The Race Between Man and Machine: Implications of Technology for Growth, Factor Shares and Employment*

Daron Acemoglu
MIT

Pascual Restrepo
Boston University

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Abstract

We examine the concerns that new technologies will render labor redundant in a framework in which tasks previously performed by labor can be automated and new versions of existing tasks, in which labor has a comparative advantage, can be created. In a static version where capital is fixed and technology is exogenous, automation reduces employment and the labor share, and may even reduce wages, while the creation of new tasks has the opposite effects. Our full model endogenizes capital accumulation and the direction of research towards automation and the creation of new tasks. If the long-run rental rate of capital relative to the wage is sufficiently low, the long-run equilibrium involves automation of all tasks. Otherwise, there exists a stable balanced growth path in which the two types of innovations go hand-in-hand. Stability is a consequence of the fact that automation reduces the cost of producing using labor, and thus discourages further automation and encourages the creation of new tasks. In an extension with heterogeneous skills, we show that inequality increases during transitions driven both by faster automation and the introduction of new tasks, and characterize the conditions under which inequality stabilizes in the long run.

Keywords: Automation, Directed Technological Change, Economic Growth, Endogenous Growth, Factor Shares, Productivity, Tasks, Technological Change.

JEL Classification: O33, O14, O31, J23, J24.

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1 Introduction

The accelerated automation of tasks performed by labor raises concerns that new technologies will make labor redundant (e.g., Brynjolfsson and McAfee, 2012, Akst, 2014, Autor, 2015). The recent decline in the labor share in national income and the employment to population ratio in the U.S. (e.g., Karabarbounis and Neiman, 2014, and Oberfield and Raval, 2014) are often interpreted as evidence for the claims that, as digital technologies, robotics and artificial intelligence penetrate the economy, workers will find it increasingly difficult to compete against machines, and their compensation will experience a relative or even absolute decline. Yet, we lack a comprehensive framework incorporating such effects, as well as potential countervailing forces.

The need for such a framework stems not only from the importance of understanding how and when automation will transform the labor market, but also from the fact that similar claims have been made, but have not always come true, about previous waves of new technologies. Keynes famously foresaw the steady increase in per capita income during the 20th century from the introduction of new technologies, but incorrectly predicted that this would create widespread technological unemployment as machines replaced human labor (Keynes, 1930). In 1965, economic historian Robert Heilbroner confidently stated that "as machines continue to invade society, duplicating greater and greater numbers of social tasks, it is human labor itself—at least, as we now think of 'labor'—that is gradually rendered redundant"(quoted in Akst, 2014). Wassily Leontief was equally pessimistic about the implications of new machines. By drawing an analogy with the technologies of the early 20th century that made horses redundant, in an interview he speculated that "Labor will become less and less important... More and more workers will be replaced by machines. I do not see that new industries can employ everybody who wants a job"(The New York Times, 1983).

This paper is a first step in developing a conceptual framework to study how machines replace human labor and why this might (or might not) lead to lower employment and stagnant wages. Our main conceptual innovation is to propose a framework in which tasks previously performed by labor are automated, while at the same time other new technologies complement labor—specifically, in our model this takes the form of the introduction of new tasks in which labor has a comparative advantage. Herein lies our answer to Leontief's analogy: the difference between human labor and horses is that humans have a comparative advantage in new and more complex tasks. Horses did not. If this comparative advantage is significant and the creation of new tasks continues, employment and the labor share can remain stable in the long run even in the face of rapid automation.

The importance of new tasks is well illustrated by the technological and organizational changes during the Second Industrial Revolution, which not only involved the replacement of the stagecoach by the railroad, sailboats by steamboats, and of manual dock workers by cranes, but also the creation of new labor-intensive tasks. These tasks generated jobs for engineers, machinists, repairmen, conductors, back-office workers and managers involved with the introduction and operation of new technologies (e.g., Landes, 1969, Chandler, 1977, and Mokyr, 1990).

Today, as industrial robots, digital technologies, computer-controlled machines and artificial intelligence replace labor, we are again witnessing the emergence of new tasks ranging from engineering and programming functions to those performed by audio-visual specialists, executive assistants, data administrators and analysts, meeting planners and social workers. Indeed, during the last 35 years, new tasks and new job titles account for a large fraction of U.S. employment growth. To document this fact, we use data from Lin (2011) to measure the share of *new job titles*—jobs in which workers perform tasks that are different from tasks in previously existing jobs—within each occupational category. In 2000, about 70% of computer software developers (an occupational category employing one million people at the time) held new job titles. Similarly, in 1990 a "radiology technician" and in 1980 a "management analyst" were new job titles. Figure 1 shows that occupations with 10 percentage points more new job titles (which is approximately the sample average in 1980) experienced 0.41% faster employment growth between 1980 and 2015. This estimate implies that about 60% of the 50 million or so jobs added during this 35 year period are associated with the additional employment growth in occupations with new job titles (relative to occupations with no new job titles).[1]

FIGURE 1: Employment growth by occupation between 1980 and 2015 (annualized) and the share of new job titles in 1980.

We start with a static model in which capital is fixed and technology is exogenous. There are two types of technological changes: automation allows firms to substitute capital for tasks

[1]The relationship shown in Figure 1 controls for the demographic composition of employment in the occupation in 1980. In Appendix B, we show that the same relationship holds between the share of new job titles in 1990 (in 2000) and employment growth from 1990 to 2015 (from 2000 to 2015), and that these patterns are present without any controls and when we control for average education in the occupation and the structural changes in the U.S. economy as well. The data for 1980, 1990 and 2000 are from the U.S. Census. The data for 2015 are from the American Community Survey. Additional information on the data and our sample is provided in Appendix B.

previously performed by labor, while the creation of new tasks enables the replacement of old tasks by new variants in which labor has a higher productivity. Our static model provides a rich but tractable framework that clarifies how automation and the creation of new tasks shape the production possibilities of the economy and determine factor prices, factor shares in national income, and employment. Automation always reduces the labor share and employment, and may even reduce wages.[2] Conversely, the creation of new tasks increases wages, employment and the labor share. These comparative statics follow because factor prices are determined by the range of tasks performed by capital and labor, and exogenous shifts in technology alter the range of tasks performed by each factor (see also Acemoglu and Autor, 2011).

We then embed this framework in a dynamic economy in which capital accumulation is endogenous, and we characterize restrictions under which the model delivers balanced growth with automation and creation of new tasks—which we take to be a good approximation to economic growth in the United States and the United Kingdom over the last two centuries. The key restrictions are that there is exponential productivity growth from the creation of new tasks and that the two types of technological changes—automation and the creation of new tasks—advance at equal rates. A critical difference from our static model is that capital accumulation responds to permanent shifts in technology in order to keep the interest rate and hence the rental rate of capital constant. As a result, the dynamic effects of technology on factor prices depend on the response of capital accumulation as well. The response of capital ensures that the productivity gains from both automation and the introduction of new tasks fully accrue to labor (the relatively inelastic factor). Although the real wage in the long run increases because of this productivity effect, automation still reduces the labor share and employment.

Our full model endogenizes the rates of improvement of these two types of technologies by marrying our task-based framework with a directed technological change setup. This full version of the model remains tractable and allows a complete characterization of balanced growth paths. If the long-run rental rate of capital is very low relative to the wage, there will not be sufficient incentives to create new tasks, and the long-run equilibrium involves full automation—akin to Leontief's "horse equilibrium." Otherwise, the long-run equilibrium involves balanced growth based on equal advancement of the two types of technologies. Under natural assumptions, this (interior) balanced growth path is stable, so that when automation runs ahead of the creation of new tasks, market forces induce a slowdown in subsequent automation and more rapid countervailing advances in the creation of new tasks. This stability result highlights a crucial new force: a wave of automation pushes down the effective cost of producing with labor, discouraging further efforts to automate additional tasks and encouraging the creation of new tasks.

[2] The effects of automation in our model contrast with the implications of factor-augmenting technologies. As we discuss in greater detail later and in particular in footnote 18, the effects of factor-augmenting technologies on the labor share depend on the elasticity of substitution between capital and labor. In addition, capital-augmenting technological improvements always increase the wage, while labor-augmenting ones also increase the wage provided that the elasticity of substitution between capital and labor is greater than the capital share in national income. This contrast underscores that it would be misleading to think of automation in terms of factor-augmenting technologies.

The stability of the balanced growth path implies that periods in which automation runs ahead of the creation of new tasks tend to trigger self-correcting forces, and as a result, the labor share and employment stabilize and could return to their initial levels. Whether or not this is the case depends on the reason why automation paced ahead in the first place. If this is caused by the random arrival of a series of automation technologies, the long-run equilibrium takes us back to the same initial levels of employment and labor share. If, on the other hand, automation surges because of a change in the innovation possibilities frontier (making automation easier relative to the creation of new tasks), the economy will tend towards a new balanced growth path with lower levels of employment and labor share. In neither case does rapid automation necessarily bring about the demise of labor.[3]

We also consider three extensions of our model. First, we introduce heterogeneity in skills, and assume that skilled labor has a comparative advantage in new tasks, which we view as a natural assumption.[4] Because of this pattern of comparative advantage, automation directly takes jobs away from unskilled labor and increases inequality, while new tasks directly benefit skilled workers and at first increase inequality as well. Over the long-run, the standardization of new tasks help low-skill workers. We characterizes the conditions under which standardization is sufficient to restore stable inequality in the long run. This extension formalizes the idea that both automation and the creation of new tasks increase inequality in the short run but standardization limits the increase in inequality in the long run.

Our second extension modifies our baseline patent structure and reintroduces the creative destruction of the profits of previous innovators, which is absent in our main model, though it is often assumed in the endogenous growth literature. The results in this case are similar, but the conditions for uniqueness and stability of the balanced growth path are more demanding.

Finally, we study the efficiency properties of the process of automation and creation of new technologies, and point to a new source of inefficiency leading to excessive automation: when the wage rate is above the opportunity cost of labor (due to labor market frictions), firms will choose automation to save on labor costs, while the social planner, taking into account the lower opportunity cost of labor, would have chosen less automation.

Our paper can be viewed as a combination of task-based models of the labor market with directed technological change models.[5] Task-based models have been developed both in the economic growth and labor literatures, dating back at least to Roy's (1955) seminal work. The first important recent contribution, Zeira (1998), proposed a model of economic growth based on capital-labor substitution. Zeira's model is a special case of our framework. Acemoglu and Zilibotti (2000) developed a simple task-based model with endogenous technology and applied it to the study of productivity

[3]Yet, it is also possible that some changes in parameters can shift us away from the region of stability to the full automation equilibrium.

[4]This assumption builds on Schultz (1965) (see also Greenwood and Yorukoglu, 1997, Caselli, 1999, Galor and Moav, 2000, Acemoglu, Gancia and Zilibotti, 2010, and Beaudry, Green and Sand, 2013).

[5]On directed technological change and related models, see Acemoglu (1998, 2002, 2003, 2007), Kiley (1999), Caselli and Coleman (2006), Gancia (2003), Thoenig and Verdier (2003) and Gancia and Zilibotti (2010).

differences across countries, illustrating the potential mismatch between new technologies and the skills of developing economies (see also Zeira, 2006, Acemoglu, 2010). Autor, Levy and Murnane (2003) suggested that the increase in inequality in the U.S. labor market reflects the automation and computerization of routine tasks.[6] Our static model is most similar to Acemoglu and Autor (2011). Our full framework extends this model not only because of the dynamic equilibrium incorporating capital accumulation and directed technological change, but also because tasks are combined with a general elasticity of substitution, and because the equilibrium allocation of tasks depends both on factor prices and the state of technology.[7]

Three papers from the economic growth literature that are related to our work are Acemoglu (2003), Jones (2005), and Hemous and Olsen (2016). The first two papers develop growth models in which the aggregate production function is endogenous and, in the long run, adapts to make balanced growth possible. In Jones (2005), this occurs because of endogenous choices about different combinations of activities/technologies. In Acemoglu (2003), this long-run behavior is a consequence of directed technological change in a model of factor-augmenting technologies. Our task-based framework here is a significant departure from this model, especially since it enables us to address questions related to automation, its impact on factor prices and its endogenous evolution. In addition, our framework provides a more robust economic force ensuring the stability of the balanced growth path: while in models with factor-augmenting technologies stability requires an elasticity of substitution between capital and labor that is less than 1 (so that the more abundant factor commands a lower share of national income), we do not need such a condition in this framework.[8] Hemous and Olsen (2016) propose a model of automation and horizontal innovation with endogenous technology, and use it to study the consequences of different types of technologies on inequality. High wages (in their model for low-skill workers) encourage automation. But unlike in our model, the unbalanced dynamics that this generates are not countered by other types of innovations in the long run. Also worth noting is Kotlikoff and Sachs (2012) who develop an overlapping generation model in which automation may have long-lasting effects. In their model, automation reduces the earnings of current workers, and via this channel, depresses savings and capital accumulation.

The rest of the paper is organized as follows. Section 2 presents our task-based framework in the context of a static economy. Section 3 introduces capital accumulation and clarifies the conditions for balanced growth in this economy. Section 4 presents our full model with endogenous

[6] Acemoglu and Autor (2011), Autor and Dorn (2013), Jaimovich and Siu (2014), Foote and Ryan (2014), Burstein and Vogel (2012), and Burstein, Morales and Vogel (2014) provide various pieces of empirical evidence and quantitative evaluations on the importance of the endogenous allocation of tasks to factors in recent labor market dynamics.

[7] Acemoglu and Autor's model, like ours, is one in which a discrete number of labor types are allocated to a continuum of tasks. Costinot and Vogel (2010) develop a complementary model in which there is a continuum of skills and a continuum of tasks. See also the recent paper by Hawkins, Ryan and Oh (2015), which shows how a task-based model is more successful than standard models in matching the co-movement of investment and employment at the firm level.

[8] The role of technologies replacing tasks in this result can also be seen by noting that with factor-augmenting technologies, the direction of innovation may be dominated by a strong market size effect (e.g., Acemoglu, 2002). Instead, in our model, it is the difference between factor prices that regulates the future path of technological change, generating a powerful force towards stability.

technology and establishes, under some plausible conditions, the existence, uniqueness and stability of a balanced growth path with two types of technologies advancing in tandem. Section 5 considers the three extensions mentioned above. Section 6 concludes. Appendix A contains the proofs of our main results, while Appendix B, which is not for publication, contains the remaining proofs, additional results, and the details of the empirical analysis presented above.

2 Static Model

We start with a static version of our model with exogenous technology, which allows us to introduce our main setup in the simplest fashion and characterize the impact of different types of technological change on factor prices, employment and the labor share.

2.1 Environment

The economy produces a unique final good Y by combining a unit measure of tasks, $y(i)$, with an elasticity of substitution $\sigma \in (0, \infty)$:

$$Y = \widetilde{B}\left(\int_{N-1}^{N} y(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\widetilde{B} > 0$. All tasks and the final good are produced competitively. The fact that the limits of integration run between $N-1$ and N imposes that the measure of tasks used in production always remains at 1. A new (more complex) task replaces or upgrades the lowest-index task. Thus, an increase in N represents the upgrading of the quality (productivity) of the unit measure of tasks.[9]

Each task is produced by combining labor or capital with a task-specific intermediate $q(i)$, which embodies the technology used either for automation or for production with labor. To simplify the exposition, we start by assuming that these intermediates are supplied competitively, and that they can be produced using ψ units of the final good. Hence, they are also priced at ψ. In Section 4 we relax this assumption and allow intermediate producers to make profits so as generate endogenous incentives for innovation.

All tasks can be produced with labor. We model the technological constraints on automation by assuming that there exists $I \in [N-1, N]$ such that tasks $i \leq I$ are *technologically automated* in the sense that it is feasible to produce them with capital. Although tasks $i \leq I$ are technologically automated, whether they will be produced with capital or not depends on relative factor prices as we describe below. Conversely, tasks $i > I$ are not technologically automated, and must be produced with labor.

The production function for tasks $i > I$ takes the form

$$y(i) = \overline{B}(\zeta)\left[\eta^{\frac{1}{\zeta}} q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)^{\frac{1}{\zeta}} \left(\gamma(i) l(i)\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}, \tag{2}$$

[9] This formulation imposes that once a new task is created at N it will be immediately utilized and replace the lowest available task located at $N-1$. This is ensured by Assumption 3 imposed below, and avoids the need for additional notation at this point. We view newly-created tasks as higher productivity versions of existing tasks.

where $\gamma(i)$ denotes the productivity of labor in task i, $\zeta \in (0, \infty)$ is the elasticity of substitution between intermediates and labor, $\eta \in (0,1)$ is the share parameter of this constant elasticity of substitution (CES) production function, and $\overline{B}(\zeta)$ is a constant included to simplify the algebra. In particular, we set $\overline{B}(\zeta) = \psi^{\eta}(1-\eta)^{\eta-1}\eta^{-\eta}$ when $\zeta = 1$, and $\overline{B}(\zeta) = 1$ otherwise.

Tasks $i \leq I$ can be produced using labor or capital, and their production function is identical to (2) except for the presence of capital and labor as perfectly substitutable factors of production:[10]

$$y(i) = \overline{B}(\zeta)\left[\eta^{\frac{1}{\zeta}} q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)^{\frac{1}{\zeta}}\left(k(i) + \gamma(i)l(i)\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}. \tag{3}$$

Throughout, we impose the following assumption:

Assumption 1 $\gamma(i)$ *is strictly increasing*

Assumption 1 implies that labor has strict *comparative advantage* in tasks with a higher index, and will guarantee that, in equilibrium, tasks with lower indices will be automated, while those with higher indices will be produced with labor.

We model the demand side of the economy using a representative household with preferences given by

$$u(C, L) = \frac{(Ce^{-\nu(L)})^{1-\theta} - 1}{1-\theta}, \tag{4}$$

where C is consumption, L denotes the labor supply of the representative household, and $\nu(L)$ designates the utility cost of labor supply, which we assume to be continuously differentiable, increasing and convex, and to satisfy $\nu''(L) + (\theta - 1)(\nu'(L))^2/\theta > 0$ (which ensures that $u(C, L)$ is concave). The functional form in (4) ensures balanced growth (see King, Plosser and Rebelo, 1988; Boppart and Krusell, 2016). When we turn to the dynamic analysis in the next section, θ will be the inverse of the intertemporal elasticity of substitution.

Finally, in the static model, the capital stock, K, is taken as given (it will be endogenized via household saving decisions in Section 3).

2.2 Equilibrium in the Static Model

Given the set of technologies I and N, and the capital stock K, we now characterize the equilibrium value of output, factor prices, employment, and the threshold task I^*.

In the text, we simplify the exposition by imposing:

Assumption 2 *One of the following two conditions holds: (i)* $\eta \to 0$*, or (ii)* $\zeta = 1$.

[10] A simplifying feature of the technology described in equation (3) is that capital has the same productivity in all tasks. This assumption could be relaxed with no change to our results in the static model, but without other changes, it would not allow balanced growth in the next section. Another simplifying assumption is that non-automated tasks can be produced with just labor. Having these tasks combine labor and capital would have no impact on our main results as we show in Appendix B.

These two special cases ensure that the demand for labor and capital is homothetic. More generally, our qualitative results are identical as long as the degree of non-homotheticity is not too extreme, though in this case we no longer have closed-form expressions and this motivates our choice of presenting these more general results in Appendix A.[11]

We proceed by characterizing the unit cost of producing each task as a function of factor prices and the automation possibilities represented by I. Because tasks are produced competitively, their price, $p(i)$, will be equal to the minimum unit cost of production:

$$p(i) = \begin{cases} \min\left\{R, \dfrac{W}{\gamma(i)}\right\}^{1-\eta} & \text{if } i \leq I, \\ \left(\dfrac{W}{\gamma(i)}\right)^{1-\eta} & \text{if } i > I, \end{cases} \tag{5}$$

where W denotes the wage rate and R denotes the rental rate of capital.

In equation (5), the unit cost of production for tasks $i > I$ is given by the effective cost of labor, $W/\gamma(i)$ (which takes into account that the productivity of labor in task i is $\gamma(i)$). The unit cost of production for tasks $i \leq I$, on the other hand, depends on $\min\left\{R, \frac{W}{\gamma(i)}\right\}$ reflecting the fact that capital and labor are perfect substitutes in the production of automated tasks. In these tasks, firms will choose whichever factor has a lower effective cost—R or $W/\gamma(i)$.

Because labor has a strict comparative advantage in tasks with a higher index, there is a (unique) threshold $\widetilde{I}$ such that

$$\frac{W}{R} = \gamma(\widetilde{I}). \tag{6}$$

This threshold represents the task for which the costs of producing with capital and labor are equal. For all tasks $i \leq \widetilde{I}$, we have $R \leq W/\gamma(i)$, and without any other constraints, these tasks will be produced with capital. However, if $\widetilde{I} > I$, firms cannot use capital all the way up to task $\widetilde{I}$ because of the constraint imposed by the available automation technology. This implies that there exists a unique equilibrium threshold task

$$I^* = \min\{I, \widetilde{I}\}$$

such that all tasks $i \leq I^*$ will be produced with capital, while all tasks $i > I^*$ will be produced with labor.[12]

Figure 2 depicts the resulting allocation of tasks to factors and also shows how, as already noted, the creation of new tasks replaces existing tasks from the bottom of the distribution.

As noted in footnote 9, we have simplified the exposition by imposing that new tasks created at N immediately replace tasks located at $N-1$, and it is therefore profitable to produce new tasks

[11]The source of non-homotheticity in the general model is the substitution between factors (capital or labor) and intermediates (the $q(i)$'s). A strong substitution creates implausible features. For example, automation, which increases the price of capital, may end up raising the demand for labor more than the demand for capital—as capital gets substituted by the intermediate inputs. Assumption 2′ in Appendix A imposes that $\left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{\max\{1,\sigma\}} \frac{1}{\left(\frac{\gamma(N)}{\gamma(N-1)}\right)^{|1-\zeta|}-1} > |\sigma - \zeta|$, which ensures that the degree of non-homotheticity is not too extreme and automation always reduces the relative demand for labor.

[12]Without loss of generality, we impose that firms use capital when they are indifferent between using capital or labor, which explains our convention of writing that all tasks $i \leq I^*$ (rather than $i < I^*$) are produced using capital.

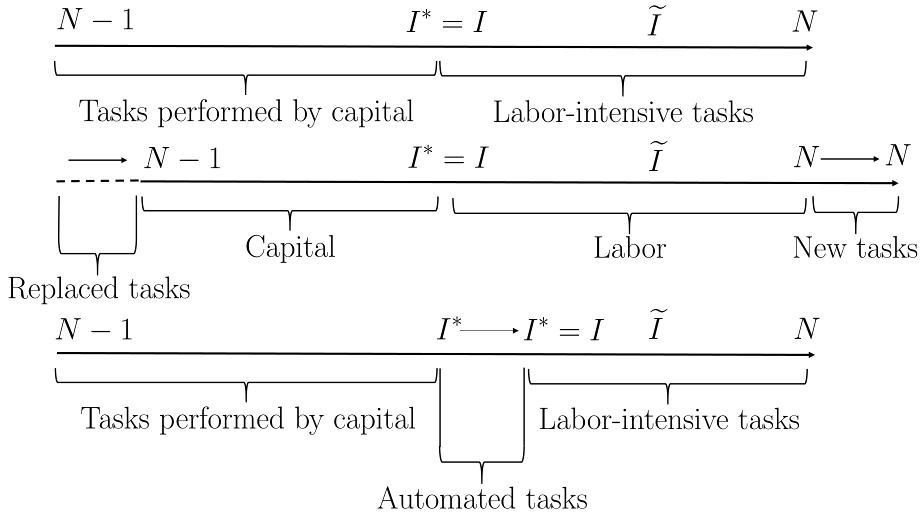

FIGURE 2: The task space and a representation of the effect of introducing new tasks (middle panel) and automating existing tasks (bottom panel).

with labor (and hence we have not distinguished N, N^* and $\widetilde{N}$). In the static model, this will be the case when the capital stock is not too large, which is imposed in the next assumption.

Assumption 3 *We have $K < \overline{K}$, where $\overline{K}$ is such that $R = \frac{W}{\gamma(N)}$.*

This assumption ensures that $R > \frac{W}{\gamma(N)}$ and consequently, new tasks will increase aggregate output and will be adopted immediately. Outside of this region, new tasks would not be utilized, which we view as the less interesting case. This assumption is relaxed in the next two sections where the capital stock is endogenous.

We next derive the demand for factors in terms of the (endogenous) threshold I^* and the technology parameter N. We choose the final good as the numeraire. Equation (1) gives the demand for task i as

$$y(i) = \widetilde{B}^{\sigma-1} Y p(i)^{-\sigma}. \tag{7}$$

Let us define $\widehat{\sigma} = \sigma(1-\eta) + \zeta\eta$ and $B = \widetilde{B}^{\frac{\sigma-1}{\hat{\sigma}-1}}$. Under Assumption 2, equations (2) and (3) yield the demand for capital and labor in each task as

$$k(i) = \begin{cases} B^{\hat{\sigma}-1}(1-\eta) Y R^{-\hat{\sigma}} & \text{if } i \leq I^*, \\ 0 & \text{if } i > I^*. \end{cases} \quad \text{and } l(i) = \begin{cases} 0 & \text{if } i \leq I^*, \\ B^{\hat{\sigma}-1}(1-\eta) Y \dfrac{1}{\gamma(i)} \left(\dfrac{W}{\gamma(i)}\right)^{-\hat{\sigma}} & \text{if } i > I^*. \end{cases}$$

We can now define a *static equilibrium* as follows. Given a range of tasks $[N-1, N]$, automation technology $I \in (N-1, N]$, and a capital stock K, a static equilibrium is summarized by a set of factor prices, W and R, threshold tasks, $\widetilde{I}$ and I^*, employment level, L, and aggregate output, Y, such that:

- $\widetilde{I}$ is determined by equation (6) and $I^* = \min\{I, \widetilde{I}\}$;

- the capital and labor markets clear, so that

$$B^{\hat{\sigma}-1}(1-\eta)Y(I^*-N+1)R^{-\hat{\sigma}}=K, \tag{8}$$

$$B^{\hat{\sigma}-1}(1-\eta)Y\int_{I^*}^{N}\frac{1}{\gamma(i)}\left(\frac{W}{\gamma(i)}\right)^{-\hat{\sigma}}di=L; \tag{9}$$

- factor prices satisfy the *ideal price index condition,*

$$(I^*-N+1)R^{1-\hat{\sigma}}+\int_{I^*}^{N}\left(\frac{W}{\gamma(i)}\right)^{1-\hat{\sigma}}di=B^{1-\hat{\sigma}}; \tag{10}$$

- labor supply satisfies $\nu'(L)=W/C$. Since in equilibrium $C=RK+WL$, this condition can be rearranged to yield the following increasing labor supply function:[13]

$$L=L^s\left(\frac{W}{RK}\right). \tag{11}$$

Proposition 1 (Equilibrium in the static model) *Suppose that Assumptions 1, 2 and 3 hold. Then a static equilibrium exists and is unique. In this static equilibrium, aggregate output is given by*

$$Y=\frac{B}{1-\eta}\left[(I^*-N+1)^{\frac{1}{\hat{\sigma}}}K^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}+\left(\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di\right)^{\frac{1}{\hat{\sigma}}}L^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}}. \tag{12}$$

Proof. See Appendix A. ■

Equation (12) shows that aggregate output is a CES aggregate of capital and labor, with the elasticity between capital and labor being $\hat{\sigma}$. The share parameters are endogenous and depend on the state of the two types of technologies and the equilibrium choices of firms. An increase in I^*—which corresponds to greater equilibrium automation—increases the share of capital and reduces the share of labor in this aggregate production function, while the creation of new tasks does the opposite.

Figure 3 illustrates the unique equilibrium described in Proposition 1. The equilibrium is given by the intersection of two curves in the (ω, I) space, where $\omega=\frac{W}{RK}$ is the wage level normalized by capital income; this ratio is a monotone transformation of the labor share and will play a central role in the rest of our analysis.[14] The upward-sloping curve represents the cost-minimizing allocation of capital and labor to tasks represented by equation (6), with the constraint that the equilibrium level of automation can never exceed I. The downward-sloping curve, $\omega(I^*, N, K)$, corresponds to the relative demand for labor, which can be obtained directly from (8), (9) and (11) as

$$\ln\omega+\frac{1}{\hat{\sigma}}\ln L^s(\omega)=\left(\frac{1}{\hat{\sigma}}-1\right)\ln K+\frac{1}{\hat{\sigma}}\ln\left(\frac{\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di}{I^*-N+1}\right). \tag{13}$$

[13] This representation clarifies that the equilibrium implications of our setup are identical to one in which an upward-sloping quasi-labor supply determines the relationship between employment and wages (and does not necessarily equate marginal cost of labor supply to the wage). This follows readily by taking (11) to represent this quasi-labor supply relationship.

[14] The increasing labor supply relationship, (11), ensures that the labor share $s_L=\frac{WL}{RK+WL}$ is increasing in ω.

As we show in Appendix A, the relative demand curve always starts above the cost minimization condition and ends up below it, so that the two curves necessarily intersect, defining a unique equilibrium, as shown in Figure 3.

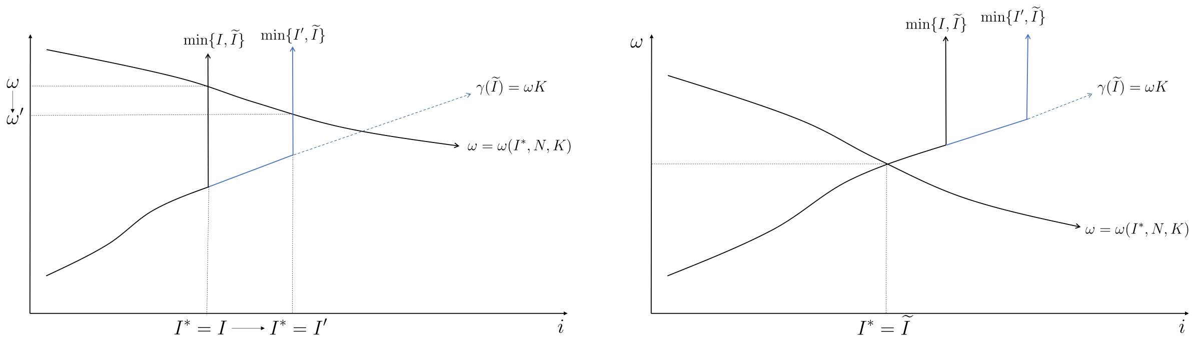

FIGURE 3: Static equilibrium. The left panel depicts the case in which $I^* = I < \widetilde{I}$ so that the allocation of factors is constrained by technology. The right panel depicts the case in which $I^* = \widetilde{I} < I$ so that the allocation of factors is not constrained by technology and is cost-minimizing. The blue curves show the shifts following an increase in I to I', which reduce ω in the left panel, but have no effect in the right panel.

The figure also distinguishes between the two cases highlighted above. In the left panel, we have $I^* = I < \widetilde{I}$ and the allocation of factors is constrained by technology, while the right panel plots the case where $I^* = \widetilde{I} < I$ and firms choose the cost-minimizing allocation given factor prices.

A special case of Proposition 1 is also worth highlighting, because it leads to a Cobb-Douglas production function with an exponent depending on the degree of automation, which is particularly tractable in certain applications.

Corollary 1 *Suppose that $\sigma = \zeta = 1$ and $\gamma(i) = 1$ for all i. Then aggregate output is*

$$Y = \frac{B}{1-\eta} K^{1-N+I^*} L^{N-I^*}.$$

The next two propositions give a complete characterization of comparative statics.[15]

Proposition 2 (Comparative statics) *Suppose that Assumptions 1, 2 and 3 hold. Let $\varepsilon_L > 0$ denote the elasticity of the labor supply schedule $L^s(\omega)$; let $\varepsilon_\gamma = \frac{d \ln \gamma(I)}{dI} > 0$ denote the semi-elasticity of the comparative advantage schedule; and let*

$$\Lambda_I = \frac{\gamma(I^*)^{\hat{\sigma}-1}}{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di} + \frac{1}{I^* - N + 1} \qquad \text{and} \qquad \Lambda_N = \frac{\gamma(N)^{\hat{\sigma}-1}}{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di} + \frac{1}{I^* - N + 1}.$$

- *If $I^* = I < \widetilde{I}$—so that the allocation of tasks to factors is constrained by technology—then:*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d \ln(W/R)}{dI} = \frac{d \ln \omega}{dI} = -\frac{1}{\hat{\sigma} + \varepsilon_L} \Lambda_I < 0, \qquad \frac{d \ln(W/R)}{dN} = \frac{d \ln \omega}{dN} = \frac{1}{\hat{\sigma} + \varepsilon_L} \Lambda_N > 0;$$

[15] In this proposition, we do not explicitly treat the case in which $I^* = I = \widetilde{I}$ in order to save on space and notation, since in this case left and right derivatives with respect to I are different.

– *and the impact of capital on relative factor prices is given by*

$$\frac{d\ln(W/R)}{d\ln K} = \frac{d\ln\omega}{d\ln K} + 1 = \frac{1+\varepsilon_L}{\hat{\sigma}+\varepsilon_L} > 0.$$

- *If $I^* = \widetilde{I} < I$—so that the allocation of tasks to factors is cost-minimizing—then:*

  – *the impact of technological change on relative factor prices is given by*

$$\frac{d\ln(W/R)}{dI} = \frac{d\ln\omega}{dI} = 0, \qquad \frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN} = \frac{1}{\sigma_{free}+\varepsilon_L}\Lambda_N > 0,$$

  where

$$\sigma_{free} = \hat{\sigma} + \frac{1}{\varepsilon_\gamma}\Lambda_I > \hat{\sigma};$$

  – *and the impact of capital on relative factor prices is given by*

$$\frac{d\ln(W/R)}{d\ln K} = \frac{d\ln\omega}{d\ln K} + 1 = \frac{1+\varepsilon_L}{\sigma_{free}+\varepsilon_L} > 0.$$

- *In all cases, the labor share and employment move in the same direction as ω: $\frac{dL}{dN} > 0$ and, when $I^* = I$, $\frac{dL}{dI} < 0$.*

Proof. See Appendix B. ■

The main implication of Proposition 2 is that the two types of technological change—automation and the creation of new tasks—have polar implications. An increase in N—the creation of new tasks—raises W/R, the labor share and employment. An increase in I—an improvement in automation technology—reduces W/R, the labor share and employment (unless $I^* = \widetilde{I} < I$ and firms are not constrained by technology in their automation choice). [16]

The reason why automation reduces employment (when $I^* = I < \widetilde{I}$) is that automation raises aggregate output per worker more than it raises wages (as we will see next, automation may even reduce wages). Thus, the negative income effect on the labor supply resulting from greater aggregate output dominates any substitution effect that might follow from the higher wages. On the other hand, the creation of new tasks always increase employment—new tasks raise wages more than aggregate output, increasing the labor supply. Although these exact results rely on the balanced growth preferences in equation (4), similar forces operate in general and create a tendency for automation to reduce employment and for new tasks to increase it.

