A Tractable Framework for Analyzing a Class of Nonstationary Markov Models*

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Abstract

Dynamic stochastic economic models normally build on the assumption of time-invariant preferences, technology and laws of motions for exogenous variables. We relax this assumption and consider a class of infinite-horizon nonstationary models in which parameters can follow both deterministic and stochastic trends: the former trends take the form of anticipated shifts and drifts, and the latter trends take the form of Markov process with time-varying transition probabilities. We introduce a quantitative framework, called *extended function path* (EFP), for calibrating, solving, simulating and estimating such nonstationary models. We establish the existence and convergence (turnpike) theorems. We apply EFP to solve a challenging nonstationary unbalanced growth model with capital augmenting technological progress. Examples of the MATLAB code are provided.

JEL classification : C61, C63, C68, E31, E52

Key Words : time-dependent models; nonstationary models; unbalanced growth; time varying parameters; anticipated shock; shooting method; parameter shift; parameter drift; regime switches; stochastic volatility; technological progress; seasonal adjustments; Fair and Taylor method; extended path; Smolyak method

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1 Introduction

Dynamic stochastic economic models are normally built on the assumption of stationary environment, namely, it is assumed that the economy's fundamentals such as preferences, technologies and laws of motions for exogenous variables do not change over time. Such models have stationary solutions in which optimal value and decision functions depend on the current state but not on time. This framework is convenient for applied work since time-invariant solutions are relatively easy to construct.

At the same time, real-world economies constantly evolve over time, experiencing population growth, technological progress, trends in tastes and habits, policy regime changes, evolution of social and political institutions, etc. Modeling such time-dependent features of the data would require us to assume that some parameters of economic models change over time, following deterministic and or stochastic trends. Some parameters changes can be modeled in a way that is consistent with an assumption of stationary environment. In particular, labor augmenting technological progress is a well-known example of a *parameter drift* (i.e., a gradual change in parameters) that leads to balanced growth and stationary solutions; see King, Plosser and Rebello (1988) for necessary conditions for the existence of a balanced growth path. Furthermore, Markov switching models are a well-known example of a *parameter shift* (i.e., a drastic change in parameters) that can be analyzed in a stationary context, e.g., Davig and Leeper (2007, 2009).

However, many interesting nostationary models do not admit stationary representations. First, parameter drifts generally lead to unbalanced growth characterized by time-varying value and decision functions. For example, growth is unbalanced under (i) investement-specific technical change used in the analysis of capital-skill complementarity of Krusell, Ohanian, Ríos-Rull and Violante's (2000); (ii) capital-augmenting progress assumed in the analysis of directed technical change of Acemoglu (2002, 2003); (iii) time trends in the volatility of output and labor-income shares documented by Mc Connel and Pérez-Quiros (2000), and Karabarbounis and Brent (2014), respectively, etc. Second, parameter shifts also produce time-dependent value and decision functions if they are anticipated. Examples of economically relevant parameters shifts that lead to anticipatory effects are deterministic seasonals (e.g., Barsky and Miron, 1989), accession of new members to the European Union (e.g., Maliar and Maliar, 2008), presidential elections with predictable outcomes, credible policy announcements, anticipated legislative changes, etc.

In the paper, we relax the restriction of stationary environment, and we consider a class of infinite-horizon dynamic stochastic economies in which preferences, technology and laws of motion for exogenous variables can change from one period to another. We assume that the model's parameters (or exogenous variables) can follow both deterministic and stochastic trends: the former trends take the form of anticipated shifts and drifts, and the latter trends take the form of Markov process with time-varying transition probabilities. The models from this class are nonstationary, and their optimal value and decision functions are time-dependent.[1]

Numerical methods that are used for stationary infinite-horizon models are not suitable

[1] Markov stochastic processes can be nonstationary even if all parameters and transition probabilities are time-invariant, for example, unit root and explosive processes are nonstationary. We do not explicitly study this kind of nonstationary processes but focus on nonstationarity that happens because the economic environment changes over time.

for analyzing nonstationary models. Indeed, in the stationary case, we construct value and decision functions to satisfy the following fixed point property: if we insert an optimal function into the right side of the Bellman or Euler equations, we obtain the same function in the left side. This is not true in the nonstationary case: if environment changes between today and tomorrow, today's optimal value and decision functions will differ from tomorrow's ones. To solve a nonstationary model, we need to construct not just one optimal value and decision functions but an infinitely long sequence (path) of such functions, with a separate function for each period of time.

We introduce a simple and tractable framework, called *extended function path* (EFP), for constructing a time path of the optimal value and decision functions in nonstationary infinite-horizon models. In particular, it can be used for calibrating, solving, simulating and estimating such models. EFP builds on a combination of backward iteration and turnpike analysis, specifically, it approximates an infinite-horizon economy with a truncated finite-horizon economy which we solve by backward iteration. EFP consists of two steps: First, we assume that in some remote period T, the economy becomes stationary and we construct the usual stationary Markov solution (this can be done by using any conventional solution method). Second, given the obtained terminal condition, we iterate backward on Bellman or Euler equation to construct a sequence of optimal value and decision functions; this can be done by using *time iteration*.[2] By construction, EFP delivers a sequence (path) of functions that makes it possible to accurately simulate the economy's path for *any* sequence of shocks and not just for one fixed sequence of shocks, as is done under the assumption of perfect foresight.

We provide theoretical foundations of the EFP framework in the context of the stylized neoclassical growth model. First, we show an *existence theorem* that establishes conditions under which a path of optimal value and decisions functions produced by EFP exists and is unique and Markov. We show that in the studied class of models, time-dependency takes a particular tractable form, namely, the optimal choices follow a Markov process with deterministic time trends and time-varying transition probabilities. Second, we prove a *turnpike theorem* that shows a uniform convergence of a truncated finite-horizon economy to the corresponding infinite-horizon nonstationary economy. This results implies that EFP is capable of approximating a solution to a nonstationary infinite-horizon model with an arbitrary degree of precision.

In our numerical examples, we demonstrate how to use EFP to analyze a collection of challenging nonstationary applications with parameters shifts and drifts. They include: (i) capital augmenting technological progress that leads to unbalanced growth; (ii) a combination of anticipated and unanticipated technological shocks; (iii) periodic anticipated seasonal adjustments; (iv) parameter drifts; (v) deterministic and stochastic trends in volatility of shocks; as well as (vi) an unbalanced growth model that is calibrated and estimated using data on the U.S. economy. In all these applications, optimal value and decision functions nontrivially change from one period to another, so that conventional solution methods for stationary models cannot be applied. Examples of the MATLAB code are provided on webpages of the authors.

Although our numerical examples are limited to problems with few state variables, we

[2] Time iteration is commonly used in the context of dynamic programing methods, as well as some Euler equation methods (e.g., Coleman, 1991, Malin, Krueger and Kubler, 2011). Also, the backward type of iteration is used for solving finite-horizon life-cycle models (e.g., Krueger and Kubler (2004, 2006) and Hasanhodzic and Kotlikoff (2013).

implement EFP in a way that makes it tractable in large-scale applications. In particular, we use sparse grids, nonproduct monomial integration and relatively inexpensive Gauss-Jacobi fixed-point iteration; see Maliar and Maliar (2014) for a survey of these and other techniques suitable for problems with high dimensionality. Furthermore, to speed up convergence of the finite-horizon solution to the infinite-horizon one, we choose the terminal condition of the finite-horizon economy as closed as possible to a T-period solution of the infinite-horizon economy, whereas standard turnpike analysis assumes a zero-capital terminal condition. Finally, to avoid the need of a numerical solver, we implement fixed-point iteration on the whole economy path at once, as in Fair and Taylor (1984), instead of conventional time-iteration.

Turnpike results on asymptotic convergence of finite-horizon to infinite-horizon economies date back to the seminal papers of Brock (1971) and McKenzie (1976) who analyzed stationary economies. There are also papers that show turnpike theorems for economies with time-varying fundamentals like ours; see Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997). However, this literature focuses exclusively on existence theorems and never attempts to propose a tractable way of constructing numerical solutions to nonstationary models.

The main novelty of the present paper consists in showing how the earlier turnpike theorems can be effectively combined with familiar backward iteration to analyze a challenging and empirically-relevant class of infinite-horizon nonstationary economic models that are either not studied in the literature yet or studied under some simplifying assumptions. Nonetheless, our simple (almost obvious) EFP framework has an important value-added in terms of applications that can be analyzed quantitatively. First of all, a large body of real business cycle literature focuses on one specific parameter drift, namely, labor augmenting technological progresses. In turn, here, we solve models with any type of technological progresses (capital, Hicks neutral, investment-specific), as well as any other parameter drifts (e.g., drifts in a depreciation rate, discount factor, utility-function parameters, etc.). Furthermore, the existing methods for analyzing parameter shifts focus either on unanticipated shifts such as Markov switching (e.g., Davig and Leeper, 2007, 2009) or on anticipated shocks of a fixed horizon (e.g., Schmitt-Grohé and Uribe, 2012). In contrast, we show how to handle any combination of unanticipated and anticipated shocks of any periodicity and duration in a fully nonlinear manner.[3] Finally, there is a literature on stochastic volatility that studies effects of uncertainty on business cycle fluctuations by assuming that volatility of shocks follows a stationary Markov process such as a first-order autoregressive process or recurring Markov regime switches (e.g., Bloom, 2009, Fernández-Villaverde and Rubio-Ramírez, 2010, and Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez, 2010). In turn, we are able to analyze models in which volatility has both stochastic and deterministic components.

Moreover, the EFP framework is a novel tool for policy modeling in stochastic economies: it allows to analyze time-dependent policies, and thus, complements the mainstream of the literature that focuses on state-dependent policies. In the time-dependent case, a policy maker commits to adopt a new policy on a certain date, independently of the economy's state (e.g., to

[3]Seasonal adjustments are a special case of anticipated parameter shifts. In the literature, they are analyzed either by constructing periodic optimal decision rules with spectral density transformations (e.g., Hansent and Sargent, 1993, 2013) or by solving for season-specific time-invariant decision rules using linearization around a seasonally-varying steady state (e.g., Christiano and Todd, 2002). EFP provides a simple and general alternative to these methods and allows us to analyze seasonal fluctuations in a fully non-linear context; it treats seasonal fluctuations just like any other combination of anticipated and unanticipated shocks.

raise the interest rate on a certain future date), whereas in the state-dependent case, a policy maker commits to adopt a new policy when economy reaches a certain state, independently of the date (e.g., to raise the interest rate when certain economic conditions are met). Both of these cases are empirically relevant and can be of interest in applications.

Two clarifying comments on the relation between time- and state-dependent models are in order. First, in principle, it is always possible to convert our time-dependent economy into a state-dependent economy by enlarging the state space, and in particular, we can add a variable "time" to the set of state variables. However, such "enlarged" models are not yet studied and it is unknown whether they are tractable or not. Second, nonstationary models with anticipated parameter shifts cannot be either reduced to or approximated by Markov switching models because regime changes come at random in such models and thus, no anticipatory effects that are emphasized by our analysis are ever observed.

Numerical analysis of nonstationary models in economics is originated from two earlier papers: one is by Lipton, Poterba, Sachs and Summers (1980), and the other is by Fair and Taylor (1983). The former uses shooting methods to characterize an optimal path of a deterministic economy. The latter introduces an extended path (EP) method for constructing an optimal path of the economy with uncertainty. The key contribution of Fair and Taylor (1983) analysis is the introduction of a certainty equivalence method for approximating expectation functions. Other path solving methods had been developed in subsequent economic literature; see, e.g., Chen (1999), Judd (2002), Grüne, Semmler and Stieler (2013), Cagliarini and Kulish (2013); see also Atolia and Buffie (2009 a,b) for a careful discussion of shooting methods.[4] A shortcoming of the certainty-equivalence approach of Fair and Taylor (1983) is that it can be insufficiently accurate in some applications. Adjemian and Juillard (2013) propose a stochastic extended path method that approximates expectation functions more accurately by constructing and averaging multiple paths. Krusell and Smith (2015) develop a related numerical method that combines approximate aggregation and perturbation of distributions along the transition path in a climate change model. In contrast, to path-solving methods, EFP deals with uncertainty in a way which is typical for global nonlinear solution methods, namely, it constructs state-contingent decision functions using deterministic integration methods, and it can accurately solve nonlinear stochastic models in which the certainty equivalence approach is either not applicable or leads to inaccurate solutions.

The rest of the paper is as follows: In Section 2, we define a class of nonstationary Markov models. In Section 3, we introduce EFP and provide its theoretical foundations. In Section 4, we describe the relation of EFP to existing literature. In Section 5, we assess the performance of EFP in a nonstationary test model with a balanced growth path. In Section 6, we show how to use EFP for analyzing anticipated parameter shifts and drifts. In Section 7, we solve a collection of challeging nonstationary applications. Finally, in Section 8, we conclude.

[4] There is also literature that studies a transition between two aggregate steady states in heterogeneous-agent economies by constructing a deterministic transition path for aggregate quantities and prices; see Conesa and Krueger (1999). Our framework exploits the same idea but we analyze a transition between two stochastic steady states by constructing a transition path of decision functions.

