn or has no skill, $E\left[R_t^i \mid \mathcal{F}_{t-1}^I(i)\right]$ is the best predictor of the return gap given all public information available to investors, including current realized asset returns, and past fund returns, and $Var\left[R_t^i \mid \mathcal{F}_t^I(i)\right]$ is the corresponding conditional variance.

If one believes that the value that fund managers add to asset management is their acquisition of superior private information, then one should examine the variance of their return gap over time, especially since first moments are highly path-dependent. Intuitively, in any given trading period, fund managers may appear more or less active because of noise in their information or time variation in expected returns: however, systematically their portfolios should deviate from their more passive counterparts. This motivates examining dynamic measures of skill, such as the empirical analogue of our S_T statistic, to measure fund manager skill in an active management.

While the S_T statistic would allow for better monitoring of fund managers, since it could relax their IC constraints, such asymmetric payoff provisions are prohibited by the SEC. Though backward-looking as a measure, if investors allocate capital to funds based on the historical S_T statistic, then managers would face forward-looking incentives for effort to achieve a higher S_T statistic, and through this channel it would dynamically enter their compensation. To see this, suppose that a new generation of investors at date T cannot distinguish between delegating investors and direct investors, and allocates their capital to funds in proportion to the signal that the fund manager has skill based on the fund's returns, according to the rule $w(S_T)$:

$$w(S_T) = (S_T - S_{crit,\alpha}) W_0,$$

where $\frac{S_{crit,\alpha}}{\sigma_0} = \Phi^{-1}(1-\alpha)$ is the $\alpha-$level of confidence under the null hypothesis, and $\Phi(\cdot)$ is the CDF of the normal distribution. [24] Interestingly, $w(S_T)$ is increasing and convex in the most recent fund return gap R_T^i through S_T, leading to a short-term convex flow-performance relation. In addition, if the compensation of fund managers is based on their final AUM at date T, so that:

$$C_T^S = \rho_0 + \rho_S \left(W_T^S - \left(R^f\right)^T W_0 + w(S_T) \right) + \rho_R' \sum_{t=1}^{T} \mathbf{R}_t,$$

which is still a symmetric performance contract, then the future flow-performance sensitivity can incentivize forward-looking fund managers to exert effort before date t to raise their fund flows at date T. Such a mechanism suggests that convex flow-performance sensitivity may be a reaction to information about manager skill through this S_T statistic, and nonlinear flow-performance sensitivity could be a tool for completing the contracting space that is restricted in direct compensation by the SEC to linear contracts. Since such implicit compensation schemes can incentivize managers to acquire information, investors may be willing to provide such rewards even though there is no direct benefit to them in attracting future investors.

[24] We assume implicitly that T is large enough that the asymptotic distribution is a reasonable approximation. Under the null hypothesis of no skill, or $R_t^i \sim iid\left(0, Var_U\left[R_t^i \mid \mathcal{F}_t^c\right]\right)$, S_T has a normal asymptotic distribution $\mathcal{N}\left(0, \sigma_0^2\right)$, where σ_0^2 is the variance of S_T when R_t^i is i.i.d.

6 Conclusion

We study an economy in which investors delegate their capital in financial markets to fund managers, and must incentivize active fund managers to exert costly effort to acquire private information about asset payoffs. Our equilibrium analysis features a novel channel by which performance-based incentives feed into the information acquisition decisions of active managers by impacting the informativeness of asset prices. This allows us to study the rich interaction between compensation incentives in the active management industry, in which benchmarking arises endogenously in our framework, and the learning and trading decisions of active managers. Our model cautions the use of existing empirical measures of skill employed in the literature, and offers a new measure that is motivated by a dynamic extension of our model. Our analysis highlights that the skill of active managers is endogenous to both the asset environment and the compensation incentives they face, and offers a potential explanation for the recent shift in fund flows to passive strategies.

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A Appendix

A.1 Equilibrium Asset Prices

Given the asset demand of direct investors and fund managers from equation (9) and Proposition 1, respectively, we are now in a position to derive equilibrium prices. Aggregating the demand of fund managers and direct investors, $\omega_1^S(j)$ and ω_1^D, respectively, the market-clearing condition reveals that:

$$\chi \frac{1}{\gamma_M \rho_S} \left(F\Omega(j) F'\right)^{-1} \left(F \int_0^1 \hat{\Theta}(j)\, di - R^f \mathbf{P}\right) - \chi \frac{1}{\rho_S} \rho_{\mathbf{R}} + (1-\chi) \frac{1}{\gamma} \left(F\Omega F'\right)^{-1} \left(F\hat{\Theta} - R^f \mathbf{P}\right) = \mathbf{x}.$$