Figure 3 illustrates the comparative statics: automation moves us along the relative labor demand curve in the technology-constrained case shown in the left panel (and has no impact in the right panel), while the creation of new tasks shifts out the relative labor demand curve in both cases.

A final implication of Proposition 2 is that the "technology-constrained" elasticity of substitution between capital and labor, $\hat{\sigma}$, which applies when $I^* = I < \widetilde{I}$, differs from the "technology-free"

[16] Throughout, by "automation" or "automation technology" we refer to I, and use "equilibrium automation" to refer to I^*.

elasticity, σ_{free} , which applies when the decision of which tasks to automate is not constrained by technology (i.e., when $I^* = \widetilde{I} < I$). This is because in the former case, as relative factor prices change, the set of tasks performed by each factor remains fixed. In the latter case, as relative factor prices change, firms reassign tasks to factors. This additional margin of adjustment implies that $\sigma_{\text{free}} > \hat{\sigma}$.

Proposition 3 (Impact of technology on productivity, wages, and factor prices) *Suppose that Assumptions 1, 2 and 3 hold, and denote the changes in productivity—the change in aggregate output holding capital and labor constant—by* $d\ln Y\mid_{K,L}$.

- *If* $I^* = I < \widetilde{I}$*—so that the allocation of tasks to factors is constrained by technology—then* $\frac{W}{\gamma(I^*)} > R > \frac{W}{\gamma(N)}$, *and*

$$d\ln Y\mid_{K,L} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I^*)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right)dI + \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.$$

  That is, both technologies increase productivity.

  Moreover, let s_L *denote the share of labor in net output. The impact of technology on factor prices in this case is given by:*

$$\begin{aligned} d\ln W =& d\ln Y\mid_{K,L} + (1-s_L)\left(\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN - \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI\right), \\ d\ln R =& d\ln Y\mid_{K,L} - s_L\left(\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN - \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI\right). \end{aligned}$$

  That is, a higher N *always increases the equilibrium wage but may reduce the rental rate of capital, while a higher* I *always increases the rental rate of capital but may reduce the equilibrium wage. In particular, there exists* $\widetilde{K} < \infty$ *such that an increase in* I *increases the equilibrium wage when* $K > \widetilde{K}$ *and reduces it when* $K < \widetilde{K}$.

- *If* $I^* = \widetilde{I} < I$*—so that the allocation of tasks to factors is not constrained by technology—then* $\frac{W}{\gamma(I^*)} = R > \frac{W}{\gamma(N)}$, *and*

$$d\ln Y\mid_{K,L} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN.$$

  That is, new tasks increase productivity, but additional automation technologies do not.

  Moreover, the impact of technology on factor prices in this case is given by:

$$\begin{aligned} d\ln W =& d\ln Y\mid_{K,L} + (1-s_L)\frac{1}{\sigma_{\mathit{free}}+\varepsilon_L}\Lambda_N dN \\ d\ln R =& d\ln Y\mid_{K,L} - s_L\frac{1}{\sigma_{\mathit{free}}+\varepsilon_L}\Lambda_N dN. \end{aligned}$$

  That is, an increase in N *(more new tasks) always increases the equilibrium wage but may reduce the rental rate, while an increase in* I *(greater technological automation) has no effect on factor prices.*

Proof. See Appendix B. ■

The most important result in Proposition 3 is that, when $I^* = I < \widetilde{I}$, automation—an increase in I—always increases aggregate output, but has an ambiguous effect on the equilibrium wage. On the one hand, there is a positive *productivity effect* captured by the term $d\ln Y\mid_{K,L}$: by substituting cheaper capital for expensive labor, automation raises productivity, and hence the demand for labor in the tasks that are not yet automated.[17] Countering this, there is a negative *displacement effect* captured by the term $\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I$. This negative effect occurs because automation contracts the set of tasks performed by labor. Because tasks are subject to diminishing returns in the aggregate production function, (1), bunching workers into fewer tasks puts downward pressure on the wage.

As the equation for $d\ln Y\mid_{K,L}$ reveals, the productivity gains depend on the cost savings from automation, which are given by the difference between the effective wage at I^*, $\frac{W}{\gamma(I^*)}$, and the rental rate, R. The displacement effect dominates the productivity effect when the gap between $\frac{W}{\gamma(I^*)}$ and R is small—which is guaranteed when $K < \widetilde{K}$. In this case, the overall impact of automation on wages is negative.

Finally, Proposition 3 shows that an increase in N always raises productivity and the equilibrium wage (recall that Assumption 3 imposed that $R > \frac{W}{\gamma(N)}$). When the productivity gains from the creation of new tasks are small, it can reduce the rental rate of capital as well.

The fact that automation may reduce the equilibrium wage while increasing productivity is a key feature of the task-based framework developed here (see also Acemoglu and Autor, 2011). In our model, automation shifts the range of tasks performed by capital and labor—it makes the production process more capital intensive and less labor intensive, and it always reduces the labor share and the wage-rental rate ratio, W/R. This reiterates that automation is very different from factor-augmenting technological changes and has dissimilar implications. The effects of labor- or capital-augmenting technology on the labor share and the wage-rental rate ratio depend on the elasticity of substitution (between capital and labor). Also, capital-augmenting technological improvements always increase the equilibrium wage, and labor-augmenting ones also do so provided that the elasticity of substitution is greater than the share of capital in national income.[18]

3 Dynamics and Balanced Growth

In this section, we extend our model to a dynamic economy in which the evolution of the capital stock is determined by the saving decisions of a representative household. We then investigate

[17] This discussion also clarifies that our productivity effect is similar to the productivity effect in models of offshoring, such as Grossman and Rossi-Hansberg (2008), Rodriguez-Clare (2010) and Acemoglu, Gancia and Zilibotti (2015), which results from the substitution of cheap foreign labor for domestic labor in certain tasks.

[18] For instance, with a constant returns to scale production function and two factors, capital and labor are $q-$complements. Thus, capital-augmenting technologies always increases the marginal product of labor. To see this, let $F(A_K K, A_L L)$ be such a production function. Then $W = F_L$, and $\frac{dW}{dA_K} = KF_{LK} = -LF_{LL} > 0$ (because of constant returns to scale).

Likewise, improvements in A_L increase the equilibrium wage provided that the elasticity of substitution between capital and labor is greater than the capital share, which is a fairly weak requirement (in other words, A_L can reduce the equilibrium wage only if the elasticity of substitution is very very low).

the conditions under which the economy admits a balanced growth path (BGP), where aggregate output, the capital stock and wages grow at a constant rate. We conclude by discussing the long-run effects of automation on wages, the labor share and employment.

3.1 Balanced Growth

We assume that the representative household's dynamic preferences are given by

$$\int_0^\infty e^{-\rho t} u(C(t), L(t))dt, \tag{14}$$

where $u(C(t), L(t))$ is as defined in equation (4) and $\rho > 0$ is the discount rate.

To ensure balanced growth, we impose more structure to the comparative advantage schedule. Because balanced growth is driven by technology, and in this model sustained technological change comes from the creation of new tasks, constant growth requires productivity gains from new tasks to be exponential.[19] Thus, in what follows we strengthen Assumption 1 to:

Assumption 1′ $\gamma(i)$ *satisfies:*

$$\gamma(i) = e^{Ai} \text{ with } A > 0. \tag{15}$$

The path of technology, represented by $\{I(t), N(t)\}$, is exogenous, and we define

$$n(t) = N(t) - I(t)$$

as a summary measure of technology, and similarly let $n^*(t) = N(t) - I^*(t)$ be a summary measure of the state of technology used in equilibrium (since $I^*(t) \leq I(t)$, we have $n^*(t) \geq n(t)$). New automation technologies reduce $n(t)$, while the introduction of new tasks increases it.

From equation (12), aggregate output net of intermediates, or simply "net output", can be written as a function of technology represented by $n^*(t)$ and $\gamma(I^*(t)) = e^{AI^*(t)}$, the capital stock, $K(t)$, and the level of employment, $L(t)$, as

$$F\left(K(t), e^{AI^*(t)}L(t); n^*(t)\right) = B\left[(1-n^*(t))^{\frac{1}{\hat{\sigma}}} K(t)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}} (e^{AI^*(t)}L(t))^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}}. \tag{16}$$

The resource constraint of the economy then takes the form

$$\dot{K}(t) = F\left(K(t), e^{AI(t)}L(t); n^*(t)\right) - C(t) - \delta K(t),$$

where δ is the depreciation rate of capital.

[19] Notice also that in this dynamic economy, as in our static model, the productivity of capital is the same in all automated tasks. This does not, however, imply that any of the previously automated tasks can be used regardless of N. As N increases, as emphasized by equation (1), the set of feasible tasks shifts to the right, and only tasks above $N-1$ remain compatible with and can be combined with those currently in use. Just to cite a few motivating examples for this assumption: power looms of the 18th and 19th century are not compatible with modern textile technology; first-generation calculators are not compatible with computers; many hand and mechanical tools are not compatible with numerically controlled machinery; and bookkeeping methods from the 19th and 20th centuries are not compatible with the modern, computerized office.

We characterize the equilibrium in terms of the employment level $L(t)$, and the normalized variables $k(t) = K(t)e^{-AI^*(t)}$, and $c(t) = C(t)e^{\frac{1-\theta}{\theta}\nu(L(t))-AI^*(t)}$. As in our static model, $R(t)$ denotes the rental rate, and $w(t) = W(t)e^{-AI^*(t)}$ is the normalized wage. These normalized variables determine factor prices as:

$$\begin{aligned}R(t) =& F_K[k(t), L(t); n^*(t)] \\ =& B(1-n^*(t))^{\frac{1}{\hat{\sigma}}}\left[(1-n^*(t))^{\frac{1}{\hat{\sigma}}} + \left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}}\left(\frac{L(t)}{k(t)}\right)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{1}{\hat{\sigma}-1}}\end{aligned}$$

and

$$\begin{aligned}w(t) =& F_L[k(t), L(t); n^*(t)] \\ =& B\left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}}\left[(1-n^*(t))^{\frac{1}{\hat{\sigma}}}\left(\frac{k(t)}{L(t)}\right)^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_0^{n^*(t)} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}}\right]^{\frac{1}{\hat{\sigma}-1}}.\end{aligned}$$

The equilibrium interest rate is $R(t) - \delta$.

Given time paths for $g(t)$ (the growth rate of $e^{AI^*(t)}$) and $n(t)$, *a dynamic equilibrium* can now be defined as a path for the threshold task $n^*(t)$, (normalized) capital and consumption, and employment, $\{k(t), c(t), L(t)\}$, that satisfies

- $n^*(t) \geq n(t)$, with $n^*(t) = n(t)$ only if $w(t) > R(t)$, and $n^*(t) > n(t)$ only if $w(t) = R(t)$;
- the Euler equation,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(F_K[k(t), L(t); n^*(t)] - \delta - \rho) - g(t); \tag{17}$$

- the endogenous labor supply condition,

$$\nu'(L(t))e^{\frac{\theta-1}{\theta}\nu(L(t))} = \frac{F_L[k(t), L(t); n^*(t)]}{c(t)}; \tag{18}$$

- the representative household's transversality condition,

$$\lim_{t\to\infty} k(t)e^{-\int_0^t (F_K[k(s),L(s);n^*(s)]-\delta-g(s))ds} = 0; \tag{19}$$

- and the resource constraint,

$$\dot{k}(t) = F(k(t), L(t); n^*(t)) - c(t)e^{-\frac{1-\theta}{\theta}\nu(L(t))} - (\delta + g(t))k(t). \tag{20}$$

We also define a *balanced growth path* (BGP) as a dynamic equilibrium in which the economy grows at a constant positive rate, factor shares are constant, and the rental rate of capital $R(t)$ is constant.

To characterize the growth dynamics implied by these equations, let us first consider a path for technology such that $g(t) \to g$ and $n(t) \to n$, consumption grows at the rate g and the Euler equation holds $R(t) = \rho + \delta + \theta g$. Suppose first that $n^*(t) = n(t) = 0$, in which case F becomes linear and $R(t) = B$. Because the growth rate of consumption must converge to g as well, the Euler equation (17) is satisfied in this case only if ρ is equal to

$$\overline{\rho} = B - \delta - \theta g. \tag{21}$$

Lemma A2 in Appendix A shows that this critical value of the discount rate divides the parameter space into two regions as shown in Figure 4. To the left of $\overline{\rho}$, there exists a decreasing curve $\widetilde{n}(\rho)$ defined over $[\rho_{min}, \overline{\rho}]$ with $\widetilde{n}(\overline{\rho}) = 0$, and to the right of $\overline{\rho}$, there exists an increasing curve $\overline{n}(\rho)$ defined over $[\overline{\rho}, \rho_{max}]$ with $\overline{n}(\overline{\rho}) = 0$, such that:[20]

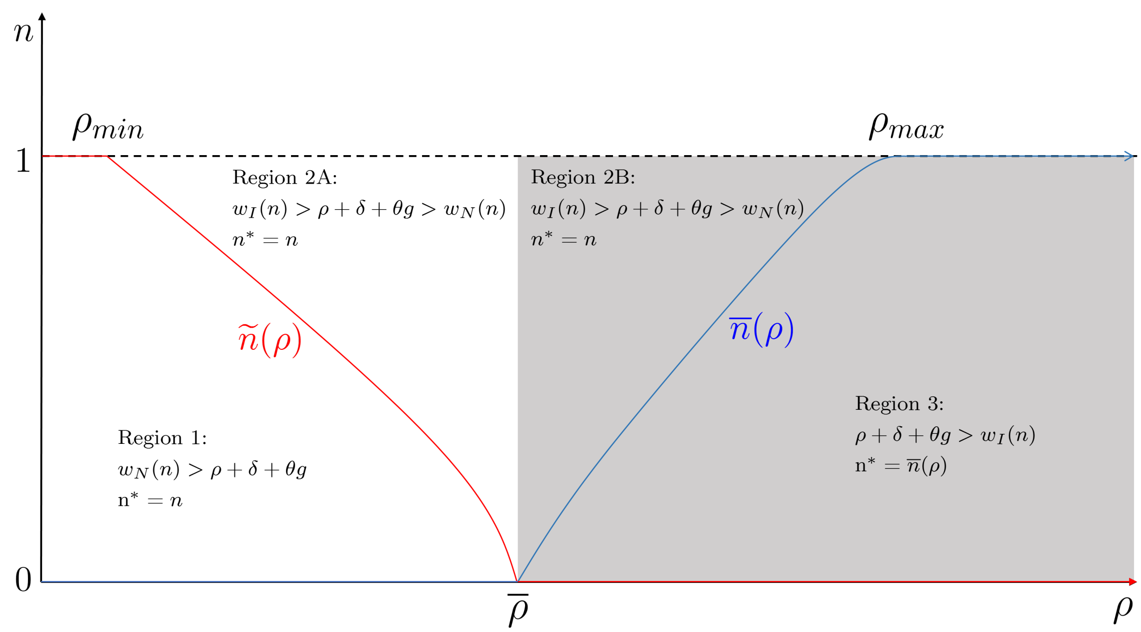

FIGURE 4: Behavior of factor prices in different parts of the parameter space.

- for $n < \widetilde{n}(\rho)$, we have $\frac{w(t)}{\gamma(N(t))} > R(t)$ and new tasks would reduce aggregate output, so are not adopted (recall that $w(t) = W(t)e^{-AI^*(t)}$);
- for $n > \widetilde{n}(\rho)$, we have $\frac{w(t)}{\gamma(N(t))} < R(t)$ and in this case, new tasks raise aggregate output and are immediately produced with labor;
- for $n > \overline{n}(\rho)$, we have $w(t) > R(t)$, as a result, automated tasks raise aggregate output and are immediately produced with capital; and

[20] The functions $w_N(n)$ and $w_I(n)$ depicted in this figure are introduced and explained below.

- for $n < \overline{n}(\rho)$, we have $w(t) < R(t)$ and additional automation would reduce aggregate output, so small changes in automation technology do not affect n^* and other equilibrium objects.

The next proposition provides the conditions under which a BGP exists, and characterizes the BGP allocations in each case. In what follows, we no longer impose Assumption 3, since depending on the value of ρ, the capital stock can become large and violate this assumption.

Proposition 4 (Dynamic equilibrium with exogenous technological change) *Suppose that Assumptions 1′ and 2 hold. The economy admits a BGP with positive growth if only if we are in one of the following cases:*

1. ***Full automation:*** *$\rho < \overline{\rho}$ and $N(t) = I(t)$ (and $B > \delta + \rho > \frac{1-\theta}{\theta}(B - \delta - \rho) + \delta$ to ensure the transversality condition). In this case, there is a unique and globally stable BGP. In this BGP, $n^*(t) = 0$ (all tasks are produced with capital), and the labor share is zero.*

2. ***Interior BGP with immediate automation:*** *$\rho \in (\rho_{min}, \rho_{max})$, $\dot{N}(t) = \dot{I}(t) = \Delta$, and $n(t) = n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ (and $\rho + (\theta - 1)A\Delta > 0$ to ensure the transversality condition). In this case, there is a unique and globally stable BGP. In this BGP, $n^*(t) = n$ and $I^*(t) = I(t)$.*

3. ***Interior BGP with eventual automation:*** *$\rho > \overline{\rho}$, $\dot{N}(t) = \Delta$ with $\dot{I}(t) \geq \Delta$, and $n(t) < \overline{n}(\rho)$ (and $\rho + (\theta - 1)A\Delta > 0$ to ensure the transversality condition). In this case, there is a unique and globally stable BGP. In this BGP, $n^*(t) = \overline{n}(\rho)$ and $I^*(t) = \widetilde{I}(t) > I(t)$.*

4. ***No automation:*** *$\rho > \rho_{max}$, and $\dot{N}(t) = \Delta$ (and $\rho + (\theta - 1)A\Delta > 0$ to ensure the transversality condition). In this case, there exists a unique and globally stable BGP. In this BGP, $n^*(t) = 1$ (all tasks are produced with labor), and the capital share is zero.*

Proof. See Appendix A. ■

The first type of BGP in Proposition 4 involves the automation of all tasks, in which case aggregate output becomes linear in capital. This case was ruled out by Assumption 3 in our static analysis, but as the proposition shows, when the discount rate, ρ, is sufficiently small, it can emerge in the dynamic model. A BGP with no automation (case 4), where growth is driven entirely by the creation of new tasks, is also possible if the discount rate is sufficiently large.

More important for our focus are the two interior BGPs where automation and the introduction of new tasks go hand-in-hand, and as a result, $n^*(t)$ is constant at some value between 0 and 1; this implies that both capital and labor perform a fixed measure of tasks. In the more interesting case where automated tasks are immediately produced with capital (case 2), the proposition also highlights that this process needs to be "balanced" itself: the two types of technologies need to advance at exactly the same rate so that $n(t) = n$.

Balanced growth with constant labor share emerges in this model because the net effect of automation and the creation of new technologies proceeding at the same rate is to augment labor while keeping constant the share of tasks performed by labor—as shown by equation (16). In this case, the gap between the two types of technologies, $n(t)$, regulates the share parameters in the resulting CES production function, while the levels of $N(t)$ and $I(t)$ determine the productivity of labor in the set of tasks that it performs. When $n(t) = n$, technology becomes purely labor augmenting on net because labor performs a fixed share of tasks, and labor becomes more productive over time in producing the newly-created tasks.[21]

To illustrate the main implication of the proposition, let us focus on part 2 with $\dot{I} = \dot{N} = \Delta$ and $n(t) = n \geq \overline{n}(\rho)$. Along such a path, $n^*(t) = n$ and $g(t) = A\Delta$. Figure 5 presents the phase diagram for the system of differential equations comprising the Euler equation (equation (17)) and the resource constraint (equation (20)). This system of differential equations determines the structure of the dynamic equilibrium and is identical to that of the neoclassical growth model with labor-augmenting technological change and endogenous labor supply (which makes the locus for $\dot{c} = 0$ downward-sloping because of the negative income effect on the labor supply).

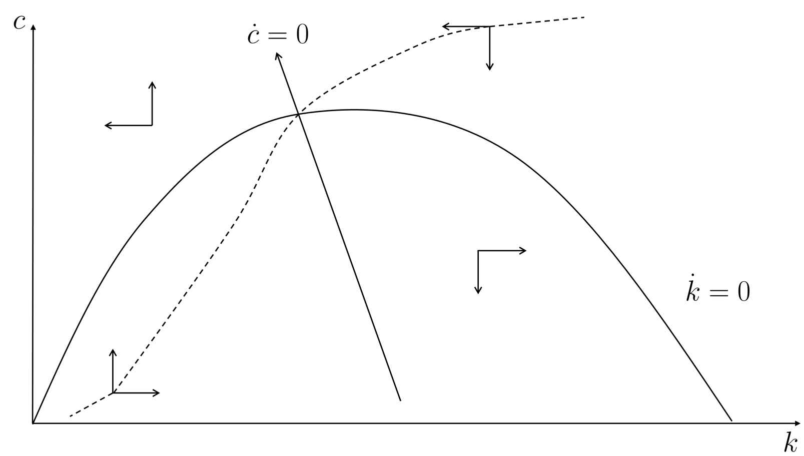

FIGURE 5: Dynamic equilibrium when technology is exogenous and satisfies $n(t) = n$ and $g(t) = A\Delta$.

3.2 Long-Run Comparative Statics

We next study the log-run implications of an unanticipated and permanent decline in $n(t)$, which corresponds to automation running ahead of the creation of new tasks. Because in the short run capital is fixed, the short-run implications of this change in technology are the same as in our static analysis in the previous section. But the fact that capital adjusts implies different long-run dynamics.

Consider an interior BGP in which $N(t) - I(t) = n \in (0, 1)$. Along this path, the equilibrium wage grows at the rate $A\Delta$. Define $w_I(n) = \lim_{t\to\infty} W(t)/\gamma(I^*(t))$ as the effective wage paid in the least complex task produced with labor and $w_N(n) = \lim_{t\to\infty} W(t)/\gamma(N(t))$ as the effective

[21]This intuition connects Proposition 4 to Uzawa's Theorem, which implies that balanced growth requires a representation of the production function with purely labor-augmenting technological change (e.g., Acemoglu, 2009, or Grossman, Helpman and Oberfield, 2016).

wage paid in the most complex task produced with labor. Both of these functions are well-defined and depend only on n. Figure 4 shows how these effective wages compare to the BGP value of the rental rate of capital, $\rho + \delta + \theta g$.

The next proposition characterizes the long-run impact of automation on factor prices, employment and the labor share in the interior BGPs.

Proposition 5 (Long-run comparative statics) *Suppose that Assumptions 1′ and 2 hold. Consider a path for technology in which $n(t) = n \in (0, 1)$, $n > \tilde{n}(\rho)$, and $g(t) = g$ (so that we are in case 2 or 3 in Proposition 4). In the unique BGP we have that $R(t) = \rho + \delta + \theta g$, and*

- *for $n < \overline{n}(\rho)$, we have that $n^*(t) = \overline{n}(\rho)$, $w_I(n) = w_I(\overline{n}(\rho))$ and $w_N(n) = w_N(\overline{n}(\rho))$. In this region, small changes in n do not affect the paths of effective wages, employment and the labor share;*

- *for $n > \overline{n}(\rho)$, we have that $n^*(t) = n$, and $w_I(n)$ is increasing and $w_N(n)$ is decreasing in n. Moreover, the asymptotic values for employment and the labor share are increasing in n. Finally, if the increase in n is caused by an increase in I, the capital stock also increases.*

Proof. See Appendix B. ■

We discuss this proposition for $n > \overline{n}(\rho)$, so that we are in the most interesting region of the parameter space where $I^* = I$ and the level of automation is constrained by technology. The long-run implications of automation now differ from its short-term impact. In the long run, automation reduces employment and the labor share, but it always increases the wage because in the long run capital per worker increases to keep the rental rate constant at $\rho + \delta + \theta g$, making sure that productivity gains accrue to the scarce factor, labor.[22]

Figure 6 illustrates the response of the economy to permanent changes in automation. It plots two potential paths for all endogenous variables. The dotted line depicts the case where $w_I(n)$ is large relative to R, so that there are significant productivity gains from automation. In this case, an increase in automation raises the wage immediately, followed by further increases in the long run. The solid line depicts the dynamics when $w_I(n) \approx R$, so that the productivity gains from automation are very small. In this case, an increase in automation reduces the wage in the short run and leaves it approximately unchanged in the long run. In contrast to the concerns that highly productive automation technologies will reduce the wage and employment, our model thus shows that it is precisely when automation fails to raise productivity significantly that it has a more detrimental impact on wages and employment. In both cases, the duration of the period

[22] This result follows because $w_N(n)$ is decreasing in n, and thus a lower n implies a higher wage level. However, because $w_I(n)$ is increasing in n, the wage increase is less than proportional to $\gamma(I(t))$. The result can also be understood by noting that the ideal price index condition implies

$$d \ln Y \mid_{K,L} = s_L d \ln W + (1 - s_L) d \ln R.$$

In general, productivity gains from technological change accrue to both capital and labor. In the long run, however, capital adjusts to keep the rental rate fixed at $R = \rho + \delta + \theta g$, and as a result, $d \ln W = \frac{1}{s_L} d \ln Y \mid_{K,L} > 0$, meaning that productivity gains accrue only to the inelastic factor—labor.

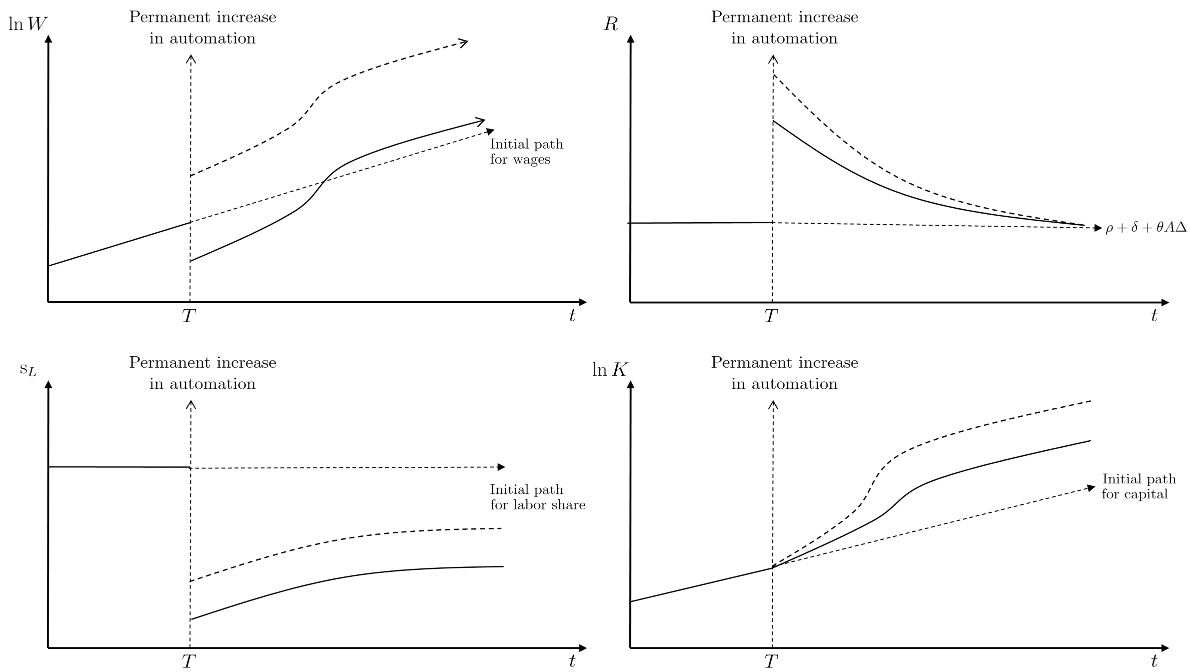

FIGURE 6: Dynamic behavior of wages ($\ln W$), the rental rate of capital (R), the labor share (s_L), and the capital stock following a permanent increase in automation.

with stagnant or depressed wages depends on θ, which determines the speed of capital adjustment following an increase in the rental rate.

The remaining panels of the figure show that automation reduces employment and the labor share, as stated in Proposition 5. If $\hat{\sigma} < 1$, the resulting capital accumulation mitigates the short-run decline in the labor share but does not fully offset it (this is the case depicted in the figure). If $\hat{\sigma} > 1$, capital accumulation further depresses the labor share—even though it raises the wage.

The long-run impact of a permanent increase in $N(t)$ can also be obtained from the proposition. In this case, new tasks increase the wage (because $w_I(n)$ is increasing in n), aggregate output, employment, and the labor share, both in the short and the long run. Because the short-run impact of new tasks on the rental rate of capital is ambiguous, so is the response of capital accumulation.

In light of these results, the recent decline in the labor share and the employment to population ratio in the United States can be interpreted as a consequence of automation outpacing the creation of new labor-intensive tasks. Faster automation relative to the creation of new tasks might be driven by an acceleration in the rate at which $I(t)$ advances, in which case we would have stagnant or lower wages in the short run while capital adjusts to a new higher level. Alternatively, it might be driven by a deceleration in the rate at which $N(t)$ advances, in which case we would also have low growth of aggregate output and wages. We return to the productivity implications of automation once we introduce our full model with endogenous technological change in the next section.

4 Full Model: Tasks and Endogenous Technologies

The previous section established, under some conditions, the existence of an interior BGP with $\dot{N} = \dot{I} = \Delta$. This result raises a fundamental question: why should these two types of technologies advance at the same rate? To answer this question we now develop our full model, which endogenizes the pace at which automation and the creation of new tasks proceeds.

4.1 Endogenous and Directed Technological Change

To endogenize technological change, we deviate from our earlier assumption of a perfectly competitive market for intermediates, and assume that (intellectual) property rights to each intermediate, $q(i)$, are held by a technology monopolist who can produce it at the marginal cost $\mu\psi$ in terms of the final good, where $\mu \in (0, 1)$ and $\psi > 0$. We also assume that this technology can be copied by a fringe of competitive firms, which can replicate any available intermediate at a higher marginal cost of ψ, and that μ is such that the unconstrained monopoly price of an intermediate is greater than ψ. This ensures that the unique equilibrium price for all types of intermediates is a limit price of ψ, and yields a per unit profit of $(1-\mu)\psi > 0$ for technology monopolists. These profits generate incentives for creating new tasks and automation technologies.

In this section, we adopt a structure of intellectual property rights that abstracts from the creative destruction of profits.[23] We assume that developing a new intermediate that automates or replaces an existing task is viewed as an infringement of the patent of the technology previously used to produce that task. For that reason, a firm must compensate the technology monopolist who owns the property rights over the production of the intermediate that it is replacing. We also assume that this compensation takes place with the new inventors making a take-it-or-leave-it offer to the holder of the existing patent.

Developing new intermediates that embody technology requires scientists.[24] There is a fixed supply of S scientists, which will be allocated to automation ($S_I(t) \geq 0$) or the creation of new tasks ($S_N(t) \geq 0$), so that

$$S_I(t) + S_N(t) \leq S.$$

When a scientist is employed in automation, she automates κ_I tasks per unit of time and receives a wage $W_I^S(t)$. When she is employed in the creation of new tasks, she creates κ_N new tasks per unit of time and receives a wage $W_N^S(t)$. We assume that automation and the creation of new tasks proceed in the order of the task index i. Thus, the allocation of scientists determines the evolution of both types of technology—summarized by $I(t)$ and $N(t)$—as

$$\dot{I}(t) = \kappa_I S_I(t), \text{ and } \dot{N}(t) = \kappa_N S_N(t). \tag{22}$$

[23] The creative destruction of profits is present in other models of quality improvements such as Aghion and Howitt (1992) and Grossman and Helpman (1991), and will be introduced in the context of our model in Section 5.

[24] An innovation possibilities frontier that uses just scientists, rather than variable factors as in the lab-equipment specifications (see Acemoglu 2009), is convenient because it enables us to focus on the direction of technological change—and not on the overall amount of technological change.

Because we want to analyze the properties of the equilibrium locally, we make a final assumption to ensure that the allocation of scientists varies smoothly when there is a small difference between $W_I^S(t)$ and $W_N^S(t)$ (rather than having discontinuous jumps). In particular, we assume that scientists differ in the cost of effort: when working in automation, scientist j incurs a cost of $\chi_I^j Y(t)$, and when working in the creation of new tasks, she incurs a cost of $\chi_N^j Y(t)$.[25] Consequently, scientist j will work in automation if $\frac{W_I^S(t)-W_N^S(t)}{Y(t)} > \chi_I^j - \chi_N^j$. We also assume that the distribution of $\chi_I^j - \chi_N^j$ among scientists is given by a smooth and increasing distribution function G over a support $[-\upsilon, \upsilon]$, where we take υ to be small enough that χ_I^j and χ_N^j are always less than $\max\left\{\frac{\kappa_N V_N(t)}{Y(t)}, \frac{\kappa_I V_I(t)}{Y(t)}\right\}$ and thus all scientists always work. For notational convenience, we also adopt the normalization $G(0) = \frac{\kappa_N}{\kappa_I+\kappa_N}$.

4.2 Equilibrium with Endogenous Technological Change

We first compute the present discounted value accruing to monopolists from automation and the creation of new tasks. Let $V_I(t)$ denote the value of automating task $i = I(t)$ (i.e., the task with the highest index that has not yet been automated, or more formally $i = I(t) + \varepsilon$ for ε arbitrarily small and positive). Likewise, $V_N(t)$ is the value of a new technology creating a new task at $i = N(t)$.