2 A class of nonstationary Markov economies

We analyze a class of infinite-horizon, nonlinear dynamic economic models in which the model's parameters (interpreted as exogenous variables) change over time. The parameters changes take forms of anticipated shifts and drifts, as well as unanticipated Markov shocks with time-varying transition probabilities. The constructed class of models is nonstationary because the optimal decision and value functions are time dependent. Our goal is to develop a tractable framework for constructing numerical solutions to such nonstationary models.

2.1 A nonstationary optimization problem

We consider a nonstationary stochastic growth model in which preferences, technology and laws of motion for exogenous variables change over time. The representative agent solves

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty}\beta^t u_t\left(c_t\right)\right] \tag{1}$$

$$\text{s.t. } c_t + k_{t+1} = \left(1-\delta\right)k_t + f_t\left(k_t, z_t\right), \tag{2}$$

$$z_{t+1} = \varphi_t\left(z_t, \varepsilon_{t+1}\right), \tag{3}$$

where $c_t \geq 0$ and $k_t \geq 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u_t : \mathbb{R}_+ \to \mathbb{R}$ and $f_t : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$ are possibly time-dependent utility function, production functions and law of motion for exogenous variable z_t, respectively; the sequence of u_t, f_t and φ_t for $t \geq 0$ is known to the agent in period $t = 0$; ε_{t+1} is i.i.d; $\beta \in (0,1)$ is the discount factor; $\delta \in [0,1]$ is the depreciation rate; and $E_t[\cdot]$ is an operator of expectation, conditional on a t-period information set.

Stationary models. A well-known special case of the general setup (1)–(3) is a stationary Markov model in which $u_t \equiv u$, $f_t \equiv f$ and $\varphi_t \equiv \varphi$. Such a model has a stationary Markov solution in which the value function $V(k_t, z_t)$ and decision functions $k_{t+1} = K(k_t, z_t)$ and $c_t = C(k_t, z_t)$ are both state-contingent and time-invariant functions; see, e.g., Stokey and Lucas with Prescott (1989, p. 391).

Nonstationary models. In a general case, a solution to the model (1)–(3) is nonstationary. The decision functions of endogenous variables c_t and k_{t+1} could be time-dependent for two reasons: first, because u_t and f_t change over time; and second, because φ_t and consequently, transition probabilities of exogenous variable z_t change over time.

Remark 1. For presentational convenience, we assume that only z_t is a random variable following a Markov process with possibly time-varying transition probabilities, while the other model's parameters evolve in a deterministic manner, i.e., the sequence of u_t, f_t and φ_t for all $t \geq 0$ is fully anticipated. However, the framework we develop can be used to solve models in which β, δ, as well as the parameters of u_t, f_t and φ_t, are all random variables following both deterministic and stochastic trends. In particular, in Section 6, we consider a model in which δ follows a Markov process with time-varying transition probabilities.

2.2 Assumptions about exogenous variable

The mainstream of related literature, including real business cycle literature, either focuses on a class of models in which exogenous variables are both stationary and Markov or it imposes restrictions, derived by King et al. (1988), that make it possible to convert a nonstationary model into stationary one as is in the case of balanced growth.

In (3), we maintain the assumption of Markov process, however, we relax the restriction of stationarity, namely, we allow for the case when transition probabilities of z_t change over time. In Appendix A0, we provide a formal description of our stochastic environment and define stationary and Markov processes using measure-theory notation. Our exposition is standard and closely follows Stokey and Lucas with Prescott (1989), Santos (1999) and Stachurski (2009). Below, we show a simple example that illustrates the notion of random processes that will be used in our analysis for modeling exogenous variables.

Example 1. Consider a first-order autoregressive process

$$z_{t+1} = \rho_t z_t + \sigma_t \varepsilon_{t+1}, \tag{4}$$

where $\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$ and $(\rho_0, \rho_1, ...)$ and $(\sigma_0, \sigma_1, ...)$ are given at $t = 0$.

We distinguish the following cases:

i). Markov process. Since the conditional probability distribution $z_{t+1} \sim \mathcal{N}(\rho_t \overline{z}_t, \sigma_t^2)$ depends only on the most recent past $z_t = \overline{z}_t$ and is independent of history $(z_t, ..., z_0)$, the process is Markov.

ii). Nonstationary transitions. If ρ_t and σ_t change over time, then the distribution $\mathcal{N}(\rho_t \overline{z}_t, \sigma_t^2)$ depends not only on the current state $z_t = \overline{z}_t$ but also on a specific period t, so that transition probabilities are not stationary, and as a result, the process is nonstationary (in particular, it does not have time-invariant unconditional probability measure). This is the case we consider in the paper; we assume $|\rho_t| < 1$ for all t.

iii). Stationary process. If $\rho_t = \rho$ and $\sigma_t = \sigma$ for all t, then the conditional probability distribution $\mathcal{N}(\rho \overline{z}_t, \sigma^2)$ depends only on state $z_t = \overline{z}_t$ but not on time, so that the transitions are stationary. If, in addition, $|\rho| < 1$, then we have the familiar stationary process.

iv). Unit root and explosive processes. The case of $|\rho_t| = 1$ for all t corresponds to a unit root process, which is nonstationary even if $\sigma_t = \sigma$ for all t; and $|\rho_t| > 1$ for all t leads to an explosive process. Unit root and explosive processes are not explicitly studied in the paper, although our analysis is still valid under these processes if the boundedness assumption A3 of the next section is satisfied. ■

Remark 2. Mitra and Nyarko (1991) refer to a class of Markov processes with nonstationary transition probabilities of type ii) as semi-Markov processes because of their certain similarity to Lévy's (1954) generalization of the Markov renewal process for the case of random arrival times; see Jansen and Manca (2006) for a review of applications of semi-Markov processes in statistics, operation research and other fields.

2.3 Assumptions about the utility and production functions

We make standard (strong) assumptions about the utility and production functions that ensure the existence, uniqueness and interiority of a solution. Concerning the utility function u_t, we assume that for each $t \geq 0$:

Assumption 1. *(Utility function). a) u_t is twice continuously differentiable on $\mathbb{R}_+$; b) $u_t' > 0$, i.e., u_t is strictly increasing on $\mathbb{R}_+$, where $u_t' \equiv \frac{\partial u_t}{\partial c}$; c) $u_t'' < 0$, i.e., u_t is strictly concave on $\mathbb{R}_+$, where $u_t'' \equiv \frac{\partial^2 u_t}{\partial c^2}$; and d) u_t satisfies the Inada conditions $\lim_{c \to 0} u_t'(c) = +\infty$ and $\lim_{c \to \infty} u_t'(c) = 0$.*

Concerning the production function f_t, we assume that for each $t \geq 0$:

Assumption 2. *(Production function). a) f_t is twice continuously differentiable on $\mathbb{R}_+^2$, b) $f_t'(k, z) > 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t' \equiv \frac{\partial f_t}{\partial k}$, c) $f_t''(k, z) \leq 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t'' \equiv \frac{\partial^2 f_t}{\partial k^2}$; and d) f_t satisfies the Inada conditions $\lim_{k \to 0} f_t'(k, z) = +\infty$ and $\lim_{k \to \infty} f_t'(k, z) = 0$ for all $z \in \mathbb{R}_+$.*

We need one more assumption. Let us define a pure capital accumulation process $\{k_t^{\max}\}_{t=0}^{\infty}$ by assuming $c_t = 0$ for all t in (2) which for each history $h_t = (z_0, ..., z_t)$, leads to

$$k_{t+1}^{\max} = f_t\left(k_t^{\max}, z_t\right), \tag{5}$$

where $k_0^{\max} \equiv k_0$. We impose an additional joint boundedness restriction on preferences and technology by using the constructed process (5):

Assumption 3. *(Objective function). $E_0\left[\sum_{t=0}^{\infty} \beta^t u_t\left(k_t^{\max}\right)\right] < \infty$.*

This assumption insures that the objective function (1) is bounded so that its maximum exists. In particular, Assumption 3 holds either (i) when u_t is bounded from above for all t, i.e., $u_t(c) < \infty$ for any $c \geq 0$ or (ii) when f_t is bounded from above for all t, i.e., $f_t(k, z_t) < \infty$ for any $k \geq 0$ and $z_t \in Z_t$. However, it also holds for economies with nonvanishing growth and unbounded utility and production functions as long as $u_t\left(k_t^{\max}\right)$ does not grow too fast so that the product $\beta^t u_t\left(k_t^{\max}\right)$ still declines at a sufficiently high rate and the objective function (1) converges to a finite limit.

2.4 Optimal program

Definition 1 *(Feasible program).* A feasible program for the economy (1)–(3) is a pair of adapted (t-measurable) processes $\{c_t, k_t\}_{t=0}^{\infty}$ such that given initial condition k_0, they satisfy $c_t \geq 0$, $k_t \geq 0$ and (2) for each possible history $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$.

We denote by $\Im^\infty$ a set of all feasible programs from given initial capital k_0. We next introduce the concept of solution to the model.

Definition 2 *(Optimal program).* A feasible program $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ is called optimal if

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\left\{u_t\left(c_t^\infty\right)-u_t\left(c_t\right)\right\}\right]\geq 0 \tag{6}$$

for every feasible process $\{c_t, k_t\}_{t=0}^\infty \in \Im^\infty$.

Stochastic models with time-varying fundamentals are studied in Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997), among others. The existence results for this class of models have been established in the literature for a general measurable stochastic environment without imposing the restriction of Markov process (3). In particular, this literature shows that, under Assumptions 1-3, there exists an optimal program $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ in the economy (1), (2), and it is both interior and unique; see Theorem 4.1 in Mitra and Nyarko (1991) and see Theorem 7 in Majumdar and Zilcha (1987). The results of this literature apply to us as well.

Remark 3. The existence of the optimal program in the economy (1)–(3) can be shown even under weaker assumptions. For example, Mitra and Nyarko (1991) use a joint boundedness restriction on preferences and technology (the so-called *Condition E*) that is less restrictive than our Assumption 3; Joshi (1997) characterizes the optimal programs in nonconvex economies by relaxing our Assumptions A2c and A2d, etc.

While the previous literature establishes the existence and uniqueness results for the constructed class of nonstationary models under very general assumptions, it does not offer a practical approach to constructing time-dependent solutions in applications. In contrast, we distinguish a tractable class of nonstationary models satisfying the Markov property (3) for which the solutions can be conveniently characterized in applications, both analytically and numerically.

3 Extended function path framework

We introduce an *extended function path* (EFP) framework for approximating an optimal program in the infinite-horizon nonstationary Markov economy (1)–(3). In Section 3.1, we present the EFP framework, and in Section 3.2, we develop its theoretical foundations.

3.1 Introducing extended function path framework

Characterizing an optimal program in the nonstationary infinite-horizon model (1)–(3) requires to construct an infinite sequence of time-varying value and decision functions. While it is not generally possible to accurately approximate an infinite sequence of functions, we will be able to accurately approximate a given number of elements of such a sequence. For example, our EFP framework can approximate the capital decision functions $K_1, ..., K_{100}$ during the first $\tau = 100$ periods with a given degree of precision.

The key idea of our EFP framework is to approximate an infinite-horizon nonstationary economy by using a truncated finite-horizon economy. Formally, this procedure is described below.

Algorithm 1: Extended function path (EFP).

Step 0. Initialization. Choose some $T \gg \tau$ and assume that for $t \geq T$, the economy becomes stationary, i.e., $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all $t \geq T$.

Step 1. Construct a stationary Markov solution for $t \geq T$, i.e., find a stationary capital function K satisfying:

$$u'(c) = \beta E\left[u'(c')(1-\delta+f'(k',\varphi(z,\varepsilon')))\right]$$
$$c = (1-\delta)k + f(k,z) - k'$$
$$c' = (1-\delta)k' + f(k',\varphi(z,\varepsilon')) - k''$$
$$k' = K(k,z) \text{ and } k'' = K(k',\varphi(z,\varepsilon')).$$

Step 2. Construct a path for capital policy functions $(K_0, ..., K_T)$ that matches the terminal condition $K_T \equiv K$ and satisfies for $t = 0, ...T-1$:

$$u_t'(c_t) = \beta E_t\left[u_{t+1}'(c_{t+1})(1-\delta+f_{t+1}'(k_{t+1},\varphi_t(z_t,\varepsilon_{t+1})))\right]$$
$$c_t = (1-\delta)k_t + f_t(k_t,z_t) - k_{t+1}$$
$$c_{t+1} = (1-\delta)k_{t+1} + f_{t+1}(k_{t+1},\varphi_t(z_t,\varepsilon_{t+1})) - k_{t+2}$$
$$k_{t+1} = K_t(k_t,z_t) \text{ and } k_{t+2} = K_{t+1}(k_{t+1},\varphi_t(z_t,\varepsilon_{t+1})).$$

The first τ functions $(K_0, ..., K_\tau)$ constitute an approximate solution and the remaining $T-\tau$ functions $(K_{\tau+1}, ..., K_T)$ are discarded.