Substituting for $\hat{\Theta}$ and $\hat{\Theta}(j)$ with equations (5) and (7), respectively, and imposing the Strong LLN, we find that:

$$\begin{aligned} \mathbf{P} &= \left(\left(\frac{\chi}{\gamma_M \rho_S}\Omega(j)^{-1} + \frac{1-\chi}{\gamma}\Omega^{-1}\right) R^f F^{-1} - \left(\frac{1-\chi}{\gamma} + \frac{\chi}{\gamma_M \rho_S}\right) \tau_x \Pi_\theta' \left(\Pi_x \Pi_x'\right)^{-1}\right)^{-1} \times \\ &\quad \left(\left(\frac{1-\chi}{\gamma} + \frac{\chi}{\gamma_M \rho_S}\right)\left(\tau_\theta \bar{\Theta} - \tau_x \Pi_\theta' \left(\Pi_x \Pi_x'\right)^{-1} \left(\Pi_0 + \Pi_x \bar{\mathbf{x}}\right)\right) + \frac{\chi}{\gamma_M \rho_S} \Sigma_j (\mathbf{e}_j)^{-1} \Theta - F'\mathbf{x} - \frac{\chi}{\rho_S} F' \rho_{\mathbf{R}}\right). \end{aligned}$$

Matching coefficients with the conjectured form of prices (4), and the imposing equation (6), we find that:

$$\Omega^{-1} = \tau_\theta Id_N + \tau_x \left(\frac{\chi}{\gamma_M \rho_S}\right)^2 \Sigma_j (\mathbf{e}_j)^{-1} \left(F'F\right)^{-1} \Sigma_j (\mathbf{e}_j)^{-1}, \tag{A1}$$

and that Π_θ, Π_x, and Π_0 are given by:

$$\Pi_\theta = \frac{1}{R^f} F \left(\begin{array}{c} \tau_\theta \left(\tau_x \left(\frac{\chi}{\gamma_M \rho_S}\right)^2 \Sigma_j (\mathbf{e}_j)^{-1} (F'F)^{-1} \Sigma_j (\mathbf{e}_j)^{-1} + \left(1 + \frac{1-\chi}{\chi}\frac{\gamma_M}{\gamma}\rho_S\right)^{-1} \Sigma_j (\mathbf{e}_j)^{-1}\right)^{-1} \\ + Id_N \end{array} \right)^{-1} \tag{A2}$$

$$\Pi_x = -\frac{\gamma_M \rho_S}{\chi} \Pi_\theta \Sigma_j (\mathbf{e}_j) F' \tag{A3}$$

$$\Pi_0 = \frac{1}{R^f} F \left(\frac{\chi}{\gamma_M \rho_S}\Omega(j)^{-1} + \frac{1-\chi}{\gamma}\Omega^{-1}\right)^{-1} \left(\begin{array}{c} \left(\frac{1-\chi}{\gamma} + \frac{\chi}{\gamma_M \rho_S}\right)\left(\tau_\theta \bar{\theta} - \tau_x \Pi_\theta' \Pi_x'^{-1} \bar{\mathbf{x}}\right) \\ -\frac{\chi}{\rho_S} F' \rho_{\mathbf{R}} \end{array} \right), \tag{A4}$$

which confirms the conjectured linear equilibrium.

Several features of the equilibrium are immediately apparent from the price coefficients.

We see, for instance, that if $\Sigma_j(\mathbf{e}_j)^{-1}$ is zero, so that fund managers have no private information, then $\Pi_\theta, \Pi_x \to 0_{N\times N}$, and prices reflect only prior information about the risky asset payoffs. In addition, the signal-to-noise ratio of prices as signals about the risky asset payoffs, $\Pi_x^{-1}\Pi_\theta = -\frac{\chi}{\gamma_M\rho_S}F'^{-1}\Sigma_j(\mathbf{e}_j)^{-1}$, depends not only on the correlation structure of asset payoffs and the effort exerted by fund managers, but also negatively on their risk aversion γ_M and the sensitivity of their compensation to the realized return of their fund, ρ_S. That these latter two features enter as $\gamma_M\rho_S$ highlights that ρ_S makes the fund manager effectively more risk-averse over his fund's performance, and, as a result, more conservative in his investment policies.