To simplify the exposition, let us assume that in this equilibrium $n(t) > \max\{\bar{n}(\rho), \widetilde{n}(\rho)\}$, so that $I^*(t) = I(t)$ and newly-automated tasks start being produced with capital immediately. The flow profits that accrue to the technology monopolist that automated task i are

$$\pi_I(t,i) = bY(t)R(t)^{\zeta-\hat{\sigma}}, \tag{23}$$

where $b = (1-\mu)B^{\hat{\sigma}-1}\eta\psi^{1-\zeta}$.[26] Likewise, the flow profits that accrue to the technology monopolist that created the labor-intensive task i are

$$\pi_N(t,i) = bY(t)\left(\frac{W(t)}{\gamma(i)}\right)^{\zeta-\hat{\sigma}}. \tag{24}$$

The take-it-or-leave-it nature of offers implies that a firm that automates task I needs to compensate the existing technology monopolist by paying her the present discounted value of the profits that her inferior labor-intensive technology would generate if not replaced. This take-it-or-leave-it offer is given by:[27]

$$b\int_t^\infty e^{-\int_0^\tau (R(s)-\delta)ds}Y(\tau)\left(\frac{W(\tau)}{\gamma(I)}\right)^{\zeta-\hat{\sigma}} d\tau.$$

Likewise, a firm that creates task N needs to compensate the existing technology monopolist by paying her the present discounted value of the profits from the capital-intensive alternative

[25] The cost of effort is multiplied by $Y(t)$ to capture the income effect on the costs of effort in a tractable manner.

[26] This expression follows because the demand for intermediates is $q(i) = B^{\hat{\sigma}-1}\eta\psi^{-\zeta}Y(t)R(t)^{\zeta-\hat{\sigma}}$, every intermediate is priced at ψ and the technology monopolist makes a per unit profit of $1-\mu$.

[27] This expression is written by assuming that the patent-holder will also turn down subsequent less generous offers in the future. Deriving it using dynamic programming and the one-step ahead deviation principle leads to the same conclusion.

technology. This take-it-or-leave-it offer is given by

$$b\int_t^{\infty} e^{-\int_0^{\tau}(R(s)-\delta)ds}Y(\tau)R(\tau)^{\zeta-\hat{\sigma}}d\tau.$$

In both cases, the patent-holders will immediately accept theses offers and reject less generous ones.

We can then compute the values of a new automation technology and a new task, respectively, as

$$V_I(t) = bY(t)\int_t^{\infty} e^{-\int_t^{\tau}(R(s)-\delta-g_y(s))ds}\left(R(\tau)^{\zeta-\hat{\sigma}} - \left(w(\tau)e^{\int_t^{\tau}g(s)ds}\right)^{\zeta-\hat{\sigma}}\right)d\tau, \tag{25}$$

and

$$V_N(t) = bY(t)\int_t^{\infty} e^{-\int_t^{\tau}(R(s)-\delta-g_y(s))ds}\left(\left(\frac{w(\tau)}{\gamma(n(t))}e^{\int_t^{\tau}g(s)ds}\right)^{\zeta-\hat{\sigma}} - R(\tau)^{\zeta-\hat{\sigma}}\right)d\tau, \tag{26}$$

where $g_y(t)$ is the growth rate of aggregate output at time t and as noted above, $g(t)$ is the growth rate of $\gamma(N(t))$.

To ensure that these value functions are well-behaved and non-negative, we impose the following assumption for the rest of the paper:

Assumption 4 $\hat{\sigma} > \zeta$.

This assumption ensures that innovations are directed towards technologies that allow firms to produce tasks by using the cheaper (or more productive) factors, and consequently, that the present discounted values from innovation are positive. This assumption is intuitive and reasonable: since intermediates embody the technology that directly works with labor or capital, they should be highly complementary with the relevant factor of production in the production of tasks.[28]

The expressions for the value functions, $V_I(t)$ and $V_N(t)$ in equations (25) and (26) are intuitive. The value of developing new automation technologies depends on the gap between the cost of producing with labor (given by the effective wage, $w(\tau)$) and the rental rate of capital (recall that $\hat{\sigma} > \zeta$). When the wage is higher, $V_I(t)$ increases and technology monopolists have greater incentives to introduce new automation technologies to substitute capital for the more expensive labor. The expression for $V_N(t)$ has an analogous interpretation, and is greater when the gap between the rental rate of capital and the cost of producing new tasks with labor $(w(\tau)/\gamma(n(t)))$ is larger.[29]

[28] The profitability of introducing an intermediate that embodies a new technology depends on its demand. As a factor (labor or capital) becomes cheaper, there are two effects on the demand for $q(i)$. First, the decline in costs allows firms to scale up their production, which increases the demand for the intermediate good. The extent of this positive scale effect is regulated by the elasticity of substitution $\hat{\sigma}$. Second, because the cheaper factor is substituted for the intermediate it is combined with, the demand for that intermediate good falls. This countervailing substitution effect is regulated by the elasticity of substitution ζ. The condition $\hat{\sigma} > \zeta$ guarantees that the former, positive effect dominates, so that prospective technology monopolists have an incentive to introduce technologies that allow firms to produce tasks with cheaper factors. When the opposite holds, i.e., $\zeta > \hat{\sigma}$, we have the paradoxical situation where technologies that work with more expensive factors are more profitable. In this case, the present discounted values from innovation are negative.

[29] There is an important difference between the value functions in (25) and (26) and those in models of directed technological change building on factor-augmenting technologies (such as in Acemoglu, 1998, or 2002). In the latter case, the direction of technological change is determined by the interplay of a market size effect favoring the more

An *equilibrium with endogenous technology* is given by paths $\{K(t), N(t), I(t)\}$ for capital and technology (starting from initial values $K(0), N(0), I(0)$), paths $\{R(t), W(t), W_I^S(t), W_N^S(t)\}$ for factor prices, paths $\{V_N(t), V_I(t)\}$ for the value functions of technology monopolists, and paths $\{S_N(t), S_I(t)\}$ for the allocation of scientists such that all markets clear, all firms and prospective technology monopolists maximize profits, the representative household maximizes its utility. Using the same normalizations as in the previous section, we can represent the equilibrium with endogenous technology by a path of the tuple $\{c(t), k(t), n(t), L(t), S_I(t), S_N(t), V_I(t), V_N(t)\}$ such that

- consumption satisfies the Euler equation (17) and the labor supply satisfies equation (18);
- the transversality condition holds

$$\lim_{t\to\infty}(k(t)+\Pi(t))e^{-\int_0^t(\rho-(1-\theta)g(s))ds}=0, \tag{27}$$

  where in addition to the capital stock, the present value of corporate profits $\Pi(t) = I(t)V_I(t)/Y(t)+N(t)V_N(t)/Y(t)$ is also part of the representative household's assets;

- capital satisfies the resource constraint

$$\dot{k}(t) = \left[1+\frac{\eta}{1-\eta}(1-\mu)\right]F\left(k(t), L(t); n^*(t)\right) - c(t)e^{-\frac{1-\theta}{\theta}\nu(L(t))} - (\delta+g(t))k(t),$$

  where recall that $F\left(k(t), L(t); n^*(t)\right)$ is net output (aggregate output net of intermediates) and $\frac{\eta}{1-\eta}(1-\mu)F\left(k(t), L(t); n^*(t)\right)$ is profits of technology monopolists from intermediates;

- competition among prospective technology monopolists to hire scientists implies that $W_I^S(t) = \kappa_I V_I(t)$ and $W_N^S(t) = \kappa_N V_N(t)$. Thus,

$$S_I(t) = SG\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right), \qquad S_N(t) = S\left[1-G\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right)\right],$$

  and $n(t)$ evolves according to the differential equation

$$\dot{n}(t) = \kappa_N S - (\kappa_N+\kappa_I)G\left(\frac{\kappa_I V_I(t)}{Y(t)} - \frac{\kappa_N V_N(t)}{Y(t)}\right)S; \tag{28}$$

- and the value functions that determine the allocation of scientists, $V_I(t)$ and $V_N(t)$, are given by (25) and (26).

abundant factor and a price effect favoring the cheaper factor. The task-based framework here, combined with the assumption on the structure of patents, makes the benefits of new technologies only a function of the factor prices—in particular, the difference between the wage rate and the rental rate. This is because factor prices determine the profitability of producing with capital relative to labor. Without technological constraints, this would determine the set of tasks that the two factors perform. In the presence of technological constraints restricting which tasks can be produced with which factor, factor prices determine the incentives for automation (to expand the set of tasks produced by capital) and the creation of new tasks (to expand the set of tasks produced by labor).

We should also note that despite this difference, the general results on absolute weak bias of technology in Acemoglu (2007) continue to hold here—in the sense that an increase in the abundance of a factor always makes technology more biased towards that factor.

As before, a BGP is given by an equilibrium in which the normalized variables $c(t), k(t)$ and $L(t)$, and the rental rate $R(t)$ are constant, except that now $n(t)$ is determined endogenously. The definition of the equilibrium shows that the profits from automation and the creation of new tasks determine the evolution of $n(t)$: whenever one of the two types of innovation is more profitable, more scientists will be allocated to that activity.

Consider an allocation where $n(t) = n \in (0, 1)$. Let us define the *normalized value functions* $v_I(n) = \lim_{t\to\infty} V_I(t)/Y(t)$ and $v_N(n) = \lim_{t\to\infty} V_N(t)/Y(t)$, which only depend on n. Equation (28) implies that $\dot{n}(t) > 0$ if and only if $\kappa_N V_N(t) > \kappa_I V_I(t)$, and $\dot{n}(t) < 0$ if and only if $\kappa_N V_N(t) < \kappa_I V_I(t)$. Thus if $\kappa_I v_I(n) \neq \kappa_N v_N(n)$, the economy converges to a corner with $n(t)$ equal to 0 or 1, and for an interior BGP with $n \in (0, 1)$ we need

$$\kappa_I v_I(n) = \kappa_N v_N(n). \tag{29}$$

The next proposition gives the main result of the paper, and characterizes different types of BGPs with endogenous technology.

Proposition 6 (Equilibrium with endogenous technological change) *Suppose that Assumptions 1′, 2, and 4 hold. There exists $\overline{S}$ such that, when $S < \overline{S}$, we have:*[30]

1 ***Full automation:*** *For $\rho < \overline{\rho}$, there is a BGP in which $n(t) = 0$ and thus all tasks are produced with capital (this case also requires $B > \delta + \rho > \frac{1-\theta}{\theta}(B - \delta - \rho) + \delta$ to ensure the transversality condition).*

For $\rho > \overline{\rho}$, all BGPs feature $n(t) = n > \overline{n}(\rho)$. Moreover, there exist $\overline{\kappa} \geq \underline{\kappa} > 0$ such that:

2 ***Unique interior BGP:*** *if $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$ there exists a unique BGP. In this BGP we have $n^*(t) = n(t) = n \in (\overline{n}(\rho), 1)$ and $\kappa_N v_N(n) = \kappa_I v_I(n)$. If, in addition, $\theta = 0$, then the equilibrium is unique everywhere and the BGP is globally (saddle-path) stable. If $\theta > 0$, then the equilibrium is unique in the neighborhood of the BGP and is asymptotically (saddle-path) stable;*

3 ***Multiple BGPs:*** *if $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$, there are multiple BGPs;*

4 ***No automation:*** *If $\underline{\kappa} > \frac{\kappa_I}{\kappa_N}$, there exists a unique BGP. In this BGP $n^*(t) = 1$ and all tasks are produced with labor. (When $\rho > \rho_{max}$, we are always in this case).*

Proof. See Appendix A. ■

This proposition provides a complete characterization of different types of BGPs. Figure 7 shows visually how different BGPs arise in parts of the parameter space.

[30] The condition $S < \overline{S}$ ensures that the growth rate of the economy is not too high. If the growth rate is above the threshold implied by $\overline{S}$, the creation of new tasks is discouraged (even if current wages are low) because firms anticipate that the wage will grow rapidly, reducing the future profitability of creating new labor-intensive tasks. This condition also allows us to use Taylor approximations of the value functions in our analysis of local stability. Finally, in parts 2-4 this condition ensures that the transversality condition holds.

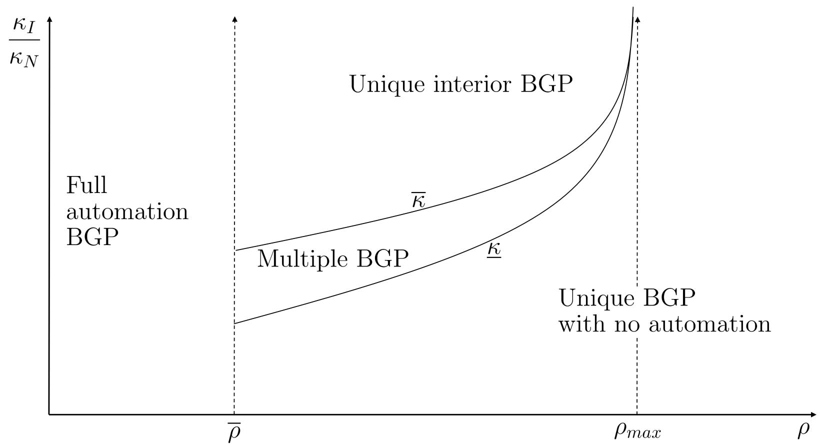

FIGURE 7: Varieties of BGPs.

Further intuition can be gained by studying the behavior $\kappa_I v_I(n)$ and $\kappa_N v_N(n)$, which we do in Figure 8. Lemma A3 shows that, for S small, the normalized value functions can be written as

$$v_I(n) = \frac{b\left((\rho+\delta+\theta g)^{\zeta-\hat{\sigma}} - w_I(n)^{\zeta-\hat{\sigma}}\right)}{\rho+(\theta-1)g} \qquad v_N(n) = \frac{b\left(w_N(n)^{\zeta-\hat{\sigma}} - (\rho+\delta+\theta g)^{\zeta-\hat{\sigma}}\right)}{\rho+(\theta-1)g}.$$

The profitability of the two types of technologies depends on the effective wages, $w_I(n)$ and $w_N(n)$. A lower value of n, which corresponds to additional automation, reduces $w_I(n)$—in other words $w_I(n)$ is increasing in n. This is because of comparative advantage: as more tasks are automated, the equilibrium wage increases less than $\gamma(I)$, and it becomes cheaper to produce the least complex tasks with labor, and thus automation becomes less profitable. Because $w_I(n)$ is increasing in n, so is $v_I(n)$ (recall that $\hat{\sigma} > \zeta$). However, $v_N(n)$ is also increasing in n: $w_N(n)$ is decreasing in n as the long-run wage increases with automation because of the productivity effect discussed in the previous section. We will see next that the fact that $v_N(n)$ is increasing in n creates a force towards multiplicity of BGPs, while the fact that $v_I(n)$ is increasing in n pushes towards uniqueness and stability.

Panel A of Figure 8 illustrates the first part of Proposition 6 (which parallels the first part of Proposition 4): when $\rho < \overline{\rho}$, $\kappa_I v_I(0)$ is above $\kappa_N v_N(0)$ for $n < \widetilde{n}(\rho)$. In this region it is not optimal to create new tasks. Consequently, there exists a BGP with full automation, meaning that all tasks will be automated and produced with capital. Reminiscent of Leontief's "horse equilibrium," in this BGP labor becomes redundant. Intuitively, as also shown in Figure 4, when $\rho < \overline{\rho}$ and $n < \widetilde{n}(\rho)$, we have $w_N(n) > \rho+\delta+\theta g$, which implies that labor is too expensive relative to capital. Utilizing and thus creating new tasks is not profitable. Economic growth in this BGP is driven by capital accumulation (because when all tasks are automated, aggregate output is linear in capital).

Panel B of the figure illustrates the remaining three types of BGPs, which apply when $\rho > \overline{\rho}$. In this case, at $n = 0$ (or at any $n \leq \bar{n}(\rho)$), $\kappa_I v_I(n)$ is strictly below $\kappa_N v_N(n)$, and thus a full automation BGP is not possible. The two curves can only intersect for $n \in (\bar{n}(\rho), 1]$, implying

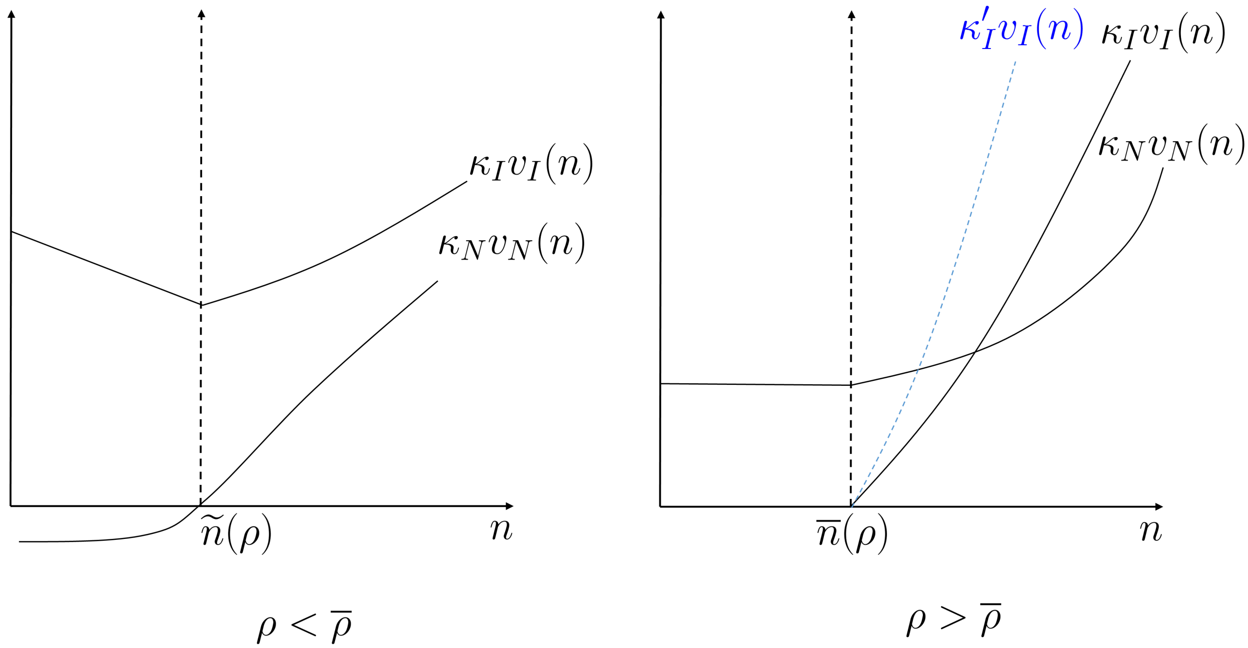

FIGURE 8: Asymptotic behavior of normalized values. In the left panel, $\kappa_I v_I(n)$ is everywhere above $\kappa_N v_N(n)$ and the BGP involves full automation. In the right panel, if κ_I/κ_N is sufficiently large, the two curves intersect and we have an interior BGP with both automation and creation of new tasks. The right panel also shows the effect of an increase in the productivity of scientists in automating tasks from κ_I to κ'_I.

that in any BGP, newly automated tasks will be immediately produced with capital. As explained above, both of these curves are increasing but their relative slopes depend on $\frac{\kappa_I}{\kappa_N}$. When $\frac{\kappa_I}{\kappa_N} < \underline{\kappa}$, $\kappa_I v_I(n)$ is not sufficiently steep relative to $\kappa_N v_N(n)$, and the two never intersect. This means that even at $n = 1$, it is not profitable to create new automation technologies, and all tasks will be produced with labor. In this BGP, capital becomes redundant, and growth is driven by endogenous technological change increasing labor's productivity as in the standard quality ladder models such as Aghion and Howitt (1992) or Grossman and Helpman (1991).

Conversely, when $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, the curve $\kappa_I v_I(n)$ is sufficiently steep relative to $\kappa_N v_N(n)$ so that the two curves necessarily intersect and can only intersect once. Hence there exists a unique interior BGP (interior in the sense that now the BGP level of n is strictly between 0 and 1, and thus some tasks are produced with labor and some with capital).

Finally, when $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$, the two curves will intersect, but will do so multiple times, leading to multiple interior BGPs.

Proposition 6 also shows that for $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, the unique interior BGP is globally stable provided that the intertemporal elasticity of substitution is infinite (i.e., $\theta = 0$), and locally stable otherwise (i.e., when $\theta > 0$). Because $\kappa_I v_I(n)$ starts below $\kappa_N v_N(n)$ at $\bar{n}(\rho)$ (reflecting the fact that at this point, new automation technologies are not immediately adopted and thus the value of creating these technologies is zero), the unique intersection must have the former curve being steeper than the former. At this point, a further increase in n always raises the value of automating an additional task, $v_I(n)$, more than the value of creating a new task, $v_N(n)$. This ensures that increases in n beyond its BGP value trigger further automation, while lower values of n encourage the creation of new tasks, ensuring the stability of the unique BGP.

The asymptotic stability of the interior BGP implies that there are powerful market forces pushing the economy towards balanced growth. An important consequence of this stability is that technological shocks that reduce n (e.g., the arrival of a series of new automation technologies) will set in motion self-correcting forces. Following such a change, there will be an adjustment process restoring the level of employment and the labor share back to their initial values.

This does not, however, imply that *all* shocks will leave the long-run prospects of labor unchanged. For one, this would not necessarily be the case in a situation with multiple steady states, and moreover, certain changes in the environment (for example, a large increase in B or a decline in ρ), can shift the economy from the region in which there is a unique interior BGP to the region with full automation, with disastrous consequences for labor. In addition, the next corollary shows that, if there is a change in the innovation possibilities frontier (in the κ's) that makes it permanently easier to develop new automation technologies, self-correcting forces still operate but will now only move the economy to a new BGP with lower employment and a lower labor share.

Corollary 2 *Suppose that $\rho > \overline{\rho}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. A one-time permanent increase in κ_I/κ_N leads to a BGP with lower n, employment and labor share.*

This corollary follows by noting that an increase in κ_I/κ_N shifts the intersection of the curves $\kappa_I v_I(n)$ and $\kappa_N v(n)$ to the left as shown by the blue dotted curve in Figure 8, leading to a lower value of n in the BGP. This triggers an adjustment process in which the labor share and employment decline over time, but ultimately settle to their new (interior) BGP values. The transition process will involve a slower rate of increase of N and a more rapid rate of increase of I than the BGP. Interestingly, if new tasks generate larger productivity gains than automation, this transition process will also be associated with a slowdown in productivity growth because automation crowds out resources that could be used to develop new tasks.[31]

In summary, Proposition 6 characterizes the different types of BGPs, and together with Corollary 2, it delineates the types of changes in technology that trigger self-correcting dynamics. Starting from the interior BGP, the effects of (small) increases in automation technology will reverse themselves over time, restoring employment and the labor share back to their initial values. Permanent changes in the ability of society to create new automation technologies trigger self-correcting dynamics as well, but these will take us towards a new BGP with lower employment and labor share,

[31] Forgone productivity gains from from slower creation of new tasks will exceed the gains from automation, causing a productivity slowdown during a transition to a higher level of automation, if $\rho > \rho_P$, where ρ_P is defined implicitly as the solution to the equation

$$\frac{1}{\sigma-1}(w_I(n)^{1-\sigma} - (\rho_P+\delta+\theta g)^{1-\sigma}) = \frac{1}{\sigma-1}((\rho_P+\delta+\theta g)^{1-\sigma} - w_N(n)^{1-\sigma}).$$

and may also involve slower productivity growth in the process.

5 Extensions

In this section we discuss three extensions. First we introduce heterogeneous skills, which allow us to analyze the impact of technological changes on inequality. Second, we study a different structure of intellectual property rights that introduces the creative destruction of profits. Finally, we discuss the welfare implications of our model.

5.1 Automation, New Tasks and Inequality

To study how automation and the creation of new tasks impact inequality, we now introduce heterogeneous skills. This extension is motivated by the observation that both automation and new tasks could increase inequality: new tasks favor high-skill workers who tend to have a comparative advantage in new and complex tasks, while automation substitutes capital for labor in lower-indexed tasks where low-skill workers have their comparative advantage.

The assumption that high-skill workers have a comparative advantage in new tasks receives support from the data. Figure 9 shows that occupations with more new job titles in 1980, 1990 and 2000 employed workers with greater average years of schooling.[32]

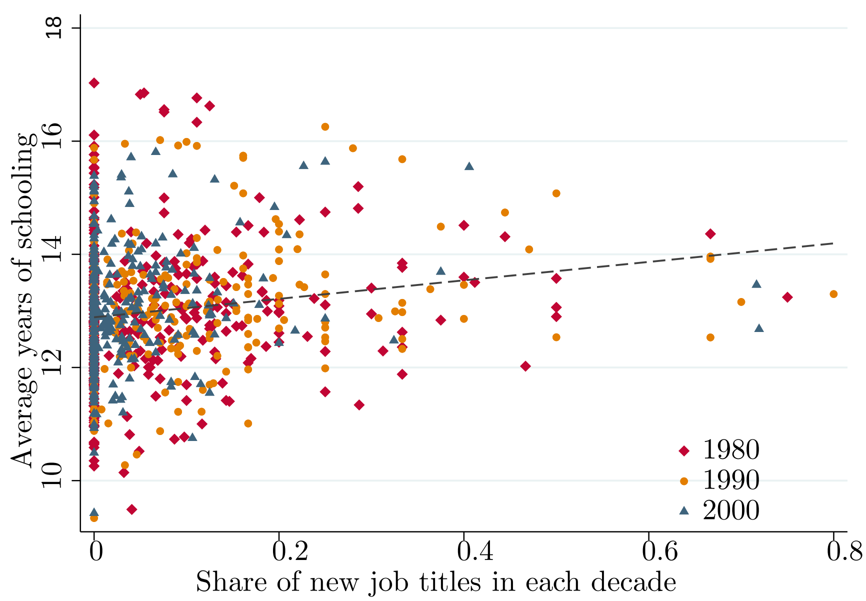

FIGURE 9: Average years of schooling among workers and the share of new job titles in 1980, 1990, and 2000. See Appendix B for data sources and detailed definitions.

To incorporate this feature, we assume that there are two types of workers: low-skill workers

[32]As in Figure 1, this figure partials out the demographic composition of employment in each occupation at the beginning of the relevant period. See Appendix B for the same relationship without these controls as well as with additional controls.

with time-varying productivity $\gamma_L(i,t)$ in task i, and high-skill workers with productivity $\gamma_H(i)$. We parametrize these productivities as follows:

Assumption 1″ *The productivities of high-skill and low-skill workers are given by*

$$\gamma_H(i) = e^{A_H i} \qquad\qquad \gamma_L(i,t) = e^{\xi A_H i}\Gamma(t - T(i)),$$

where Γ *is increasing with* $\lim_{x\to\infty}\Gamma(x) = 1$, $\xi \in (0,1]$, *and* $T(i)$ *denotes the time when task i was first introduced.*

Assumption 1″ is similar to but extends Assumption 1′ in several dimensions. The ratio $\frac{\gamma_H(i)}{\gamma_L(i,t)}$ is increasing in i, which implies that high-skill workers have a comparative advantage in higher-indexed tasks. But in addition, we also let the productivity of low-skill workers in a task increase over time, as captured by the increasing function Γ. This captures the idea that as new tasks become "standardized," they can be more productively performed by less skilled workers (e.g., Acemoglu, Gancia and Zilibotti, 2010), or that workers adapt to new technologies by acquiring human capital through training, on-the-job learning and schooling (e.g., Schultz, 1965, Nelson and Phelps, 1966, Galor and Moav, 2000, Beaudry, Green and Sand, 2013, and Goldin and Katz, 2008). Since the function Γ limits to 1 over time, the parameter ξ determines whether this standardization effect is complete or incomplete. When $\xi < 1$, the productivity of low-skill workers relative to high-skill workers converges to $\gamma_L(i,t)/\gamma_H(i) = \gamma_H(i)^{\xi-1}$, and limits to zero as more and more advanced tasks are introduced. In contrast, when $\xi = 1$, the relative productivity of low-skill workers converges to 1 for tasks that have been around for a long time.

The structure of comparative advantage ensures that there exists a threshold task M such that high-skill labor performs tasks in $[M,N]$, low-skill labor performs tasks in (I^*,M), and capital performs tasks in $[N-1,I^*]$. In what follows, we denote the wages of high and low-skill labor by W_H and W_L, respectively, and to simplify the discussion, we focus on the economy with exogenous technology and assume that the supply of high-skill labor is fixed at H and the supply of low-skill labor is fixed at L.

Proposition 7 (Automation, new tasks and inequality) *Suppose Assumptions 1″ and 2 hold. Suppose also that technology evolves exogenously with* $\dot{N} = \dot{I} = \Delta$ *and* $n(t) = n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ *(and* $A_H(1-\theta)\Delta < \rho$*). Then, there exists a unique BGP. Depending on the value of* ξ *this BGP takes one of the following forms:*

1. *If* $\xi < 1$, *in the unique BGP we have* $\lim_{t\to\infty} W_H(t)/W_L(t) = \infty$, *the share of tasks performed by low-skill workers converges to zero, and capital and high-skill workers perform constant shares of tasks.*

2. *If* $\xi = 1$, *in the unique BGP* $W_H(t)$ *and* $W_L(t)$ *grow at the same rate as the economy, the wage gap,* $W_H(t)/W_L(t)$, *remains constant, and capital, low-skill and high-skill workers perform constant shares of tasks. Moreover,* $\lim_{t\to\infty} W_H(t)/W_L(t)$ *is decreasing in* n. *Consequently,*

a permanent increase in N raises the wage gap $W_H(t)/W_L(t)$ in the short run, but reduces it in the long run, while a permanent increase in I raises the wage gap in both the short and the long run.

Like all remaining proofs in the paper, the proof of this proposition is in Appendix B.

When $\xi < 1$, this extension confirms the pessimistic scenario about the implications of new technologies for wage inequality and the employment prospects of low-skill workers—both automation and the creation of new tasks increase inequality, the former because it displaces low-skill workers ahead of high-skill workers, and the latter because it directly benefits high-skill workers who have a comparative advantage in newer, more complex tasks relative to low-skill workers. As a result, low-skill workers are progressively squeezed into a smaller and smaller set of tasks, and wage inequality grows without bound.

However, our extended model also identifies a countervailing force, which becomes particularly potent when $\xi = 1$. Because new tasks become standardized, they can over time be as productively used by low-skill workers. In this case, automation and the creation of new tasks still reduce the relative earnings of low skill-workers in the short run, but their long-run implications are very different. In the long run, inequality is decreasing in n (because a higher n translates into a greater range of tasks for low-skill workers). Consequently, automation increases inequality both in the short and the long run. The creation of new tasks, which leads to a permanently higher level of n, increases inequality in the short run but reduces it in the long run. These observations suggest that inequality may be high following a period of adjustment in which the labor share first declines (due to increases in automation), and then recovers (due to the introduction and later standardization of new tasks).

5.2 Creative Destruction of Profits

In this subsection, we modify our baseline assumption on intellectual property rights and revert to the classical setup in the literature in which new technologies do not infringe the patents of the products that they replace (Aghion and Howitt, 1992, and Grossman and Helpman, 1991). This assumption introduces the creative destruction effects—the destruction of profits of previous inventors by new innovators. We will see that this alternative structure has similar implications for the BGP, but necessitates more demanding conditions to guarantee its uniqueness and stability.

Let us first define $V_N(t,i)$ and $V_I(t,i)$ as the time t values for technology monopolist with, respectively, new task and automation technologies. These value functions satisfy the following Bellman equations:

$$r(t)V_N(t,i) - \dot{V}_N(t,i) = \pi_N(t,i) \qquad\qquad r(t)V_I(t,i) - \dot{V}_I(t,i) = \pi_I(t,i).$$

Here $\pi_I(t,i)$ and $\pi_N(t,i)$ denote the flow profits from automating and creating new tasks, respectively, which are given by the formulas in equations (23) and (24).

For a firm creating a new task i, let $T^N(i)$ denote the time at which it will be replaced by a technology allowing the automation of this task. Likewise, let $T^I(i)$ denote the time at which an automated task i will be replaced by a new task using labor. Since firms anticipate these deterministic replacement dates, their value functions also satisfy the boundary conditions $V_N(T^N(i), i) = 0$ and $V_I(T^I(i), i) = 0$. Together with these boundary conditions, the Bellman equations solve for

$$V_N^{CD}(t) = V_N(N(t), t) = b \int_t^{T^N(N(t))} e^{-\int_t^\tau (R(s)-\delta)ds} Y(\tau) \left(\frac{W(\tau)}{\gamma(N(t))} \right)^{\zeta - \hat{\sigma}} d\tau,$$
$$V_I^{CD}(t) = V_I(I(t), t) = b \int_t^{T^I(I(t))} e^{-\int_t^\tau (R(s)-\delta)ds} Y(\tau) \left(\min \left\{ R(\tau), \frac{W(\tau)}{\gamma(I(t))} \right\} \right)^{\zeta - \hat{\sigma}} d\tau.$$

For reasons that will become evident, we modify the innovation possibilities frontier to

$$\dot{I}(t) = \kappa_I \iota(n(t)) S_I(t), \text{ and } \dot{N}(t) = \kappa_N S_N(t) \tag{30}$$

Here, the function $\iota(n(t))$ is included and assumed to be nondecreasing to capture the possibility that automating tasks closer to the frontier (defined as the highest-indexed task available) may be more difficult.

Let us again define the normalized value functions as $v_I^{CD}(n) = \lim_{t \to \infty} \frac{V_I^{CD}(t)}{Y(t)}$ and $v_N^{CD}(n) = \lim_{t \to \infty} \frac{V_N^{CD}(t)}{Y(t)}$. In a BGP, the normalized value functions only depend on n because newly-created tasks are automated after a period of length $T^N(N(t)) - t = \frac{n}{\Delta}$, and newly-automated tasks are replaced by new ones after a period of length $T^I(I(t)) - t = \frac{1-n}{\Delta}$, where $\Delta = \frac{\kappa_I \kappa_N \iota(n)}{\kappa_I \iota(n) + \kappa_N} S$ is the endogenous rate at which N and I grow. The endogenous value of n in an interior BGP satisfies

$$\kappa_I \iota(n) v_I^{CD}(n) = \kappa_N v_N^{CD}(n).$$

The next proposition focuses on interior BGPs and shows that, because of creative destruction, we must impose additional assumptions on the function $\iota(n)$ to guarantee stability.