There are two steps: First, we assume that in some remote period T, the economy becomes stationary, i.e., the fundamentals do not change any more, $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all $t \geq T$. We call this economy a *T-period stationary economy.* The convenient feature of the T-period stationary economy is that its optimal program is simple to characterize. Indeed, since the economy (1)–(3) becomes stationary at T, the optimal program is stationary Markov for $t \geq T$, and the usual stationary Markov equilibrium can be constructed by using any conventional solution method. Second, given the terminal condition, produced by the T-period stationary economy, we can use backward induction (also known as time iteration), to construct a path for value and decision functions for $t = 0, ..., T-1$ that satisfy $T-1$ Euler equations. Namely, given the capital function K_T, we use the Euler equation to compute the capital function K_{T-1} at $T-1$; given K_{T-1}, we use it to compute K_{T-2}; and we proceed until the entire path $(K_T, ..., K_0)$ is constructed. In the next section, we show that under our assumptions A1-A3, the path of functions produced by EFP exists, is unique and can approximate arbitrary well the time-varying decision functions of the corresponding nonstationary model.

Let us show a graphical illustration to the solution produced by EFP. To implement EFP, we use a combination of three techniques. First, to approximate decision functions, we use Smolyak (sparse) grids. Second, to approximate expectation functions, we use a nonproduct monomial integration rule. Finally, to solve for coefficients of the policy functions, we use a Gauss-Jacobi method, which is a derivative-free fixed-point-iteration method in line with Fair and Taylor (1983). Further implementation details are described in Section 5 and Appendix B.[5]

[5] In the paper, we focus on a version of EFP that constructs global nonlinear approximations. A similar

In Figure 1, we illustrate a sequence of functions (a function path) produced by EFP for a version of the model (1)–(3) with exogenous labor augmenting technological progress (the model's parameterization and implementation details are described in Section 5).

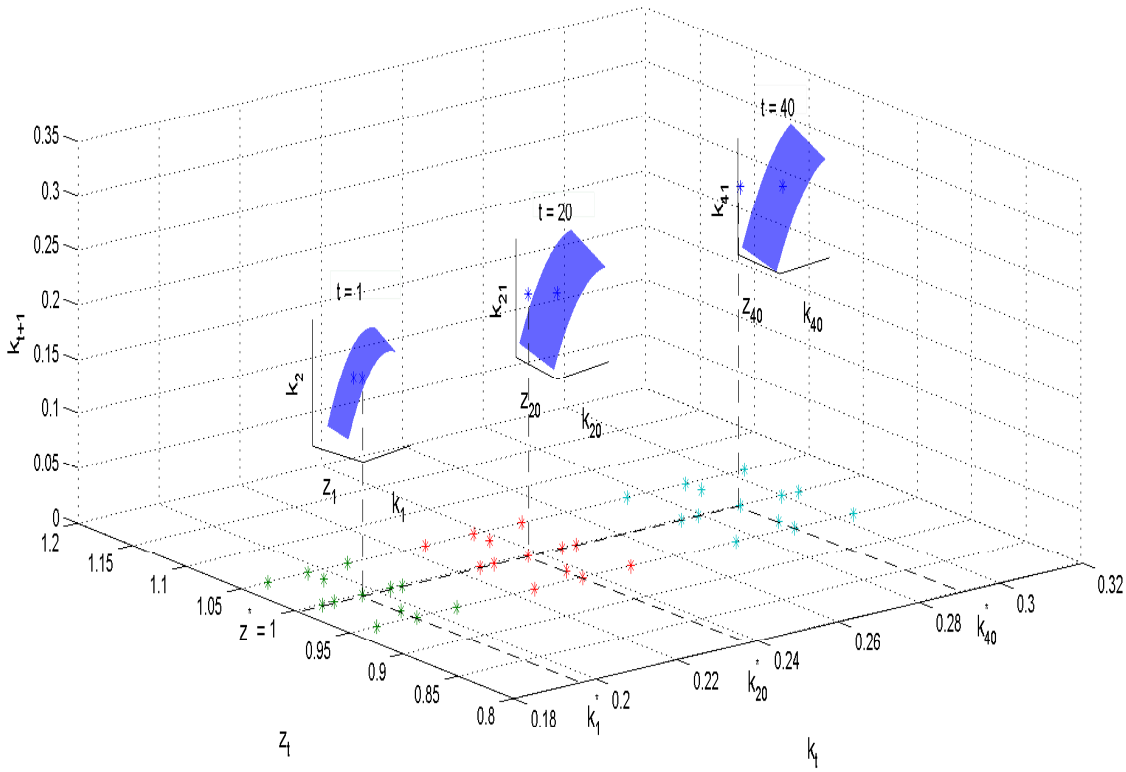

Figure 1. Function path, produced by EFP, for a growth model with technological progress

We plot the capital functions for periods 1, 20 and 40, (i.e., $k_2 = K_1(k_1, z_1)$, $k_{21} = K_{20}(k_{20}, z_{20})$ and $k_{41} = K_{40}(k_{40}, z_{40})$) which we approximate using Smolyak (sparse) grids. In Step 1 of the algorithm, we construct the capital function K_{40} by assuming that the economy becomes stationary in period $T = 40$; and in Step 2, we construct a path of the capital functions that $(K_1, ...K_{39})$ that matches the corresponding terminal function K_{40}. The Smolyak grids are shown by stars in the horizontal $k_t \times z_t$ plane. The domain for capital (on which Smolyak grids are constructed) and the range of the constructed capital function grow at the rate of labor augmenting technological progress.

Finally, let us explain the choice of the name *extended function path* for the framework we propose. *Extended path (EP)* method of Fair and Taylor (1983) constructs a path of variables for larger time horizon T than the number of periods τ for which an approximate solution is actually needed. As we will see, by taking a sufficiently large T, we can ensure that our today's

path of decision functions can be constructed using local perturbation techniques. The conventional EP method of Fair and Taylor (1983) is incorporated in the dynare software platform, and possibly, a perturbation-based version of EFP can be included there as well; for a description of the dynare platform, see the manual of Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011).

decision functions accurately approximate the corresponding functions in the infinite-horizon economy independently of how we choose the terminal condition. In this respect, the EFP and EP frameworks are similar. In turn, the wording *path* versus *function path* highlights the key difference between the EP and EFP methods: the former constructs a path for variables (by using certainty equivalence), whereas the latter constructs a path for decision functions (by using more accurate integration methods such as Monte Carlo, Gauss-Hermite quadrature and monomials methods). As a result, EFP can also accurately solve those models in which EP is insufficiently accurate. Another important difference between EFP and EP is that EFP produces a sequence of functions that makes it possible to simulate the model for any sequence of exogenous shocks, while EP produces a solution that is valid only for one fixed sequence of shocks. In Appendix C, we discuss the relation between EFP and EP in more details.

3.2 Theoretical foundations of the EFP framework

We now develop theoretical foundations of the EFP framework. We prove two results: Theorem 1 shows that the optimal program in the T-period stationary economy is given by a Markov process with possibly time-varying transition probabilities; and Theorem 2 shows that the optimal program of the T-period stationary economy uniformly converges to the optimal program of the original nonstationary Markov economy (1)–(3) as T increases.

Theorem 1 *(Optimal program of the T-period stationary economy). In the T-period stationary economy (1)–(3), the optimal program is given by a Markov process with possibly time-varying transition probabilities.*

Proof. Under Assumptions 1-3, first-order conditions (FOCs) are necessary for optimality. We will show that FOCs are also sufficient both to identify the optimal program and to establish its Markov structure. Our proof is constructive: it relies on backward induction and includes two steps that correspond to Steps 1 and 2 of Algorithm 1, respectively.

Step 1. At T, the economy becomes stationary and remains stationary forever, i.e., $u_t \equiv u$, $f_t \equiv f$ and $\varphi_t \equiv \varphi$ for all $t \geq T$. Thus, the model's equations and decision functions are time invariant for $t \geq T$. It is well known that under Assumptions 1-3, there is a unique stationary Markov capital function K that satisfies the optimality conditions that are listed in Step 1 of Algorithm 1; see, e.g., Stokey and Lucas with Prescott (1989, p. 391).

Step 2. Given the constructed T-period capital function $K_T \equiv K$, we define the capital functions $K_{T-1}, ..., K_0$ in previous periods by using backward induction. As a first step, we write the Euler equation for period $T-1$,

$$u'_{T-1}(c_{T-1}) = \beta E_{T-1}\left[u'_T(c_T)(1-\delta+f'_T(k_T, z_T))\right], \tag{7}$$

where c_{T-1} and c_T are related to k_T and k_{T+1} in periods T and $T-1$ by

$$\begin{aligned} c_{T-1} &= (1-\delta)\,k_{T-1} + f_{T-1}(k_{T-1}, z_{T-1}) - k_T, & (8)\\ c_T &= (1-\delta)\,k_T + f_T(k_T, z_T) - k_{T+1}. & (9) \end{aligned}$$

By assumption (3), z_T follows a Markov process, i.e., $z_T = \varphi_T(z_{T-1}, \varepsilon)$. Furthermore, by construction of the decision function K in Step 1, we have that $k_{T+1} = K_T(k_T, z_T)$ is a Markov decision function. By substituting these two results into (7)–(9), we obtain a functional equation that defines k_T for each possible state (k_{T-1}, z_{T-1}). The existence and uniqueness of the solution to this functional equation under our assumptions is established in the previous literature; see Theorem 4.1 in Mitra and Nyarko (1991) and Theorem 7 in Majumdar and Zilcha (1987). Therefore, capital choices at period $T-1$ are described by a state-contingent function $k_T = K_{T-1}(k_{T-1}, z_{T-1})$, i.e., capital choices today are independent of history that leads to the current state. However, the constructed decision functions depend on the parameters of the utility and production functions and the law of motions for shocks in periods $T-1$ and T, and it is not generally true that $K_{T-1} \neq K_T$. By proceeding iteratively backward, we construct a sequence of state-contingent and time-dependent capital functions $K_{T-1}(k_{T-1}, z_{T-1}), ..., K_0(k_0, z_0)$ that satisfies (7)–(9) for $t = 0, ..., T-1$ and that matches terminal function $K_T(k_T, z_T)$. Hence, k_{t+1} follows a Markov process with possibly time-varying transition probabilities. ■

Remark 4. We analyze a variant of EFP that constructs time-dependent capital functions $(K_0, ..., K_\tau)$ that satisfy the Euler equation. Similarly, we can formulate a variant of EFP that constructs time-dependent value functions $(V_0, ..., V_\tau)$ by iterating on the Bellman equation. First, we solve for $V_T = V_{T+1} = V$ for the T-period stationary economy and then we solve for a path $(V_{T-1}, ..., V_0)$ that satisfies the sequence of the Bellman equations for $t = 0, ..., T$ and that meets the terminal condition V_T of the T-period stationary economy. This extension is straightforward.

We next show that the optimal program of the constructed T-period stationary economy approximates arbitrary well the optimal program of the nonstationary economy (1)–(3) as T increases. Our analysis is related to the literature that shows asymptotic convergence of the optimal program of the finite horizon economy to that of the infinite horizon economy; see Brock (1971) and McKenzie (1976) for early contributions, and see Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997) for the convergence results for economies with time-varying fundamentals like ours. This kind of convergence results is referred to as *turnpike theorems*.

Turnpike means a highway. The name *turnpike theorem* emphasizes the idea that a highway is often the fastest route between two points even if it is not a direct route. Specifically, when we drive to some remote destination (e.g., a small town), we often get on a highway as soon as possible, stay on the highway as long as possible and get off highway as close as possible to our final destination.[6] In Figure 2, we show that the same kind of behavior is observed in our model if we interpret the infinite-horizon and finite-horizon economies as a turnpike and our final destination, respectively.

[6] Turnpike theorems, in which initial-period decisions are insensitive to terminal conditions, are classified as early turnpike theorem in the literature; there are also medium and late turnpike theorems corresponding for different variations in the initial and terminal conditions; see McKenzie (1976) and Joshi (1997) for discussion. We do not analyze these turnpike theorems since they are not directly related to our EFP framework.

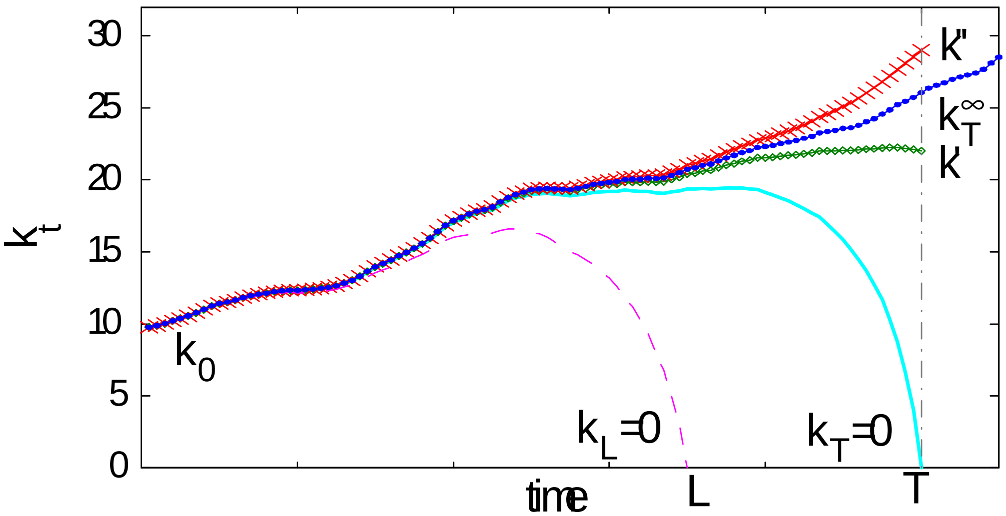

Figure 2. Convergence of the optimal program of T-period stationary economy

Let us again consider a version of the model with labor augmenting technological progress described in Section 5); we fix the same initial condition and realization of shocks in all experiments; k_t^∞ denotes the true solution to the infinite-horizon nonstationary model (1)–(3), and k_L, k_T, k' and k'' denote the corresponding solutions to the finite-horizon models characterized by different terminal conditions. We see that the optimal program of the T-period stationary economy $\{k_t^T\}$ follows for a long time the optimal program of the nonstationary economy $\{k_t^\infty\}$ (turnpike) and it gets off the turnpike only at the end in order to meet a given terminal condition (i.e. the final destination).