A.2 Proof of Proposition 1

Assuming the linear contract, the IC constraint of the fund manager, conditional on an effort choice $\mathbf{e}$, reduces to the mean-variance optimization problem:

$$\sup_{\omega_1^S(j)}\left\{\begin{array}{c}\rho_0+\rho_S\omega_1^S(j)'\left(F\hat{\Theta}(j)-R^f\mathbf{P}\right)+\rho_{\mathbf{R}}'\left(F\hat{\Theta}(j)-R^f\mathbf{P}\right)\\ -\frac{\gamma_M}{2}\left(\rho_S\omega_1^S(j)+\rho_{\mathbf{R}}\right)'F\Omega(j)F'\left(\rho_S\omega_1^S(j)+\rho_{\mathbf{R}}\right)\end{array}\right\},$$

given its CARA-normal structure. It then follows from the FOC for ω_1^S at interior solution that:

$$\omega_1^S(j)=\frac{1}{\gamma_M\rho_S}\left(F\Omega(j)F'\right)^{-1}\left(F\hat{\Theta}(j)-R^f\mathbf{P}\right)-\frac{1}{\rho_S}\rho_{\mathbf{R}}.$$

Substituting this optimal portfolio choice into the manager's utility, the IC constraint when choosing effort level $\mathbf{e}$ becomes:

$$\mathbf{e}\in\operatorname{argsup}_{\mathbf{e}_j\in\mathbb{R}_+^N}\left\{E\left[-\exp\left(\begin{array}{c}H(\mathbf{e}_j)-\gamma_M\rho_0\\ -\frac{1}{2}\left(\hat{\Theta}(j)-R^fF^{-1}\mathbf{P}\right)'\Omega(j)^{-1}\left(\hat{\Theta}(j)-R^fF^{-1}\mathbf{P}\right)\end{array}\right)\right]\right\}.$$

To solve for the optimal level of effort for fund managers, we invoke the law of iterated expectations and first find the expected utility of a fund manager conditional on having observed market prices. The optimal choice of effort conditional on having observed market prices is independent of the specific realization of prices. As a result, the optimal effort of fund managers conditional on observing prices is also a measurable strategy for fund managers before observing prices. Since unconditional strategies cannot improve on strategies that condition on more information, this optimal effort ex-post must also be optimal ex-ante.

Recognizing that $\mathbf{s}(j)\mid\mathcal{F}_0^c\sim\mathcal{N}\left(\hat{\Theta},\Omega+\Sigma_j(\mathbf{e}_j)\right)$, and that

$$\hat{\Theta}(j) - R^f F^{-1}\mathbf{P} = \hat{\Theta} - R^f F^{-1}\mathbf{P} + \Omega(j)\,\Sigma_j(\mathbf{e}_j)^{-1}\left(\mathbf{s}(j) - \hat{\Theta}\right),$$

where

$$\Omega(j)^{-1} = \Omega^{-1} + \Sigma_j(\mathbf{e}_j)^{-1},$$

by completing the square for normal random variables, the expected utility of fund manager i given only the market beliefs and effort $\mathbf{e}'$ $E\left[\sup_{\omega\in\mathbb{R}\mathbb{R}^N} E\left[u\left(C_0^S;\omega',\mathbf{e}'\right) \mid \mathcal{F}_j\right] \mid \mathcal{F}^c\right]$ is

$$\begin{aligned}
& E\left[\sup_{\omega\in\mathbb{R}^N} E\left[u\left(C_0^S;\omega',\mathbf{e}'\right) \mid \mathcal{F}_j\right] \mid \mathcal{F}_0^c\right] \\
= & -E\left[\exp\begin{pmatrix} \frac{1}{2}h\left((\mathbf{e}')'\mathbf{1}_{N\times 1}\right) - \gamma_M\rho_0 \\ -\frac{1}{2}\left(\hat{\Theta}(j) - R^f F^{-1}\mathbf{P}\right)'\Omega(j)^{-1}\left(\hat{\Theta}(j) - R^f F^{-1}\mathbf{P}\right) \end{pmatrix} \mid \mathcal{F}_0^c\right] \\
= & -\frac{(2\pi)^{-\frac{N}{2}}}{\left|\Omega + \Sigma_j(\mathbf{e}_j)\right|^{1/2}}\int_{-\infty}^{\infty}\exp\begin{pmatrix} \frac{1}{2}h\left((\mathbf{e}')'\mathbf{1}_{N\times 1}\right) - \gamma_M\rho_0 \\ -\frac{1}{2}\left(\hat{\Theta}(j) - R^f F^{-1}\mathbf{P}\right)'\Omega(j)^{-1}\left(\hat{\Theta}(j) - R^f F^{-1}\mathbf{P}\right) \\ -\frac{1}{2}\left(\mathbf{s}(j) - \hat{\Theta}\right)'\left(\Omega + \Sigma_j(\mathbf{e}_j)\right)^{-1}\left(\mathbf{s}(j) - \hat{\Theta}\right) \end{pmatrix} d\mathbf{s}(j)
\end{aligned}$$