Proposition 8 (Equilibrium with creative destruction) *Suppose that $\rho > \overline{\rho}$, Assumptions 1′, 2 and 4 hold and there is creative destruction of profits. Then:*

1. *There exist $\overline{\iota}$ and $\underline{\iota} < \overline{\iota}$ such that if $\iota(0) < \underline{\iota}$ and $\iota(1) > \overline{\iota}$, then there is at least one locally stable interior BGP with $n(t) = n \in (\overline{n}(\rho), 1)$.*

2. *If $\iota(n)$ is constant, there is no stable interior BGP (with $n(t) = n \in (\overline{n}(\rho), 1)$). Any stable BGP involves $n(t) \to 0$ or $n(t) \to 1$.*

The first part of the proposition follows from an analogous argument to that in the proof of Proposition 6, with the only difference being that, because of the presence of the function $\iota(n)$ in equation (30), the key condition that pins down n becomes $\kappa_I \iota(n) v_I^{CD}(n) = \kappa_N v_N^{CD}(n)$.

The major difference with our previous analysis is that creative destruction introduces a new source of instability. Unlike the previous case with no creative destruction, we now have that $v_I^{CD}(n)$ is decreasing in n. As more tasks are automated, the rental rate remains unchanged and newly-automated tasks will be replaced less frequently (recall that newly-automated tasks are replaced after $(1-n)/\Delta$ units of time). As a result, automating more tasks renders further automation more profitable. Moreover, $v_N^{CD}(n)$ continues to be increasing in n. This is for two reasons: first, as before, the productivity effect ensures that the effective wage in new tasks, $w_N(n)$, is decreasing in n; and second, because newly-created tasks are automated after $\frac{n}{\Delta}$ units of time, an increase in n increases the present discounted value of profits from new tasks. These observations imply that, if $\iota(n)$ were constant, the intersection between the curves $\kappa_N v_N^{CD}(n)$ and $\kappa_I \iota(n) v_I^{CD}(n)$ would correspond to an unstable BGP.

Economically, the instability is a consequence of the fact that, in contrast to our baseline model (and the socially planned economy which we describe in the next subsection), here innovation incentives depend on the total revenue that a technology generates rather than its *incremental value* created (the difference between these revenues and the revenues that the replaced technology generated). In our baseline model, the key force ensuring stability is that incentives to automate are shaped by the cost difference between producing a task with capital or with labor—by lowering the effective wage at the next tasks to be automated, current automation reduces the incremental value of additional automation. This force is absent when innovators destroy the profits of previous technology monopolists because they no longer care about the cost of production with the technology that they are replacing.

5.3 Welfare

We study welfare from two complementary perspectives. First, in Appendix B we discuss the socially optimal allocation in the presence of endogenous technology and characterize how this allocation can be decentralized. One of the main insights from Proposition 6 is that the expected path for factor prices determines the incentives to automate and create new tasks. We show that a planner would also allocate scientists according to the same principle—guided by the *cost savings* that each technology grants to firms. Although similar to the efficient allocation of scientists in this regard, the decentralized equilibrium is typically inefficient because the technology monopolists neither capture the full benefits from the new tasks they create nor internalize how their innovation affects other existing and future technology monopolists.

The second perspective is more novel and relevant to current debates about automation reducing employment and its policy implications. We examine whether an exogenous increase in automation could reduce welfare. Even though automation expands productivity—a force which always raises welfare—it also reduces employment. When the labor market is fully competitive as in our baseline model, this reduction in employment has no first-order welfare cost for the representative household (who sets the marginal cost of labor supply equal to the wage). Consequently, automation increases overall welfare. Next suppose that there are labor market frictions. In particular, suppose that there exists an upward-sloping quasi–labor supply schedule, $L_{qs}(\omega)$, which constrains the level of employment, so that $L \leq L_{qs}(\omega)$ (see Appendix B for a microfoundation). This quasi-labor supply schedule then acts in a very similar fashion to the labor supply curve derived in (11) in Section 2, except that the marginal cost of labor supply is no longer equated to the wage. Crucially, the reduction in employment resulting from automation now has a negative impact on welfare, and this negative effect can exceed the positive impact following from the productivity gains, turning automation, on net, into a negative for welfare.

The next proposition provides the conditions under which automation can reduce welfare in the context of our static model with exogenous technology. Our focus on the static model is for transparency. The same forces are present in the dynamic model and also in the full model with endogenous technology.

Proposition 9 (Welfare implications of automation) *Consider the static economy and suppose that Assumptions 1, 2 and 3 hold, and that $I^* = I < \widetilde{I}$. Let $\mathcal{W} = u(C, L)$ denote the welfare of representative household.*

1. *Consider the baseline model without labor market frictions, where the representative household chooses the amount of labor without constraints and thus $\frac{W}{C} = \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) > 0,$$

$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) > 0.$$

2. *Suppose that there are labor market frictions, so that employment is constrained by a quasi-labor supply curve $L \leq L_{qs}(\omega)$. Suppose also that the quasi-labor supply schedule $L_{qs}(\omega)$ is increasing in ω, has an elasticity $\widetilde{\varepsilon}_L > 0$, and is binding in the sense that $\frac{W}{C} > \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) - L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_I\right] \lesseqgtr 0.$$

$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}} \left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) + L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_N\right] > 0.$$

The first part of the proposition shows that both types of technological improvements increase welfare when the labor market has no frictions. In this case, automation increases productivity by substituting cheaper capital for human labor, and this leads to less work for workers, but since they were previously choosing labor supply optimally, a small reduction in employment does not have a first-order impact on welfare, and overall welfare increases. The implications of the creation of new tasks are similar.

The situation is quite different in the presence of labor market frictions, however, as shown in the second part. Automation again increases productivity and reduces employment. But now, because workers are constrained in their labor supply choices, the lower employment that results from automation has a first-order negative effect on their welfare. Consequently, automation can reduce welfare if the productivity gains, captured by the first term, are not sufficiently large to compensate for the second, negative term. Interestingly, in this case new tasks increase welfare even more than before, because they not only raise productivity but also expand employment, and by the same logic, the increase in labor supply has a welfare benefit for the workers (since they were previously constrained in their employment).

An important implication of this analysis emphasized further in Appendix B is that when labor market frictions are present and the direction of technological change is endogenized, there will be a force towards excessive automation. In particular, in this case, assuming that labor market frictions also constrain the social planner's choices, the decentralized equilibrium involves too much effort being devoted to improving automation relative to what she would like—because the social planner recognizes that additional automation has a negative effect through employment.

6 Conclusion

As automation, robotics and AI technologies are advancing rapidly, concerns that new technologies will render labor redundant have intensified. This paper develops a comprehensive framework in which these forces can be analyzed and contrasted. At the center of our model is a task-based framework. Automation is modeled as the (endogenous) expansion of the set of tasks that can be performed by capital, replacing labor in tasks that it previously produced. The main new feature of our framework is that, in addition to automation, there is another type of technological change complementing labor. In our model, this takes the form of the introduction of new, more complex versions of existing tasks, and it is assumed that labor has a comparative advantage in these new tasks. We characterize the structure of equilibrium in such a model, showing how, given factor prices, the allocation of tasks between capital and labor is determined both by available technology and the endogenous choices of firms between producing with capital or labor.

One attractive feature of task-based models is that they highlight the link between factor prices and the range of tasks allocated to factors: when the equilibrium range of tasks allocated to capital increases (for example, as a result of automation), the wage relative to the rental rate and the labor share decline, and the equilibrium wage rate may also fall. Conversely, as the equilibrium range of tasks allocated to labor increases, the opposite result obtains. In our model, because the supply of labor is elastic, automation tends to reduce employment, while the creation of new tasks increases employment. These results highlight that, while both types of technological changes undergird economic growth, they have very different implications for the factor distribution of income and employment.

Our full model endogenizes the direction of research towards automation and the creation of new tasks. If in the long run capital is very cheap relative to labor, automation technologies will advance rapidly and labor will become redundant. However, when the long-run rental rate of capital is not so low relative to labor, our framework generates a BGP in which both types of innovation go hand-in-hand. Moreover in this case, under reasonable assumptions, the dynamic equilibrium is unique and converges to the BGP. Underpinning this stability result is the impact of relative factor prices on the direction of technological change. The task-based framework—differently from the standard models of directed technological change based on factor-augmenting technologies—implies that as a factor becomes cheaper, this not only influences the range of tasks allocated to it, but also generates incentives for the introduction of technologies that allow firms to utilize this factor more intensively. These economic incentives then imply that by reducing the effective cost of labor in the least complex tasks, automation discourages further automation and generates a self-correcting force towards stability.

We show in addition that, though market forces ensure the stability of the BGP, they do not necessarily generate the efficient composition of technology. If the elastic labor supply relationship results from rents (so that there is a wedge between the wage and the opportunity cost of labor), there is an important new distortion: because firms make automation decisions according to the wage rate, not the lower opportunity cost of labor, there is a natural bias towards excessive automation.

Several commentators are further concerned about the inequality implications of automation and related new technologies. We study this question by extending our model so that high-skill labor has a comparative advantage in new tasks relative to low-skill labor. In this case, both automation (which squeezes out tasks previously performed by low-skill labor) and the creation of new tasks (which directly benefits high-skill labor) increase inequality. Nevertheless, the long-term implications of the creation of new tasks could be very different, because they are later standardized and used by low-skill labor. If this standardization effect is sufficiently powerful, there exists a BGP in which not only the factor distribution of income (between capital and labor) but also inequality between the two skill types stays constant.

We consider our paper to be a first step towards a systematic investigation of different types of technological changes that impact capital and labor differentially. Several areas of research appear fruitful based on this first step. First, our model imposes that it is always the tasks at the bottom that are automated; in reality, it may be those in the middle (e.g., Acemoglu and Autor, 2001). Incorporating the possibility of such "middling tasks" being automated is an important generalization, though ensuring a pattern of productivity growth consistent with balanced growth in this case is more challenging. Second, there may be technological barriers to the automation of certain tasks and the creation of new tasks across industries (e.g., Polanyi, 1966, Autor, Levy and Murnane, 2003). An interesting step is to construct realistic models in which the sectoral composition of tasks performed by capital and labor as well as technology evolves endogenously and is subject to industry-level technological constraints (e.g., on the feasibility or speed of automation). Third, in this paper we have focused on the creation of new labor-intensive tasks as the type of technological change that complements labor and plays a countervailing role against automation. Another interesting area is to theoretically and empirically investigate different types of technologies that may complement labor. Fourth, our analysis of the creation of new tasks and standardization abstracted from the need for workers to acquire new skills to work in such tasks. In practice, the acquisition of new skills may need to go hand-in-hand with workers shifting to newer tasks, and the inability of the educational system to adapt to the requirements of these new tasks could become a bottleneck and prevent the rebound in the demand for labor following a wave of automation. Finally, and perhaps most importantly, our model highlights the need for additional empirical evidence on how automation impacts employment and wages (which we investigate in Acemoglu and Restrepo, 2017a) and how the incentives for automation and the creation of new tasks respond to policies, factor prices and supplies (some aspects of which are studied in Acemoglu and Restrepo, 2017b).

Appendix A: Proofs

General Model

The analysis in the text was carried out under Assumption 2, which imposed $\eta \to 0$ or $\zeta = 1$, and significantly simplified some of the key expressions. Throughout the Appendix, we relax Assumption 2 and replace it with:

Assumption 2′ *One of the following three conditions holds: (i)* $\eta \to 0$*; (ii)* $\zeta = 1$*; or (iii)*

$$|\sigma - \zeta| < \left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{\max\{1,\sigma\}} \frac{1}{\left(\frac{\gamma(N)}{\gamma(N-1)}\right)^{|1-\zeta|} - 1}. \tag{A1}$$

All of our qualitative results remain true and will be proved under this more general assumption. Intuitively, the conditions in Assumption 2 ensured homotheticity (see footnote 11). Assumption 2′ , on the other hand, requires that the departure from homotheticity is small relative to the inverse of the productivity gains from new tasks (where $\gamma(N)/\gamma(N-1)$ measures these productivity gains).

Task prices in this more general case are given by

$$p(i) = \begin{cases} c^u\left(\min\left\{R, \frac{W}{\gamma(i)}\right\}\right) = \left[\eta\psi^{1-\zeta} + (1-\eta)\min\left\{R, \frac{W}{\gamma(i)}\right\}^{1-\zeta}\right]^{\frac{1}{1-\zeta}} & \text{if } i \le I, \\ c^u\left(\frac{W}{\gamma(i)}\right) = \left[\eta\psi^{1-\zeta} + (1-\eta)\left(\frac{W}{\gamma(i)}\right)^{1-\zeta}\right]^{\frac{1}{1-\zeta}} & \text{if } i > I. \end{cases} \tag{A2}$$

Here $c^u(\cdot)$ is the unit cost of production for task i, derived from the task production functions, (2) and (3). Naturally, this equation simplifies to (5) under Assumption 2.

From equations (5) and (7), equilibrium levels of task production are

$$y(i) = \begin{cases} B^{\hat{\sigma}-1} Y c^u\left(\min\left\{R, \frac{W}{\gamma(i)}\right\}\right)^{-\sigma} & \text{if } i \le I, \\ B^{\hat{\sigma}-1} Y c^u\left(\frac{W}{\gamma(i)}\right)^{-\sigma} & \text{if } i > I. \end{cases}$$

Combining this with equations (2) and (3), we obtain the task-level demands for capital and labor as

$$k(i) = \begin{cases} B^{\hat{\sigma}-1}(1-\eta) Y c^u(R)^{\zeta-\sigma} R^{-\zeta} & \text{if } i \le I^*, \\ 0 & \text{if } i > I^*. \end{cases}$$

and

$$l(i) = \begin{cases} 0 & \text{if } i \leq I^*, \\ B^{\hat{\sigma}-1}(1-\eta)Y\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta} & \text{if } i > I^*. \end{cases}$$

Aggregating the preceding two equations across tasks, we obtain the following capital and labor market-clearing equations,

$$B^{\hat{\sigma}-1}(1-\eta)Y(I^*-N+1)c^u(R)^{\zeta-\sigma}R^{-\zeta} = K, \tag{A3}$$

and

$$B^{\hat{\sigma}-1}(1-\eta)Y\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}di = L^s\left(\frac{W}{RK}\right). \tag{A4}$$

Finally, from the choice of aggregate output as the numeraire, we obtain a generalized version of the ideal price condition,

$$(I^*-N+1)c^u(R)^{1-\sigma} + \int_{I^*}^{N}c^u\left(\frac{W}{\gamma(i)}\right)^{1-\sigma}di = B^{1-\hat{\sigma}}, \tag{A5}$$

which again simplifies to the ideal price index condition in the text, (10), under Assumption 2.

Proofs from Section 2

Proof of Proposition 1: We prove Proposition 1 under the more general Assumption 2′.

To prove the existence and uniqueness of the equilibrium, we proceed in three steps. First, we show that I^*, N and K, determine unique equilibrium values for R, W and Y, thus allowing us to define the function $\omega(I^*, N, K)$ representing the relative demand for labor, which was introduced in the text. Second, we prove a lemma which ensures that $\omega(I^*, N, K)$ is decreasing in I^* (and increasing in N). Third, we show that $\min\{I, \widetilde{I}\}$ is nondecreasing in ω and conclude that there is a unique pair $\{\omega^*, I^*\}$ such that $I^* = \min\{I, \widetilde{I}\}$ and $\omega^* = \omega(I^*, N, K)$. This pair uniquely determines the equilibrium relative factor prices and the range of tasks that get effectively automated.

Step 1: Consider I^*, N and K such that $I^* \in (N-1, N)$. Then, R, W and Y satisfy the system of equations given by capital and labor market-clearing, equations (A3) and (A4), and the ideal price index, equation (A5).

Taking the ratio of (A3) and (A4), we obtain

$$\frac{\int_{I^*}^{N}\gamma(i)^{\zeta-1}c^u\left(\frac{W}{\gamma(i)}\right)^{\zeta-\sigma}W^{-\zeta}di}{L^s\left(\frac{W}{RK}\right)(I^*-N+1)c^u(R)^{\zeta-\sigma}R^{-\zeta}} = \frac{1}{K}. \tag{A6}$$

In view of the fact that L^s is increasing and the function $c^u(x)^{\zeta-\sigma}x^{-\zeta}$ is decreasing in x (as it can be verified directly by differentiation), it follows that the left-hand side is decreasing in W and increasing in R . Therefore, (A6) defines an upward-sloping relationship between W and R, which we refer to as the relative demand curve (because it traces the combinations of wage and rental rate consistent with the demand for labor relative to capital being equal to the supply of labor divided by the capital stock).

On the other hand, inspection of equation (A5) readily shows that this equation gives a downward-sloping locus between R and W as shown in Figure A1, which we refer to as the ideal price curve.

The unique intersection of the relative demand and ideal price curves determines the equilibrium factor prices for given I^*, N and K. Because the relative demand curve is upward sloping and the ideal price index curve is downward sloping, there can be at most one intersection. To prove that there always exists an intersection, observe that $\lim_{x\to 0} c^u(x)^{\zeta-\sigma}x^{-\zeta} = \infty$, and that $\lim_{x\to\infty} c^u(x)^{\zeta-\sigma}x^{-\zeta} = 0$. These observations imply that as $W \to 0$, the numerator of (A6) limits to infinity, and so must the denominator, i.e., $R \to 0$. This proves that the relative demand curve starts from the origin. Similarly, as $W \to \infty$, the numerator of (A6) limits to zero, and so must the denominator (i.e., $R \to \infty$). This then implies that the relative demand curve goes to infinity as $R \to \infty$. Thus, the upward-sloping relative demand curve necessarily starts below and ends above the ideal price curve, which ensures that there always exists an intersection between these curves. The unique intersection defines the equilibrium values of W and R, and therefore the function $\omega(I^*, N, K) = \frac{W}{RK}$.

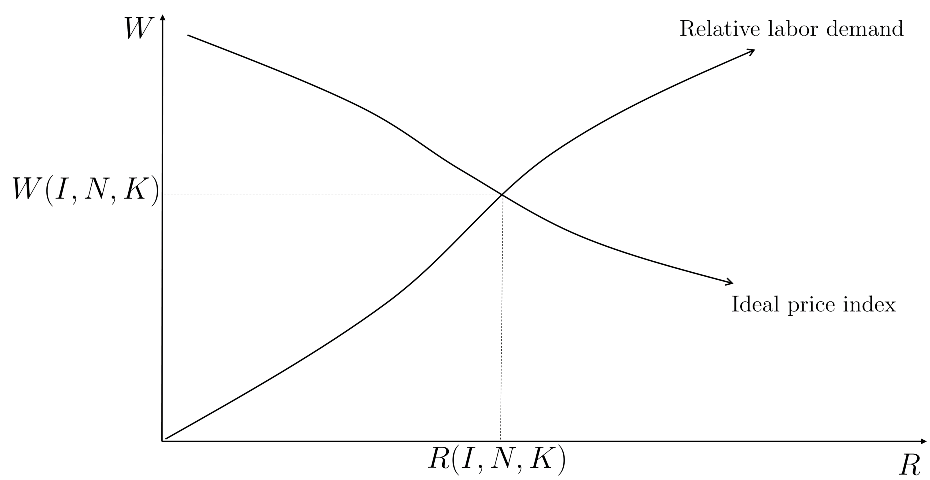

FIGURE A1: Construction of the function $\omega(I^*, N, K)$.

Step 2: This step follows directly from the following lemma, which we prove in Appendix B.

Lemma A1 *Suppose that Assumption 2′ holds,* $K < \overline{K}$ *and* $I^* \leq \widetilde{I}$. *Then* $\omega(I^*, N, K)$ *is decreasing in* I^* *and is increasing in* N.

Although the general proof for this lemma is long (and thus relegated to Appendix B), the lemma is trivial under Assumption 2. In that case, equation (A6) yields:

$$\omega(I^*, N, K)^{\hat{\sigma}} L^s(\omega(I^*, N, K)) = \frac{\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di}{I^* - N + 1} K^{1-\hat{\sigma}}.$$

Taking logs, we obtain equation (13) in the main text, which implies that $\omega(I^*, N, K)$ is increasing in N and decreasing in I^*.

Step 3: We now show that $I^* = \min\{I, \widetilde{I}\}$ is uniquely defined. Because $\gamma(\widetilde{I}) = \omega K$, we have that $I^* = \min\{I, \widetilde{I}\}$ is increasing in ω and has a vertical asymptote at I.

Consider the pair of equations $\omega = \omega(I^*, N, K)$ and $I^* = \min\{I, \widetilde{I}\}$ plotted in Figure 3. Because $\omega = \omega(I^*, N, K)$ is decreasing in I^* for $I^* \leq \widetilde{I}$ and $I^* = \min\{I, \widetilde{I}\}$ is increasing in ω, there exists at most a unique (ω, I^*) satisfying these two equations (or a unique intersection in the figure).

To prove existence, we again verify the appropriate boundary conditions. Suppose that $I^* \to N - 1$. Then from (A3), $R \to 0$, while $W > 0$, and thus $\omega \to \infty$. This ensures that the curve $\omega(I^*, N, K)$ starts above $I^* = \min\{I, \widetilde{I}\}$ in Figure 3. Since $I^* = \min\{I, \widetilde{I}\}$ has a vertical asymptote at $I < N$, the two curves must intersect. This observation completes the proof of the existence and uniqueness of the equilibrium.

When Assumption 2 holds we can explicitly solve for aggregate output. In this case, the market-clearing conditions, (8) and (9), become

$$R = \left(B^{\hat{\sigma}-1}(1-\eta)(I^* - N + 1)\frac{Y}{K} \right)^{\frac{1}{\hat{\sigma}}} \qquad W = \left(B^{\hat{\sigma}-1}(1-\eta)\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di \frac{Y}{L} \right)^{\frac{1}{\hat{\sigma}}},$$

which combined with (10) yields (12), completing the proof of Proposition 1. ■

Proofs from Section 3

Lemma A2 (Derivation of Figure 4) *Suppose that Assumptions 1′ and 2′ hold. Consider a path of technology where $n(t) \to n$ and $g(t) \to g$, consumption grows at the rate g and the Euler equation* (17) *holds. Then, there exist $\rho_{min} < \overline{\rho} < \rho_{max}$ such that:*

1. *If $\rho \in [\rho_{min}, \overline{\rho}]$, there is a decreasing function $\widetilde{n}(\rho) : [\rho_{min}, \overline{\rho}] \to (0, 1]$ such that for all $n > \widetilde{n}(\rho)$ we have $w_I(n) > \rho + \delta + \theta g > w_N(n)$ and $\rho + \delta + \theta g = w_N(\widetilde{n}(\rho))$. Moreover, $\widetilde{n}(\rho_{min}) = 1$ and $\widetilde{n}(\overline{\rho}) = 0$.*

2. *If $\rho \in [\overline{\rho}, \rho_{max}]$, there is an increasing function $\overline{n}(\rho) : [\overline{\rho}, \rho_{max}] \to (0, 1]$ such that for all $n > \overline{n}(\rho)$, we have $w_I(n) > \rho + \delta + \theta g > w_N(n)$ and $\rho + \delta + \theta g = w_I(\overline{n}(\rho))$. Moreover, $\overline{n}(\rho_{max}) = 1$ and $\overline{n}(\overline{\rho}) = 0$.*

3. *If $\rho > \rho_{max}$, for all $n \in [0, 1]$ we have $\rho + \delta + \theta g > w_I(n) \geq w_N(n)$, which implies that automation is not profitable for any $n \in [0, 1]$.*

4. *If $\rho < \rho_{min}$, for all $n \in [0, 1]$ we have $w_I(n) \geq w_N(n) > \rho + \delta + \theta g$, which implies that new tasks do not increase aggregate output and will not be adopted for any $n \in [0, 1]$.*

Proof. Because consumption grows at the rate g and the Euler equation (17) holds, we have

$$R(t) = \rho + \delta + \theta g.$$

The effective wages $w_I(n)$ and $w_N(n)$ are then determined by the generalized ideal price index condition, (A5), as

$$B^{1-\hat{\sigma}} = (1-n)c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^n c^u\left(\frac{w_I(n)}{\gamma(i)}\right)^{1-\sigma} di \tag{A7}$$
$$= (1-n)c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^n c^u(\gamma(i)w_N(n))^{1-\sigma} di.$$

Differentiating these expressions, we obtain

$$\frac{w_I'(n)}{w_I(n)} = \frac{1}{1-\sigma}\left(c^u(\rho+\delta+\theta g)^{1-\sigma} - c^u(w_N(n))^{1-\sigma}\right)\frac{1}{\int_0^n c^{u\prime}(w_N(n)\gamma(i))c^u(w_N(n)\gamma(i))^{-\sigma}w_N(n)\gamma(i)di}$$
$$\frac{w_N'(n)}{w_N(n)} = \frac{1}{1-\sigma}\left(c^u(\rho+\delta+\theta g)^{1-\sigma} - c^u(w_I(n))^{1-\sigma}\right)\frac{1}{\int_0^n c^{u\prime}(w_N(n)\gamma(i))c^u(w_N(n)\gamma(i))^{-\sigma}w_N(n)\gamma(i)di}. \tag{A8}$$

To prove part 1, define ρ_{min} as

$$\rho_{min} + \delta + \theta g = w_N(1),$$

and define $\overline{\rho} > \rho_{min}$ as

$$c^u(\overline{\rho}+\delta+\theta g)^{1-\sigma} = B^{1-\hat{\sigma}}.$$

(When Assumption 2 holds we get $\overline{\rho} = B - \delta - \theta g$, as claimed in the main text).

To show that $\overline{\rho} > \rho_{min}$, note that

$$\begin{aligned}\frac{1}{1-\sigma}c^u(\rho_{min}+\delta+\theta g)^{1-\sigma} &= \frac{1}{1-\sigma}\int_0^1 c^u(\rho_{min}+\delta+\theta g)^{1-\sigma}di\\ &= \frac{1}{1-\sigma}\int_0^1 c^u(w_N(1))^{1-\sigma}di\\ &< \frac{1}{1-\sigma}\int_0^1 c^u(w_N(1)\gamma(i))^{1-\sigma}di\\ &= \frac{1}{1-\sigma}B^{1-\hat{\sigma}}\\ &= \frac{1}{1-\sigma}c^u(\overline{\rho}+\delta+\theta g)^{1-\sigma}.\end{aligned}$$

Because the function $\frac{1}{1-\sigma}c^u(x)^{1-\sigma}$ is increasing, we have $\overline{\rho} > \rho_{min}$.

Using the generalized ideal price index condition, (A5), we define $\widetilde{n}(\rho)$ implicitly as

$$B^{1-\hat{\sigma}} = (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\widetilde{n}(\rho)} c^u(\gamma(i)(\rho+\delta+\theta g))^{1-\sigma}di.$$

Differentiating this expression with respect to ρ shows that $\widetilde{n}(\rho)$ is decreasing. Moreover, $\widetilde{n}(\rho_{min}) = 1$ and $\widetilde{n}(\overline{\rho}) = 0$, so $\widetilde{n}(\cdot)$ is well-defined for $\rho \in [\rho_{min}, \overline{\rho}]$.

For $n = \widetilde{n}(\rho)$, we have $w_I(\widetilde{n}(\rho)) > \rho+\delta+\theta g = w_N(\widetilde{n}(\rho))$. Thus, the formulas for $w_I'(n)$ and $w_N'(N)$ show that, for $\rho \in [\rho_{min}, \overline{\rho}]$ and starting at $\widetilde{n}(\rho)$, the curve $w_N(n)$ is decreasing in n and the curve $w_I(n)$ is increasing in n. Thus, for all $n > \widetilde{n}(\rho)$, we have

$$w_I(n) > w_I(\widetilde{n}(\rho)) > \rho+\delta+\theta g = w_N(\widetilde{n}(\rho)) > w_N(n),$$

as claimed. On the other hand, for all $n < \widetilde{n}(\rho)$, we have $w_N(n) > \rho + \delta + \theta g$.

To prove part 2, define $\rho_{max} > \overline{\rho}$ as

$$\rho_{max} + \delta + \theta g = w_I(1).$$

To show that $\overline{\rho} < \rho_{max}$, a similar argument establishes

$$\begin{aligned}
\frac{1}{1-\sigma} c^u(\rho_{max} + \delta + \theta g)^{1-\sigma} &= \frac{1}{1-\sigma} \int_0^1 c^u(\rho_{max} + \delta + \theta g)^{1-\sigma} di \\
&> \frac{1}{1-\sigma} \int_0^1 c^u(w_I(1)/\gamma(i))^{1-\sigma} di \\
&= \frac{1}{1-\sigma} c^u(\overline{\rho} + \delta + \theta g)^{1-\sigma}.
\end{aligned}$$

Because the function $\frac{1}{1-\sigma} c^u(x)^{1-\sigma}$ is increasing, we have $\overline{\rho} < \rho_{max}$.

Using (A7), we define the function $\overline{n}(\rho)$ implicitly as

$$B^{1-\hat{\sigma}} = (1 - \overline{n}(\rho)) c^u(\rho + \delta + \theta g)^{1-\sigma} + \int_0^{\overline{n}(\rho)} c^u((\rho + \delta + \theta g)/\gamma(i))^{1-\sigma} di.$$

Differentiating this expression with respect to ρ shows that $\overline{n}(\rho)$ is increasing in ρ on $[\overline{\rho}, \rho_{max}]$. Moreover, $\overline{n}(\rho_{max}) = 1$ and $\overline{n}(\overline{\rho}) = 0$, so $\overline{n}(\cdot)$ is well-defined for all $\rho \geq \overline{\rho}$.

For $n = \overline{n}(\rho)$, we have $w_I(\widetilde{n}(\rho)) = \rho + \delta + \theta g > w_N(\widetilde{n}(\rho))$. Thus, the formulas for $w_I'(n)$ and $w_N'(n)$ show that, for $\rho \in [\overline{\rho}, \rho_{max}]$ and starting at $\overline{n}(\rho)$, $w_N(n)$ is decreasing in n and $w_I(n)$ is increasing in n. Thus, for all $n > \overline{n}(\rho)$, we have

$$w_I(n) > w_I(\overline{n}(\rho)) = \rho + \delta + \theta g > w_N(\overline{n}(\rho)) > w_N(n),$$

as claimed. On the other hand, for all $n < \overline{n}(\rho)$, we have $w_I(n) < \rho + \delta + \theta g$. In this region we have $n^* = \overline{n}(\rho) > n$, and not all automated tasks are produced with capital.

To prove part 3, note that for $\rho > \rho_{max}$, we have

$$\rho + \delta + \theta g > w_I(1) > w_N(1).$$

The expressions for $w_I'(n)$ and $w_N'(n)$ show that in this region, as n decreases, so does $w_I(n)$. Thus $\rho + \delta + \theta g > w_I(n) > w_N(n)$, and for all these values we have $n^* = 1$, and no task will be produced with capital.

To prove part 4, note that for $\rho < \rho_{min}$, we have

$$\rho + \delta + \theta g < w_N(1) < w_I(1).$$

The expressions for $w_I'(n)$ and $w_N'(N)$ show that, in this region, as n decreases, both $w_N(n)$ $w_N(n)$ increase. Thus $\rho + \delta + \theta g < w_N(n) < w_I(n)$ and for these values of ρ, new tasks do not raise aggregate output. ■

Proof of Proposition 4: We prove this proposition under the more general Assumption $2'$.

We start by deriving necessary conditions on $N(t)$ and $I(t)$ such that the economy admits a BGP, and then show that these are also sufficient for establishing the existence of a unique and globally stable BGP.

The capital market-clearing condition implies that:

$$c^u(R(t))^{\sigma-\zeta}R(t)^\zeta\frac{K(t)}{Y(t)} = B^{\hat{\sigma}-1}(1-\eta)(1-n^*(t)).$$

Because in BGP the rental rate of capital, $R(t)$, and the capital to aggregate output ratio, $\frac{K(t)}{Y(t)}$, are constant, we must have $n^*(t) = n$, or in other words, labor and capital must perform constant shares of tasks.

Lemma A2 shows that we have four possibilities corresponding to the four cases in Proposition 4, each of which we now discuss in turn.

1. All tasks are automated: $n^*(t) = n = 0$. Because in this case capital performs all tasks, Lemma A2 implies that we must have $\rho < \overline{\rho}$ and $I(t) = N(t)$. In this part of the parameter space, net output is given by $A_K K$, and the economy grows at the rate $\frac{A_K-\delta-\rho}{\theta}$. The transversality condition, (19), is satisfied if and only if $A_K - \delta > \frac{A_K-\delta-\rho}{\theta}$—or $r > g$. Moreover, positive growth imposes $A_K > \delta + \rho$. The generalized ideal price index condition, equation (A5), then implies that $R = c^{u^{-1}}(B^{\frac{1-\hat{\sigma}}{1-\sigma}})$, and thus $A_K = c^{u^{-1}}(B^{\frac{1-\hat{\sigma}}{1-\sigma}})$. Under Assumption 2, this last expression further simplifies to $A_K = B$ as claimed in the text.

We now show that these necessary conditions are sufficient to generate balanced growth. Suppose $\rho < \overline{\rho}$ and $I(t) = N(t)$ so that $n^*(t) = 0$. Because all tasks are produced with capital, we also have $F_L = 0$, and thus the representative household supplies zero labor. Consequently, the dynamic equilibrium can be characterized as the solution to the system of differential equations

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} =& \frac{1}{\theta}(A_K - \delta - \rho)\\ \dot{K}(t) =& (A_K - \delta)K(t) - C(t)e^{\nu(0)\frac{\theta-1}{\theta}},\end{aligned}$$

together with the initial condition, $K(0) > 0$ and the trasversality condition, (19). We next show that there is a unique solution to the system, and this solution converges to the full automation BGP described in the proposition.

Define $\widetilde{c} = \frac{C}{K}$. The behavior of $\widetilde{c}$ is governed by the differential equation,

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = \frac{1}{\theta}(A_K - \delta - \rho) - (A_K - \delta) + \widetilde{c}(t)e^{\nu(0)\frac{\theta-1}{\theta}}.$$

This differential equation has a stable rest point at zero and an unstable rest point at $c_B = \left(A_k - \delta - \frac{1}{\theta}(A_K - \delta - \rho)\right)e^{\nu(0)\frac{1-\theta}{\theta}} > 0$. There are therefore three possible equilibrium paths for $\widetilde{c}(t)$: (i) it immediately jumps to c_B and stays there; (ii) it starts at $[0, c_B)$ and converges to zero; (iii) it starts at (c_B, ∞) and diverges. The second and third possibilities violate, respectively, the transversality condition (19) (because the capital stock would grow at the rate $A_k - \delta$, implying $r = g$), and the resource constraint (when $\widetilde{c}(t) = \infty$). The first possibility, on the other hand, satisfies the transversality condition (since asymptotically it involves $r > g$), and yields an equilibrium path. In this path the economy converges to a unique BGP in which $C(t)$ and $K(t)$ grow at a constant rate $\frac{A_K-\delta-\rho}{\theta}$, thus also establishing uniqueness and global stability.