We also make the following important observation. While the path of the T-period stationary economy converges to that of the nonstationary economy under all terminal conditions considered, the convergence is faster under terminal conditions κ' and κ'', that are located relatively close to the true T-period capital of the nonstationary economy $\{k_T^\infty\}$, than under a zero terminal condition that is located farther away from the true solution. In particular, it is clear that a zero-capital terminal condition, commonly used in the turnpike literature, is not an efficient choice for approximating an infinite horizon nonstationary economy with growth. Indeed, in the infinite-horizon economy, capital continues to grow, while in the finite horizon economy, capital needs to turn down near the terminal date to meet a zero-capital terminal condition, which slows down the convergence.

While the speed of convergence plays no role in the asymptotic convergence theorems established in the turnpike literature, it plays an important role in the accuracy and speed of numerical solution methods developed in the present paper. To attain the fastest possible convergence, we must choose the terminal condition $\{k_T^T\}$ of the T-period stationary economy as close as possible to the T-period capital stock of the infinite-horizon nonstationary economy $\{k_T^\infty\}$. To this purpose, we provide a version of the turnpike theorem that holds for an arbitrary terminal condition, while the existing theorems for economies with time varying fundamentals are shown for a zero-capital terminal conditions; see Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997).

To show our turnpike theorem, we fix some initial condition (k_0, z_0) and consider any history $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$. We then construct the productivity levels $\{z_t\}_{t=0}^T$ using (3) and use the sequence of capital functions of the T-period stationary economy $K_0(k_0, z_0), ..., K_T(k_T, z_T)$ to generate the optimal program $\left\{c_t^T, k_t^T\right\}_{t=0}^T$ such that

$$k_{t+1}^T = K_t\left(k_t^T, z_t\right), \tag{10}$$

where $k_0^T = k_0$, and c_t^T satisfies the budget constraint (2) for all $t \geq 0$. Then, we have the following result.

Theorem 2 *(Turnpike theorem): For any real number $\varepsilon > 0$ and any natural number τ, there exists a threshold terminal date $T(\varepsilon, \tau)$ such that for any $T \geq T(\varepsilon, \tau)$,* we have

$$\left|k_t^\infty - k_t^T\right| < \varepsilon, \quad \text{for all } t \leq \tau, \tag{11}$$

where $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ is the optimal program in the nonstationary economy (1)–(3), and $\left\{c_t^T, k_t^T\right\}_{t=0}^T$ is the optimal program (10) in the T-period stationary economy.

Proof. The proof is shown in Appendix A4, and it relies on three lemmas presented in Appendices A1-A3. In Appendix A1, we construct a limit program of a finite horizon economy with a terminal condition $k_T = 0$; this construction is standard in the turnpike analysis, see Majumdar and Zilcha (1987), Mitra and Nyarko (1991), Joshi (1997), and it is shown for the sake of completeness. In Appendix A2, we prove a new result about convergence of the optimal program of the T-period stationary economy with arbitrary terminal capital stock k_T^T to the limiting program of the finite horizon economy with a zero terminal condition $k_T = 0$. Finally, in Appendix A3, we show that the limit program of the finite horizon economy with a zero terminal condition $k_T = 0$ is also an optimal program for the infinite horizon nonstationary economy (1)–(3); in the proof, we also follow the previous turnpike literature. Thus, our main theoretical contribution is contained in Appendix A2. ■

The convergence is uniform: Our turnpike theorem states that for all $T \geq T(\varepsilon, \tau)$, the constructed nonstationary Markov approximation $\left\{k_t^T\right\}$ is guaranteed to be within a given ε-accuracy range of the true solution $\{k_t^\infty\}$ during the initial τ periods for any history of shocks $h_\infty = (\varepsilon_0, \varepsilon_1 ...)$ (for periods $t > \tau$, our approximation may become insufficiently accurate and exit the ε-accuracy range). For example, in Figure 1, the trajectories with a zero-capital terminal condition deviate dramatically from the infinite-horizon solution when we approach the terminal date.

Remark 5. The property, which is essential for our analysis, is that the optimal decision functions are Markov at T. In our baseline implementation of EFP, we attain this property by constructing a supplementary T-period stationary economy in which preferences, technology and laws of motion for exogenous variables do not change starting from $t = T$, i.e., $u_t = u_T$, $f_t = f_T$ and $\varphi_t = \varphi_T$ for all $t \geq T$. Instead, we can use any other assumptions that lead to Markov decision functions at T, for example, we can assume that the economy switches to balanced growth with a stationary Markov representation at T or that it arrives at a zero capital stock at T with the corresponding trivial Markov solution $k_t = 0$ for all $t \geq T$. Also, we can construct a T-period Markov decision function $K(k, z)$ without specifying the underlying economic model that generates this terminal condition. Our turnpike theorem implies that the decision functions in initial periods are insensitive to a specific terminal condition provided that the time horizon T is sufficiently large.

4 Relation of EFP to the literature

In the previous section, we discussed a relation of EFP to early theoretical literature that establishes existence and turnpike theorems for nonstationary models. The key difference of our analysis from that literature is that we provide a tractable framework for constructing numerical solutions, while the early literature is purely existential. We now focus on the relation of EFP to two other streams of the literature: one that develops path-solving numerical methods for nonstationary models and the other that constructs decision functions for stationary models.

4.1 Methods solving for path in nonstationary models

Numerical analysis of nonstationary models in economics is dated back to the work of Lipton, Poterba, Sachs and Summers (1980) and Fair and Taylor (1983). The former paper applies shooting methods to characterize transition path of deterministic economy, and the latter paper introduces an extended path (EP) method for constructing transition path of economy with uncertainty. Both papers explicitly state that path solving methods can be used in the context of nonstationary problems (although they do not provide any numerical nonstationary examples). In particular, Lipton, Poterba, Sachs and Summers (1980, p.2) say "... we allow for a possibility that F [model's equations] may be time dependent (i.e., non-autonomous)". Fair and Taylor (1983) also use time dependent notation for the model's equations. We provide a detailed description of the shooting and extended path methods in Appendix C.

Other path-solving methods in the literature include a continuous time analysis of Chen (1999); a parametric path method of Judd (2002) that approximates a deterministic path using a family of polynomial functions; an EP method using a Newton-style solver of Heer and Maußner (2010); a framework for analyzing time-dependent linear rational expectation models of Cagliarini and Kulish (2013); and a nonlinear predictive control method for value function iteration of Grüne, Semmler and Stieler (2013). Applications of path methods in economics are numerous, e.g., Chen, Imrohoroğlu and Imrohoroğlu (2006), Bodenstein, Erceg and Guerrieri (2009), Coibion, Gorodnichenko and Wieland (2011), Braun and Körber (2011) and Hansen and Imrohoroğlu (2013). There is also literature that uses path-solving methods for the analysis of heterogeneous agent models, in particular, Conesa and Krueger (1999) show how to solve life-cycle models in which the aggregate economy's path is deterministic but there is idiosyncratic uncertainty; see also Krueger and Ludwig (2007). Finally, Krusell and Smith (2015) develop a related numerical method that combines approximate aggregation and perturbation of distributions to solve for a transition path in a multi-region climate change model.

Adjemian and Juillard (2013) propose a stochastic extended path method that improves on certainty equivalence approach of the baseline Fair and Taylor's (1983) method. They construct and analyze a tree of all possible future paths for exogenous state variables. Although the number of tree branches and paths grows exponentially with the path length, the authors propose a clever way of reducing the cost by restricting attention to paths that have highest probability of occurrence. However, the implementation of this method is nontrivial, in particular, in models with multiple state variables.

EFP differs from the above literature both in the object it constructs and in the way it deals with uncertainty. Namely, EFP constructs a sequence of Markov state-contingent decision functions that include stochastic shocks as one of the arguments rather than solving for a path for variables under fixed sequences of shocks. In this respect, EFP is similar to conventional solution methods that construct decision functions for stationary Markov models. Since EFP produces decision functions, the simulation of the solutions is cheap under any sequence of shocks, in contrast to path solving methods in which the solution and simulation steps are combined together so that the model must be separately solved for each new sequence of shocks.

4.2 Methods solving for decision functions in stationary models

Conventional methods for constructing stationary Markov solutions are not directly applicable to analyzing nonstationary applications. However, the techniques used in the context of conventional methods can be used as ingredients of EFP. First, to construct decision functions, we can use a variety of grids, integration rules, approximation methods, iteration schemes, etc. that are used by conventional solution methods. Second, to construct a function path, we can use any numerical method that can solve a system of nonlinear equations, including Newton-style solvers as well as Gauss-Siedel or Gauss-Jacobi iteration. Since EFP constructs not just one set of decision functions but a possibly long sequence of such functions, the computational expense can be high. To make EFP tractable not only in small-scale but also in large-scale applications, we use numerical techniques whose cost does not rapidly increase with the dimensionality of the problem, including sparse, simulated, cluster and epsilon-distinguishable-set grids; nonproduct monomial and simulation based integration methods; and derivative-free solvers; see Maliar and Maliar (2014) for a survey of techniques that are designed for dealing with large-scale applications.

There are three groups of methods for stationary problems that EFP is particularly close to. First, EFP is related to methods that construct decision functions in finite-horizon problems such as life-cycle models studied in Krueger and Kubler (2004, 2006) and Hasanhodzic and Kotlikoff (2013). The decision functions in such models change from one generation to another, and the sequence of the generation-specific decision functions resembles a function path constructed by EFP; see Ríos-Rull (1999) and Nishiyama and Smetters (2014) for reviews of the literature on life-cycle economies. Also, backward-style iteration is commonly used for constructing numerical solutions to stationary dynamic economic models, in particular, it is used under conventional dynamic programing approaches, as well as some Euler equation methods, e.g., Coleman (1991), Mirman, Morand and Reffett (2008), Malin, Krueger and Kubler (2011). Our key novelty relative to this literature is to show that the familiar backward iteration method, combined with the turnpike analysis, can be used to analyze a class of infinite-horizon nonstationary models that is not yet well studied in the literature.

Second, EFP is related to the literature on balanced growth. However, this class of models is very limited; for example, models with labor augmenting technological progress are generally consistent with a balanced growth path but not models with either capital augmenting or Hicks neutral technological progress; see King, Plosser and Rebello (1988) for restrictions on preferences and technology that are consistent with a balanced growth path. There are examples of constructing a balanced growth path in some models that do not satisfy the restrictions in King, Plosser and Rebello (1988) but they are also limited.[7]

Finally, EFP is related to the literature that incorporates certain kinds of nonstationarity by augmenting the economic models to include additional state variables. In particular, Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010), among others, argue that the behavior of real-world economies is affected by degrees of uncertainty and introduce models with stochastic volatility. Furthermore, Davig and Leeper (2009), Farmer, Waggoner and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013) and Zhong (2015), among others, advocate periodic unanticipated changes in regimes. Finally, a recent paper of Schmitt-Grohé and Uribe (2012) proposes a quantitative framework that allows for anticipated exogenous shocks of a fixed periodicity and length. The key difference of our analysis from this literature in that we allow for time-dependence of the model, while the above literature expands the state space of time-invariant models. As a result, the EFP framework can handle any combination of unanticipated and anticipated shocks of any periodicity and duration.

5 Assessing EFP accuracy in a test model with balanced growth

In this section, we assess the quality of EFP approximations in a version of the model (1)–(3) with labor augmenting technological progress and balanced growth. We choose this model as a test application because in this special case, a nonstationary model that can be converted into stationary model and can be accurately solved by using conventional solution methods, so that we have a high-quality benchmark solution for comparison. We parameterize the model (1)–(3) by Cobb-Douglas utility and production functions,

$$u_t(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{and} \quad f_t(k,z) = zk^{\alpha}A_t^{1-\alpha}, \tag{12}$$

where $\rho \in (-1,1)$, $\sigma \in (0,\infty)$, and $A_t = A_0 g_A^t$ represents labor augmenting technological progress with an exogenous constant growth rate $g_A \geq 1$; and the productivity level in (3) follows $\ln z_{t+1} = \rho \ln z_t + \sigma\varepsilon_{t+1}$, where $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$, $\rho \in (-1,1)$ and $\sigma \in (0,\infty)$.