$$\begin{aligned}
= & -\frac{(2\pi)^{-\frac{N}{2}}}{\left|\Omega + \Sigma_j(\mathbf{e}_j)\right|^{1/2}}\int_{-\infty}^{\infty}\exp\begin{pmatrix} \frac{1}{2}h\left((\mathbf{e}')'\mathbf{1}_{N\times 1}\right) - \gamma_M\rho_0 \\ -\frac{1}{2}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)'\Omega^{-1}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right) \\ -\frac{1}{2}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P} + \mathbf{s}(j) - \hat{\Theta}\right)'\Sigma_j(\mathbf{e}_j)^{-1}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P} + \mathbf{s}(j) - \hat{\Theta}\right) \end{pmatrix} d\mathbf{s}(j) \\
= & -\exp\begin{pmatrix} \frac{1}{2}h\left((\mathbf{e}')'\mathbf{1}_{N\times 1}\right) - \frac{1}{2}\log\left|Id_N + \Sigma_j(\mathbf{e}')^{-1}\Omega\right| - \gamma_M\rho_0 \\ -\frac{1}{2}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)'\Omega^{-1}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right) \end{pmatrix}.
\end{aligned}$$

A similar result can be found by applying results for the moment-generating function of the non-central chi-square random variables. As one can see, the optimal choice of effort enters the conditional expected utility only through the $-\frac{1}{2}\log\left|Id_N + \Sigma_j(\mathbf{e}_j)^{-1}\Omega\right|$ term. Since fund managers are price-takers and the conditional variance of market beliefs Ω is known ex-ante, we find that:

$$\begin{aligned}
E\left[\sup_{\omega\in\mathbb{R}^N} E\left[u\left(C_0^S;\omega',\mathbf{e}'\right)\right]\right] = & -\exp\left(\frac{1}{2}h\left((\mathbf{e}')'\mathbf{1}_{N\times 1}\right) - \frac{1}{2}\log\left|Id_N + \Sigma_j(\mathbf{e}')^{-1}\Omega\right| - \gamma_M\rho_0\right) \\
& \times E\left[\exp\left(-\frac{1}{2}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)'\Omega^{-1}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)\right)\right].
\end{aligned}$$

We first consider the case when the participation constraint does not bind. The optimization program for the effort of fund manager is then equivalent to:

$$\mathbf{e} \in \text{argsup}_{\mathbf{e}' \in \mathbb{R}^N_+} \left\{ \log \left| \Omega^{-1} + \Sigma_j \left(\mathbf{e}'\right)^{-1} \right| - h\left(\left(\mathbf{e}'\right)' \mathbf{1}_{N\times 1} \right) \right\}.$$

Recognizing that $\Sigma_j \left(\mathbf{e}_j\right)^{-1} = M \cdot Id_N + diag\left(\mathbf{e}\right)$, and invoking results of the matrix calculus, the FOC for the optimal level of effort $\mathbf{e}_i$ is:

$$Tr\left[\left(\Omega^{-1} + M \cdot Id_N + diag\left(\mathbf{e}\right) \right)^{-1} J_i \right] - h'\left(\mathbf{e}'\mathbf{1}_{N\times 1} \right) \leq 0 \;\; (= \;\; if \;\; e_i > 0).$$

where J_i is the $N \times N$ matrix with entry $J_{ii} = 1$ and zero otherwise. Since Tr is a linear operator, we can stack all the FOCs to arrive at:

$$Diag\left[\left(\Omega^{-1} + M \cdot Id_N + diag\left(\mathbf{e}\right) \right)^{-1} \right] - h'\left(\mathbf{e}'\mathbf{1}_{N\times 1} \right) \mathbf{1}_{N\times 1} \leq \mathbf{0}_{N\times 1},$$

where $Diag$ is the operator that stacks the diagonal of a matrix into a vector. Furthermore, the second-order derivative of $\log \left| \Omega^{-1} + \Sigma_j \left(\mathbf{e}'\right)^{-1} \right|$ is:

$$\partial^2_{\mathbf{e}_i \mathbf{e}_i} \log \left| \Omega^{-1} + \Sigma_j \left(\mathbf{e}'\right)^{-1} \right| = -\left(\Omega^{-1} + M \cdot Id_N + diag\left(\mathbf{e}\right) \right)^{-1} J_i \left(\Omega^{-1} + M \cdot Id_N + diag\left(\mathbf{e}\right) \right)^{-1}.$$

Since $h'\left(\cdot\right)$ is a (weakly) convex function, the optimization program is concave in $\mathbf{e}$, and therefore the FOC is both necessary and sufficient for the optimal $\mathbf{e}$.