2. Interior equilibrium in which automated tasks are immediately produced with capital: $n^*(t) = n(t) = n \in (0,1)$. Because capital performs all automated tasks, Lemma A2 implies that $n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ and $\dot{N}(t) = \dot{I}(t)$. Moreover, because in this candidate BGP $R(t)$ is constant, the general form of the generalized ideal price index condition, (A5), implies that $W(t)/\gamma(I(t))$ must be constant too, and this is only possible if $\dot{I}(t) = \Delta$. Consequently, the growth rate of aggregate output is $A\Delta$. Finally, the transversality condition, (19), is satisfied given the condition $\rho + (\theta - 1)A\Delta > 0$ in this part of the proposition. Lemma A2 then verifies that $n^*(t) = n > \overline{n}(\rho)$. Substituting the market-clearing conditions for capital and labor, (A3) and (A4), into (1), (2) and (3) and then subtracting the costs of intermediates, we obtain net output as $F(k, L; n)$. (When Assumption 2 holds, $F(k, L; n)$ is given by the CES aggregate in equation (16)). $F(k, L; n)$ exhibits constant returns to scale, and because factor markets are competitive, we also have $R(t) = F_K(k(t), L(t); n)$ and $w(t) = F_L(k(t), L(t); n)$.

To establish uniqueness, let w_B denote the BGP value of the wage rate, k_B the BGP value of the normalized capital stock, c_B the BGP value of normalized consumption, L_B the BGP value of employment, and R_B the BGP value of the rental rate of capital. These variables are, by definition, all constant. Then, the Euler equation, (17), implies $R_B = \rho + \delta + \theta g$, and because $R_B = F_K(k_B, L_B; n)$, we must also have $\frac{k_B}{L_B} = \phi$, where ϕ is the unique solution to

$$F_K(\phi, 1; n) = \rho + \delta + \theta g.$$

Lemma B1 in Appendix B shows that, for $n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$, $F(\phi, 1; n)$ satisfies the following Inada conditions,

$$\lim_{\phi \to 0} F_K(\phi, 1; n) > \rho + \delta + \theta g \qquad \lim_{\phi \to \infty} F_K(\phi, 1; n) < \rho + \delta + \theta g,$$

which ensure that ϕ is well-defined. Combining the labor supply condition, (18), with the resource constraint, (20), we obtain $(F(\phi, 1; n) - (\delta + g)\phi)L_B = \frac{F_L(\phi,1)}{\nu'(L_B)}$. The left-hand side of this equation is linear and increasing in L (the concavity of F in k implies that $F(\phi, 1; n) > \phi F_K(\phi, 1; n) > (\delta+g)\phi$), while the right-hand side is decreasing in L. This ensures that there exists a unique value $L_B > 0$ that satisfies this equation, and also pins down the value of the normalized capital stock as $k_B = \phi L_B$. Finally, c_B is uniquely determined from the resource constraint, (20), as

$$c_B = (F(\phi, 1; n) - (\delta + g)\phi)L_B e^{\nu(L_B)\frac{1-\theta}{\theta}}.$$

Note also that there cannot be any BGP with $L_B = 0$, since this would imply $c_B = 0$ from the resource constraint, (20). But then we would have $\nu'(0)e^{\frac{\theta-1}{\theta}\nu(0)} < \frac{F_L(\phi,1;n)}{c_B}$, which contradicts the labor supply optimality condition, (18). Hence, the only possible BGP is one in which $k(t) = k_B$, $c(t) = c_B$ and $L(t) = L_B > 0$. Moreover, in view of the fact that $\rho + (\theta - 1)A\Delta > 0$, this candidate BGP satisfies the transversality condition (19), and is indeed the unique BGP. The proof of the global stability of this unique BGP is similar to the analysis of global stability of the neoclassical growth model with endogenous labor supply, and for completeness, we provide the details in Appendix B.

3. Interior equilibrium in which automated tasks are eventually but not immediately produced with capital: $n^*(t) = \overline{n}(\rho) > n(t)$. Because capital does not immediately perform all automated tasks, Lemma A2 implies that $n(t) < \overline{n}(\rho)$ and $\rho > \overline{\rho}$. Moreover, because $R(t)$ is constant, the ideal price index condition, (A5), implies that $W(t)/\gamma(I^*(t))$ must be constant too. Thus, to generate constant growth of wages we must have $\dot{I}^*(t) = \Delta \leq I(t)$, so that the growth rate of the economy is given by $A\Delta$. Because $n^*(t) = \overline{n}(\rho)$, this also implies that $\dot{N}(t) = \Delta$. Finally, the transversality condition, (19), is satisfied in view of the fact that this part of the proposition imposes $\rho+(\theta-1)A\Delta > 0$. The uniqueness and global stability of the BGP follow from an identical arguments to part 2, with the only modification that $\overline{n}(\rho)$ plays the role of n in the preceding proof.

4. All tasks are always produced with labor: $n^*(t) = 1$. Because labor performs all tasks, Lemma A2 now implies $\rho > \rho_{max}$ and $n(t) \geq 1$, while the ideal price index condition, (A5), imposes that $W(t)/\gamma(N(t))$ must be constant. Thus, to generate a constant wage, aggregate output and capital growth, we must have $\dot{N}(t) = \Delta$, with $\rho + (\theta-1)\Delta > 0$ (where the last condition again ensures transversality). To show sufficiency of these conditions for balanced growth, let w_B denote the BGP value of the normalized wage, which is defined by

$$\int_0^1 c^u(w_B/\gamma(i))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

Consequently, net output is given by $F(k, L; n) = w_B\gamma(N(t)-1)L(t)$, and thus depends linearly on labor and is independent of capital. This implies $K(t) = 0$ and $C(t) = w_B\gamma(N(t)-1)L(t)$. The representative household's labor supply condition, (18), implies that in this BGP

$$\nu'(L(t)) = \frac{w_B\gamma(N(t)-1)}{C(t)} = \frac{1}{L(t)},$$

which uniquely defines a BGP employment level L_B. Because this allocation also satisfies the transversality condition (in view of the fact that $\rho + (\theta-1)A\Delta > 0$), it defines a unique BGP. Its global stability follows by noting that starting with any positive capital stock, $K(0) > 0$, the representative household chooses zero investment and converges to this path. ■

Proofs from Section 4

All of the results in this section apply and will be proved, under Assumption 2′ .

Lemma A3 (Asymptotic behavior of the normalized value functions) *Suppose that Assumptions 1′ , 2′ and 4. Let* $g = A\frac{\kappa_I\kappa_N}{\kappa_I+\kappa_N}S$ *denote the growth rate of the economy in a BGP. Then there exists a threshold* $\widetilde{S}$ *such that for* $S < \widetilde{S}$*, we have* $\rho + (\theta-1)g > 0$*, and*

- *if* $n \geq \max\{\overline{n}, \widetilde{n}\}$*, both* $v_N(n)$ *and* $v_I(n)$ *are positive and increasing in* n*;*

- *if $n \leq \overline{n}(\rho)$ (and $\rho > \overline{\rho}$), we have $\kappa_N v_N(n) > \kappa_I v_I(n) = \mathcal{O}(g)$ (meaning that it goes to zero as $g \to 0$);*

- *If $n < \widetilde{n}(\rho)$ (and $\rho < \overline{\rho}$), we have $\kappa_I v_I(n) > 0 > \kappa_N v_N(n)$. Moreover, in this region, $v_I(n)$ is decreasing and $v_N(n)$ is increasing in n.*

Proof. See Appendix B. ■

Proof of Proposition 6: We first show that all the scenarios described in the proposition are BGPs with endogenous technology. We then turn to analyzing the stability of interior BGPs.

Part 1: Characterization of the BGPs with endogenous technology.

Suppose that $S < \widetilde{S}$ so that Lemma A3 applies. We consider the two cases described in the proposition separately.

1. $\rho < \overline{\rho}$**:** Suppose that $n < \widetilde{n}(\rho)$. As depicted in the left panel of Figure 8 and shown in Lemma A3, in this region $v_I(n)$ is positive and decreasing in n, and $v_N(n)$ is negative and increasing in n. Thus, the only possible BGP in this region must be one with $n(t) = 0$. No interior BGP exists with $n \in (0, \widetilde{n}(\rho))$. Proposition 4 shows that for $\rho < \overline{\rho}$, a path for technology with $n(t) = 0$ yields balanced growth. Moreover, along this path all tasks are produced with capital, which implies that $V_I(t) = V_N(t) = 0$. Thus, a path for technology in which $n(t) = 0$ is consistent with the equilibrium allocation of scientists. The resulting BGP is an equilibrium with endogenous technology.

2. $\rho > \overline{\rho}$: Suppose $n(t) \leq \overline{n}(\rho)$. Then, we have $n^*(t) = \overline{n}(\rho)$ and therefore $v_N(n) = v_N(\overline{n}(\rho))$ and $v_I(n) = v_I(\overline{n}(\rho))$. Moreover, Lemma A3 implies that $\kappa_N v_N(\overline{n}(\rho)) > \kappa_I v_I(\overline{n}(\rho))$ and $v_I(\overline{n}(\rho)) = \mathcal{O}(g)$ with g small (again because $S < \widetilde{S}$). Therefore, in this region this region we always have that all scientists will be employed to create new tasks, and thus $\dot{n} > 0$ (and is uniformly bounded away from zero). But this contradicts $n(t) < \overline{n}(\rho)$. Suppose, instead, that $n(t) > \overline{n}(\rho)$. Then, Proposition 4 shows that the economy admits a BGP only if $n(t) = n$. Thus, a necessary and sufficient condition for an interior BGP is (29) in the text. Consequently, each interior BGP corresponds to a solution to this equation in $(\overline{n}(\rho), 1)$. Lemma A3 shows that at $\overline{n}$, $\kappa_N v_N(\overline{n})$ is above $\kappa_I v_I(\overline{n})$, and $\kappa_I v_I(\overline{n}) = O(g)$. Moreover, when $\frac{\kappa_I}{\kappa_N} = 0$, the entire curve $\kappa_N v_N(n)$ is above $\kappa_I v_I(n)$. As this ratio increases, the curve $\kappa_I v_I(n)$ rotates up, and eventually crosses $\kappa_N v_N(n)$ at a point to the right of $\overline{n}(\rho)$. This defines the threshold $\underline{\kappa}$. Above this threshold, there exists another threshold $\overline{\kappa}$ such that if $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, there is a unique intersection of $\kappa_I v_I(n)$ and $\kappa_N v_N(n)$. (Note that one could have $\underline{\kappa} = \overline{\kappa}$). By continuity, there exists $\hat{S}$ such that, the thresholds $\underline{\kappa}$ and $\overline{\kappa}$ are defined for all $S < \hat{S}$ (recall that $g = A\frac{\kappa_I \kappa_N}{\kappa_I + \kappa_N} S$). It then follows that for $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$, there exists a unique BGP, which is interior and satisfies $n(t) = n^*(t) = n_B \in (\overline{n}, 1)$. For $S < \min\{\widetilde{S}, \hat{S}\}$ and $\underline{\kappa} < \frac{\kappa_I}{\kappa_N} < \overline{\kappa}$ (provided that $\underline{\kappa} < \overline{\kappa}$), the economy admits multiple BGPs with endogenous technology. Finally, for $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} < \underline{\kappa}$, the only potential BGP is the corner one with $n(t) = 1$ as in part 4 of Proposition 4. Because $\kappa_N v_N(1) > \kappa_I v_I(1)$, this path for technology is consistent with the equilibrium allocation of scientists and provides a BGP with endogenous technology.

Part 2: Stability analysis.

The stability analysis applies to the case in which $\rho > \overline{\rho}$, $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. In this case, the economy admits a unique BGP defined by $n_B \in (\overline{n}(\rho), 1)$. We denote by c_B, k_B and L_B the values of (normalized) consumption and capital, and employment in this BGP.

Proof of global stability when $\theta = 0$: Because $\theta = 0$, we also have $R = \rho + \delta$, and capital adjusts immediately and its equilibrium stock only depends on n, which becomes the unique state variable of the model.

Let $v = \kappa_I v_I - \kappa_N v_N$. Now starting from any $n(0)$, an equilibrium with endogenous technology is given by the path of (n, v) such that the evolution of the state variable is given by

$$\dot{n} = \kappa_N S - (\kappa_N + \kappa_I) G(v) S;$$

and the evolution of the difference of the normalized value functions, v, satisfies the forward-looking differential equation

$$\rho v - \dot{v} = b\kappa_I \left(c^u (\rho + \delta)^{\zeta - \sigma} - c^u (w_I)^{\zeta - \sigma} \right) - b\kappa_N \left(c^u (w_N)^{\zeta - \sigma} - c^u (\rho + \delta)^{\zeta - \sigma} \right) + \mathcal{O}(g)$$

together with the transversality condition (27) holds.

When $g = 0$, the locus for $\dot{v} = 0$ crosses zero from below at a unique point (recall that we are in the parameter region where there is a unique BGP). By continuity there exists a threshold $\breve{S}$ such that, for $S < \breve{S}$, the locus for $\dot{v} = 0$ crosses zero from below at a unique point n_B, which denotes the BGP value for $n(t)$ derived from (29).

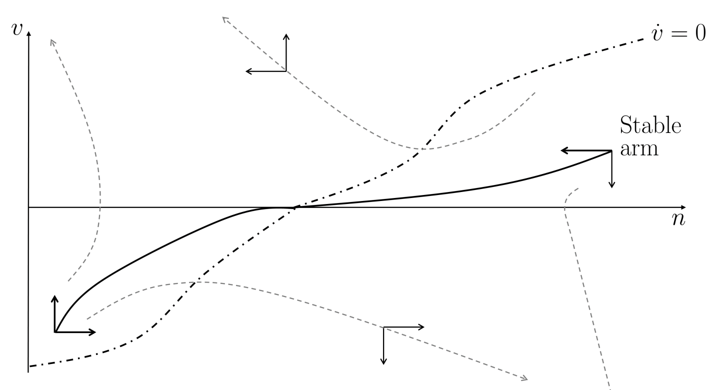

FIGURE A2: Phase diagram and global saddle path stability when $\theta = 0$. The figure plots the locus for $\dot{v} = 0$ and the locus for $\dot{n} = 0$. The unique BGP is located at their interception.

We now analyze the stability properties of the system and show that the BGP is globally saddle-path stable. Figure A2 presents the phase diagram of the system in (v, n). The locus for $\dot{v} = 0$ crosses $v = 0$ at n_B from below only once. This follows from the fact that $\kappa_I v_I(n)$ cuts $\kappa_N v_N(n)$ from below at n_B as shown in Figure 8. The laws of motion of the two variables, v and n, take the form shown in the phase diagram.[33] This implies the existence of the unique stable arm, and also

[33]This can also be verified locally from the fact that the behavior of n and v near the BGP can be approximated by the linear system $\dot{n} = -(\kappa_N + \kappa_I) G'(0) S v$ and $\dot{v} = \rho v - Q$, where $Q > 0$ denotes the derivative of $-M\kappa_I c^u (w_I)^{\zeta - \sigma} +$

establishes that there are no equilibrium paths that are not along this stable arm. In particular, all paths above the stable arm feature $\dot{v} > 0$ and eventually $n \to 0$ and $v \to \infty$, and since v_N is positive, $v_I \to \infty$. But this violates the transversality condition, (27). Similarly, all paths below the stable arm feature $\dot{v} < 0$ and eventually $n \to 1$ and $v \to -\infty$, and thus $v_N \to \infty$, once again violating the transversality condition.

Proof of local stability of the unique BGP when $\theta > 0$: Appendix B shows that there exists a threshold $\check{S}$ such that the BGP in this case is locally stable for $S < \check{S}$, and thus the conclusions of the proposition follow setting $\overline{S} = \min\{\widetilde{S}, \hat{S}, \breve{S}, \check{S}\}$. ■

A1 References

Acemoglu, Daron (1998) "Why Do New Technologies Complement Skills? Directed Technical Change and Wage Inequality," *Quarterly Journal of Economics,* 113(4): 1055-1089.

Acemoglu, Daron (2002) "Directed Technical Change," *Review of Economic Studies,* 69(4): 781–810.

Acemoglu, Daron (2003) "Labor- and Capital-Augmenting Technical Change," *Journal of European Economic Association,* 1(1): 1–37.

Acemoglu, Daron (2007) "Equilibrium Bias of Technology," *Econometrica,* 75(5): 1371–1410.

Acemoglu, Daron (2010) "When Does Labor Scarcity Encourage Innovation?" *Journal of Political Economy,* 118(6): 1037–1078.

Acemoglu, Daron and David Autor (2011) "Skills, tasks and technologies: Implications for employment and earnings," *Handbook of Labor Economics*, 4: 1043–1171.

Acemoglu, Daron, Gino Gancia and Fabrizio Zilibotti (2010) "Competing Engines of Growth: Innovation and Standardization," *Journal of Economic Theory,* 147(2): 570–601.

Acemoglu, Daron, Gino Gancia and Fabrizio Zilibotti (2015) "Offshoring and Directed Technical Change," forthcoming *American Economic Journal: Macro.*

Acemoglu, Daron and Fabrizio Zilibotti (2001) "Productivity Differences," *Quarterly Journal of Economics,* 116(2): 563–606.

Acemoglu, Daron and Pascual Restrepo (2017a) "Robots and Jobs: Evidence from US Labor Markets" NBER Working Paper No. 23285.

Acemoglu, Daron and Pascual Restrepo (2017b) "Demographics and Robots" mimeo.

Aghion, Philippe and Peter Howitt (1992) "A Model of Growth Through Creative Destruction," *Econometrica,* 60(2): 323–351.

Akst, Daniel (2013) "What Can We Learn From Past Anxiety Over Automation?" Wilson Quarterly.

$M\kappa_N c^u (w_N)^{\zeta-\sigma}$ with respect to n (this derivative is positive because $\kappa_I v_I(n)$ cuts $\kappa_N v_N(n)$ from below at n_B). Because the product of the eigenvalues of the characteristic polynomial of this system is $-Q(\kappa_N + \kappa_I)G'(0)S < 0$, there is one positive and one negative eigenvalue (and their sum is $\rho > 0$, so the positive one is larger in absolute value).

Allen, Robert C. (2009) "Engels' Pause: Technical Change, Capital Accumulation, and Inequality in the British Industrial Revolution," *Explorations in Economic History,* 46(4): 418–435.

Autor, David H. and David Dorn (2013) "The Growth of Low-Skill Service Jobs and the Polarization of the U.S. Labor Market," *American Economic Review,* 103(5): 1553–97.

Autor, David H., Frank Levy and Richard J. Murnane (2003) "The Skill Content of Recent Technological Change: An Empirical Exploration," *The Quarterly Journal of Economics,* 118(4): 1279–1333.

Beaudry, Paul, David A. Green, Benjamin M. Sand (2013) "The Great Reversal in the Demand for Skill and Cognitive Tasks," NBER Working Paper No. 18901.

Brynjolfsson, Erik and Andrew McAfee (2014) *The Second Machine Age: Work, Progress, and Prosperity in a Time of Brilliant Technologies*, W. W. Norton & Company.

Burstein, Ariel, and Jonathan Vogel (2014) "International Trade, Technology, and the Skill Premium," Society for Economic Dynamics, Meeting Papers. No. 664.

Burstein, Ariel, Eduardo Morales and Jonathan Vogel (2014) "Accounting for Changes in Between-Group Inequality," NBER Working Paper No. 20855

Caselli, Francesco (1999) "Technological Revolutions," *American Economic Review,* 89(1): 78–102.

Caselli, Francesco, and Wilbur John Coleman (2006) "The World Technology Frontier," *American Economic Review,* 96(3): 499–522.

Chandler, Alfred D. (1977) *The Visible Hand: The Managerial Revolution in American Business,* Harvard University Press: Cambridge, MA.

Costinot, Arnaud and Jonathan Vogel (2010) "Matching and Inequality in the World Economy," *Journal of Political Economy,* 118(4): 747–786.

Foote, Christopher L. and Richard W. Ryan (2014) "Labor-Market Polarization Over the Business Cycle," *NBER Macroeconomics Annual 2014, Volume 29,* NBER.

Galor, Oded and Omer Moav (2000) "Ability-Biased Technological Transition, Wage Inequality, and Economic Growth," *The Quarterly Journal of Economics,* 115(2): 469–497.

Goldin, Claudia and Lawrence F. Katz (2008) *The Race between Education and Technology*, Harvard University Press: Cambridge.

Greenwood, Jeremy and Mehmet Yorukoglu (1997) "1974," *Carnegie-Rochester Conference Series on Public Policy*, 46: 49–95

Grossman, Gene M. and Elhanan Helpman (1991) "Quality Ladders in the Theory of Growth," *The Review of Economic Studies,* 58(1): 43–61.

Grossman, Gene M. and Esteban Rossi-Hansberg (2008) "Trading Tasks: A Simple Theory of Offshoring," *American Economic Review*, 98(5): 1978–1997.

Grossman, Gene M., Elhanan Helpman and Ezra Oberfield (2016) "Balanced Growth Despite Uzawa," NBER Working Paper No. 21861.

Hawkins, William, Ryan Michaels and Jiyoon Oh (2015) "The Joint Dynamics of Capital and Employment at the Plant Level," Mimeo, Yeshiva University.

Hemous, David, and Morten Olsen (2016) "The Rise of the Machines: Automation, Horizontal Innovation and Income Inequality," Unpublished manuscript, University of Zurich.

Jaimovich, Nir and Henry E. Siu (2014) "The Trend is the Cycle: Job Polarization and Jobless Recoveries," NBER Working Paper No. 18334.

Keynes John Maynard (1930) "Economic Possibilities for Our Grandchildren," in *Essays in Persuasion*, New York: Norton & Co.

Kotlikoff, Laurence J. and Jeffrey D. Sachs (2012) "Smart Machines and Long Term Misery," NBER Working Paper No. 18629.

Jones, Charles I. (2005) "The Shape of Production Functions and the Direction of Technical Change," *The Quarterly Journal of Economics,* 120(2): 517–549.

Karabarbounis, Loukas and Brent Neiman (2014) "The Global Decline of the Labor Share," *The Quarterly Journal of Economics*, 129(1): 61–103.

Kiley, Michael T. (1999) "The Supply of Skilled Labor and Skill-Biased Technological Progress," *The Economic Journal,* 109(458): 708–724.

Landes, David (1969) *The Unbound Prometheus,* Cambridge University Press: New York.

Lin, Jeffrey (2011) "Technological Adaptation, Cities, and New Work" *Review of Economics and Statistics,* 93(2): 554–574.

Mokyr, Joel (1990) *The Lever of Riches: Technological Creativity and Economic Progress,* Oxford University Press: New York.

Nelson, Richard R. and Edmund S. Phelps (1966) "Investment in Humans, Technological Diffusion, and Economic Growth," *The American Economic Review,* 56(1): 69–75.

Oberfield, Ezra and Devesh Raval (2014) "Micro Data and Macro Technology," NBER Working Paper No. 20452.

Rodriguez-Clare, Andres (2010) "Offshoring in a Ricardian World," *American Economic Journal: Macroeconomics,* 2: 227–258.

Roy, A. D. (1951) "Some Thoughts on the Distribution of Earnings," *Oxford Economic Papers,* 3(2): 135–146.

Schultz, Theodore (1975) "The Value of the Ability to Deal with Disequalibria." Journal of Economic Literature. 13:827–846.

The New York Times (1983) "Machines vs. Workers," by Charlotte Curtis, February 8th edition.

Thoenig, Mathias, and Thierry Verdier (2003) "A Theory of Defensive Skill-Biased Innovation and Globalization," *American Economic Review,* 93(3): 709–728.

Zeira, Joseph (1998) "Workers, Machines, and Economic Growth," *Quarterly Journal of Economics,* 113(4): 1091–1117.

Zeira, Joseph (2006) "Machines as Engines of Growth," Dynamics, Economic Growth, and International Trade Conference Papers. No. 011059.

Appendix B (Not-For-Publication): Omitted Proofs and Additional Results

Details of the Empirical Analysis

This section provides information about the data used in constructing Figures 1 and 9. We also provide a regression analysis documenting the robustness of the patterns illustrated in these figures.

Data: We use data on employment counts for 304 occupational categories that we can track consistently over time, from 1980 until 2015. Our occupational categories roughly match the 330 categories proposed by David Dorn (see `http://www.ddorn.net/data.htm`). We aggregate some of these categories to account for merged occupational codes in recent waves of the American Community Survey. The details of our approach can be found in the replication files that accompany this paper.

We use data from the Census for 1980, 1990, and 2000, as well as the American Community Survey for 2010 and 2015 (Ruggles et al. 2017). Using these data, we compute for each of our 304 occupational categories the total employment count and the demographic characteristics of its workers, including their gender, age, education, race and whether they are foreign born (we focus on workers between 16 and 64 years of age). We also compute the share of jobs in each occupational category that are in manufacturing, the primary sector (agriculture, forestry, fisheries and mining), and services (retail trade, finance, business and repair services, personal services, entertainment services, professional services, and public administration).

Our measure of new job titles comes from Lin (2011), who computes the total amount of job titles and new job titles in each occupational category for 1980, 1990 and 2000.[34] Lin identifies new job titles by comparing changes across waves of the *Dictionary of Occupational Titles*, and also by comparing the *1990 Census Index of Occupations* with its 2000 counterpart. Importantly, Lin uses official documentation to avoid labeling as new those jobs that were simply reclassified or divided because of reasons unrelated to the type of work people performed (i.e., because of administrative changes in U.S. statistical agencies). Instead, Lin's measure counts a job as new if workers perform a different set of tasks in this job than in any previously existing jobs. The data on new and total job titles can be matched consistently to 303 of our occupations in 1980 and 1990, and to all of our occupations in 2000.

Detailed Analysis for Figure 1: In addition to Figure 1 in the main text, in Figure B1 we pool the 1980-2015 changes together with the 1990-2015 and 2000-2015 changes. In this case, the share of new job titles refers to this variable measured at the beginning of each time window (i.e., 1980, 1990 or 2000). A very similar positive relationship is visible in the figure.

We further document this relationship and probe its robustness by estimating the following regression:

$$\Delta \ln E_{it} = \beta N_{it} + \delta_t + \varPhi_t X_{it} + u_{it}. \tag{B1}$$

[34] The data are available from Jeffrey Lin's website `https://sites.google.com/site/jeffrlin/newwork`

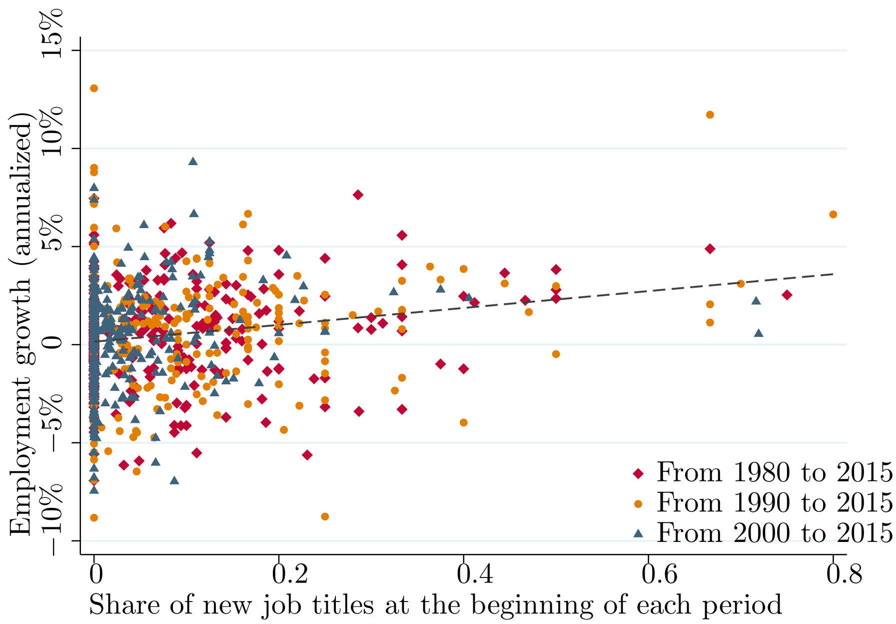

FIGURE B1: Employment growth by occupation over different time periods (annualized), plotted against the share of new job titles at the beginning of each period in each occupation.

Here, the dependent variable is the (annualized) growth in employment in occupation i. The key explanatory variable is N_{it}—the share of new job titles in occupational category i at the beginning of the period.

We start in Table B1 with the 1980-2015 change as in Figure 1. In this case, there is only one observation per occupation, and we report standard errors that are robust against heteroscedasticity.

Column 1 shows the raw correlation without any covariates, which is positive and statistically significant.

Column 2 includes the initial level of employment and total number of job titles in each occupation. This leads to a larger and more precisely estimated coefficient on the share of new job titles: 3.953 (standard error = 1.080). Column 3, which is our baseline specification shown in Figure 1, in addition controls for the demographic composition of employment in each occupation—in particular, allowing for differential growth by average age, fraction male, share foreign-born, fraction black and fraction Hispanic in the occupation in 1980. Now the coefficient on the share of new job titles is 4.153 (standard error = 1.143). Using this estimate, we compute that if there had been no additional employment growth in occupations with more new job titles in 1980, total employment growth between 1980 and 2015 would have been 24% rather than 60%. This is the basis of our claim in the text that about 60% of employment growth between 1980 and 2015 is associated with faster employment growth in occupations with more new job titles.

In column 3, we do not control for average education in the occupation, since, as we discuss further below, occupations with more new job titles attract more educated workers, making this variable a "bad control." Nevertheless, column 4 shows that controlling for it does not alter the qual-

Table B1: Long-differences estimates of employment growth in occupational categories with a higher baseline share of new job titles.

	Dependent variable: percent change in employment growth between 1980-2015 (annualized).					
	(1)	(2)	(3)	(4)	(5)	(6)
Share of new job titles in 1980	2.602**	3.953***	4.153***	3.425***	3.254***	3.334***
	(1.264)	(1.080)	(1.143)	(1.059)	(1.014)	(1.146)
log of employment in 1980		-0.001	-0.002	-0.002	-0.003**	-0.003***
		(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
log of job titles in 1980		-0.705***	-0.647***	-0.484***	-0.272*	-0.128
		(0.112)	(0.120)	(0.114)	(0.156)	(0.121)
Average age of workers in 1980			-0.055	-0.063	-0.022	-0.042
			(0.044)	(0.044)	(0.042)	(0.038)
Share of male workers in 1980			-1.035*	-0.484	0.270	0.151
			(0.540)	(0.533)	(0.601)	(0.547)
Share of foreign workers in 1980			2.725	-8.628*	1.455	2.887
			(4.838)	(4.685)	(4.786)	(4.759)
Share of black workers in 1980			-2.904	1.531	-2.044	-2.350
			(2.931)	(2.635)	(2.367)	(2.706)
Share of Hispanic workers in 1980			-3.964	17.642***	2.959	1.887
			(5.913)	(5.888)	(5.873)	(5.872)
Average years of education in 1980				0.562***		
				(0.101)		
Share of manufacturing jobs in 1980					-1.756**	-2.342***
					(0.698)	(0.696)
Share of primary sector jobs in 1980					-0.012	-0.393
					(0.699)	(0.904)
Share of service sector jobs in 1980					2.160***	1.831***
					(0.575)	(0.584)
R-squared	0.01	0.18	0.20	0.27	0.32	0.31
Observations	303	303	303	303	303	303
Covariates and estimation:						
Total job titles and occupation size		✓	✓	✓	✓	✓
Demographics			✓	✓	✓	✓
Educational level				✓		
Sectoral composition					✓	✓
Robust regression						✓

Notes: The table presents long-difference estimates of the relationship between the share of new job titles in an occupational category in 1980 and subsequent employment growth between 1980-2015 (annualized). The table also reports the coefficients estimated for the covariates included in each model. Finally, in column 6 we present robust-regression estimates following Li (1985). Standard errors that are robust against heteroscedasticity are presented in parentheses.

itative relationship between share of new job titles and employment growth, though the coefficient now declines modestly to 3.425 (standard error = 1.059).

In column 5 we add share of manufacturing, primary sector and service job titles in each occupation in 1980 to the specification of column 3. These variables enable us to control for the general structural change in the economy away from manufacturing and primary sector jobs towards service jobs. This also leads to a somewhat lower estimate, which still remains precisely estimated: 3.254 (standard error = 1.014). Finally, in column 6, we estimate a robust regression down-weighing outliers and excessively influential observations (following Li, 1985). The results are very similar.

In Table B2, we estimate the same models now exploiting variation in the share of new job titles at the beginning of each decade between 1980 and 2000. Panel A looks at a sample consisting of stacked differences for 1980-2015, 1990-2015 and 2000-2015. Panel B is for 1980-2010, 1990-2010 and 2000-2010. Finally, Panel C focuses on decadal changes, 1980-1990, 1990-2000 and 2000-2010. In each case, the share of new job titles refers to this variable measured at the beginning of the period for the relevant time window. In addition, we control for a full set of period dummies and the standard errors are now robust against heteroscedasticity and serial correlation at the occupation level. The results are very similar to those reported in Table B1 in all cases.

Table B2: Stacked-differences estimates of employment growth in occupational categories with a higher baseline share of new job titles.