5.1 Methodology of numerical analysis

In this section, we describe the methodology of our numerical analysis and outline some implementation details.

[7] One example is the paper of Maliar and Maliar (2004) which shows the existence of a balanced growth path in a model with endogenous growth and cycles by removing a common stochastic trend representing randomly arriving technological innovations. Another example is Maliar and Maliar (2010) who construct a balanced growth path in a model with capital-skill complementarity and several types of technical progress by imposing additional restrictions on the growth rates of variables.

Four solution methods We solve the nonstationary growth model (1)–(3), (12) by using four alternative solution methods which we call *exact*, *EFP*, *Fair and Taylor* and *naive* solution methods.

i). Exact solution method. Under the assumptions in (12), the nonstationary model (1)–(3) is consistent with balanced growth and can be converted into a stationary model; see King, Plosser and Rebelo (1988). We first accurately solve the stationary model by using a conventional projection method, specifically, we use a Smolyak projection method in line with Krueger and Kubler (2004) and Judd, Maliar, Maliar and Valero (2014). We then recover a solution to the original nonstationary model; see Appendix D for details. The resulting numerical solution is very accurate, namely, the unit-free maximum residuals in the model's equations of order 10^{-6} on a stochastic simulation of 10,000 observations. We refer to this numerical solution as *exact*, and we use it as a benchmark for comparison with other numerical solutions.

ii). EFP solution method. EFP solves a nonstationary model directly, without converting it into stationary, by following the steps outlined in Algorithm 1 of Section 3.1; see Appendix B for implementation details. The solution produced by EFP for this model is shown in Figures 1 and 2.

iii). Fair and Taylor (1983) solution method. Fair and Taylor's (1983) method also solves a nonstationary model directly, without converting it into stationary. It constructs a path for the model's variables (not functions!) under one given sequence of shocks by using the certainty equivalence approach for approximating expectation functions. The implementation of Fair and Taylor's (1983) method is described in Appendix C.

iv). Naive solution method. A naive method replaces a nonstationary model with a sequence of stationary models and solves such models one by one, independently of one another. Similar to EFP, the naive method constructs a path of decision functions for $t = 0, ..., T$ but it differs from EFP in that it neglects the connections between the decision functions in different time periods. A comparison of the EFP and naive solutions will allow us to appreciate the importance of anticipatory effects.

Growth path for the EFP method The implementation details of these solution methods are described in Appendices B, C and D. Here, we focus on one important implementation issue, namely, how to construct a sequence of grids for $t = 0, ..., T$ on which a sequence (path) of the EFP decision functions will be approximated. In a stationary model, we typically center the grid in the deterministic steady state. However, in a nonstationary model, a steady state does not exist. To address this case, we define an analogue of steady state as a path for the model's variables that solves an otherwise equivalent deterministic model when shocks are shut down. We refer to such a solution as a *growth path,* and we denote it by a superscript "$*$". For example, in Figure 1, we show a growth path for capital k_1^*, k_{20}^* and k_{40}^* for periods 1, 20 and 40, respectively; see the centers of Smolyak grids in a (k_t, z_t) plane. In the special case of the balanced growth model (12), the growth path can be constructed analytically. Namely, in the stationary economy, the steady state capital is given by $k_0^* \equiv A_0 \left(\dfrac{g_A^\gamma - \beta + \delta\beta}{\alpha\beta} \right)^{1/(\alpha-1)}$, and in the growing economy, it evolves as $k_t^* = k_0^* g_A^t$ for $t = 1, ..., T$. In unbalanced growth models, the growth path must be in general constructed numerically; see Section 7.1 for an example and discussion.

Parameterization, software and hardware For all experiments, we fix $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$ and $\rho = 0.95$. The remaining parameters are set in the benchmark case at $\gamma = 5$, $\sigma_\varepsilon = 0.03$, $g_A = 1.01$ and $T = 200$ but we vary these parameters across experiments. For all simulations, we use the same initial condition and the same sequence of productivity shocks for all methods considered. Our code is written in MATLAB 2013a, and we use a desktop computer with Intel(R) Core(TM) i7-2600 CPU (3.40 GHz) with RAM 12GB.

5.2 Comparison results for four solution methods

In the left panel of Figure 3, we plot the growing time-series solutions for the four solution methods, as well as the (steady state) growth path for capital. In the right panel, we display the time series solutions after detrending the growth path.

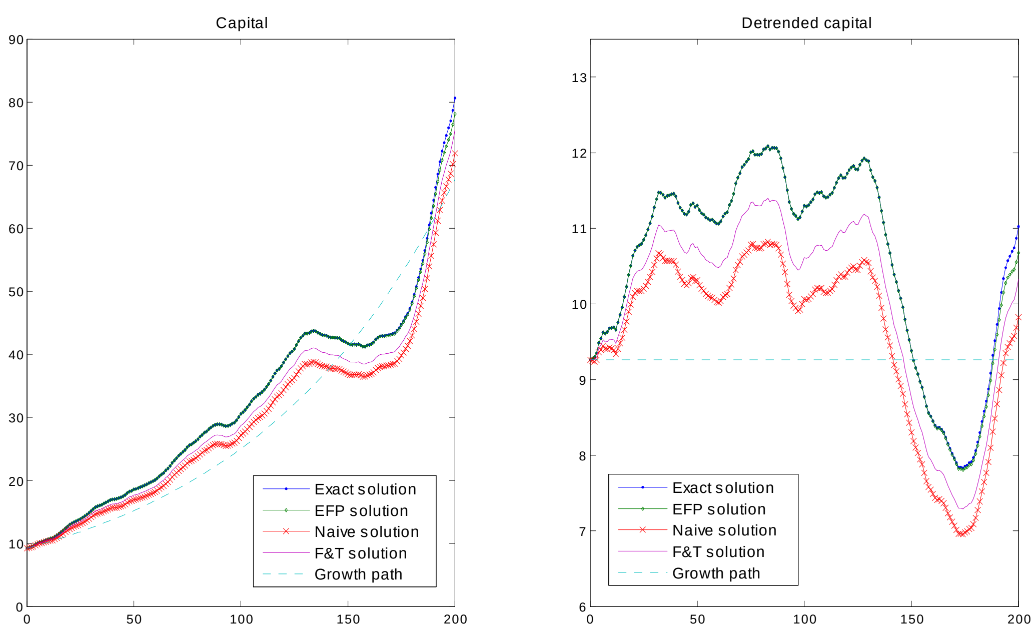

Figure 3. Comparison of the solution methods for the test model with balanced growth

As is evident from both panels, the EFP solution and the exact solution are visually indistinguishable except at the end of the time horizon – the last 10-15 periods. The difference at the end between the EFP and exact solutions are a consequence of different terminal conditions used: in the former case, we assume that the economy becomes stationary (i.e., stops growing) at T, whereas in the latter case, the growth continues forever. If we use the same terminal condition for the EFP as the exact solution at T, then the EFP solution would be indistinguishable from the exact solution everywhere in the figure. However, Fair and Taylor's (1983) and naive methods are far less accurate; they produce solutions that are systematically lower than the exact solution everywhere in the figure; and the naive solution is the least accurate of all.

We next evaluate the accuracy of the four constructed solutions numerically. We first simulate each of the four solutions 100 times and we then compute the mean and maximum absolute differences in log 10 units between the exact solution and the remaining three solutions across 100 simulations for the intervals $[0, 50]$, $[0, 100]$, $[0, 150]$, $[0, 175]$, and $[0, 200]$. This kind of statistics shows how the accuracy of numerical solutions deteriorates, as we move closer to the terminal period. The accuracy results are reported n Table 1, as well as the time needed for computing and simulating 100 solutions of length T (in seconds). We observe that in most

Table 1: Comparison of four solution methods.

	Fair-Taylor (1983) method, $\tau = 1$		Naive method	EFP method $\tau = 1$			EFP method $\tau = 200$		
Terminal condition	Steady state	Steady state	-	Balanced growth	T-period stationary		Balanced growth	T-period stationary	
T	200	400	200	200	200	400	200	200	400
Mean errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.60	-1.60	-1.36	-7.30	-6.97	-7.15	-7.23	-6.75	-7.01
$t \in [0, 100]$	-1.42	-1.42	-1.19	-7.06	-6.81	-6.98	-7.03	-6.19	-6.81
$t \in [0, 150]$	-1.34	-1.35	-1.11	-6.96	-6.73	-6.91	-6.94	-5.47	-6.73
$t \in [0, 175]$	-1.32	-1.32	-1.09	-6.93	-6.71	-6.89	-6.91	-5.09	-6.70
$t \in [0, 200]$	-1.30	-1.31	-1.07	-6.91	-6.69	-6.87	-6.90	-4.70	-6.68
Maximum errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.29	-1.29	-1.04	-6.83	-6.63	-6.81	-6.82	-6.01	-6.42
$t \in [0, 100]$	-1.18	-1.18	-0.92	-6.69	-6.42	-6.68	-6.68	-4.39	-5.99
$t \in [0, 150]$	-1.14	-1.14	-0.89	-6.66	-6.39	-6.67	-6.66	-2.89	-5.98
$t \in [0, 175]$	-1.14	-1.13	-0.89	-6.66	-6.40	-6.66	-6.66	-2.10	-5.98
$t \in [0, 200]$	-1.14	-1.13	-0.89	-6.66	-6.37	-6.66	-6.66	-1.45	-5.92
Running time, in seconds									
Solution	1.2(+4)	6.1(+4)	28.9	216.5	8.6(+3)	1.9(+4)	104.9	99.1	225.9
Simulation	-	-	2.6	2.6	2.6	5.8	2.6	2.8	5.7
Total	1.2(+4)	6.1(+4)	31.5	219.2	8.6(+3)	1.9(+4)	107.6	101.9	231.6

Notes: "Mean errors" and "Maximum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by a method in the column. The difference between the solutions is computed across 100 simulations.

implementations, the approximation errors of EFP do not exceed $10^{-6} = 0.0001\%$, while the errors produced by Fair and Taylor's (1983) and naive methods can be as large as $10^{-1.6} \approx 2.5\%$ and $10^{-0.89} \approx 12\%$. We explain these findings below in more details.

5.2.1 EFP method

In Table 1, we provide the results under three alternative implementations of EFP that illustrate how the properties of the EFP solutions depend on the choices of the terminal condition, K_T, time horizon T and parameter τ.

The role of the terminal condition: better terminal condition gives more accurate solutions. EFP requires us to specify a terminal condition in the form of T-period decision functions. What terminal condition do we choose? Again, for a special case of the balanced growth model, it is possible to infer the "exact" terminal condition from the solution to the stationary model; see Appendix D for details (this terminal condition is referred to as "balanced growth" terminal condition in Table 1). If we use the exact balanced-growth terminal condition, the EFP approximation is very accurate everywhere independently of τ and T, namely, the difference between the exact and EFP solutions is less than $10^{-6} = 0.0001\%$.

However, a stationary representation of the model and the exact terminal condition are generally unknown. To assess the role of the terminal condition in the accuracy of the EFP solutions, we also implement EFP by using a solution to the T-period stationary model as a terminal condition (this terminal condition is referred to as "T-period stationary" in Table 1). We observe that with this terminal condition, the accuracy critically depends on the choice of τ and T.

The choice of τ: a trade-off between accuracy and cost. We analyze two different values of τ such as $\tau = 1$ and $\tau = 200$. Under $\tau = 1$, EFP constructs a path of function in the same way as Fair and Taylor's (1983) method constructs a path of variables. First, given the terminal capital function K_T, EFP solves for decision functions for $t = 0, ..., T-1$, stores K_0 and discards the rest of the functions. Next, given K_{T+1}, EFP solves for decision functions for $t = 1, ..., T$, stores K_1 and discards the rest of the functions. It proceeds forward until the function path of a desired length is constructed.

As we see from the table, the EFP method with $\tau = 1$ is very accurate independently of T and a specific terminal condition used, namely, the EFP and exact solutions again differ by less than $10^{-6} = 0.0001\%$. This result illustrates the implication of the turnpike theorem that the effect of any terminal condition on the very first element of the path $\tau = 1$ is negligible if the time horizon T is sufficiently large.

A shortcoming of the version of EFP with $\tau = 1$ is its high computational expense: the running time under $T = 200$ and $T = 400$ is 15 and 30 minutes, respectively. The cost is high because we need to entirely recompute a sequence of decision functions each time when we extend the path by one period ahead. Effectively, we implement EFP T times and not just once, which is costly.

The choice of T: making EFP cheap. Our turnpike theorem suggests a cheaper version of EFP in which we construct a longer path (i.e., we use $\tau > 1$) but we do it just once; the results for this version of the EFP method are provided in the last three columns of Table 1. For $\tau = 200$, the terminal condition plays a critical role in the accuracy of solutions near the tail. Namely, if we use the terminal condition from the T-period stationary economy, and consider $T = 200$, than the approximation errors near the tail reach $10^{-1.45} \approx 4\%$.