If F is diagonal, so that asset payoffs are independent, then Ω^{-1} is also diagonal, and the above condition reduces to:

$$\frac{1}{\Omega^{-1}_{ii} + M + \mathbf{e}_i} \leq h'\left(\mathbf{e}'\mathbf{1}_{N\times 1} \right) \;\; \forall \; i \in \{1, ..., N\}.$$

When instead the participation constraint binds, then effort is chosen such that the level of active manager utilty is u_0, or:

$$\begin{aligned} \frac{1}{2} h\left(\left(\mathbf{e}'\right)' \mathbf{1}_{N\times 1} \right) - \frac{1}{2} \log \left| Id_N + \Sigma_j \left(\mathbf{e}'\right)^{-1} \Omega \right| &= \log\left(-u_0\right) + \gamma_M \rho_0 \\ - \log E\left[\exp\left(-\frac{1}{2} \left(\hat{\Theta} - R^f F^{-1} \mathbf{P} \right)' \Omega^{-1} \left(\hat{\Theta} - R^f F^{-1} \mathbf{P} \right) \right) \right] \end{aligned} \tag{A5}$$

A.3 Proof of Proposition 2

Define

$$G = Tr\left[X^{-1} J_i \right] - h'\left(\mathbf{e}'\mathbf{1}_{N\times 1} \right) = 0,$$

where $X = \left((\tau_\theta + M) \cdot Id_N + k \left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right) \left(F'F\right)^{-1} \left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right) + diag\left(\mathbf{e}\right)\right) J_{ii}$, and $k = \tau_x \left(\frac{\chi}{\gamma_M \rho_S}\right)^2$. By the implicit function theorem,

$$\partial_z \mathbf{e}_i = -\frac{\partial_z G}{\partial_{\mathbf{e}_i} G},$$

for parameter z. Recognizing that $\partial\left(X^{-1}\right) = X^{-1}\left(\partial X\right) X^{-1}$, taking the derivative under the Tr operator since the Tr operator is linear and the trace is bounded, it follows that:

$$\partial_{\mathbf{e}_i} G = Tr\left[AJ_i\right] = \mathbf{v}_i' A \mathbf{v}_i - h''\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right),$$

where J_i is the $N \times N$ matrix with entry $J_{ii} = 1$ and zero otherwise, $\mathbf{v}_i$ is the Euclidian $N \times N$ basis vector in the i^{th} direction, and

$$\begin{aligned} A &= -X^{-1}\left(k\left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right) F^{-1} F'^{-1} + k\left(F'F\right)^{-1}\left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right) + Id_N\right) X^{-1} \\ &\quad -h''\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right) Id_N. \end{aligned}$$

Given that F is a lower triangular matrix with entries of 1 on the diagonal, $F'F$ is a positive definite (PD) matrix since $\det(AB) = \det(A)\det(B)$. Since $F'F$ is a positive definite (PD) matrix, it follows that $\left(F'F\right)^{-1}$ is a PD matrix, since the eigenvalues of $\left(F'F\right)$ are the inverse of the eigenvalues of $F'F$. Since $\left(F'F\right)^{-1}$ is a PD matrix, X is also a PD matrix, and it follows that A is a negative definite (ND) matrix. Therefore, it follows since $\mathbf{v}_i$ has non-negative entries and $h\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right)$ is convex that:

$$\partial_{\mathbf{e}_i} G = \mathbf{v}_i' A \mathbf{v}_i - h''\left(\mathbf{e}'\mathbf{1}_{N\times 1}\right) < 0.$$

Consequently,

$$\partial_z \mathbf{e}_i = \frac{\partial_z G}{\left|\partial_{\mathbf{e}_i} G\right|} = \frac{\partial_z Tr\left[X^{-1}\right]}{\left|\partial_{\mathbf{e}_i} G\right|},$$

and the sign of $\frac{\partial \mathbf{e}_i}{\partial z}$ is the same as the sign of $\partial_z Tr\left[X^{-1}\right]$. Differentiating under the Tr operator again, it follows that:

$$\partial_z G = -Tr\left[X^{-1}\partial_z\left((\tau_\theta + M) \cdot Id_N + k\left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right)\left(F'F\right)^{-1}\left(M \cdot Id_N + diag\left(\mathbf{e}\right)\right)\right) X^{-1} J_i\right].$$

For $z = \tau_\theta$, it is straightforward to verify that $\partial_{\tau_\theta} G > 0$, since $Tr\left[X^{-1}\tau_\theta X^{-1} J_i\right] = \tau_\theta Tr\left[X^{-1} X^{-1} J_i\right]$ and X is PD, and therefore:

$$\partial_{\tau_\theta} \mathbf{e}_i < 0.$$

Similarly, it follows that:

$$\partial_k \mathbf{e}_i < 0.$$

The results for elements of k then follow by the chain rule.