	Dependent variable: percent change in employment growth (annualized).					
	(1)	(2)	(3)	(4)	(5)	(6)
	Panel A: stacked differences for 1980-2015, 1990-2015, and 2000-2015.					
Share of new job titles at t	4.284***	4.399***	4.448***	3.499***	3.562***	3.150***
	(0.894)	(0.881)	(0.909)	(0.954)	(1.021)	(0.696)
R-squared	0.05	0.16	0.18	0.23	0.29	0.30
Observations	910	910	910	910	910	910
	Panel B: stacked differences for 1980-2010, 1990-2010, and 2000-2010.					
Share of new job titles at t	4.297***	4.490***	4.593***	3.428***	3.537***	3.310***
	(1.009)	(0.968)	(1.003)	(1.047)	(1.122)	(0.712)
R-squared	0.04	0.17	0.20	0.25	0.34	0.36
Observations	910	910	910	910	910	910
	Panel C: stacked differences for 1980-1990, 1990-2000, and 2000-2010.					
Share of new job titles at t	3.809**	4.223**	4.277**	3.088*	3.224*	3.406***
	(1.804)	(1.683)	(1.737)	(1.849)	(1.839)	(0.866)
R-squared	0.04	0.13	0.16	0.19	0.23	0.35
Observations	910	910	910	910	910	910
Covariates and estimation:						
Decadal dummies	✓	✓	✓	✓	✓	✓
Total job titles and occupation size		✓	✓	✓	✓	✓
Demographics			✓	✓	✓	✓
Educational level				✓		
Sectoral composition					✓	✓
Robust regression						✓

Notes: The table presents stacked-differences estimates of the relationship between the share of new job titles in an occupational category and subsequent employment growth (annualized). In Panel A, we stack the data for the periods from 1980-2015, 1990-2015 and 2000-2015. In Panel B, we stack the data for the periods from 1980-2010, 1990-2010 and 2000-2010. In Panel C, we stack the data for the three decades 1980-1990, 1990-2000 and 2000-2010. All models include a full set of decadal effects. In addition, we introduce the covariates indicated at the bottom rows, but do not report their coefficients. In column 2 we control for the log of total job titles and employment in each occupational category. In column 3 we also control for the demographic characteristics of workers in each occupation (average age, gender, place of birth, and race). In column 4 we further add average years of schooling in the occupation. In column 5 we control for the sectoral composition of jobs in each occupation (share of jobs in manufacturing, the primary sector, and services). All these covariates are allowed to have time-varying coefficients. Finally, in column 6 we present robust-regression estimates following Li (1985). Standard errors that are robust against heteroscedasticity and serial correlation within occupations are presented in parentheses.

Detailed Analysis for Figure 9: We now briefly present regression evidence documenting that the pattern shown in Figure 9 is robust. In particular, we report estimates of the following equation

$$H_{it} = \beta N_{it} + \delta_t + \Phi_t X_{it} + u_{it}, \tag{B2}$$

where the left-and side variable is the average years of schooling (or the share of workers with college) among workers employed in occupation i at time t, while N_{it} is again the share of new job titles in occupational category i at time t. The regressions always include period dummies and standard errors are robust against heteroscedasticity and serial correlation at the occupation level. The five columns in Table B3 correspond to columns 1-3 and 5-6 of Table B1 (because the left-hand side variable is average years of schooling, we do not control for it on the right-hand side). The results show that in all specifications there is a significant positive association between the share of new job titles in an occupation and the average years of schooling of workers in the subsequent decades. The relationship shown in Figure 9 corresponds to column 3, where we control for differential demographic trends.

Table B3: Estimates of the education level of workers in occupational categories with a higher baseline share of new job titles.

	Dependent variable: educational attainment of workers within each occupational category.				
	(1)	(2)	(3)	(4)	(5)
	Panel A: Average years of education among workers.				
Share of new job titles at t	3.144***	3.082***	1.685***	1.324***	1.221***
	(0.522)	(0.543)	(0.363)	(0.335)	(0.304)
R-squared	0.05	0.17	0.67	0.71	0.75
Observations	910	910	910	910	910
	Panel B: Share of workers with a college degree.				
Share of new job titles at t	0.420***	0.414***	0.215***	0.160**	0.131**
	(0.085)	(0.088)	(0.068)	(0.065)	(0.053)
R-squared	0.03	0.12	0.55	0.60	0.64
Observations	910	910	910	910	910
Covariates and estimation:					
Decadal dummies	✓	✓	✓	✓	✓
Total job titles and occupation size		✓	✓	✓	✓
Demographics			✓	✓	✓
Sectoral composition				✓	✓
Robust regression					✓

Notes: The table presents estimates of the relationship between the share of new job titles in an occupational category and the educational level of its workers. In these estimates, we pool data from 1980, 1990 and 2000. In Panel A, we use the average years of schooling among workers in an occupational category as the dependent variable. In Panel B, we use the share of workers with college degree as dependent variable. All models include a full set of decadal effects. In addition, we introduce the covariates indicated at the bottom rows, but do not report their coefficients. In column 2 we control for the log of total job titles and employment in each occupational category. In column 3 we also control for the demographic characteristics of workers in each occupation (average age, gender, place of birth, and race). In column 4 we also control for the sectoral composition of jobs in each occupation (share of jobs in manufacturing, the primary sector, and services). All these covariates are allowed to have time-varying coefficients. Finally, in column 5 we present robust-regression estimates following Li (1985). Standard errors that are robust against heteroscedasticity and serial correlation within occupations are presented in parentheses

Remaining Proofs from Section 2

We start with the proof of Lemma A1.

Proof of Lemma A1. The assumption that $K < \overline{K}$ and $I^* \leq \widetilde{I}$ implies that:

$$\frac{W}{\gamma(N)} < R \leq \frac{W}{\gamma(I^*)}. \tag{B3}$$

We first show that $\omega(I^*, N, K)$ is (strictly) decreasing in I^*. To do so, we compute $\omega_I^*(I^*, N, K)$ and show that Assumption $2'$ is sufficient to ensure it is negative.

Log-differentiating equations (A3) and (A4), we have

$$\varepsilon_K \frac{d\ln R}{dI^*} = \frac{d\ln Y}{dI^*} + \frac{1}{I^* - N + 1}, \tag{B4}$$

$$\varepsilon_L \frac{d\ln W}{dI^*} = \frac{d\ln Y}{dI^*} - \xi(I^*), \tag{B5}$$

where

$$\varepsilon_K = \zeta + (\sigma - \zeta)\varsigma_K,$$

$$\varepsilon_L = \int_{I^*}^{N} \xi(i)(\zeta + (\sigma - \zeta)\varsigma_L(i))di,$$

and $\varsigma_K \in [0,1]$ is the share of capital in tasks produced with capital, $\varsigma_L(i) \in [0,1]$ is the share of labor in task i, and $\xi(i) \in [0,1]$ is the share of total payments to labor earned by workers in task i (in particular, we have $\int_{I^*}^{N} \xi(i)di = 1$).

Differentiating equation (A5), we get

$$\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} = s_K \frac{d\ln R}{dI^*} + s_L \frac{d\ln W}{dI^*}. \tag{B6}$$

where $s_K = \frac{RK}{Y} \in [0,1]$ and $s_L = \frac{WL}{Y} \in [0,1]$ are respectively the capital and the labor shares in national income.

Solving the system of equations determined by (B4), (B5), and (B6) yields

$$\begin{aligned}
\frac{\omega_{I^*}(I^*, N, K)}{\omega(I^*, N, K)} =& \frac{d\ln W}{dI^*} - \frac{d\ln R}{dI^*}, \\
=& -\frac{s_L + s_K}{\varepsilon_K s_L + \varepsilon_L s_K}\left(\frac{1}{I^* - N + 1} + \xi(I^*)\right) \\
&+ \frac{\varepsilon_K - \varepsilon_L}{\varepsilon_K s_L + \varepsilon_L s_K}\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}}.
\end{aligned}$$

Therefore, $\omega(I^*, N, K)$ is (strictly) decreasing in I^* if and only if

$$(\varepsilon_K - \varepsilon_L)\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} < (s_L + s_k)\left(\frac{1}{I^* - N + 1} + \xi(I^*)\right).$$

Let $\varsigma_{max} = \max_{i\in[I^*,N]}\{\varsigma_L(i)\}$ and $\varsigma_{min} = \min_{i\in[I^*,N]}\{\varsigma_L(i)\}$. Inequality (B3) implies that $\varsigma_K \in [\varsigma_{min}, \varsigma_{max}]$. Thus:

$$\varepsilon_K - \varepsilon_L = (\sigma - \zeta)\left(\varsigma_K - \int_{I^*}^{N} \xi(i)\varsigma_L(i)di\right) < |\sigma - \zeta|(\varsigma_{max} - \varsigma_{min}).$$

In addition, $s_L + s_K > \varsigma_{min}$, because the share of capital or labor in every task is at least ς_{min}. Thus, the inequality

$$|\sigma - \zeta|\frac{\varsigma_{max} - \varsigma_{min}}{\varsigma_{min}}\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} < \frac{1}{I^* - N + 1} + \xi(I^*) \tag{B7}$$

suffices to ensure that $\omega(I^*, N, K)$ is (strictly) decreasing in I^*.

We now show that Assumption $2'$ implies (B7).

If $\eta \to 0$, then $\varsigma_{max} = \varsigma_{min} = 1$ and (B7) holds. Likewise, if $\zeta = 1$, $\varsigma_{max} = \varsigma_{min} = 1 - \eta$ and (B7) holds. To complete the proof we show that (B7) holds under (A1). This follows from the following sequence of inequalities:

- If $\zeta < 1$,

$$\begin{aligned}\frac{\varsigma_{max} - \varsigma_{min}}{\varsigma_{min}} &= \frac{\frac{(W/\gamma(I^*))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(I^*))^{1-\zeta}} - \frac{(W/\gamma(N))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(N))^{1-\zeta}}}{\frac{(W/\gamma(N))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(N))^{1-\zeta}}}, \\ &= \frac{((W/\gamma(I^*))^{1-\zeta} - (W/\gamma(N))^{1-\zeta})\eta\psi^{1-\zeta}}{(W/\gamma(N))^{1-\zeta}(\eta\psi^{1-\zeta} + (1-\eta)(W/\gamma(I^*))^{1-\zeta})}, \\ &< \frac{(W/\gamma(I^*))^{1-\zeta} - (W/\gamma(N))^{1-\zeta}}{(W/\gamma(N))^{1-\zeta}}, \\ &= \left(\frac{\gamma(N)}{\gamma(I^*)}\right)^{1-\zeta} - 1, \\ &< \left(\frac{\gamma(N)}{\gamma(N-1)}\right)^{1-\zeta} - 1.\end{aligned}$$

If, on the other hand, if $\zeta > 1$,

$$\begin{aligned}\frac{\varsigma_{max} - \varsigma_{min}}{\varsigma_{min}} &= \frac{\frac{(W/\gamma(N))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(N))^{1-\zeta}} - \frac{(W/\gamma(I^*))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(I^*))^{1-\zeta}}}{\frac{(W/\gamma(I^*))^{1-\zeta}}{\eta\psi^{1-\zeta}+(1-\eta)(W/\gamma(I^*))^{1-\zeta}}}, \\ &= \frac{((W/\gamma(N))^{1-\zeta} - (W/\gamma(I^*))^{1-\zeta})\eta\psi^{1-\zeta}}{(W/\gamma(I^*))^{1-\zeta}(\eta\psi^{1-\zeta} + (1-\eta)(W/\gamma(N))^{1-\zeta})}, \\ &< \frac{(W/\gamma(N))^{1-\zeta} - (W/\gamma(I^*))^{1-\zeta}}{(W/\gamma(I^*))^{1-\zeta}}, \\ &= \left(\frac{\gamma(I^*)}{\gamma(N)}\right)^{1-\zeta} - 1, \\ &< \left(\frac{\gamma(N-1)}{\gamma(N)}\right)^{1-\zeta} - 1.\end{aligned}$$

Thus,

$$\frac{\varsigma_{max} - \varsigma_{min}}{\varsigma_{min}} < \left(\frac{\gamma(N)}{\gamma(N-1)}\right)^{|1-\zeta|} - 1. \tag{B8}$$

- The function $f(x) = \frac{1}{1-\sigma}x^{1-\sigma}$ is concave. Because $\frac{W}{\gamma(I^*)} \geq R$, we also have

$$\begin{aligned}\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} &< \frac{c^u(R)^{-\sigma}(c^u(W/\gamma(I^*)) - c^u(R))}{B^{1-\sigma}}, \\ &< \frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}}\frac{c^u(W/\gamma(I^*))}{c^u(R)}, \\ &< \frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}}\frac{\gamma(N)}{\gamma(N-1)}.\end{aligned}$$

  In the last line we used the fact that:

$$\frac{c^u(W/\gamma(I^*))}{c^u(R)} < \frac{c^u(W/\gamma(N-1))}{c^u(W/\gamma(N))} \leq \frac{\gamma(N)}{\gamma(N-1)},$$

  which follows from observing that $c^u(x)/x$ is decreasing in x.

  Finally, the ideal price index condition in equation (A5) implies that

$$\frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}} < \frac{c^u(R)^{1-\sigma}}{(I^*-N+1)c^u(R)^{1-\sigma}} < \frac{1}{I^*-N+1} + \xi(I^*).$$

  This inequality implies

$$\frac{1}{1-\sigma}\frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} < \left(\frac{1}{I^*-N+1} + \xi(I^*)\right)\frac{\gamma(N)}{\gamma(N-1)}. \qquad \text{(B9)}$$

- Multiplying inequalities (A1), (B8), and (B9), we obtain the sufficient condition (B7). This shows that Assumption $2'$ implies (B7), and ensures that $\omega_I(I^*, N, K)$ is negative.

We now show that $\omega(I^*, N, K)$ is (strictly) increasing in N. To do so, we compute $\omega_N(I^*, N, K)$ and show that Assumption $2'$ is sufficient to ensure it is positive.

Log-differentiating equations (A3), (A4), and (A5), and solving for the change in wages and rental rates, we have

$$\begin{aligned}\frac{\omega_N(I^*, N, K)}{\omega(I^*, N, K)} =& \frac{d\ln W}{dN} - \frac{d\ln R}{dN}, \\ =& \frac{s_L + s_K}{\varepsilon_K\lambda_L + \varepsilon_L\lambda_K}\left(\frac{1}{I^*-N+1} + \xi(N)\right) \\ &+ \frac{\varepsilon_K - \varepsilon_L}{\varepsilon_K s_L + \varepsilon_L s_K}\frac{1}{1-\sigma}\frac{c^u(R)^{1-\sigma} - c^u(W/\gamma(N))^{1-\sigma}}{B^{1-\sigma}}.\end{aligned}$$

Therefore, $\omega(I^*, N, K)$ is (strictly) increasing in N if and only if

$$(\varepsilon_L - \varepsilon_K)\frac{1}{1-\sigma}\frac{c^u(R)^{1-\sigma} - c^u(W/\gamma(N))^{1-\sigma}}{B^{1-\sigma}} < (s_L + s_K)\left(\frac{1}{I^*-N+1} + \xi(N)\right).$$

Inequality (B3) implies that $\varsigma_K \in [\varsigma_{min}, \varsigma_{max}]$. Thus

$$\varepsilon_L - \varepsilon_K = (\sigma - \zeta)\left(\int_{I^*}^N \xi(i)\varsigma_L(i)di - \varsigma_K\right) < |\sigma - \zeta|(\varsigma_{max} - \varsigma_{min}).$$

In addition, $s_L + s_K > \varsigma_{min}$, because the share of capital or labor in every task is at least ς_{min}. Thus,the inequality

$$|\sigma - \zeta| \frac{\varsigma_{max} - \varsigma_{min}}{\varsigma_{min}} \frac{1}{1-\sigma} \frac{c^u(R)^{1-\sigma} - c^u(W/\gamma(N))^{1-\sigma}}{B^{1-\sigma}} < \left(\frac{1}{I^* - N + 1} + \xi(N) \right) \tag{B11}$$

suffices to ensure that $\omega(I^*, N, K)$ is (strictly) increasing in N.

We now show that Assumption $2'$ implies (B11).

If $\eta \to 0$ then $\varsigma_{max} = \varsigma_{min} = 1$ and (B11) holds. Likewise, if $\zeta = 1$, $\varsigma_{max} = \varsigma_{min} = 1 - \eta$ and (B11) holds. To complete the proof we show that (B11) holds under (A1). This follows from the next three steps:

- Following the same steps as before, we have that (B8) holds.
- The function $f(x) = \frac{1}{1-\sigma} x^{1-\sigma}$ is concave. Because $\frac{W}{\gamma(N)} < R$,

$$\begin{aligned} \frac{1}{1-\sigma} \frac{c^u(R)^{1-\sigma} - c^u(W/\gamma(N))^{1-\sigma}}{B^{1-\sigma}} &< \frac{c^u(W/\gamma(N))^{-\sigma}(c^u(R) - c^u(W/\gamma(N)))}{B^{1-\sigma}}, \\ &< \frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}} \frac{c^u(R)^{\sigma}}{c^u(W/\gamma(N))^{\sigma}}, \\ &< \frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}} \left(\frac{\gamma(N)}{\gamma(N-1)} \right)^{\sigma}, \end{aligned}$$

  where the last inequality follows because $c^u(x)/x$ is decreasing and thus

$$\frac{c^u(R)}{c^u(W/\gamma(N))} < \frac{c^u(W/\gamma(N-1))}{c^u(W/\gamma(N))} \le \frac{\gamma(N)}{\gamma(N-1)}.$$

- Finally, the ideal price index condition in equation (A5) implies

$$\frac{c^u(R)^{1-\sigma}}{B^{1-\sigma}} < \frac{c^u(R)^{1-\sigma}}{(I^* - N + 1)c^u(R)^{1-\sigma}} < \frac{1}{I^* - N + 1} + \xi(N).$$

  This inequality implies

$$\frac{1}{1-\sigma} \frac{c^u(W/\gamma(I^*))^{1-\sigma} - c^u(R)^{1-\sigma}}{B^{1-\sigma}} < \left(\frac{1}{I^* - N + 1} + \xi(N) \right) \left(\frac{\gamma(N)}{\gamma(N-1)} \right)^{\sigma}. \tag{B12}$$

- Multiplying inequalities (A1), (B8), and (B12), we obtain the sufficient condition (B11). This shows that Assumption $2'$ implies (B11), and ensures that $\omega_N(I^*, N, K)$ is positive.

■

Proof of Proposition 2: We first formulate a more general version of this proposition, which holds under Assumption $2'$, and then derive the tighter characterization presented in the text (under Assumption 2). In this proof, $\frac{\partial \omega}{\partial I^*}$, $\frac{\partial \omega}{\partial N}$ and $\frac{\partial \omega}{\partial K}$ denote the partial derivatives of the function $\omega(I^*, N, K)$ with respect to its arguments.

Proposition B1 (Comparative statics in the general model) *Suppose that Assumptions 1, 2′ and 3 hold. Let $\varepsilon_L > 0$ denote the elasticity of the labor supply schedule $L^s(\omega)$ with respect to ω; let $\varepsilon_\gamma = \frac{d\ln\gamma(I)}{dI} > 0$ denote the semi-elasticity of the comparative advantage schedule.*

- *If $I^* = I < \widetilde{I}$—so that the allocation of tasks to factors is constrained by technology—then:*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d\ln(W/R)}{dI} = \frac{d\ln\omega}{dI} = \frac{1}{\omega}\frac{\partial\omega}{\partial I^*} < 0,$$
$$\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN} = \frac{1}{\omega}\frac{\partial\omega}{\partial N} > 0$$

  - *the impact of capital on relative factor prices is given by*

$$\frac{d\ln(W/R)}{d\ln K} = \frac{d\ln\omega}{d\ln K} + 1 = \frac{1+\varepsilon_L}{\sigma_{cons}+\varepsilon_L} > 0,$$

  where $\sigma_{cons} \in (0,\infty)$ is the elasticity of substitution between labor and capital that applies when technology constraints the allocation of factors to tasks. This elasticity is given by a weighted average of σ and ζ.

- *If $I^* = \widetilde{I} < I$—so that the allocation of tasks to factors is cost-minimizing—then*
  - *the impact of technological change on relative factor prices is given by*

$$\frac{d\ln(W/R)}{dI} = \frac{d\ln\omega}{dI} = 0,$$
$$\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN} = \frac{\sigma_{cons}+\varepsilon_L}{\sigma_{free}+\varepsilon_L}\frac{1}{\omega}\frac{\partial\omega}{\partial N} > 0,$$

  - *and the impact of capital on relative factor prices is given by*

$$\frac{d\ln(W/R)}{d\ln K} = \frac{d\ln\omega}{d\ln K} + 1 = \left(\frac{1+\varepsilon_L}{\sigma_{free}+\varepsilon_L}\right) > 0,$$

  where

$$\sigma_{free} = (\sigma_{cons}+\varepsilon_L)\left(1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}\right) - \varepsilon_L > \hat{\sigma};$$

- *In both parts of the proposition, the labor share and employment move in the same direction as ω.*

- *Finally, under Assumption 2, we have*

$$\frac{1}{\omega}\frac{\partial\omega}{\partial I^*} = -\frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I \qquad\qquad \frac{1}{\omega}\frac{\partial\omega}{\partial N} = \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N,$$

  and the elasticities of substitution are

$$\sigma_{cons} = \hat{\sigma} \qquad\qquad \sigma_{free} = \hat{\sigma} + \frac{1}{\varepsilon_\gamma}\Lambda_I.$$

Note: In this proposition, we do not explicitly treat the case in which $I^* = I = \widetilde{I}$ in order to save on space and notation, since in this case left and right derivatives with respect to I are different.

Proof. We first establish the comparative statics of ω with respect to I, N and K when both $I^* = I < \widetilde{I}$ and $I^* = \widetilde{I} < I$.

Comparative statics for K: The curve $I^* = \min\{I, \widetilde{I}\}$ does not depend on K, all comparative statics are determined by the effect of capital on $\omega(I^*, N, K)$. An increase in K shifts up the relative demand locus in Figure A1 (this does not affect the ideal price index condition, which simplifies the analysis in this case), and thus increases W and reduces R . The impact on $\omega = \frac{W}{RK}$ depends on whether the initial effect on W/R has elasticity greater than one (since K is in the denominator).

Notice that the function $\omega(I^*, N, K)$ *already* incorporates the equilibrium labor supply response. To distinguish this supply response from the elasticity of substitution determined by factor demands, we define $\omega^L(I^*, N, K, L)$ as the static equilibrium for a fixed level of the labor supply L.

The definition of σ_{cons} implies that $\frac{\partial \omega^L}{\partial K}\frac{K}{\omega^L} = \frac{1}{\sigma_{\text{cons}}} - 1$ and $-\frac{\partial \omega^L}{\partial L}\frac{L}{\omega^L} = \frac{1}{\sigma_{\text{cons}}}$. Thus, when $I^* = I < \widetilde{I}$, we have

$$d\ln(W/R) = d\ln\omega + 1 = \left(\frac{1}{\sigma_{\text{cons}}} - 1\right) d\ln K - \frac{1}{\sigma_{\text{cons}}}\varepsilon_L d\ln\omega + d\ln K = \frac{1+\varepsilon_L}{\sigma_{\text{cons}} + \varepsilon_L} d\ln K,$$

where we have used the fact that $\omega(I^*, N, K) = \omega^L(I^*, N, K, L^s(\omega))$. This establishes the claims about the comparative statics with respect to K when $I^* = I < \widetilde{I}$.

For the case where $I^* = \widetilde{I} < I$, we have that the change in K also changes the threshold task $I^* = \widetilde{I}$. In particular, $dI^* = \frac{1}{\varepsilon_\gamma} d\ln\omega$. Thus,

$$d\ln(W/R) = \frac{1+\varepsilon_L}{\sigma_{cons} + \varepsilon_L} d\ln K + \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma} d\ln(W/R) = \frac{1+\varepsilon_L}{\sigma_{cons} + \varepsilon_L}\frac{1}{1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I}\frac{1}{\varepsilon_\gamma}} d\ln K = \frac{1+\varepsilon_L}{\sigma_{\text{free}} + \varepsilon_L} d\ln K,$$

where we define σ_{free} as in the proposition.

Comparative statics with respect to I**:** The relative demand locus $\omega = \omega(I^*, N, K)$ does not directly depend on I . Thus, the comparative statics are entirely determined by the effect of changes in I on the $I^* = \min\{I, \widetilde{I}\}$ schedule depicted in Figure 3. When $I^* = \widetilde{I} < I$, small changes in I have no effect as claimed in the proposition. Suppose next that $I^* = I < \widetilde{I}$. In this case, an increase in I shifts the curve $I^* = \min\{I, \widetilde{I}\}$ to the right in Figure 3. Lemma A1 implies that $\omega(I^*, N, K)$ is decreasing in I^*. Thus, the shift in I increases I^* and reduces ω—as stated in the proposition. Moreover, because $I^* = I$, we have

$$\frac{d\ln(W/R)}{dI} = \frac{d\ln\omega}{dI^*} = \frac{1}{\omega}\frac{\partial\omega}{\partial I^*} < 0,$$

where $\frac{\partial\omega}{\partial I^*}$ denotes the partial derivative of $\omega(I^*, N, K)$ with respect to I^*.

Comparative statics for N: From Lemma A1, changes in N only shift the relative demand curve up in Figure 3. Hence, when $I^* = I < \widetilde{I}$, we have

$$\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN} = \frac{1}{\omega}\frac{\partial\omega}{\partial N} > 0,$$

where $\frac{\partial \omega}{\partial N}$ denotes the partial derivative of $\omega(I^*, N, K)$ with respect to N.

Turning next to the case where $I^* = \widetilde{I} < I$, note that the threshold task is given by $\gamma(I^*) = \omega K$. Therefore, $dI^* = \frac{1}{\varepsilon_\gamma} d\ln\omega$ (where recall that ε_γ is the semi-elasticity of the γ function as defined in the proposition). Therefore, $\frac{d\ln(W/R)}{dN} = \frac{d\ln\omega}{dN}$, and we can compute this total derivative as claimed in proposition:

$$\frac{d\ln\omega}{dN} = \frac{1}{\omega}\frac{\partial\omega}{\partial N} + \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}\frac{d\ln\omega}{dN} = \frac{\frac{1}{\omega}\frac{\partial\omega}{\partial N}}{1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}} = \frac{\sigma_{\text{ cons}} + \varepsilon_L}{\sigma_{\text{free}} + \varepsilon_L}\frac{1}{\omega}\frac{\partial\omega}{\partial N}.$$

To conclude the proposition, we specialize to the case in which Assumption 2 holds. The expressions for the partial derivative $\frac{\partial\omega}{\partial I^*}$, $\frac{\partial\omega}{\partial N}$ and $\hat{\sigma}$ presented in the proposition follow directly from differentiating equation (13) in the main text. Finally, the definition of σ_{free} in the proposition implies that in this case,

$$\sigma_{\text{free}} = (\hat{\sigma} + \varepsilon_L)\left(1 - \frac{1}{\omega}\frac{\partial\omega}{\partial I^*}\frac{1}{\varepsilon_\gamma}\right) - \varepsilon_L = \hat{\sigma} + \frac{1}{\varepsilon_\gamma}\Lambda_I,$$

which proofs the claims in Proposition 2 in the main text. ■

Proof of Proposition 3: The formulas provided for $d\ln Y\mid_{K,L}$ in this proposition hold under Assumption 2, and we impose this assumption in this proof.

We start by deriving the formulas for $d\ln Y\mid_{K,L}$ in the case in which technology binds and $I^* = I < \widetilde{I}$. To do so, we first consider a change in dN and totally differentiate equation (12) in the main text:

$$\begin{aligned} d\ln Y\mid_{K,L} =& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\gamma(N)^{\hat{\sigma}-1}\left(\frac{\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di}{L}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \left(\frac{I^*-N+1}{K}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dN \\ =& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\gamma(N)^{\hat{\sigma}-1}\left(\frac{B^{1-\hat{\sigma}}W^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \left(\frac{B^{1-\hat{\sigma}}R^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dN \\ =& B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN. \end{aligned}$$

Likewise, following a change in dI^*, we have

$$\begin{aligned} d\ln Y\mid_{K,L} =& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\left(\frac{I^*-N+1}{K}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \gamma(I)^{\hat{\sigma}-1}\left(\frac{\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}di}{L}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dI \\ =& \frac{B}{(1-\eta)Y}\left[\frac{Y(1-\eta)}{B}\right]^{\frac{1}{\hat{\sigma}}}\frac{1}{\hat{\sigma}-1}\left(\left(\frac{B^{1-\hat{\sigma}}R^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}} - \gamma(I)^{\hat{\sigma}-1}\left(\frac{B^{1-\hat{\sigma}}W^{\hat{\sigma}}}{(1-\eta)Y}\right)^{\frac{1-\hat{\sigma}}{\hat{\sigma}}}\right)dI \\ =& B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right)dI. \end{aligned}$$

We now derive the formulas for the impact of technology on factor prices. Let s_L denote the labor share in net output. Because $WL + RK = (1-\eta)Y$, we obtain

$$s_L d\ln W + (1-s_L)d\ln R = d\ln Y\mid_{K,L}. \qquad \text{(B13)}$$

Moreover, Proposition 2 implies

$$d\ln W - d\ln R = \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_N dN - \frac{1}{\hat{\sigma}+\varepsilon_L}\Lambda_I dI. \tag{B14}$$

Solving the system of equations given by (B13) and (B14), we obtain the formulas for $d\ln W$ and $d\ln R$ in the proposition.

To establish the existence of the threshold $\widetilde{K}$, we substitute $1-s_L = (I^*-N+1)B^{\hat{\sigma}-1}R^{1-\hat{\sigma}}$—this is the share of capital in output net of intermediates—in the formula for $\frac{d\ln W}{dI}$ given in the proposition. We find that automation reduces wages if and only if:

$$\frac{1}{1-\hat{\sigma}}\left[\left(\frac{W}{R}\frac{1}{\gamma(I^*)}\right)^{1-\hat{\sigma}} - 1\right] < (I^*-N+1)\Lambda_I.$$

Let $\underline{K}$ be the level of capital at which $\frac{W}{\gamma(I^*)} = R$. For $K > \underline{K}$, we have that $\frac{W}{\gamma(I^*)} \geq R$, and thus $I^* = I < \widetilde{I}$. At $\underline{K}$, the above inequality holds. Also, the left-hand side of the above inequality is a continuous and increasing function of W/R. This implies that there exists a threshold $\widetilde{K} > \underline{K}$ such that, the above inequality holds for $K \in (\underline{K}, \widetilde{K})$ but is reversed for $K > \widetilde{K}$.

Consider next the case where $I^* = \widetilde{I} < I$. In this case we have:

$$\begin{aligned} d\ln Y\mid_{K,L} &= B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN + B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(\widetilde{I})}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right)dI^* \\ &= B^{\hat{\sigma}-1}\frac{1}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right)dN. \end{aligned}$$

Thus, changes in I^* do not affect aggregate output because the marginal firm at $\widetilde{I}$ is indifferent between producing with capital or producing with labor. On the other hand, because I is not binding, changes in I do not affect aggregate output.

We derive the formulas for the impact of technology on factor prices as before, except that equation (B14) now becomes

$$d\ln W - d\ln R = \frac{1}{\sigma_{\text{free}} + \varepsilon_L}\Lambda_N dN.$$

■

Remaining Proofs from Section 3

We start by providing an additional lemma showing that, for a path of technology in which $g(t) = g$ and $n > \max\{\overline{n}, \widetilde{n}(\rho)\}$, the resulting production function $F(k, L; n)$ satisfies the Inada conditions required in a BGP.

Lemma B1 (Inada conditions) *Suppose that Assumptions 1′ and 2 hold. Consider a path of technology in which $n(t) \to n$ and $g(t) \to g$. Let $F(k, L; n)$ denote net output introduced in the proof of Proposition 4. If $\rho \in (\rho_{min}, \rho_{max})$ and $n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$ we have that F satisfies the Inada conditions*

$$\lim_{\phi\to 0} F_K(\phi, 1; n) > \rho + \delta + \theta g \qquad \lim_{\phi\to\infty} F_K(\phi, 1; n) < \rho + \delta + \theta g.$$

Proof. Let $\phi = \frac{k}{L}$. Let $MPK(\phi) = F_K(\phi, 1; n)$ and $w(\phi) = F_L(\phi, 1; n)$ denote the rental rate of capital and the wage at this ratio, respectively.

When $n > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$, these factor prices satisfy the system of equations given by the ratio of the market-clearing conditions (A3) and (A4),

$$\phi = \frac{(1-n)c^u(MPK(\phi))^{\zeta-\sigma}MPK(\phi)^{-\zeta}}{\int_0^n \gamma(i)^{\zeta-1}c^u(w(\phi)/\gamma(i))^{\zeta-\sigma}w(\phi)^{-\zeta}},$$

together with the generalized ideal price index condition (A5), which we can rewrite succinctly as:

$$B^{1-\hat{\sigma}} = (1-n)c^u(MPK(\phi))^{1-\sigma} + \int_0^n c^u(w(\phi)/\gamma(i))^{1-\sigma}di. \tag{B15}$$

We start by considering the limit case in which $\phi = 0$. The factor-demand equation requires that either (i) $MPK(\phi) = \infty$, or (ii) $w(\phi) = 0$. In the first case, we have $MPK(\phi) > \rho + \delta + \theta g$ as claimed. In the second case we have:

$$c^u(0) = \begin{cases} 0 & \text{if } \zeta \geq 1 \\ c_0^u & \text{if } \zeta < 1. \end{cases}$$

We show that in both cases $MPK(0) > \rho + \delta + \theta g$:

1. Suppose that $\zeta \geq 1$. For the ideal price index condition in (B15) to hold, we require $\sigma < 1$ (otherwise the right-hand side diverges). Moreover, the ideal price index condition in (B15) implies that $MPK(0)$ is implicitly given by:

   $$(1-n)c^u(MPK(0))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

   First, suppose that $\rho \leq \overline{\rho}$. We have that

   $$c^u(MPK(0))^{1-\sigma} > (1-n)c^u(MPK(0))^{1-\sigma} = B^{1-\hat{\sigma}} = c^u(\overline{\rho} + \delta + \theta g)^{1-\sigma}.$$

   Here we have used the fact that $n > 0$ and the definition of $\overline{\rho}$ introduced in Lemma A2. Because $\sigma < 1$, the above inequality implies $MPK(0) > \overline{\rho} + \delta + \theta g \geq \rho + \delta + \theta g$ as claimed.

   Finally, suppose that $\rho > \overline{\rho}$. Because $n > \overline{n}(\rho)$, we have:

   $$\begin{aligned}(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} &> (1-n)c^u(MPK(0))^{1-\sigma} \\ &= B^{1-\hat{\sigma}} \\ &= (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\overline{n}(\rho)} c^u((\rho+\delta+\theta g)/\gamma(i))^{1-\sigma}di \\ &> (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma}.\end{aligned}$$

   Here we have also used the definition of $\overline{n}(\rho)$ introduced in Lemma A2. Because $\sigma < 1$, the above inequality implies $MPK(0) > \rho+\delta+\theta g$ as claimed (recall that in this region $\overline{n}(\rho) < 1$).