However, the approximation errors can be dramatically reduced by increasing the time horizon T, as shows the last column of Table 1. Namely, if we construct a path of length $T = 400$, however, use only the first $\tau = 200$ decision functions and discrard the remaining path, the solution for the first $\tau = 200$ periods is almost as accurate as that produced under

$\tau = 1$. This is true even if we use the terminal condition from the T-period stationary economy that is far away from the exact terminal condition. Importantly, the construction of a longer path is relatively inexpensive: the running time increases from about 2 minutes to 4 minutes when we increase the time horizon from $T = 200$ to $T = 400$, respectively.

5.2.2 Fair and Taylor's (1983) method

As Table 1 shows, EFP improves upon Fair and Taylor's (1983) method in both accuracy and speed. Fair and Taylor's (1983) method has relatively low accuracy (namely, approximation errors of $10^{-1.6} \approx 2.5\%$) because the certainty equivalence approach does not produce sufficiently accurate approximation to conditional expectations under the given parameterization. A comparison of $T = 200$ and $T = 400$ shows that the accuracy cannot be increased by increasing the time horizon. Fair and Taylor's (1983) method is more accurate with a smaller variance of shocks and /or smaller degrees of nonlinearities. For example, in the model with $\gamma = 1$, $\sigma_\varepsilon = 0.01$, $g_A = 1.01$ and $T = 200$, the difference between the exact solution and Fair and Taylor's (1983) solutions is around 0.1% (this experiment is not reported).

The high cost of Fair and Taylor's (1983) method is explained by two factors. First, $\tau = 1$ is the only possible choice for Fair and Taylor's (1983) method. To solve for variables of period $t = 0$, this method assumes that productivity shocks are all zeros starting from period $t = 1$, so that the path for $t = 1, ..., T$ has no meaning other than helping to approximate the variables of period $t = 0$. In contrast, EFP can use any τ as long as the the resulting solution is sufficiently accurate, which reduces the cost.

Second, for Fair and Taylor's (1983) method, the cost of simulating the model is very high because the solution and simulation steps are combined together: in order to produce a new simulation, it is necessary to entirely recompute the solution under a different sequence of shocks. In contrast, for EFP, the simulation cost is very low: we construct a path of decision functions just once, and we can use the constructed functions to produce as many simulations as we need under different sequences of shocks. For example, the time that EFP needs to compute a solution and simulate it 100 times is about 2 and 4 minutes for $T = 200$ and $T = 400$, respectively, while the corresponding times for Fair and Taylor's (1983) method are 20 and 60 minutes, respectively.

5.2.3 Naive method

For the naive method, we report the solution only for $T = 200$ since neither time horizon nor terminal condition are relevant for this method. The performance of the naive method is poor: the difference between the exact and naive solutions can be as large as $10^{-0.89} \approx 12\%$. The naive solution is so inaccurate because the naive method neglects anticipatory effects. In each time period t, this method computes a stationary solution under the assumption that technology will remain at the levels $A_t = A_0 g_A^t$ and $A_{t+1} = A_0 g_A^{t+1}$ forever, meanwhile the true nonstationary economy continues to grow. Since the naive agent is "unaware" about the future permanent productivity growth, the expectations of such an agent are systematically more pessimistic than those of the agent who is aware of future productivity growth. It was pointed out by Cooley, Leroy and Raymon (1984) that naive-style solution methods are logically inconsistent and contradict to rational expectation paradigm: agents are unaware about a possibility of parameter changes when they solve their optimization problems, however, they are confronted with parameter changes in simulations. Our analysis suggests that naive solutions are particularly inaccurate in growing economies. We conclude that approximating expectation functions accurately is critical for constructing accurate solutions to nonstationary growth models.

5.2.4 Sensitivity analysis

On the basis of the results in Table 1, we advocate the version of EFP that constructs a sufficiently long path $\tau > 1$ by using $T \gg \tau$. We assess the accuracy and cost of this preferred EFP version by using $\tau = 200$ and $T = 400$ under several alternative parameterizations for $\{\gamma, \sigma_\varepsilon, g_A\}$ such that $\gamma \in \{0.1; 1; 5; 10\}$, $\sigma_\varepsilon \in \{0.01; 0.03\}$ and $g_A \in \{1; 1.01; 1.05\}$. As a terminal condition, we use decision rules produced by the T-period stationary economy. These sensitivity results are provided in Table 2 of Appendix E.

The accuracy and cost of EFP in these experiments are similar to those reported in Table 1 for the benchmark parameterization. The difference between the exact and EFP solutions varies from $10^{-7} = 0.00001\%$ to $10^{-6} = 0.0001\%$ and the running time varies from 155 to 306 seconds. The exception is the model with a low degree of risk aversion $\gamma = 0.1$ for which the running time increases to 842 seconds. (We find that with a low degree of risk aversion, the convergence of EFP is more fragile, so that we had to use a larger degree of damping for iteration, decreasing thus the speed of convergence). Overall, our sensitivity results show that the EFP method can solve nonstationary growth models both accurately and reliably in a wide range of the model's parameters at a relatively low cost.

6 Modeling anticipated parameter shifts and drifts

In this section, we show how to use the EFP framework for modeling anticipated parameter shifts and drifts. For comparison, we also show naive solutions in which agents fail to take anticipatory effects into consideration.

6.1 A nonstationary model with a parameter shift

Parameter shifts (also referred to as regime switches) are drastic changes in the model's parameters. Parameter shifts can be either anticipated or not by the agents. Our analysis will focus on anticipated parameter shifts.

6.1.1 Literature on parameter shifts

The case of unanticipated regime switches is well studied. The literature assumes that regimes come at random with some probabilities, and it uses the same random process for regime switches for simulation; see Sims and Zha (2006), Davig and Leeper (2007, 2009), Farmer, Waggoner, and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013) and Zhong (2015), among others. Such a framework addresses the shortcomings of naive solution methods and provides a logically consistent way of modeling unanticipated regime switches.

However, there are real-world situations in which regime switches are anticipated by agents, for example, presidential elections with predictable outcomes, credible policy announcements, anticipated legislative changes, deterministic seasonals. The idea that anticipated shocks play an important role in business cycle fluctuations goes back to Pigou (1927); and it is also advocated in, e.g., Cochrane (1994), Beaudry and Portier (2006), and Schmitt-Grohé and Uribe (2012). An empirically relevant example of an anticipated shock is an accession of new members to the European Union that was announced many years in advance and that resulted in large anticipatory inflows of foreign direct investments; see Garmel, Maliar and Maliar (2008) for a quantitative analysis of the EU accession in a three-country general equilibrium model.

The case of anticipated regime switches is more challenging to analyze, unless the environment is fully deterministic. Here, the optimal decision rules change from one period to another driven by anticipatory effects. Schmitt-Grohé and Uribe (2012) propose a perturbation-based computational approach that allows us to deal with anticipated parameter shifts of a fixed time horizons in the context of stationary Markov models (e.g., shocks that happen each fourth or eight periods). In turn, EFP allows us to handle any combination of unanticipated and anticipated shocks of any periodicity and duration in a fully nonlinear manner.

6.1.2 EFP versus naive solutions with parameter shifts

We consider a version of the model (1)–(3), (12) in which the technology level A_t can take two values, $\underline{A} = 1$ (low) and $\bar{A} = 1.2$ (high). A special case of this setup is a model in which $\underline{A}$ and $\bar{A}$ are unanticipated and randomly drawn from a given probability distribution. Such a model has a stationary Markov solution that can be studied using the approaches described in the literature on regime switches, e.g., Davig and Leeper (2007, 2009).

In contrast, we focus on the case when the regime switches are anticipated by the agent from the beginning. As an example, we consider a scenario when the economy starts with $\underline{A}$ at $t = 0$, switches to $\bar{A}$ at $t' = 250$ and then switches back to $\underline{A}$ at $t'' = 550$, for example, the U.K. joined the EU in 1973 and existed in 2016. We show the technology profile in the upper panel of Figure 4.

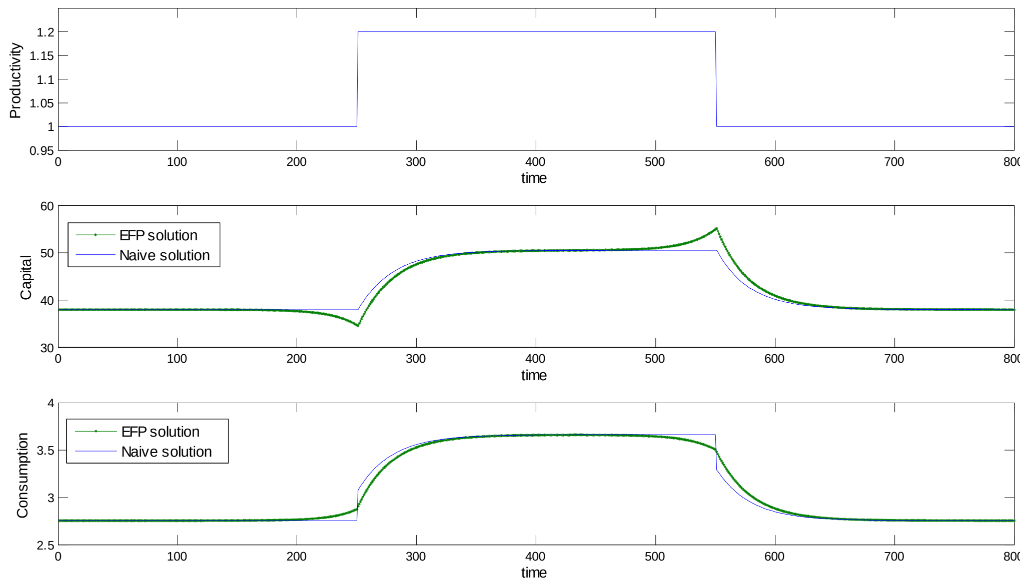

Figure 4. EFP versus naive solutions in the model with parameter shifts

We parameterize this model by $T = 900$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$. In simulation, we set $z_t = 1$ for all t to make the anticipatory effects more pronounced.

For a naive agent, regime switches are unexpected. We construct two stationary naive solutions under $\underline{A}$ and $\bar{A}$. The naive agent follows the first solution until the first switch at $t' = 250$, then the agent follows the second solution until the second switch at $t'' = 550$ and finally, the agent goes back to the first solution for the rest of the simulation. In contrast, a rational agent is assumed to anticipate regime switches. We use EFP to solve the utility-maximization problem at $t = 0$ given the technology profile.

Remarkably, under the EFP solution, we observe a strong anticipatory effect: about 50 periods before the switch from $\underline{A}$ and $\bar{A}$ takes place, the agent starts gradually increasing her consumption and decreasing her capital stock in order to bring some part of the benefits from future technological progress to present. When a technology switch actually occurs, it has only a minor effect on consumption. (The tendencies reverse when there is a switch from $\bar{A}$ to $\underline{A}$). In contrast, consumption-smoothing anticipatory effects are absent for the naive solution. Here, unexpected technology shocks lead to large jumps in consumption in the exact moment of technology switches. The difference in the solutions is quantitatively significant under our empirically plausible parameter choice. In the Appendix F, we plot the simulated solution by considering both deterministic switches in the level of technology and stochastic productivity shocks that follow an AR(1) process (32); see Figure 10. Anticipatory effects are well pronounced in those experiments as well.

6.2 A nonstationary model with a parameter drift

Parameter drifts (time trends) are gradual changes in the model's parameters. Like parameter shifts, parameter drifts can be either anticipated or not by the agents. Our analysis will again focus on anticipated parameter drifts.

6.2.1 Literature on parameter drifts

There is ample evidence on parameter drifts, see, e.g., Clarida, Galí and Gertler (2000), Lubick and Schorfheide (2004), Cogley and Sargent (2005), Goodfriend and King (2009). Also, Galí (2006) argues that regime changes with gradual policy variations are empirically relevant. Again, the literature focuses primarily on the case of unanticipated drifts by assuming that the model's parameters follow a stationary autoregressive process; see, e.g., Fernández-Villaverde and Rubio-Ramírez (2007), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010).

However, there are empirically relevant parameter drifts that are anticipated, in particular, population growth and different types of technological progress. Labor augmenting technological progress is a well-known example of a parameter drift that leads to balanced growth and stationary solutions; see King, Plosser and Rebello (1988). However, parameters drifts generally lead to unbalanced growth in which optimal the decision rules nontrivially change from one period to another. Below, we use EFP to analyze a parameter drift that includes both deterministic and stochastic trends.

6.2.2 EFP versus naive solutions with a parameter drift

We now assume that technology does not switch to a higher/lower level in one period but increases/decreases gradually. To be specific, we assume that technology is at a low level $\underline{A}$ for the first 200 periods; it increases linearly to a high level $\overline{A}$ for the next 100 periods; it stays constant for the following 300 periods; it decreases linearly back to a low level $\underline{A}$ for the last

200 periods; and finally, it stays there for the remaining periods; see Figure 5.