A.4 Proof of Proposition 3

Substituting for W_2^I and C_0^S, the utility of investors is:

$$
\begin{aligned}
& V\left(W_2^I, C_0^S\right) \\
= \quad & -\exp\left(-\gamma\left(\begin{array}{c}
R^f W_0 - y_i\rho_0 + y_i\frac{1-\rho_S}{\gamma_M\rho_S}\left(s_j - \hat{\Theta}\right)'\Sigma_j\left(\mathbf{e}_j\right)^{-1}\left(\Theta - R^f F^{-1}\mathbf{P}\right) \\
+y_i\frac{1}{\rho_S}\left(\frac{1-\rho_S}{\gamma_M}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)'\left(\Omega^{-1} + \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right) - \rho_R' F\right)\left(\Theta - R^f F^{-1}\mathbf{P}\right) \\
+\left(1-y_i\right)\frac{1}{\gamma}\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right)'\Omega^{-1}\left(\Theta - R^f F^{-1}\mathbf{P}\right)
\end{array}\right)\right),
\end{aligned}
$$

Importantly, $\mathbf{e}_j$ is independent of the realization of Θ. To find expected investor utility when investing with fund managers, we recognize by the law of iterated expectations that $E\left[V\left(W_2^I, C_0^S\right)\right] = E\left[E\left[V\left(W_2^I, C_0^S\right) \mid \mathcal{F}^c\right]\right]$, and that $E\left[V\left(W_2^I, C_0^S\right) \mid \mathcal{F}^c\right] = E\left[E\left[V\left(W_2^I, C_0^S\right) \mid \Theta, x\right]\right.$ Taking conditional expectations with respect to the realized shocks, and integrating over the idiosyncratic signal noise of fund managers, we find:

$$
\begin{aligned}
& E\left[V\left(W_2^I, C_0^S\right)|\Theta, x\right] \\
= \quad & -\exp\left(-\gamma\left(\begin{array}{c}
R^f W_0 - y_i\rho_0 + y_i\frac{1-\rho_S}{\gamma_M\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\left(\Theta - \hat{\Theta}\right)'\Sigma_j\left(\mathbf{e}_j\right)^{-1}\left(\Theta - R^f F^{-1}\mathbf{P}\right) \\
+\left(\mathbf{Z}'\left(\left(\left(1-y_i\right)\frac{1}{\gamma}y_i\frac{1-\rho_S}{\gamma_M\rho_S}\right)\Omega^{-1} + y_i\frac{1-\rho_S}{\gamma_M\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right) - y_i\frac{1}{\rho_S}\rho_R' F\right) \\
\times\left(\Theta - R^f F^{-1}\mathbf{P}\right)
\end{array}\right)\right),
\end{aligned}
$$

where $\mathbf{Z} = \hat{\Theta} - R^f F^{-1}\mathbf{P}$. Taking conditional expectations with respect to the market beliefs, we then arrive at:

$$
\begin{aligned}
& E\left[V\left(W_2^I, C_0^S\right)|F^c\right] \\
= \quad & -\frac{\exp\left(\begin{array}{c}
\gamma y_i\rho_0 - \gamma R^f W_0 - \frac{1}{2}\mathbf{Z}'\Omega^{-1}\mathbf{Z} \\
+\frac{1}{2}\left[\mathbf{Z}'\left(1 - y_i + y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S} - 1\right)\Omega^{-1} - y_i\frac{\gamma}{\rho_S}\rho_{\mathbf{R}}' F\right] \\
\times\left[\Omega^{-1} + 2y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right]^{-1} \\
\times\left[\left(1 - y_i + y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S} - 1\right)\Omega^{-1}\mathbf{Z} - y_i\frac{\gamma}{\rho_S}F\rho_{\mathbf{R}}\right]
\end{array}\right)}{\left|Id_N + 2y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right|^{1/2}},
\end{aligned}
$$

From an ex ante perspective, $\mathbf{Z} \sim \mathcal{N}(\mu, \Omega_Z)$. Taking unconditional expectations, we arrive at:

$$
\begin{aligned}
& E\left[V\left(W_2^I, C_0^S\right)\right] \\
= & -\frac{\exp\left(\begin{array}{c} \gamma y_i \rho_0 - \gamma R^f W_0 + \frac{1}{2}\left(y_i \frac{\gamma}{\rho_S}\right)^2 \rho_{\mathbf{R}}' F\left[\Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right]^{-1} F \rho_{\mathbf{R}} \\ + \frac{1}{2} G' Q^{-1} G - \frac{1}{2} \mu' \Omega_Z^{-1} \mu \end{array}\right)}{\left|\Omega_Z\right|^{1/2}\left|Id_N + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Omega \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right|^{1/2}|Q|^{1/2}},
\end{aligned}
$$

where

$$
\begin{aligned}
Q &= \Omega_Z^{-1} + \Omega^{-1} - y_i^2\left(1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right)^2 \Omega^{-1}\left(\Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)^{-1} \Omega^{-1}, \\
G &= y_i^2 \frac{\gamma}{\rho_S}\left(1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Omega^{-1}\left(\Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)^{-1} F' \rho_R + \Omega_Z^{-1} \mu_Z.
\end{aligned}
$$