2. Suppose that $\zeta < 1$. We have that $0 < c_0^u < c^u(x)$ for all $x > 0$. The ideal price index condition in (B15) implies that $MPK(0)$ is implicitly given by:

$$(1-n)c^u(MPK(0))^{1-\sigma} + n c_0^{u\,1-\sigma} = B^{1-\hat{\sigma}}.$$

When $\sigma < 1$, we have the following series of inequalities:

$$\begin{aligned}(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma} &> (1-n)c^u(MPK(0))^{1-\sigma} + n c_0^{u\,1-\sigma} \\ &= B^{1-\hat{\sigma}} \\ &= (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} \\ &+ \int_0^{\overline{n}(\rho)} c^u((\rho+\delta+\theta g)/\gamma(i))^{1-\sigma} di \\ &> (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma}.\end{aligned}$$

Here, we have used the fact that $n > \overline{n}(\rho)$ and $0 < c_0^u < c^u(x)$ for all $x > 0$, and the definition of $\overline{n}(\rho)$ introduced in Lemma A2. Because $\sigma < 1$, the above inequality implies $MPK(0) > \rho + \delta + \theta g$ as claimed (recall that in this region $\overline{n}(\rho) < 1$).

When $\sigma > 1$, the previous inequalities are reversed, and thus

$$(1-\overline{n}(\rho))c^u(MPK(0))^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma} < (1-\overline{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \overline{n}(\rho)c_0^{u\,1-\sigma}.$$

Because $\sigma > 1$, the above inequality implies $MPK(0) > \rho + \delta + \theta g$ as claimed.

We next consider the limit case in which $\phi = \infty$. With a slight abuse of notation, we define $MPK(\infty) = \lim_{\phi\to\infty} MPK(\phi)$ and $w(\infty) = \lim_{\phi\to\infty} w(\phi)$. The factor-demand equation requires that either (i) $MPK(\infty) = 0$, or (ii) $w(\infty) = \infty$. In the first case, $MPK(\infty) < \rho + \delta + \theta g$. In the second case, we have

$$c^u(\infty) = \begin{cases} \infty & \text{if } \zeta \leq 1 \\ c_\infty^u & \text{if } \zeta > 1. \end{cases}$$

We show that in both cases $MPK(\infty) < \rho + \delta + \theta g$.

1. Suppose that $\zeta \leq 1$. For the ideal price index condition in (B15) to hold, we require $\sigma > 1$ (otherwise the right-hand side diverges). Moreover, the ideal price index condition in (B15) implies that $MPK(\infty)$ is implicitly given by

$$(1-n)c^u(MPK(\infty))^{1-\sigma} = B^{1-\hat{\sigma}}.$$

First, suppose that $\rho \geq \overline{\rho}$. Then

$$c^u(MPK(\infty))^{1-\sigma} > (1-n)c^u(MPK(\infty))^{1-\sigma} = B^{1-\hat{\sigma}} = c^u(\overline{\rho}+\delta+\theta g)^{1-\sigma}.$$

Here we have used the fact that $n > 0$ and the definition of $\overline{\rho}$ introduced in Lemma A2. Because $\sigma > 1$, the above inequality implies $MPK(\infty) < \overline{\rho} + \delta + \theta g \leq \rho + \delta + \theta g$ as claimed.

Finally, suppose that $\rho < \overline{\rho}$. Because $n > \widetilde{n}(\rho)$, we have

$$\begin{aligned}
(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} &> (1-n)c^u(MPK(\infty))^{1-\sigma} \\
&= B^{1-\hat{\sigma}} \\
&= (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \int_0^{\widetilde{n}(\rho)} c^u((\rho+\delta+\theta g)\gamma(i))^{1-\sigma} di \\
&> (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma}.
\end{aligned}$$

Here we have also used the definition of $\widetilde{n}(\rho)$ introduced in Lemma A2. Because $\sigma > 1$, the above inequality implies $MPK(\infty) < \rho + \delta + \theta g$ as claimed (recall that in this region $1 > \widetilde{n}(\rho)$).

2. Suppose that $\zeta > 1$. We have that $0 < c^u(x) < c^u_\infty$ for all $x < \infty$. The ideal price index condition in (B15) implies that $MPK(\infty)$ is implicitly given by

$$(1-n)c^u(MPK(\infty))^{1-\sigma} + n{c^u_\infty}^{1-\sigma} = B^{1-\hat{\sigma}}.$$

When $\sigma < 1$, we also have

$$\begin{aligned}
(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma} &< (1-n)c^u(MPK(\infty))^{1-\sigma} + n{c^u_\infty}^{1-\sigma} \\
&= B^{1-\hat{\sigma}} \\
&= (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} \\
&+ \int_0^{\widetilde{n}(\rho)} c^u((\rho+\delta+\theta g)\gamma(i))^{1-\sigma} di \\
&< (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma}.
\end{aligned}$$

Here, we have used the fact that $n > \widetilde{n}(\rho)$ and $0 < c^u(x) < c^u_\infty$ for all $x < \infty$, and the definition of $\widetilde{n}(\rho)$ introduced in Lemma A2. Because $\sigma < 1$, this series of inequalities implies $MPK(\infty) < \rho + \delta + \theta g$ as claimed (recall that in this region $\widetilde{n}(\rho) < 1$).

When $\sigma > 1$, the previous inequalities are reversed, and

$$(1-\widetilde{n}(\rho))c^u(MPK(\infty))^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma} > (1-\widetilde{n}(\rho))c^u(\rho+\delta+\theta g)^{1-\sigma} + \widetilde{n}(\rho){c^u_\infty}^{1-\sigma}.$$

Because $\sigma > 1$, this inequality implies $MPK(\infty) < \rho + \delta + \theta g$, completing the proof.

■

Proof of Global Stability for Part 2 of Proposition 4: Here we provide the details of global stability of the interior equilibrium where all automated tasks are immediately produced with capital (part 2 of Proposition 4). In particular, we show that the BGP given by $k(t) = k_B$, $c(t) = c_B$ and $L(t) = L_B$ is globally stable.

For a given level of capital and consumption, we can define the equilibrium labor supply schedule, $L^E(k,c)$, implicitly as the solution to the first-order condition

$$\nu'(L^E(k,c))e^{\nu(L^E(k,c))\frac{\theta-1}{\theta}} = \frac{F_L(k, L^E(k,c))}{c}.$$

The left-hand side of this equation is increasing in L^E. Thus, the optimal labor supply $L^E(k,c)$ is increasing in k (because of the substitution effect) and is decreasing in c (because of the income effect). In addition, because F_L is homogeneous of degree zero, one can verify that $\frac{L}{k} > L^E_k > 0$, so that labor responds less than one-to-one to an increase in capital.

Any dynamic equilibrium must solve the system of differential equations

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} =& \frac{1}{\theta}(F_K(k(t), L^E(k(t), c(t)); n) - \delta - \rho) - g\\ \dot{k}(t) =& F(k(t), L^E(k(t), c(t)); n) - (\delta + g)k(t) - c(t)e^{\nu(L^E(k(t),c(t)))\frac{\theta-1}{\theta}},\end{aligned}$$

together with the transversality condition in equation (19).

We analyze this system in the (c,k) space. We always have one of the two cases portrayed in Figure B2; either $\lim_{c\to 0} L^E(k,c) = \overline{L}$ or $\lim_{c\to 0} L^E(k,c) = \infty$.

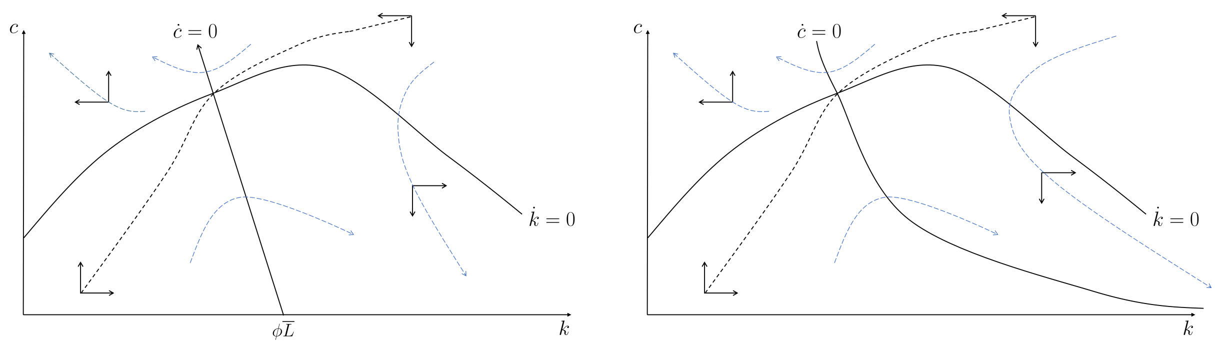

FIGURE B2: The left panel shows the phase diagram of the equilibrium system when $\lim_{c\to 0} L^E(k,c) = \overline{L}$. The right panel shows the phase diagram of the equilibrium system when $\lim_{c\to 0} L^E(k,c) = \infty$.

The locus for $\dot{k} = 0$ yields a curve that defines the maximum level of consumption that can be sustained at each level of capital. This level is determined implicitly by

$$F(k, L^E(k,c); n) - (\delta + g)k = ce^{\nu(L^E(k,c))\frac{\theta-1}{\theta}}.$$

The locus for $\dot{c} = 0$ is given by $k = \phi L^E(k,c)$, which defines a decreasing curve between c and k. Depending on whether $\nu'(L)$ has a vertical asymptote or not, as $c \to 0$, this locus converges to $k = \phi\overline{L}$ (left panel in figure B2) or $k = \infty$ (right panel in figure B2).

Importantly, we always have that, as $c \to 0$, the locus for $\dot{k} = 0$ is above the locus for $\dot{c} = 0$. This is clearly the case when $\lim_{c\to 0} L^E(k,c) = \overline{L}$. To show this when $\lim_{c\to 0} L^E(k,c) = \infty$, consider a point (c_0, k_0) in the locus for $\dot{k} = 0$. We have

$$F_K\left(1, \frac{L^E(k_0,c_0)}{k_0}\right) < F\left(1, \frac{L^E(k_0,c_0)}{k_0}\right) = \delta + g + \mathcal{O}(c_0)$$

Thus, for $c_0 \to 0$, the condition $\rho + (\theta - 1)g > 0$ implies

$$F_K\left(1, \frac{L^E(k_0, c_0)}{k_0}\right) < \rho + \delta + \theta g.$$

This inequality implies that $\frac{L^E(k_0,c_0)}{k_0} < \frac{1}{\phi}$, which is equivalent to the point (c_0, k_0) being in the northeast region of the locus for $\dot{c} = 0$.

As shown in Appendix A, both when $\lim_{c\to 0} L^E(k,c) = \overline{L}$ or $\lim_{c\to 0} L^E(k,c) = \infty$, we have a unique interior equilibrium at (c_B, k_B). Moreover, because as $c \to 0$ the locus for $\dot{c} = 0$ is below the locus for $\dot{k} = 0$, we must have that the locus for $\dot{c} = 0$ always cuts the locus for $\dot{k} = 0$ from above at (c_B, k_B). Thus, as shown in the phase diagrams in Figure B2, the unique interior equilibrium at (c_B, k_B) is saddle-path stable.

One could also establish local saddle-path stability as follows. Around the interior BGP, the system of differential equations that determines the equilibrium can be linearized as (suppressing the arguments of the derivatives of the production function)

$$\begin{aligned}
\dot{k}(t) =& \left(\rho + (\theta - 1)g + \frac{1}{\theta}F_L L_k^E\right)(k(t) - k_B) + \left(-e^{\nu(L_B)\frac{\theta-1}{\theta}} + \frac{1}{\theta}F_L L_c^E\right)(c(t) - c_B) \\
\dot{c}(t) =& \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)(k(t) - k_B) + \frac{c_B}{\theta}F_{KL}L_c^E(c(t) - c_B).
\end{aligned}$$

The characteristic matrix of the system is therefore given by

$$M_{\text{exog}} = \begin{pmatrix} \rho + (\theta - 1)g + \frac{1}{\theta}F_L L_k^E & -e^{\nu(L_B)\frac{\theta-1}{\theta}} + \frac{1}{\theta}F_L L_c^E \\ \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E) & \frac{c_B}{\theta}F_{KL}L_c^E \end{pmatrix}.$$

To analyze the properties of this matrix, we will use two facts: (i) $F_L L_k^E + c_B F_{KL} L_c^E = 0$ and (ii) $F_{KK} + F_{KL}L_k^E < 0$. First, (i) follows by implicitly differentiating the optimality condition for labor, which yields:

$$L_k^E = \frac{\frac{1}{c}F_{Lk}}{e^{\nu(L)\frac{\theta-1}{\theta}}\left(\nu''(L) + \frac{\theta-1}{\theta}\nu'^2\right) - \frac{1}{c}F_{LL}} \qquad L_c^E = -\frac{\frac{1}{c^2}F_L}{e^{\nu(L)\frac{\theta-1}{\theta}}\left(\nu''(L) + \frac{\theta-1}{\theta}\nu'^2\right) - \frac{1}{c}F_{LL}}.$$

Next, (ii) follows by noting that, because $L_k^E < \frac{L^E}{k}$, we have

$$F_{KK} + F_{KL}L_k^E < F_{KK} + F_{KL}\frac{L^E}{k} = 0.$$

Using these facts, we can compute the trace of M_{exog} as

$$\text{Tr}(M_{\text{exog}}) = \rho + (\theta - 1)g + \frac{1}{\theta}F_L L_k^E + \frac{c_B}{\theta}F_{KL}L_c^E = \rho + (\theta - 1)g > 0.$$

In addition, the determinant of M_{exog} is given by:

$$\begin{aligned}
\text{Det}(M_{\text{exog}}) =& \frac{c_B}{\theta}F_{KL}L_c^E\left(\rho + (\theta - 1)g + \frac{1}{\theta}F_L L_k^E\right) \\
&- \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L_B)\frac{\theta-1}{\theta}}\right) < 0 \qquad \text{(B16)}
\end{aligned}$$

The inequality follows by noting that $F_{KL}L^E_c < 0$, $\rho + (\theta-1)g + \frac{1}{\theta}F_L L^E_k > 0$, $F_{KK} + F_{KL}L^E_k < 0$, and $\frac{1}{\theta}F_L L^E_c - e^{\nu(L)\frac{\theta-1}{\theta}} < 0$.

(The negative determinant is equivalent to the fact established above that the curve for $\dot{c} = 0$ cuts the curve for $\dot{k} = 0$ from above. Moreover, the algebra here shows that, at the intersection (c_B, k_B), the locus for $\dot{k}$ is increasing).

The sign of the trace and the determinant imply that the matrix has one positive and real eigenvalue and one negative and real eigenvalue. Theorem 7.19 in Acemoglu (2009) shows that, locally, the economy with exogenous technology is saddle-path stable as wanted.

To show the global stability of the unique BGP (c_B, k_B), we need to rule out two types of paths: the candidate paths that converge to zero capital, which we will show are not feasible, and the candidate paths that converge to zero consumption, which we will show are not optimal.

To rule out the paths that converge to zero capital, note that such paths converge to an allocation with $k(t) = 0$ and $c(t) > \underline{c}$. Here $\underline{c} \geq 0$ is the maximum level of consumption that can be sustained when $k = 0$, which is given by:

$$F(0, L^E(0, \underline{c})) = \underline{c}e^{\nu(L^E(0,\underline{c}))\frac{\theta-1}{\theta}}.$$

To rule out the paths that converge to zero consumption, we show that they violate the transversality condition in equation (19). In all these paths we have $c(t) \to 0$. There are two possible paths for capital. Either capital converges to $\overline{k}$—even at zero consumption the economy only sustains a finite amount of capital—, or capital grows with no bound. In the first case, note that:

$$F_K\left(1, \frac{L^E(k,c)}{k}\right) \leq F\left(1, \frac{L^E(k,c)}{k}\right) = \delta + g.$$

Thus, the transversality condition in (19) does not hold. In the second case, we have that capital grows at an asymptotic rate of $F\left(1, \frac{L^E(k,c)}{k}\right) - \delta - g$. This is greater than or equal to the discount rate used in the transversality condition in equation (19), which is $F_K\left(1, \frac{L^E(k,c)}{k}\right) - \delta - g$. Thus, the transversality condition does not hold in this case either. ■

Proof of Proposition 5: We prove the proposition in the more general case in which Assumption 2′ holds.

Proposition 4 shows that for this path of technology the economy admits a unique BGP.

If $n < \overline{n}(\rho)$, we have that in the BGP $n^*(t) = \overline{n}(\rho) > n$. Thus, small changes in n do not affect the BGP equations; n does not affect effective wages, employment, or the labor share.

If $n > \overline{n}(\rho)$, we have that in the BGP $n^*(t) = n$. In this case, the behavior of the effective wages follows from the formulas for $w_I'(n)$ and $w_N'(N)$ in equation (A8), whose signs can be determined from Lemma A2.

To characterize the behavior of employment, note that we can rewrite the first-order condition for the BGP level of employment in equation (18) as

$$\frac{1}{L\nu'(L)e^{\frac{\theta-1}{\theta}\nu(L(t))}} = \frac{c}{wL} = \frac{1}{s_L}\frac{\rho + (\theta-1)g}{\rho+\delta+\theta g} + \frac{\delta+g}{\rho+\delta+\theta g}.$$

It follows that, asymptotically, there is an increasing relationship between employment and the labor share (recall that the joint concavity of the utility function requires $\nu'(L)e^{\frac{\theta-1}{\theta}\nu(L(t))}$ to increase in L). Thus, the BGP level of employment is given by the increasing function $L^{LR}(\omega)$, whose elasticity we denote by ε_L^{LR}.

To characterize the behavior of the labor share we use Lemma A1. This lemma was derived for the static model when the labor supply was given by $L^s(\omega)$, but we can use it here to describe the asymptotic behavior of the economy when the supply of labor is given by $L^{LR}(\omega)$.

We consider two cases. First, suppose that $\sigma_{\text{const}} \leq 1$. Let $k_I(n)$ denote the BGP value for $K(t)/\gamma(I(t))$. Recall that the function $\omega(I^*, N, K)$ yields the value of $\omega = \frac{W}{RK}$ when the level of technology is given by I^*, N, and the stock of capital is given by K. Thus, the definition of $w_I(n)$ and $k_I(n)$ implies that:

$$\omega(0, n, k_I(n)) = \frac{w_I(n)}{(\rho + \delta + \theta g)k_I(n)}.$$

Differentiating this expression, we obtain

$$k_I'(n) = \frac{w_I'(n)\frac{1}{Rk} - \frac{\partial \omega}{\partial N}}{\frac{\omega}{k}\frac{1+\varepsilon_L^{LR}}{\sigma_{\text{const}}+\varepsilon_L^{LR}}}.$$

Using this expression for $k_I'(n)$, it follows that the total effect of technology on ω is given by

$$\begin{aligned}\frac{d\omega}{dn} =& \frac{\partial \omega}{\partial N} + \frac{\partial \omega}{\partial K}k_I'(n)\\ =& \frac{\partial \omega}{\partial N}\left(\frac{\sigma_{\text{const}} + \varepsilon_L^{LR}}{1+\varepsilon_L^{LR}}\right) + \frac{w_I'(n)}{Rk}\left(\frac{1-\sigma_{\text{const}}}{1+\varepsilon_L^{LR}}\right).\end{aligned}$$

Because $\frac{\partial \omega}{\partial N} > 0$ and $w_I'(n) > 0$, we have that, whenever $\sigma_{\text{const}} \leq 1$, ω is increasing in n. Moreover, because the BGP level of employment is given by the increasing function $L^{LR}(\omega)$, n raises employment too.

Next suppose that $\sigma_{\text{const}} > 1$. Let $k_N(n)$ denote the BGP value for $K(t)/\gamma(N(t))$. Using an analogous reasoning as before, we get

$$\omega(-n, 0, k_N(n)) = \frac{w_N(n)}{(\rho + \delta + \theta g)k_N(n)}.$$

Differentiating this expression, we have

$$k_N'(n) = \frac{w_N'(n)\frac{1}{Rk} + \frac{\partial \omega}{\partial I^*}}{\frac{\omega}{k}\frac{1+\varepsilon_L^{LR}}{\sigma_{\text{const}}+\varepsilon_L^{LR}}} < 0.$$

Using this expression for $k_N'(n)$, it follows that the total effect of technology on ω is given by

$$\begin{aligned}\frac{d\omega}{dn} =& -\frac{\partial \omega}{\partial I^*} + \frac{\partial \omega}{\partial K}k_N'(n)\\ =& -\frac{\partial \omega}{\partial I^*}\left(\frac{\sigma_{\text{const}} + \varepsilon_L^{LR}}{1+\varepsilon_L^{LR}}\right) + \frac{w_N'(n)}{Rk}\left(\frac{1-\sigma_{\text{const}}}{1+\varepsilon_L^{LR}}\right).\end{aligned}$$

Because $\frac{\partial \omega}{\partial I^*} < 0$ and $w_N'(n) < 0$, we have that, whenever $\sigma_{\text{const}} \geq 1$, ω is increasing in n. Moreover, because the BGP level of employment is given by the increasing function $L^{LR}(\omega)$, n raises employment too.

The previous observations show that automation reduces the labor share in the long run. In addition, we have shown that $k_N'(n) < 0$, which implies that in response to automation, capital increases above its trend. The induced capital accumulation implies that the impact of automation on the labor share worsens over time if $\sigma_{\text{const}} > 1$ and eases if $\sigma_{\text{const}} < 1$.

When Assumption 2 holds the capital share is given by $(1-n)\left(\frac{R(t)}{B}\right)^{1-\hat{\sigma}}$. In this case, n reduces the capital share and thus increases the labor share. This expression also shows that when the rental rate returns to its BGP level, the induced capital accumulation will cause a further decline in the labor share when $\hat{\sigma} > 1$ and a partial offset when $\hat{\sigma} < 1$. ■

Remaining Proofs from Section 4

Proof of Lemma A3: In a BGP we have that the economy grows at the rate $g = A\frac{\kappa_I \kappa_N}{\kappa_I + \kappa_N} S$.

Suppose that $n \geq \max\{\overline{n}, \widetilde{n}\}$. In this case, we can write the value functions in the BGP as

$$v_N(n) = b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(w_N(n)e^{g\tau}\right)^{\zeta-\sigma} - c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}\right]d\tau,$$
$$v_I(n) = b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left(w_I(n)e^{g\tau}\right)^{\zeta-\sigma}\right]d\tau.$$

Thus, the value functions only depend on the unit cost of labor $w_N(n)$ and $w_I(n)$, and on the rental rate, which is equal to $\rho+\delta+\theta g$ in the BGP.

Now consider Taylor expansions of both of these expressions (which are continuously differentiable) around $S = 0$—so that the growth rate of the economy is small. Thus,

$$v_N(n) = \frac{b}{\rho}\left[c^u\left(w_N(n)\right)^{\zeta-\sigma} - c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}\right] + \mathcal{O}(g), \tag{B17}$$
$$v_I(n) = \frac{b}{\rho}\left[c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma} - c^u\left(w_I(n)\right)^{\zeta-\sigma}\right] + \mathcal{O}(g).$$

Because $\mathcal{O}(g) \to 0$ as $S \to 0$, we can approximate the above integrals when S is small with the explicit expressions evaluated at $g = 0$.

Differentiating the value functions in (B17) establishes that they are both strictly increasing in n. This follows from the result established in Proposition 5 that, in this region, $w_I(n)$ increases in n and $w_N(n)$ decreases in n. Moreover, as $S \to 0$, both $v_N(n)$ and $v_I(n)$ are positive. Thus, there exists $\widetilde{S}_1$ such that for $S < \widetilde{S}_1$, both $v_N(n)$ and $v_I(n)$ are positive ans strictly increasing in n.

Now suppose that $n \leq \overline{n}(\rho)$ (this case requires that $\rho > \overline{\rho}$). In this region we have $n^*(t) = \overline{n}$. Therefore, newly automated tasks are not immediately produced with capital, and thus

$$v_I(n) = b\int_0^\infty e^{-(\rho-(1-\theta)g)\tau}\left[c^u\left(\min\left\{\rho+\delta+\theta g, \frac{w_I(\overline{n}(\rho))}{\gamma(\overline{n}(\rho)-n)}e^{g\tau}\right\}\right)^{\zeta-\sigma}\right.$$
$$\left. -c^u\left(\frac{w_I(\overline{n}(\rho))}{\gamma(\overline{n}(\rho)-n)}e^{g\tau}\right)^{\zeta-\sigma}\right]d\tau.$$

The min operator $\min\left\{\rho+\delta+\theta g, \frac{w_I(\overline{n}(\rho))}{\gamma(\overline{n}(\rho)-n)}e^{g\tau}\right\}$ captures the fact that a task that is automated at time t will only generate profits in the future starting at a time $\tau > t$ such that $I^*(\tau) = I(t)$. At this point in time, $\frac{w_I(\overline{n}(\rho))}{\gamma(\overline{n}(\rho)-n)}e^{g\tau} = \rho+\delta+\theta g$, and it becomes profitable to use capital to produce the automated task.

In addition, $w_I(\overline{n}(\rho)) = \rho+\delta+\theta g$. Thus, $\lim_{g\to 0} v_I(n) = 0$ and we have $v_I(n) = \mathcal{O}(g)$ for all $n \leq \overline{n}(\rho)$. On the other hand $v_N(n)$ remains bounded away from zero as $S \to 0$. Thus, there exists $\widetilde{S}_2 > 0$ such that for $S < \widetilde{S}_2$, $\kappa_N v_N(\overline{n}) > \kappa_I v_I(\overline{n}) > 0$ as claimed, and $v_I(\overline{n}) = \mathcal{O}(g)$.

Finally, consider the case where $n < \widetilde{n}(\rho)$ (this case requires that $\rho < \overline{\rho}$). Because $w_N(n)e^{g\tau} > \rho+\delta+\theta g$ and $w_I(n)e^{g\tau} > \rho+\delta+\theta g$ for all $\tau \geq 0$, it follows that, in this region, $v_I(n) > 0 > v_N(n)$ as claimed. Moreover, the derivatives for $w_I(n)$ and $w_N(n)$ in equation (A8) imply that, in this region, both $w_I(n)$ and $w_N(n)$ are decreasing in n. Thus, in this region, $v_I(n)$ is decreasing and $v_N(n)$ is increasing in n.

To complete the proof of this lemma, we simply take $\widetilde{S} = \min\{\widetilde{S}_1, \widetilde{S}_2\}$ if $\theta \geq 1$, and $\widetilde{S} = \min\{\widetilde{S}_1, \widetilde{S}_2, \frac{\rho(\kappa_I+\kappa_N)}{(1-\theta)A\kappa_I\kappa_N}\}$ if $\theta < 1$. This choice also ensures that $\rho + (\theta-1)g > 0$ as required in the Lemma. ■

Proof of local stability for the unique BGP when $\theta > 0$: The local stability analysis applies to the case where $\rho > \bar{\rho}$ and $S < \min\{\widetilde{S}, \hat{S}\}$ and $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$. In this case, the economy admits a unique BGP.

When $\rho > \bar{\rho}$ and $S < \min\{\widetilde{S}, \hat{S}\}$, we can simplify the characterization of equilibrium. In particular, in this case, starting with initial conditions $n(0) \geq 0$ and $k(0) > 0$, the equilibrium with endogenous technology can be summarized by paths for $\{c(t), k(t), n(t), v(t), S_I(t)\}$ such that:

1. The normalized consumption satisfies the Euler equation:
$$\frac{\dot{c}}{c} = \frac{1}{\theta}(F_K(k,L;n) - \delta - \rho) + \mathcal{O}(g).$$
2. The endogenous labor supply is given by $L^E(k,c;n)$, and is defined implicitly by:
$$c\nu'^{\nu(L)\frac{\theta-1}{\theta}} \geq F_L(k,L;n),$$
with equality if $L^E(k,c;n) > 0$.
3. The capital stock satisfies the resource constraint:
$$\dot{k} = F(k,L;n) + X(k,L;n) - \delta k - ce^{\nu(L)\frac{\theta-1}{\theta}} + \mathcal{O}(g).$$
Here, $X(k,L;n) = b(1-n^*)yc^u(F_K)^{\zeta-\sigma} + by\int_0^{n^*} c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di$ are the profits from the intermediate sales.
4. The transversality condition
$$\lim_{t\to\infty}(k(t)+\pi(t))e^{-\int_0^t F_K(k(s),L(s);n(s))-\delta-\mathcal{O}(g))ds} = 0$$
holds, where $\pi(t) = I(t)v_I(t) + N(t)v_N(t)$ are (the normalized) corporate profits.

5. Technology evolves endogenously according to:

$$\dot{n} = \kappa_N S - (\kappa_I + \kappa_N) G(v) S.$$

6. The value function, $v = \kappa_I v_I - \kappa_N v_N$, satisfies

$$(F_K - \delta - g)v - \dot{v} = b\kappa_I \pi_I(k, L; n) - b\kappa_N \pi_N(k, L; n) + \mathcal{O}(g). \tag{B18}$$

Around $g = 0$, the above system of differential equations is Lipschitz continuous (on their right-hand side, the equations for $\dot{c}$, $\dot{k}$, $\dot{n}$ and $\dot{v}$ have bounded derivatives around the BGP $\{c_B, k_B, n_B, v_B\}$; this can be seen from the matrix containing these derivatives M_{endog}, which we present below). Thus, from the theorem of the continuous dependence of trajectories of a dynamical system on parameters (e.g., Walter, 1998, page 146, Theorem VI), there exists a neighborhood of $g = 0$ and a threshold $\overline{S}_1$ such that for $S < \overline{S}_1$, the trajectories that solve the above system have the same direction as the trajectories of the system evaluated at $g = 0$. In particular, for $S < \overline{S}_1$, the BGP is locally saddle-path stable if and only if it is also locally saddle path stable in the limit in which $g = 0$.

The previous argument shows that, to analyze the local stability of the BGP when $S < \overline{S}_1$, it is sufficient to analyze the limit case in which $g = 0$. In what follows we focus on this limit.

As in the proof of Proposition 4, the Euler equation and the resource constraint can be linearized around the BGP (denoted with the subscript B) as follows:

$$\begin{aligned}
\dot{c} =& \frac{c_B}{\theta}(F_{Kn} + F_{KL}L_n^E)(n - n_B) + \frac{c_B}{\theta}F_{KL}L_c^E(c - c_B) + \frac{c_B}{\theta}(F_{KK} + F_{KL}L_k^E)(k - k_B) \\
\dot{k} =& \left(F_n + \frac{1}{\theta}F_L L_n^E + X_n + X_L L_n^E\right)(n - n_B) + \left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)(c - c_B) \\
&+ \left(F_K + \frac{1}{\theta}F_L L_k^E - \delta + X_k + X_L L_k^E\right)(k - k_B).
\end{aligned}$$

Let us denote by X_k, X_L and X_n the partial derivatives of the function $X(k, L; n)$ with respect to each of its arguments.

We now show that, under Assumption 4 (which requires $\sigma > \zeta$), we have $X_k > 0$ and $X_L > 0$—that is, the demand for intermediates is increasing in K and L (which also implies that capital and labor are complements to intermediates). We first show this for X_L. Let us rewrite X as

$$X = \frac{b}{B^{\hat{\sigma}-1}(1-\eta)}kR^{\zeta} + by\int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di,$$

which implies

$$\begin{aligned}
X_L =& \frac{b}{B^{\hat{\sigma}-1}(1-\eta)}k\zeta R^{\zeta-1}F_{KL} \\
&+ by_L\int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} di + (\zeta - \sigma)by\int_0^n c^u \left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma} \varsigma_L(i)\frac{F_{LL}}{F_L}di > 0.
\end{aligned}$$

Here, $\varsigma_L(i)$ is the share of labor in the production of task i. The above inequality follows from the fact that $y_L > 0$, $F_{KL} > 0$, and $F_{LL} < 0$.

We now show that $X_k > 0$. Differentiating the labor market-clearing condition yields

$$\begin{aligned}\frac{y_k}{y} =&(\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}\varsigma_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}di}+\zeta\frac{F_{KL}}{F_L}+\frac{L_k^E}{L^E}\\ >&(\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}\varsigma_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\gamma(i)^{\zeta-1}di}.\end{aligned}$$

An application of Chebyshev's inequality then implies

$$\frac{y_k}{y} > (\sigma-\zeta)\frac{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\varsigma_L(i)\frac{F_{KL}}{F_L}di}{\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}di}. \tag{B19}$$

(Chebyshev's inequality applies because when $\zeta > 1$, both $\gamma(i)^{\zeta-1}$ and $\varsigma_L(i)$ are increasing in i, and when $\zeta < 1$, both are decreasing in i.)

Therefore,

$$\begin{aligned}X_k =&b(1-n)y_k c^u(F_K)^{\zeta-\sigma}+(\zeta-\sigma)b(1-n^*)yc^u(F_K)^{\zeta-\sigma}\varsigma_K\frac{F_{KK}}{F_K}\\ &+by_k\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}di+by(\zeta-\sigma)\int_0^n c^u\left(\frac{F_L}{\gamma(i)}\right)^{\zeta-\sigma}\varsigma_L(i)\frac{F_{KL}}{F_L}di>0,\end{aligned}$$

where ς_K denotes the share of capital in the production of automated tasks. The inequality then follows from the fact that $y_k > 0$ and $F_{KK} < 0$ (recall that $\sigma > \zeta$ under Assumption 4), and the inequality in equation (B19) derived above.

(When Assumption 2 holds, $X = \frac{\eta}{1-\eta}F(k, L; n)$, and it is clear that $X_k > 0$ and $X_L > 0$).