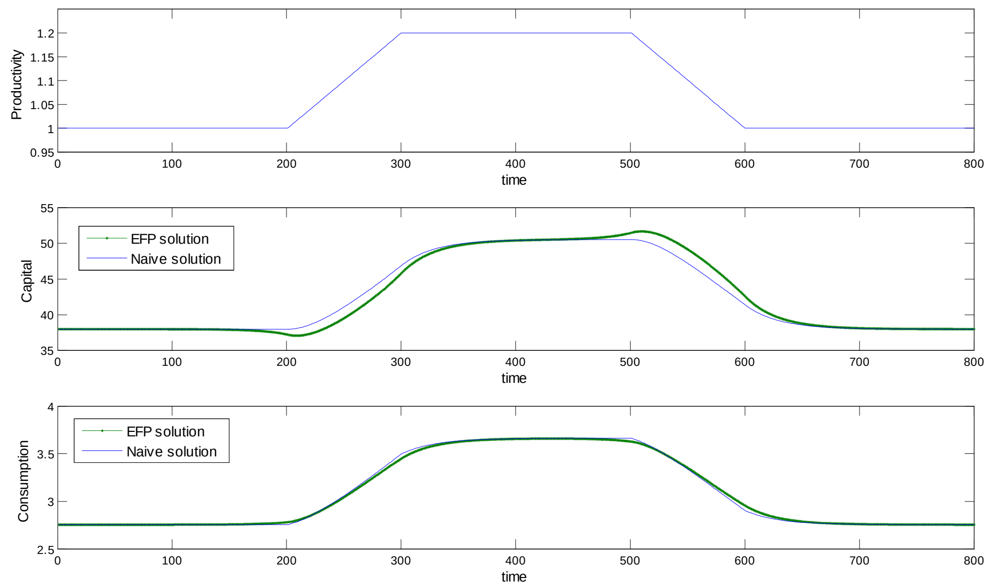

Figure 5. EFP versus naive solutions in a model with a parameter drift

To calibrate the model, we use the same parameters as in the model with the parameter shift.

In Figure 5, we plot the EFP time-series solution to the model with a parameter drift (see the middle and lower panels). For comparison, we also provide a naive solution in which the drift is always unanticipated. To produce the naive solution, we solve a stationary model under each of 100 levels of technology that occurs during the parameter drift, and we switch from one stationary naive solution to another after each technology change. Again, to simulate the naive and EFP solutions, we set $z_t = 1$ for all t for a better visibility of anticipatory effects.

Under the EFP solution, we observe a well-pronounced pattern of consumption smoothing at the cost of anticipatory adjustments of capital. In particular, the consumption path with an expected parameter drift is smoother than the one in the naive solution in those places where the parameter shift begins / ends and we observe the kink. In the Appendix F, we provide a plot of the simulated solution with both deterministic productivity shifts and stochastic productivity shocks; see Figure 11. Anticipatory effects are well pronounced in that figure as well.

7 Numerical analysis of nonstationary and unbalanced growth applications

We present a collection of numerical examples that illustrate how EFP can be used for calibrating, solving, estimating and simulating nonstationary models. Our examples include infinite-horizon stochastic growth models with unbalanced growth, seasonal adjustments and deterministically changing volatility, as well as an example of calibrating and estimating parameters in an unbalanced growth model using the data on the U.S. economy.

7.1 Application 1: Unbalanced growth model with a CES production function and capital-augmenting technological progress

Real business cycle literature heavily relies on the assumption of labor augmenting technological progress leading to balanced growth. However, Acemoglu (2002) argues that technical change is not always directed to the same fixed factors of production but to those factors of production that give the largest improvement in the efficiency of production.[8] One implication of this argument is that technical change can be directed to either capital or labor depending on the economy's state. Furthermore, Acemoglu (2003) explicitly incorporates capital augmenting technological progress into a deterministic model of endogenous technical change by allowing for innovations in both capital and labor. Evidence in support of capital augmenting technical change is provided in, e.g., Klump, Mc Adam and Willman (2007), and León-Ledesma León-Ledesma, Mc Adam and Wilman (2015).[9]

Constant elasticity of substitution production function In line with this literature, we consider the stochastic growth model (1)–(3) with a constant elasticity of substitution (CES) production function, and we allow for both labor and capital augmenting types of technological progress

$$F(k_t, \ell_t) = \left[\alpha (A_{k,t} k_t)^v + (1-\alpha)(A_{\ell,t}\ell_t)^v\right]^{1/v}, \tag{13}$$

where $A_{k,t} = A_{k,0} g_{A_k}^t$; $A_{\ell,t} = A_{\ell,0} g_{A_\ell}^t$; $v \leq 1$; $\alpha \in (0,1)$; and g_{A_k} and g_{A_ℓ} are the rates of capital and labour augmenting technological progresses, respectively. We assume that labor is supplied inelastically and normalize it to one $\ell_t = 1$ for all t, and we denote the corresponding production function by $f(k_t) \equiv F(k_t, 1)$.

A growth path for economy with unbalanced growth Our first goal is to define a growth path around which the sequence of EFP grids will be centered. For constructing the growth path, we shut down uncertainty by assuming that $z_t = 1$ for all t (similar to what we do in a model with balanced growth) and we rewrite the model's equations in the way that is convenient for identifying the growth path.

First, the Euler equation of period t, evaluated on the steady state path, is

$$1 = \beta \left[\frac{u'(c_{t+1}^*)}{u'(c_t^*)} (1 - \delta + f' \left\{\alpha A_{k,t+1}^v (k_{t+1}^*)^{v-1} \left[\alpha (A_{k,t+1} k_{t+1}^*)^v + (1-\alpha) A_{\ell,t+1}^v\right]^{(1-v)/v}\right\}\right],$$

where c_t^* and k_t^* are the variables on the growth path. From the last equation, we express k_{t+1}^* as

$$k_{t+1}^* = (1-\alpha)^{1/v} \frac{A_{\ell,t+1}}{A_{k,t+1}} \left[\left(\frac{(g_{u',t+1})^{-1} - \beta + \delta\beta}{\alpha\beta \cdot A_{k,t+1}}\right)^{v/(1-v)} - \alpha\right]^{1/v}, \tag{14}$$

[8] Namely, endogenous technical change is biased toward a relatively more scarce factor when the elasticity of substitution is low (because this factor is relatively more expensive); however, it is biased toward a relatively more abundant factor when the elasticity of substitution is high (because technologies using such a factor have a larger market).

[9] There are other empirically relevant types of technological progress that are inconsistent with balanced growth, for example, investement-specific technological progress considered in Krusell, Ohanian, Ríos-Rull and Violante (2000).

where $g_{u',t+1} \equiv \frac{u'(c_{t+1}^*)}{u'(c_t^*)}$ follows from the budget constraints (2) for t and $t+1$:

$$g_{u',t+1} = \frac{u'\left[(1-\delta)\,k_{t+1}^* + \left[\alpha(A_{k,t+1}k_{t+1}^*)^v + (1-\alpha)A_{\ell,t+1}^v\right]^{1/v} - k_{t+2}^*\right]}{u'\left[(1-\delta)\,k_t^* + \left[\alpha(A_{k,t}k_t^*)^v + (1-\alpha)A_{\ell,t}^v\right]^{1/v} - k_{t+1}^*\right]}. \tag{15}$$

Thus, we obtain a system of $T-1$ equations (14) with $T+1$ unknowns $k_0^*, ..., k_{T+1}^*$. This system does not have a unique solution unless we impose additional restrictions.

Identifying restrictions on initial and terminal conditions There are many possible ways to impose identifying restrictions on the solution of system (14), (15). In this specific application, we restrict the initial and terminal capital stocks, k_0^* and k_{T+1}^*. Namely, we restrict k_0^* by assuming that the capital growth rate is the same in the first two periods $\frac{k_1^*}{k_0^*} = \frac{k_2^*}{k_1^*}$, and we restrict k_{T+1}^* by assuming such a growth rate is the same in the last two periods $\frac{k_T^*}{k_{T-1}^*} = \frac{k_{T+1}^*}{k_T^*}$. These assumptions pin down the initial and terminal capital stocks on the growth path in terms of $k_1^*, ..., k_T^*$,

$$k_0^* = \frac{(k_1^*)^2}{k_2^*} \qquad \text{and} \qquad k_{T+1}^* = \frac{(k_T^*)^2}{k_{T-1}^*}. \tag{16}$$

The model satisfies the assumptions of King, Plosser and Rebelo (1988) if there is only labor augmenting technological progress, i.e., $A_{\ell,t}$ grows at a constant, exogenously given rate g_{A_ℓ} and $A_{k,t} = A_k$ for all t. In this special case, the model has a balanced growth path on which all variables grow at a constant rate g_{A_ℓ} and this is in particular true for initial and terminal periods, i.e., condition (16) is satisfied exactly.

In the case of capital augmenting technological progress, the growth rate of endogenous variables changes over time in an unbalanced manner even if we assume that $A_{k,t}$ grows at a constant, exogenously given growth rate g_{A_k} and $A_{\ell,t} = A_\ell$ for all t. By imposing two additional restrictions in (16), we determine a specific sequence $k_0^*, ..., k_{T+1}^*$ satisfying (14), (15). In our applications, the changes in the growth path had only a minor effect on the resulting approximations. This is because a specific growth path does not identify the solution itself but only a set of points in which the Smolyak grids are centered. Centering a grid in a slightly different point will not significantly affect the properties of solution in a typical application. The assumption in (16) can be modified if needed.

Results of numerical experiments For numerical experiments, we assume $T = 260$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$, $v = -0.42$; the last value is taken in line with Antrás (2004) who estimated the elasticity of substitution between capital and labor to be in a range $[0.641, 0.892]$ that corresponds to $v \in [-0.12, -0.56]$. We solve two models: the model with labor augmenting progress parameterized by $A_{\ell,0} = 1.1130$, $g_{A_\ell} = 1.00153$ and $A_{k,0} = g_{A_k} = 1$ and the model with capital augmenting progress parameterized by $A_{k,0} = 1$, $g_{A_k} = 0.9867$ and $A_{\ell,0} = g_{A_\ell} = 1$. (The parameters $A_{\ell,0}$, g_{A_ℓ}, $A_{k,0}$, g_{A_k} for both models are chosen to approximately match the initial and terminal capital stocks for time-series solutions of both models).

Figure 6 plots the time-series solutions of the models with labour and capital augmenting technological progresses, as well as their growth paths.

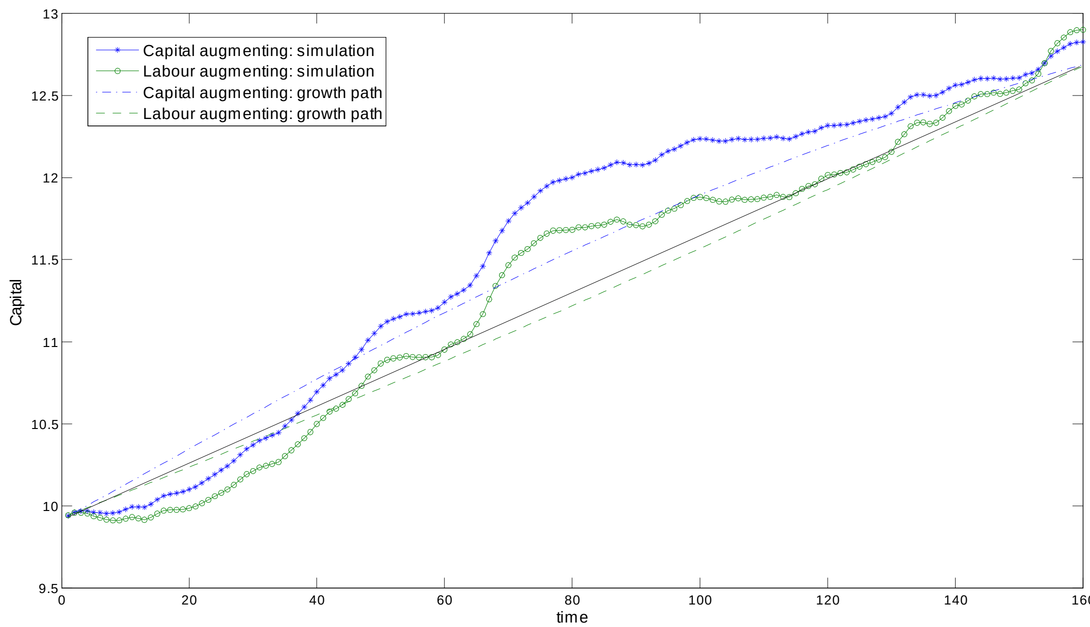

Figure 6: Technological progress in the model with a CES production function

The model with labor augmenting technological progress is well known. There is an exponential growth path with a constant growth rate and cyclical fluctuations around the growth path. (In the figure, the growth path in the model with labor augmenting technological progress is situated slightly below the linear growth path shown by a solid line). In contrast, the model with capital augmenting technological progress is not studied yet in the literature (to the best of our knowledge). Here, we observe a pronounced concave growth pattern indicating that the rate of return to capital decreases as the economy grows (In the figure, the growth path in the model with capital augmenting technological progress is situated above the linear growth path shown by a solid line). The cyclical properties of both models look similar (provided that growth is detrended).

7.2 Application 2: Seasonal adjustments

Growth model with seasonal adjustment is another empirically-relevant application that can be analyzed by using EFP. An important role of seasonal fluctuations in the total variation in aggregate economic variables is well documented in the literature; see, e.g., Barsky and Miron (1989). Ignoring seasonality when estimating dynamic stochastic general equilibrium models may lead to substantial errors in the estimated parameters; see, e.g., Saijo (2013).