Investors in fund managers are used to solve the optimization problem:

$$
\begin{aligned}
V_0 &= \sup_{\{\rho_0, \rho_S, \rho_{\mathbf{R}}\}} E\left[V\left(W_2^I, C_0^S\right)\right] \\
s.t. &: \quad Diag\left[\left(\Omega^{-1} + M \cdot Id_N + diag\left(\mathbf{e}_j\right)\right)^{-1} J_i\right] - h'\left(\mathbf{e}_j' \mathbf{1}_{N \times 1}\right) \leq 0 \ \forall\ i \in \{1, ..., N\} \ \text{(optimal } \mathbf{e}_j\text{)}.
\end{aligned}
$$

Importantly, the FOC for the optimal choice of active manager effort is independent of the contract from the perspective of investors.

The FOC for $\rho_{\mathbf{R}}$ is:

$$
\begin{aligned}
0 = & \frac{\gamma}{\rho_S} \rho_{\mathbf{R}}' F \Omega Q + \left(1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \mu' \Omega_Z^{-1} \\
& + y_i^2 \frac{\gamma}{\rho_S}\left(1 - \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right)^2 \rho_{\mathbf{R}}' F\left(\Omega^{-1} + 2 y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)^{-1} \Omega^{-1}
\end{aligned}
$$

which, with some manipulation, gives:

$$
\rho_{\mathbf{R}} = -\left(\frac{\rho_S}{\gamma} - \frac{1-\rho_S}{\gamma_M}\right) F'^{-1}\left(\Omega_Z + \Omega\right)^{-1} \mu.
$$

Recognizing that we can rewrite $F'^{-1}(\Omega_Z + \Omega)^{-1} \mu$ as $\left(F'(\Omega_Z + \Omega) F\right)^{-1} F\mu$, where $F\mu$ is the unconditional expected excess return on the risky assets, it follows that $F'^{-1}(\Omega_Z + \Omega)^{-1} \mu$ is a portfolio allocation chosen before prices are observed that accounts for both the overall uncertainty of excess returns and the uncertainty given common prices Ω augmented by the uncertainty over the realization of prices Ω_Z.

Defining $\omega^0 = \frac{1}{\gamma} F'^{-1} \left(\Omega_Z + \Omega\right)^{-1} \mu$ to represent this "naive" portfolio, we can express $\rho_{\mathbf{R}}$ as:

$$\rho_{\mathbf{R}} = -\left(\rho_S - \frac{\gamma}{\gamma_M}\left(1-\rho_S\right)\right)\omega^0.$$

Furthermore, by the law of total variance:

$$\begin{aligned}
Var\left(\Theta - R^f F^{-1}\mathbf{P}\right) &= E\left[Var\left(\Theta - R^f F^{-1}\mathbf{P} \mid \mathcal{F}^c\right)\right] + Var\left(E\left[\Theta - R^f F^{-1}\mathbf{P} \mid \mathcal{F}^c\right]\right) \\
&= E\left[\Omega\right] + Var\left(\hat{\Theta} - R^f F^{-1}\mathbf{P}\right) \\
&= \Omega + \Omega_Z.
\end{aligned}$$

Therefore, ω^0 can be expressed as:

$$\omega^0 = \frac{1}{\gamma} F'^{-1} Var\left(\Theta - R^f F^{-1}\mathbf{P}\right)^{-1} E\left[\Theta - R^f F^{-1}\mathbf{P}\right],$$

which is the ex-ante mean-variance efficient portfolio.

The optimal choices of ρ_S satisfies the FONC:

$$\rho_S = \arg\sup_{\rho_S}\left\{\begin{array}{c}
-\frac{1}{2}\left(y_i\frac{\gamma}{\rho_S}\right)^2 \rho_{\mathbf{R}}{}' F\left[\Omega^{-1} + 2y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right]^{-1} F'\rho_{\mathbf{R}} \\
+\frac{1}{2}\log\left|Id_N + 2y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - y_i\frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right| + \frac{1}{2}\log\left|Q\right| - \frac{1}{2}G'Q^{-1}G
\end{array}\right\}$$