Let Q_n, Q_k and $Q_c > 0$ denote the derivatives of the right-hand side of (B18) with respect to n, k and c. We can then write the Jacobian of the system of differential equations in terms of the derivatives $\{Q_n, Q_k, Q_c\}$, the derivatives $\{X_n, X_k, X_L\}$, and the derivatives of the function F as follows:

$$\begin{pmatrix} 0 & -(\kappa_I+\kappa_N)G'(0)S & 0 & 0\\ -Q_n & \rho & -Q_c & -Q_k\\ \frac{c^*}{\theta}(F_{Kn}+F_{KL}L_n^E) & 0 & \frac{c^*}{\theta}F_{KL}L_c^E & \frac{c^*}{\theta}(F_{KK}+F_{KL}L_k^E)\\ F_n+\frac{1}{\theta}F_L L_n^E+X_n+X_L L_n^E & 0 & \frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}}+X_L L_c^E & \rho+\frac{1}{\theta}F_L L_k^E+X_k+X_L L_k^E \end{pmatrix}.$$

Denote this matrix by M_{endog}, and its eigenvalues by $\lambda_1, \lambda_2, \lambda_3$, and λ_4. These eigenvalues satisfy the following properties:

- The trace satisfies

$$\mathrm{Tr}(M_{\text{endog}}) = \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2\rho + X_k + X_L L_k^E > 0.$$

  The last inequality follows from the fact that $X_L > 0$ and $L_k^E > 0$

- The determinant satisfies

$$\begin{aligned}\text{Det}(M_{\text{endog}}) =&(\kappa_I+\kappa_N)G'(0)S\times\\ &\left[\frac{c_B}{\theta}\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)\left(Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E)\right)\right.\\ &+\frac{c_B}{\theta}\left(\rho+\frac{1}{\theta}F_L L_k^E + X_k + X_L L_k^E\right)\left(Q_c(F_{Kn}+F_{KL}L_n^E) - Q_n F_{KL}L_c^E\right)\\ &\left.+\frac{c_B}{\theta}\left(F_n+\frac{1}{\theta}F_L L_n^E + X_n + X_L L_n^E\right)\left(Q_k F_{KL}L_c^E - Q_c(F_{KK}+F_{KL}L_k^E)\right)\right].\end{aligned}$$

The expression for the determinant can be further simplified by noting that $Q_c(F_{KK}+F_{KL}L_k^E) = Q_k F_{KL} L_c^E$. To show this, note that the impact of k, c on Q—the relative incentives for automation—depends on the ratio $k/L^E(k,c;n)$. For a given value of n, this ratio determines factor prices and hence Q. Let $\phi = \frac{k}{L^E(k,c;n)}$. Then

$$Q_k = Q_\phi\left(\frac{1}{L} - \frac{k}{L^2}L_k^E\right) Q_c = -Q_\phi \frac{k}{L^2}L_c^E.$$

These equations then imply

$$Q_c = -Q_k\frac{kL_c^E}{L-kL_k^E} = Q_k F_{KL}L_c^E\frac{1}{\frac{kL_k^E-L}{k}F_{KL}} = Q_k F_{KL}L_c^E\frac{1}{F_{KK}+F_{KL}L_k^E},$$

which gives the desired identity

Replacing this expression for Q_c in the determinant, we get

$$\begin{aligned}\text{Det}(M_{\text{endog}}) =&(\kappa_I+\kappa_N)G'(0)S\times\\ &\left[\frac{c_B}{\theta}\left(\frac{1}{\theta}F_L L_c^E - e^{\nu(L)\frac{\theta-1}{\theta}} + X_L L_c^E\right)\left(Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E)\right)\right.\\ &\left.+\frac{c_B}{\theta}F_{KL}L_c^E\left(\rho+\frac{1}{\theta}F_L L_k^E + X_k + X_L L_k^E\right)\left(Q_k\frac{F_{Kn}+F_{KL}L_n^E}{F_{KK}+F_{KL}L_k^E} - Q_n\right)\right].\end{aligned}$$

Because $\kappa_I v_I(n)$ cuts (i.e., is steeper than) $\kappa_N v_N(n)$ from below, we have

$$Q_n - Q_k\frac{F_{Kn}+F_{KL}L_n^E}{F_{KK}+F_{KL}L_k^E} > 0.$$

(Note that this expression is equivalent to the derivative of the profit function Q with respect to n when the capital adjusts to keep the interest rate constant. This derivative is positive when $\kappa_I v_I(n)$ cuts $\kappa_N v_N(n)$ from below). Because $F_{KK}+F_{KL}L_k^E < 0$ (as shown in the proof of Proposition 4), we also have that

$$Q_n(F_{KK}+F_{KL}L_k^E) - Q_k(F_{Kn}+F_{KL}L_n^E) < 0.$$

Thus, both terms in the determinant are positive and $\text{Det}(M_{\text{endog}}) > 0$ (recall that $L_c^E < 0$ and $L_k^E > 0$, and $X_L, X_k > 0$). Because $\text{Det}(M_{\text{endog}}) = \lambda_1\lambda_2\lambda_3\lambda_4$, this implies that the four eigenvalues have either zero, two or four negative real parts.

Let us define $Z(M_{\text{endog}}) = \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2$. We also have

$$Z(M_{\text{endog}}) = \rho\,\text{Det}(M_{\text{exog}}) + \mathcal{O}(S).$$

From equation (B16), we know $\text{Det}(M_{\text{exog}}) < 0$ and this determinant does not depend on S. Thus, there exists $\overline{S}_2 > 0$ such that for $S < \overline{S}_2$, $Z(M_{\text{endog}}) < 0$. This implies that we cannot have four eigenvalues with positive real parts.[35]

But we also have that $\text{Tr}(M_{\text{endog}}) > 0$ as shown above, and thus not all four eigenvalues can have negative real parts.

These observations show that when $S < \overline{S}_2$, M_{endog} has exactly two eigenvalues with negative real parts. Theorem 7.19 in Acemoglu (2009) shows that, in the limit case in which $g \to 0$, the economy with endogenous technology is saddle-path stable. Let $\check{S} = \min\{\overline{S}_1, \overline{S}_2\}$. Thus, for $S < \check{S}$, the unique BGP is locally stable. ■

Proofs from Section 5

Proof of Proposition 7: We prove this proposition under Assumption 2. Consider an exogenous path for technology in which $\dot{N} = \dot{I} = \Delta$ (with $\rho + (\theta - 1)A_H\Delta > 0$) and suppose that $n(t) > \max\{\overline{n}(\rho), \widetilde{n}(\rho)\}$. This implies that in any candidate BGP $I^*(t) = I(t)$ and $n^*(t) = n(t)$.

Define $M \in [I, N]$ as in the main text. Two equations determine M. First, because firms are indifferent between producing task M with low-skill or high-skill workers, we have

$$\frac{W_H(t)}{W_L(t)} = \frac{\gamma_H(M(t))}{\gamma_L(M(t), t)} = \frac{\gamma_H(M(t))^{1-\xi}}{\Gamma(t - T(M(t)))}.$$

In addition, the relative demand for high-skill and low-skill labor yields

$$\frac{L}{H}\frac{\int_{M(t)}^{N(t)} \gamma_H(i)^{\hat{\sigma}-1} di}{\int_{I(t)}^{M(t)} \gamma_L(i,t)^{\hat{\sigma}-1} di} = \left(\frac{W_H(t)}{W_L(t)}\right)^{\hat{\sigma}}.$$

Combining these two equations, we obtain the equilibrium condition

$$\frac{L}{H}\frac{\int_{M(t)}^{N(t)} \gamma_H(i)^{\hat{\sigma}-1} di}{\int_{I(t)}^{M(t)} \gamma_L(i,t)^{\hat{\sigma}-1} di} = \left(\frac{\gamma_H(M(t))^{1-\xi}}{\Gamma(t - T(M(t)))}\right)^{\hat{\sigma}}.$$

Let $m(t) = M(t) - I(t)$ and $n = N(t) - I(t)$. Using the formula for $\gamma_L(i,t)$ and the change of variables $i = N - i'$ to rewrite the integrals in the previous equations we get that $m(t)$ is uniquely

[35] This is independent of whether these eigenvalues are real or complex. For example, if we had two positive real eigenvalues, $\lambda_1, \lambda_2 > 0$, and a conjugate pair of complex eigenvalues with positive real part, λ and $\overline{\lambda}$, then

$$Z(M_{\text{endog}}) = 2\Re(\lambda)\lambda_1\lambda_2 + |\lambda|^2(\lambda_1 + \lambda_2),$$

which cannot be negative. If we had two conjugate pairs of eigenvalues with positive real parts, $\lambda_1, \overline{\lambda}_1$ and $\lambda_2, \overline{\lambda}_2$, then

$$Z(M_{\text{endog}}) = 2\Re(\lambda_1)|\lambda_2|^2 + 2\Re(\lambda_2)|\lambda_1|^2,$$

which again cannot be negative.

pinned down by

$$\frac{L}{H}\frac{\int_0^{n-m(t)}\gamma_H(i)^{1-\hat{\sigma}}di}{\int_{n-m(t)}^{n}\gamma_H(i)^{\xi(1-\hat{\sigma})}\Gamma\left(\frac{i}{\Delta}\right)^{\hat{\sigma}-1}di}=\frac{\gamma_H(N(t))^{1-\xi}}{\gamma_H(n-m(t))^{\sigma(1-\xi)}\Gamma\left(\frac{n-m(t)}{\Delta}\right)^{\hat{\sigma}}}. \tag{B20}$$

This expression also uses the fact that, because both $\dot{N}=\dot{I}=\Delta$, we have $t-T(i)=\frac{N(t)-i}{\Delta}$. The left-hand side of equation (B20)—the relative demand curve—is decreasing in $m(t)$, converges to zero as $m(t)\to n$, and converges to infinity as $m(t)\to 0$. Moreover, the right-hand side—the comparative advantage schedule—is increasing in $m(t)$. Thus, this equation uniquely determines $m(t)$ as a function of $N(t)$ and n.

To prove the first part of the proposition, consider the case in which $\xi<1$. Taking the limit as $t\to\infty$, we have that the right-hand side of equation (B20) converges to infinity. To maintain the equality, we must have $m(t)\to 0$, which implies that asymptotically $M(t)=I(t)$ and no tasks are allocated to low-skill workers. Moreover, we have that inequality grows without bound, since

$$\frac{W_H(t)}{W_L(t)}\to\frac{\gamma_H(N(t))^{1-\xi}}{\gamma(n)^{1-\xi}\Gamma\left(\frac{n}{\Delta}\right)}\to\infty.$$

To prove the second part of the proposition, consider the case where $\xi=1$. We now show that there is a BGP in which $m(t)=m$ and $\frac{W_H(t)}{W_L(t)}$ is constant. Equation (B20) shows that, in this case, m only depends on n as claimed. Moreover, the wage gap is also constant over time and given by

$$\frac{W_H(t)}{W_L(t)}=\frac{1}{\Gamma\left(\frac{n-m}{\Delta}\right)}.$$

Now, consider an increase in n, and let $s=n-m$ denote the measure of tasks performed by high-skill workers. Holding s constant, the left-hand side of equation (B20) is decreasing in n. Because the left-hand side of equation (B20) is increasing in s and its right-hand side is decreasing in s, we must have that s is also increasing in n. This implies that, as stated in the proposition, the wage gap, which is a decreasing function of s, declines with n. ■

Proof of Proposition 8: We prove this result under the more general Assumption 2′.

From the Bellman equations provided in the main text, it follows that along a BGP we have

$$v_N(n)=b\int_0^{\frac{n}{\Delta}}e^{-(\rho-(1-\theta)g)\tau}c^u\left(w_N(n)e^{g\tau}\right)^{\zeta-\sigma}d\tau,$$
$$v_I(n)=b\int_0^{\frac{1-n}{\Delta}}e^{-(\rho-(1-\theta)g)\tau}c^u\left(\rho+\delta+\theta g\right)^{\zeta-\sigma}d\tau.$$

Here $\Delta=\frac{\kappa_I\kappa_N\iota(n^D)}{\kappa_I\iota(n^D)+\kappa_N}S$ is the endogenous rate at which both technologies grow in a BGP and $g=A\Delta$. As before, a BGP requires that n satisfies

$$\kappa_I\iota(n)v_I(n)=\kappa_Nv_N(n).$$

Using these formulas, the proof of the proposition follows from the properties of the effective wages derived in Proposition 5. Following the same steps as in the proof of Proposition 6, we also obtain

that the equilibrium in this case is locally stable whenever $\kappa_I \iota(n) v_I(n)$ cuts $\kappa_N v_N(n)$ from below.
■

We now turn to Proposition 9. We prove a similar statement in the more general case in which Assumption $2'$ holds. In particular, we show that:

Proposition B2 (Welfare implications of automation in the general model) *Consider the static economy and suppose that Assumptions 1, $2'$ and 3 hold, and that $I^* = I < \widetilde{I}$. Let $\mathcal{W} = u(C, L)$ denote the welfare of households and let $F(K, L; I, N)$ denote the net output when the amount of labor supplied is L and capital is K.*

1. *Consider the baseline model without labor market frictions, so that the representative household chooses the amount of labor without constraints, and thus $\frac{W}{C} = \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_I}{F} > 0,$$
$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_N}{F} > 0.$$

2. *Suppose that there are labor market frictions, so that employment is constrained by a quasi-labor supply curve $L \leq L_{qs}(\omega)$. Suppose also that the quasi-labor supply schedule $L_{qs}(\omega)$ is increasing in ω, has an elasticity $\widetilde{\varepsilon}_L > 0$, and is binding in the sense that $\frac{W}{C} > \nu'(L)$. Then:*

$$\frac{d\mathcal{W}}{dI} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{F_I}{F} + L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\omega} \frac{\partial \omega}{\partial I^*}\right] \lesseqgtr 0.$$
$$\frac{d\mathcal{W}}{dN} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{F_N}{F} + L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\omega} \frac{\partial \omega}{\partial N}\right] > 0.$$

Proof. The unconstrained allocation of employment solves

$$\mathcal{W} = \max_{L \geq 0} u(F(K, L; I, N), L).$$

Thus, the envelope theorem implies

$$\mathcal{W}_I = u_C F_I = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_I}{F} > 0$$

(recall that $F_I > 0$ because we assumed $I^* = I$) and also

$$\mathcal{W}_N = u_C F_N = \left(Ce^{-\nu(L)}\right)^{1-\theta} \frac{F_N}{F} > 0$$

(recall that $F_N > 0$ because we imposed Assumption 3).

Now suppose that $L \leq L_{qs}(\omega)$. The allocation of employment now solves:

$$\mathcal{W} = \max_{L \geq 0} u(F(K, L; I, N), L) + \lambda(L_{qs}(\omega) - L),$$

where $\lambda = u_c F_L + u_L = cu_c\left(\frac{F_L}{c} - \nu'(L)\right) > 0$ is the multiplier on the employment constraint (by assumption this constraint is binding). Using the envelope theorem,

$$\mathcal{W}_I = u_C F_I + \lambda L'_{qs}(\omega) \frac{\partial \omega}{\partial I^*} = \left(Ce^{-\nu(L)}\right)^{1-\theta} \left[\frac{F_I}{F} + L\left(\frac{W}{C} - \nu'(L)\right) \frac{\widetilde{\varepsilon}_L}{\omega} \frac{\partial \omega}{\partial I^*}\right] \lesseqgtr 0,$$

and

$$\mathcal{W}_N = u_C F_N + \lambda L'_{qs}(\omega)\frac{\partial \omega}{\partial N} = \left(Ce^{-\nu(L)}\right)^{1-\theta}\left[\frac{F_N}{F} + L\left(\frac{W}{C} - \nu'(L)\right)\frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial N}\right] > 0.$$

The expressions presented in Proposition 9 follow from the previous two equations because when Assumption 2 holds, we also have

$$\frac{F_I}{F} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right) \qquad \frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial I} = -\frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_I$$

$$\frac{F_N}{F} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left(R^{1-\hat{\sigma}} - \left(\frac{W}{\gamma(N)}\right)^{1-\hat{\sigma}}\right) \qquad \frac{\widetilde{\varepsilon}_L}{\omega}\frac{\partial \omega}{\partial N} = \frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_N.$$

■

Properties of the constraint efficient allocation:

We now derive the constrained efficient allocation both when the labor market is frictionless and when there is a friction as the one introduced in Proposition 9. We focus on the case in which Assumption 2 holds, although similar insights apply in general.

First the planner removes markups. This implies that net output is given by

$$F^p(K, L; I^*, N) = \mu^{\frac{\eta}{\eta-1}} B\left[(I^* - N + 1)^{\frac{1}{\hat{\sigma}}} K^{\frac{\hat{\sigma}-1}{\hat{\sigma}}} + \left(\int_{I^*}^{N} \gamma(i)^{\hat{\sigma}-1} di\right)^{\frac{1}{\hat{\sigma}}} L^{\frac{\hat{\sigma}-1}{\hat{\sigma}}}\right]^{\frac{\hat{\sigma}}{\hat{\sigma}-1}},$$

$$= \mu^{\frac{\eta}{\eta-1}} F(K, L; I^*, N).$$

Using this expression, we can write the planner's problem as:

$$\max_{C(t),L(t),S_I(t),S_N(t)} \int_0^\infty e^{-\rho t}\frac{[C(t)e^{-\nu(L(t))}]^{1-\theta} - 1}{1-\theta}dt$$

Subject to

$$\dot{K}(t) = \mu^{\frac{\eta}{\eta-1}} F(K, L; I^*, N) - \delta K(t) - C(t).$$

Let $\mu_N(t)$ denote the marginal value of new tasks (increasing N) in terms of the final good. Let $\mu_I(t)$ denote the marginal value of automation (increasing I) in terms of the final good. These marginal values are the social counterparts to $V_N(t)$ and $V_I(t)$ in the decentralized economy. Assuming that the planner operates in the region where $I^*(t) = I(t)$, we can write these marginal values as

$$\mu_N(t) = (1-\eta)\mu^{\eta(1-\hat{\sigma})}\int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds}\frac{\hat{\sigma}}{1-\hat{\sigma}}Y(\tau)\left(R(\tau)^{1-\hat{\sigma}} - \gamma(n(\tau))^{\hat{\sigma}-1}w(\tau)^{1-\hat{\sigma}}\right)d\tau,$$

$$\mu_I(t) = (1-\eta)\mu^{\eta(1-\hat{\sigma})}\int_t^\infty e^{-\int_t^\tau (R(s)-\delta)ds}\frac{\hat{\sigma}}{1-\hat{\sigma}}Y(\tau)\left(w(\tau)^{1-\hat{\sigma}} - R(\tau)^{1-\hat{\sigma}}\right)d\tau.$$

With some abuse of notation and to maximize the parallel with the decentralized expressions for V_N and V_I, we are using $R(t)$ to denote the marginal product of capital $\mu^{\frac{\eta}{\eta-1}}F_K$ and $w(t)$ to denote the (normalized) marginal product of labor $\mu^{\frac{\eta}{\eta-1}}F_L e^{-AI^*(t)}$.

These observations show that the efficient allocation satisfies similar conditions to the decentralized economy in our main model in Section 4. The only difference is that now, the allocation of scientists is guided by $\mu_N(t)$ and $\mu_I(t)$ and satisfies:

$$S_I(t) = SG\left(\frac{\kappa_I\mu_I(t) - \kappa_N\mu_N(t)}{Y(t)}\right), \quad S_N(t) = S\left[1 - G\left(\frac{\kappa_I\mu_I(t) - \kappa_N\mu_N(t)}{Y(t)}\right)\right],$$

so that in the efficient allocation, $n(t)$ changes endogenously according to:

$$\dot{n}(t) = \kappa_N S - (\kappa_N + \kappa_I)G\left(\frac{\kappa_I\mu_I(t) - \kappa_N\mu_N(t)}{Y(t)}\right)S.$$

One of the key insights from Proposition 6 is that the expected path for factor prices determines the incentives to automate and create new tasks. The equations for μ_N and μ_I show that a planner would also allocate scientists to developing both types of technologies following a similar principle; guided by the cost savings that each technology grants to firms. However, the fact that $\mu_N \neq V_N$ and $\mu_I \neq V_I$ shows that the decentralized allocation is not necessarily efficient. The inefficiency arises because technology monopolists do not earn the full gains that their technology generates, nor internalize how their innovations affect other existing and future technology monopolists.

We now show that labor market frictions change the planner's incentives to allocate scientists. By contrast, conditional on the wage level, such frictions do not change the market incentives to automate or create new tasks.

Without frictions, the efficient level of labor satisfies

$$\left(\mu^{\frac{\eta}{\eta-1}}F_L - c\nu'(L)\right)\mu_K \leq 0,$$

with equality if $L > 0$.

Now suppose that there is an exogenous constraint on labor that requires $L \leq L_{qs}(\omega)$. Let μ_L be the multiplier of this constraint. We have that:

$$\mu_L = \begin{cases} \left(\mu^{\frac{\eta}{\eta-1}}F_L - c\nu'(L)\right)\mu_K > 0 & \text{if } L = L_{qs}(\omega) \\ 0 & \text{if } L < L_{qs}(\omega) \end{cases}$$

Because the planner takes into account the first-order effects from changes in the employment level, the values for μ_N and μ_I change to:

$$\begin{aligned}\mu_N(t) = (1-\eta)\mu^{\eta(1-\hat{\sigma})}\int_t^\infty e^{-\int_t^\tau(R(s)-\delta)ds}&\left[\frac{\hat{\sigma}}{1-\hat{\sigma}}Y(\tau)\left(R(\tau)^{1-\hat{\sigma}} - \gamma(n(\tau))^{\hat{\sigma}-1}w(\tau)^{1-\hat{\sigma}}\right)\right. \\ &\left. + \left(\mu^{\frac{\eta}{\eta-1}}F_L - c\nu'(L)\right)L\frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_N\right]d\tau, \\ \mu_I(t) = (1-\eta)\mu^{\eta(1-\hat{\sigma})}\int_t^\infty e^{-\int_t^\tau(R(s)-\delta)ds}&\left[\frac{\hat{\sigma}}{1-\hat{\sigma}}Y(\tau)\left(w(\tau)^{1-\hat{\sigma}} - R(\tau)^{1-\hat{\sigma}}\right)\right. \\ &\left. - \left(\mu^{\frac{\eta}{\eta-1}}F_L - c\nu'(L)\right)L\frac{\widetilde{\varepsilon}_L}{\hat{\sigma}+\widetilde{\varepsilon}_L}\Lambda_I\right]d\tau.\end{aligned}$$

Thus, when the level of employment is below its unconstrained optimum, the planner values the introduction of new tasks more because they raise the marginal product of labor and ease the constraint on total employment. Likewise, the planner values automation less because she recognizes that by reducing employment automation has a first-order cost on workers. Importantly, the market does not recognize the first-order costs from automation or the first-order benefits from introducing new tasks. As the expressions for V_N and V_I show, only factor prices—not the extent of frictions in the labor market— determine the incentives to introduce these technologies.

When New Tasks Also Use Capital

In our baseline model, new tasks use only labor. This simplifying assumption facilitated our analysis, but is not crucial or even important for our results. Here we outline a version of the model where new tasks also use capital and show that all of our results continue to hold in this case. Suppose, in particular, that the production function for non-automated tasks is

$$y(i) = \left[\eta q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)\left(B_\nu(\gamma(i)l(i))^\nu k(i)^{1-\nu}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}, \tag{B21}$$

where $k(i)$ is the capital used in the production of the task (jointly with labor), $\nu \in (0,1)$, and $B_\nu = \nu^{-\nu}(1-\nu)^{-(1-\nu)}$ is a constant that is re-scaled to simplify the algebra.

Automated tasks $i \leq I$ can be produced using labor or capital, and their production function takes the form

$$y(i) = \left[\eta q(i)^{\frac{\zeta-1}{\zeta}} + (1-\eta)\left(k_A(i) + B_\nu(\gamma(i)l(i))^\nu k(i)^{1-\nu}\right)^{\frac{\zeta-1}{\zeta}}\right]^{\frac{\zeta}{\zeta-1}}. \tag{B22}$$

Here $k_A(i)$ is the amount of capital used in an automated task, while $k(i)$ is the amount of capital used to produce a task with labor. Comparing these production functions to those in our baseline model (2) and (3), we readily see that the only difference is the requirement that labor has to be combined with capital in all tasks (while automated tasks continue not to use any labor). Note also that when $\nu \to 1$, we recover the model in the main text as a special case. It can be shown using a very similar analysis to that in our main model that most of the results continue to hold with minimal modifications. For example, there will exist a threshold $\widetilde{I}$ such that tasks below $I^* = \min\{I, \widetilde{I}\}$ will be produced using capital and the remaining more complex tasks will be produced using labor. Specifically, whenever $R \in \arg\min\left\{R, R^{1-\nu}\left(\frac{W}{\gamma(i)}\right)^\nu\right\}$ and $i \leq I$, the relevant task is produced using capital, and otherwise it is produced using labor. Since $\gamma(i)$ is strictly increasing, this implies that there exists a threshold $\widetilde{I}$ at which, if technologically feasible, firms would be indifferent between using capital and labor. Namely, at task $\widetilde{I}$, we have $R = W/\gamma(\widetilde{I})$, or

$$\frac{W}{R} = \gamma(\widetilde{I}).$$

This threshold represents the index up to which using capital to produce a task yields the cost-minimizing allocation of factors. However, if $\widetilde{I} > I$, firms will not be able to use capital all the way up to task $\widetilde{I}$ because of the constraint imposed by the available automation technology. For this reason, the equilibrium threshold below which tasks are produced using capital is given by

$$I^* = \min\{I, \widetilde{I}\},$$

meaning that $I^* = \widetilde{I} < I$ when it is technologically feasible to produce task $\widetilde{I}$ with capital, and $I^* = I < \widetilde{I}$ otherwise.

The demand curves for capital and labor are similar, with the only modification that the demand for capital also comes from non-automated tasks. In particular, the market-clearing conditions become:

$$K = Y(1-\nu)(1-\eta)\int_{I^*}^{N} R^{(1-\nu)(1-\zeta)-1}\left(\frac{W}{\gamma(i)}\right)^{\nu(1-\zeta)} c^u\left(R^{1-\nu}\left(\frac{W}{\gamma(i)}\right)^{\nu}\right)^{\zeta-\sigma} di \tag{B23}$$
$$+ Y(1-\eta)(I^* - N + 1)c^u(R)^{\zeta-\sigma}R^{-\zeta}.$$

$$L = Y\nu(1-\eta)\int_{I^*}^{N} \frac{1}{\gamma(i)} R^{(1-\nu)(1-\zeta)}\left(\frac{W}{\gamma(i)}\right)^{\nu(1-\zeta)-1} c^u\left(R^{1-\nu}\left(\frac{W}{\gamma(i)}\right)^{\nu}\right)^{\zeta-\sigma} di. \tag{B24}$$

Following the same steps as in the text, we can then establish analogous results. This requires the more demanding Assumption $2''$, which guarantees that the demand for factors above is homothetic:

Assumption $2''$**:** One of the following three conditions holds:

- $\eta \to 0$;
- $\zeta \to 1$;
- or $\sigma - \zeta \to 0$.

Proposition B3 (Equilibrium in the static model when $\nu \in (0,1)$) *Suppose that Assumption $1''$ holds. Then, for any range of tasks $[N-1, N]$, automation technology $I \in (N-1, N]$, and capital stock K, there exists a unique equilibrium characterized by factor prices, W and R, and threshold tasks, $\widetilde{I}$ and I^*, such that: (i) $\widetilde{I}$ is determined by equation* (6) *and $I^* = \min\{I, \widetilde{I}\}$; (ii) all tasks $i \leq I^*$ are produced using capital and all tasks $i > I^*$ are produced combining labor and capital; (iii) the capital and labor market-clearing conditions, equations* (B23) *and* (B24)*, are satisfied; and (iv) factor prices satisfy the ideal price index condition:*

$$(I^* - N + 1)c^u(R)^{1-\sigma} + \int_{I^*}^{N} c^u\left(R^{1-\nu}\left(\frac{W}{\gamma(i)}\right)^{\nu}\right)^{1-\sigma} di = 1. \tag{B25}$$

Proof. The proof follows the same steps as Proposition 1. ■

Comparative statics in this case are also identical to those in the baseline model (as summarized in Proposition 2) and we omit them to avoid repetition. The dynamic extension of this more general model is also very similar, and in fact, Proposition 4 applies identically, and is also omitted. One can also define $\overline{\rho}$, $\overline{n}(\rho)$ and $\widetilde{n}(\rho)$ in an analogous nway as we did in the proof of Lemma A2. To highlight the parallels, we just present the equivalent of Proposition 6.

Proposition B4 (Equilibrium with endogenous technology when $\nu \in (0,1)$) *Suppose that Assumptions $1'$, $2'$, and 4 hold. Then, there exists $\overline{S}$ such that, when $S < \overline{S}$, we have:*

1 ***(Full automation)*** *For $\rho < \overline{\rho}$, there is a BGP in which $n(t) = 0$ and all tasks are produced with capital.*

For $\rho > \overline{\rho}$, all BGPs feature $n(t) = n > \overline{n}(\rho)$. Moreover, there exist $\overline{\kappa} > \underline{\kappa} > 0$ such that:

*2 **(Unique interior BGP)** if $\frac{\kappa_I}{\kappa_N} > \overline{\kappa}$ there exists a unique BGP. In this BGP we have $n(t) = n \in (\overline{n}(\rho), 1)$ and $\kappa_N v_N(n) = \kappa_I v_I(n)$. If, in addition, $\theta = 0$, then the equilibrium is unique everywhere and the BGP is globally (saddle-path) stable. If $\theta > 0$, then the equilibrium is unique in the neighborhood of the BGP and is asymptotically (saddle-path) stable;*

*3 **(Multiple BGPs)** if $\overline{\kappa} > \frac{\kappa_I}{\kappa_N} > \underline{\kappa}$, there are multiple BGPs;*

*4 **(No automation)** If $\underline{\kappa} > \frac{\kappa_I}{\kappa_N}$, there exists a unique BGP. In this BGP $n(t) = 1$ and all tasks are produced with labor.*

Proof. The proof of this result closely follows that of Proposition 6, especially exploiting the fact that the behavior of profits of automation and the creation of new tasks behave identically to those in the baseline model, and thus the value functions behave identically also. ■

Microfoundations for the Quasi-Labor Supply Function

We provide various micro-foundations for the quasi-labor supply expression used in the main text, $L^s\left(\frac{W}{RK}\right)$.

Efficiency wages: Our first micro-foundation relies on an efficiency wage story. Suppose that, when a firm hires a worker to perform a task, the worker could shirk and, instead of working, use her time and effort to divert resources away from the firm.

Each firm monitors its employees, but it is only able to detect those who shirk at the flow rate q. If the worker is caught shirking, the firm does not pay wages and retains its resources. Otherwise, the worker earns her wage and a fraction of the resources that she diverted away from the firm.

In particular, assume that each firm holds a sum RK of liquid assets that the worker could divert, and that if uncaught, a worker who shirks earns a fraction $u(i)$ of this income. We assume that the sum of money that the worker may be able to divert is RK to simplify the algebra. In general, we obtain a similar quasi-supply curve for labor so long as these funds are proportional to total income $Y = R\ K + WL$.

In this formulation, $u(i)$ measures how untrustworthy worker i is, and we assume that this information is observed by firms. $u(i)$ is distributed with support $[0, \infty)$ and has a cumulative density function G. Moreover, we assume there is a mass L of workers. A worker of type $u(i)$ does not shirk if and only if:

$$W \geq (1-q)[W + u(i)RK] \rightarrow \frac{W}{RK}\frac{q}{1-q} \geq u(i).$$

Thus, when the market wage is W, firms can only afford to hire workers who are sufficiently trustworthy. The employment level is therefore given by:

$$L^s = G\left(\frac{W}{RK}\frac{q}{1-q}\right)L.$$

When $q = 1$—so that there is no monitoring problem—, we have $G\left(\frac{W}{RK}\frac{q}{1-q}\right) = 1$, and the supply of labor is fixed at L for all wages $W \geq 0$. However, when $q < 1$—so that there is a monitoring problem—, we have $L^s < L$. Even though all workers would rather work than stay unemployed, the monitoring problem implies that not all of them can be hired at the market wage. Notice that, though it is privately too costly to hire workers with $u(i) > \frac{W}{RK}\frac{q}{1-q}$, these workers strictly prefer employment to unemployment.

Alternatively, one could also have a case in which firms do not observe $u(i)$, which is private information. This also requires that firms do not learn about workers. To achieve that, we assume that workers draw a new value of $u(i)$ at each point in time.

When the marginal value of labor is W, firms are willing to hire workers so long as the market wage $\widetilde{W}$ satisfies:

$$(W - \widetilde{W})G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) - (1-q)\left(\widetilde{W} + RK\int_{\frac{\widetilde{W}}{RK}\frac{q}{1-q}}^{\infty} udG(u)\right) \geq 0.$$

This condition guarantees that the firm makes positive profits from hiring an additional worker, whose type is not known.

Competition among firms implies that the equilibrium wage at each point in time satisfies:

$$(W - \widetilde{W})G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) - (1-q)\left(\widetilde{W} + RK\int_{\frac{\widetilde{W}}{RK}\frac{q}{1-q}}^{\infty} udG(u)\right) = 0.$$

This curve yields an increasing mapping from $\frac{W}{RK}$ to $\frac{\widetilde{W}}{RK}$, which we denote by

$$\frac{\widetilde{W}}{RK} = h\left(\frac{W}{RK}\right).$$

Therefore, the effective labor supply in this economy, or the quasi-supply of labor, is given by

$$L^s = G\left(\frac{\widetilde{W}}{RK}\frac{q}{1-q}\right) = G\left(h\left(\frac{W}{RK}\right)\frac{q}{1-q}\right)L.$$

As in the previous model, even though the opportunity cost of labor is zero, the economy only manages to use a fraction of its total labor.

Minimum wages: Following Acemoglu (2003), another way in which we could obtain a quasi-labor supply curve is if there is a binding minimum wage. Suppose that the government imposes a (binding) minimum wage $\widetilde{W}$ and indexes it to the income level (or equivalently the level of consumption):

$$\widetilde{W} = \varrho \cdot (RK + WL),$$

with $\varrho > 0$. Here, $RK + WL$ represents the total income in the economy (net of intermediate goods' costs).

Suppose that the minimum wage binds. Then:

$$L = \frac{1}{\varrho}s_L,$$

which defines the quasi-labor supply in this economy as an increasing function of the labor share.

Additional References

Acemoglu, Daron (2009) *Introduction to Modern Economic Growth,* Princeton University Press.

Li, Guoying (1985) "Robust Regression," in *Exploring Data Tables, Trends, and Shapes*, pp. 281–343.

Ruggles, Steven, Katie Genadek, Ronald Goeken, Josiah Grover, and Matthew Sobek (2017) *Integrated Public Use Microdata Series: Version 7.0 [dataset]*, Minneapolis: University of Minnesota.

Walter, Wolfgang (1998) *Ordinary Differential Equations,* Springer; Graduate Texts in Mathematics.