Applying matrix calculus, we derive the FONC for the optimal choice of ρ_S:

$$0 = \left\{\begin{array}{c}
\frac{1}{\rho_S}\left(y_i\frac{\gamma}{\rho_S}\right)^2 \rho_{\mathbf{R}}{}' F A F' \rho_{\mathbf{R}} \\
-y_i\frac{\gamma}{\gamma_M}\left(y_i\frac{\gamma}{\rho_S^2}\right)^2\left(1 - y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right) Tr\left[AF'\rho_{\mathbf{R}}\rho_{\mathbf{R}}{}' F A \Sigma_j\left(\mathbf{e}_j\right)^{-1}\right] \\
-y_i\frac{1}{\rho_S^2}\frac{\gamma}{\gamma_M}\left(1 - y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right) Tr\left[A\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right] \\
-Tr\left[\frac{y_i^2\gamma\left(1-\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)}{\gamma_M\rho_S^2} Q^{-1}\Omega^{-1}A\Omega^{-1}\left[Id_N + y_i\left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\left(1 - y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}A\Omega^{-1}\right]\right] \\
-Tr\left[\frac{y_i^2\gamma\left(1-\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)}{\gamma_M\rho_S^2} Q^{-1}GG'Q^{-1}\Omega^{-1}A\Omega^{-1}\left[Id_N + y_i\left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\left(1 - y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}A\Omega^{-1}\right]\right] \\
y_i^2\frac{\gamma}{\rho_S}G'Q^{-1}\left[\frac{1}{\rho_S}\left(1 - \frac{\gamma}{\gamma_M}\left(\frac{2}{\rho_S} - 1\right)\right)Id_N - 2y_i\frac{1}{\rho_S^2}\frac{\gamma}{\gamma_M}\left(1 - y_i\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega^{-1}A\Sigma_j\left(\mathbf{e}_j\right)^{-1}\Omega\right]\Omega^{-1}AF'\rho_{\mathbf{R}}
\end{array}\right\}$$

where:

$$A = \left(\Omega^{-1} + 2y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Sigma_j \left(\mathbf{e}_j\right)^{-1}\right)^{-1}$$

$$B' = \left(Id_N + 2y_i \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S} \left(1 - y_i \frac{1}{2} \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\right) \Omega\Sigma_j \left(\mathbf{e}_j\right)^{-1}\right)^{-1} = \Omega^{-1} A$$

Suppose that we conjecture that $\rho_S = \frac{\gamma}{(\gamma+\gamma_M)}$, its value under perfect risk-sharing. Then $\rho_R = \mathbf{0}$, and substituting this into the FOC (A.4), we find that:

$$\frac{d\log(-V_0)}{d\rho_S} = \left\{ -x_0 \frac{(\gamma+\gamma_M)^2}{\gamma\gamma_M} (1-x_0)\, Tr\left[\left(Id_N + 2x_0\left(1 - \tfrac{1}{2}x_0\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right)^{-1} \Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right] \right\} < 0. \tag{A6}$$

Assuming the program for the investor is concave in ρ_S, it then follows that $\rho_S \leq \frac{\gamma}{\gamma+\gamma_M}$.

Assuming the program of investors is concave in y_i, the FOC for y_i is given by:

$$\begin{aligned}
\rho_0 \;=\; & -\frac{1}{\gamma} x_0 \frac{1}{\rho_S^2} \rho_R' F \left[\Omega^{-1} + 2x_0 \frac{\gamma}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - x_0 \frac{1}{2}\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right]^{-1} F'\rho_R \\
& + \frac{1}{\gamma_M} \frac{1-\rho_S}{\rho_S}\left(1 - x_0 \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\left(x_0\left(\frac{\gamma}{\rho_S}\right)^2 Tr\left[AF'\rho_R\rho_R' FA\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right] + Tr\left[A\Sigma_j\left(\mathbf{e}_j\right)^{-1}\right]\right) \\
& + \frac{1}{\gamma} x_0 \left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)^2 Tr\left[H^{-1}\Omega^{-1}A\Omega^{-1}\left[x_0 \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - x_0\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}A\Omega^{-1} - Id_N\right]\right] \\
& + \frac{1}{\gamma} x_0 \left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)^2 Tr\left[H^{-1}GG'H^{-1}\Omega^{-1}A\Omega^{-1}\left[x_0 \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - x_0\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega\Sigma_j\left(\mathbf{e}_j\right)^{-1}A\Omega^{-1} - Id_N\right]\right] \\
& + x_0 \frac{1}{\rho_S}\left(1 - \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right) G'H^{-1}\left[x_0 \frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\left(1 - x_0\frac{\gamma}{\gamma_M}\frac{1-\rho_S}{\rho_S}\right)\Omega^{-1}A\Sigma_j\left(\mathbf{e}_j\right)^{-1}\Omega - Id_N\right]\Omega^{-1}AF'\rho_R
\end{aligned}$$

Since $y_i = \chi$ in equilibrium, this pins down ρ_0.