Two approaches have been proposed in the literature to model seasonality. Hansen and Sargent (1993, 2013) characterize seasonality in terms of the spectral density of variables. They assume that seasonality comes either from exogenous shock processes with spectral peaks at seasonal frequencies or from propagation mechanisms determined by preferences and technology (e.g., seasonal habit persistence) or from seasonal periodicity in the parameters of the preferences and technologies; in these cases, the optimal decision rules are periodic. Second, Christiano and Todd (2002) develop a model in which an investment process is period-specific and requires four quarters to complete; to solve such a model, they linearize the model around its seasonally varying steady state growth path and solve for four distinct decision rules. Both of these approaches are developed for linear economies. In contrast, EFP provides a simple and general alternative to these methods and allows us to analyze seasonal fluctuations in a fully non-linear context, just like any other nonstationary model with a combination of anticipated and unanticipated shocks.

As an example, we study a growth model with exogenous shock processes that peaks at seasonal frequencies in line with Hansen and Sargent (1993, 2013). Specifically, we assume that every forth period, A_t takes a high value $\overline{A}$, and the rest of the periods, it takes a low value $\underline{A}$, which yields the following sequence of technology levels: $\underline{A}, \underline{A}, \underline{A}, \overline{A}, \underline{A}, \underline{A}, \underline{A}, \overline{A}, \ldots$. For example, this pattern can be observed in a country on a seacoast in which there is a high productivity season in summer. In addition to seasonal changes, the agent faces conventional productivity shocks, so that the resulting path for the productivity level is given by a composition of expected seasonal changes in A_t and unexpected stochastic changes in the productivity levels given by a stationary autoregressive process. The parameters are the same as in the previous model except that we use $\gamma = 2$, $\beta = 0.97$, $\underline{A} = 0.98$ and $\overline{A} = 1.06$ (these parameters are fixed for expositional convenience). In Figure 7, we plot time series for productivity, capital and consumption (we normalize the initial values of all series to one).

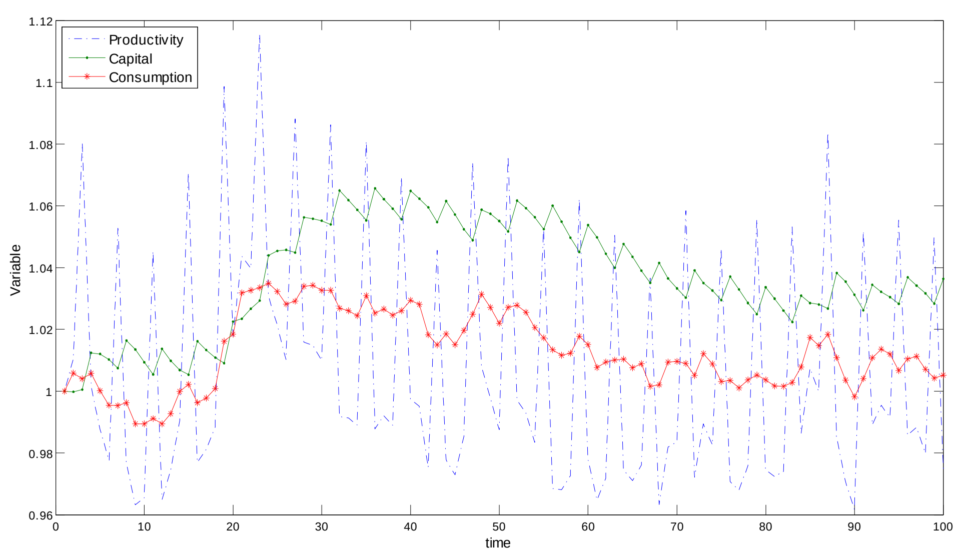

Figure 7. Seasonal adjustments

An interesting finding in Figure 7 is that the size of seasonal consumption and capital fluctuations is very small compared to the size of seasonal productivity fluctuations. A consumption-smoothing agent knows that the seasonal shock is temporary and that it does not pay to react much on the impact of such a shock. Instead, the agent adjusts her capital and consumption to take advantage of seasonal productivity growth on average, as permanent consumption hypothesis suggests. A magnitude of seasonal fluctuations in the model's variables would be far larger and comparable in size to seasonal productivity fluctuations in a naive solution in which agents would fail to take into account anticipatory effects (we do not provide the naive solution to avoid a clutter).

7.3 Application 3: Diminishing volatility

Recent literature on stochastic volatility documents the importance of degree of uncertainty for business cycle fluctuations; see, e.g., Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde, Guerrón-Quintana and Rubio-Ramírez (2010). This literature assumes that the standard deviation of exogenous shocks either follows a stationary Markov process or experiences recurring Markov regime switches with stationary transition probabilities. In the former case, a regime is an additional state variable, while in the latter case, volatility is an additional state variable; in both cases, the environment is stationary.

However, there is ample evidence that the volatility of output has a pronounced time trend. For example, Mc Connel and Pérez-Quiros (2000) document a monotone decline in the volatility of real GDP growth in the U.S. economy. In turn, Blanchard and Simon (2001) find another pattern: there was a steady decline in the volatility from the 1950s to 1970, then there was a stationary pattern and finally, there was another decline in the late 1980s and the 1990s. Finally, Stock and Watson (2003) document a sharp reduction in volatility in the first quarter of 1984. This kind of evidence cannot be reconciled in a model in which stochastic volatility follows a stationary Markov process with time-invariant parameters. Here, we show how to use EFP to analyze economies in which the volatility has both a stochastic and deterministic components.

We specifically consider the standard neoclassical stochastic growth model, modified to include a diminishing volatility of the productivity shock:

$$\ln z_t = \rho \ln z_{t-1} + \sigma_t \varepsilon_t, \qquad \sigma_t = \frac{B}{t^{\rho_\sigma}}, \quad \varepsilon_t \sim \mathcal{N}(0,1), \tag{17}$$

where B is a scaling parameter, and ρ_σ is a parameter that governs the volatility of z_t. The standard deviation of the productivity shock $B\sigma/t^{\rho_\sigma}$ decreases over time, reaching zero in the limit, $\lim_{t\to\infty} \frac{B\sigma}{t^{\rho_\sigma}} = 0$.

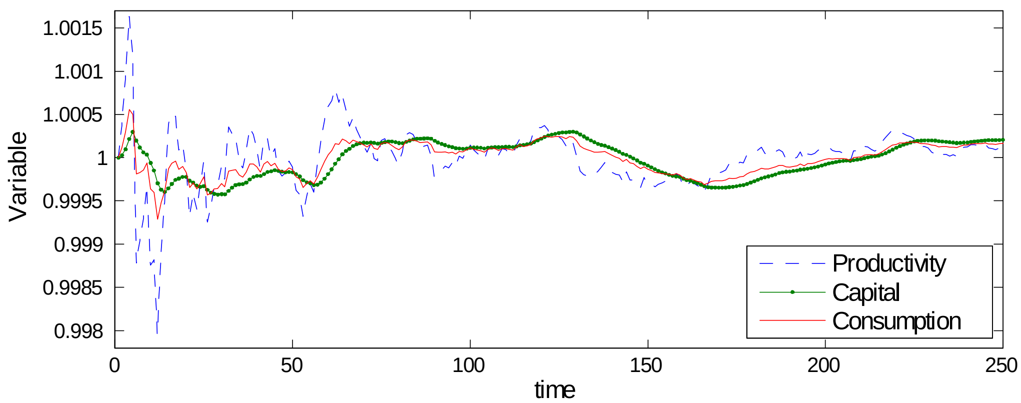

Figure 8. Diminishing volatility

In our numerical example, we use $T = 500$, $\gamma = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\varepsilon = 0.01$, $B = 1$ and $\rho_\sigma = 1.05$. In Figure 8, we plot a sequence of simulated productivity levels. Initially, there are large productivity fluctuations but gradually, these fluctuations become smaller. As expected, fluctuations in capital and consumption also decrease in amplitude in response to diminishing volatility.

7.4 Application 4: Calibrating an unbalanced growth model with a parameter drift to unbalanced U.S. data

There is a large body of econometric methods which estimate and calibrate economic models by constructing numerical solutions explicitly, including a simulated method of moments (e.g., Canova (2007)); a Bayesian estimation method (e.g., Smets and Wouters (2003), and Del Negro, Schorfheide, Smets and Wouters (2007)); and a maximum likelihood method (e.g., Fernández-Villaverde and Rubio-Ramírez (2007)). Normally, the related literature imposes restrictions on the model that lead to a balanced growth path. However, the real world data are not always consistent with the assumption of balanced growth, in particular, different variables might grow at different and possibly time-varying rates; see, e.g. Krusell et al. (2000). In this section, we illustrate how EFP can be used to calibrate and estimate parameters in an unbalanced growth model by using the data on the U.S. economy.

7.4.1 The model with a depreciation rate drift

We analyze the aggregate time series data on the U.S. economy over the period 1964:Q1 - 2011:Q4 including investment, consumption, output and capital; see Appendix G for a description of the data used. While the constructed data are grossly consistent with Kaldor's (1961) hypothesis, we still observe visible differences in growth rates across variables. We do not test whether or not such differences in growth rates are statistically significant but formulate and estimate an unbalanced growth model in which different variables can grow at differing rates. To this purpose, we extend the model (1)–(3) to include time-varying depreciation rate of capital,

$$\max_{\{c_t,k_{t+1}\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{18}$$

$$\text{s.t. } c_t + k_{t+1} = A_t z_t k_t^{\alpha} + (1 - d_t \delta_t) k_t, \tag{19}$$

$$\ln \delta_t = \rho_\delta \ln \delta_{t-1} + \varepsilon_{\delta,t}, \qquad \varepsilon_{\delta,t} \sim \mathcal{N}\left(0, \sigma_{\varepsilon_d}^2\right), \tag{20}$$

$$\ln z_t = \rho_z \ln z_{t-1} + \varepsilon_{z,t}, \qquad \varepsilon_{z,t} \sim \mathcal{N}\left(0, \sigma_{\varepsilon_z}^2\right), \tag{21}$$

where $d_t\delta_t$ stands for a time-varying depreciation rate with d_t being a trend component of depreciation, $d_t = d_0 g_d^t$, and δ_t being a stochastic shock to depreciation. Our assumption of a time trend in the depreciation rate is consistent with recent empirical findings. In particular, Karabarbounis and Brent (2014) argue that the aggregate depreciation rate can change over time because the composition of aggregate capital changes over time even if the depreciation rates of each type of capital are constant. In turn, shocks to the depreciation rate can result from economic obsolescence of capital, see e.g., Liu, Waggoner and Zha (2011), Gourio (2012) and Zhong (2015); in particular, this literature argues that a shock to the capital depreciation rate plays an important role in accounting for business cycle fluctuations.

7.4.2 Fitted time series

The details of our EFP-based calibration-estimation procedure are described in Appendix G. Figure 9 presents the simulated time-series solution for capital, output, investment and consumption in comparison with the corresponding time series from the U.S. economy data. To appreciate the differences in the growth rates, we normalized all four panels to have the same percentage change in the y axis.

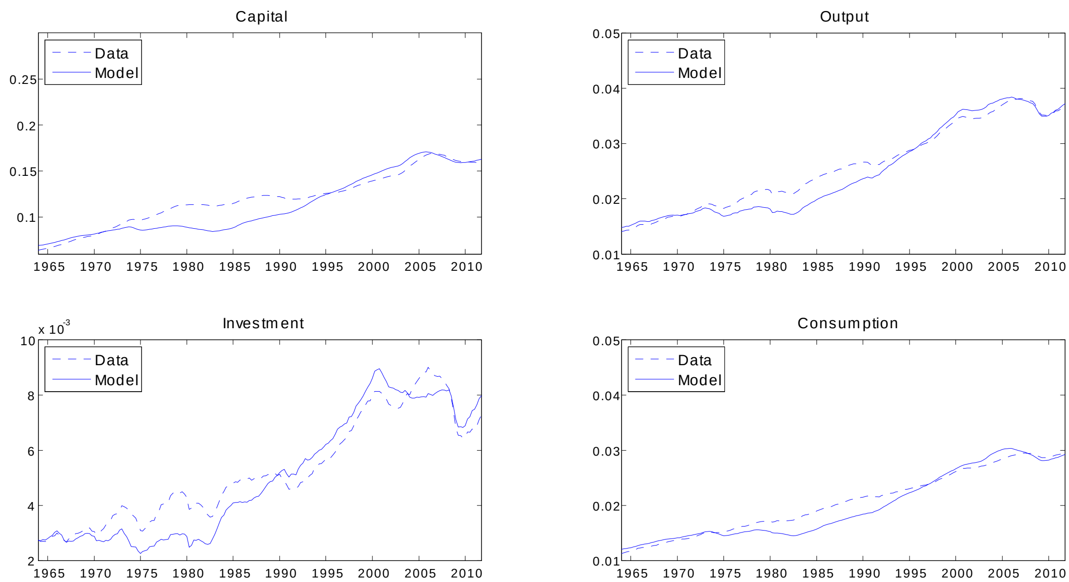

Figure 9. Matching nonstationary macroeconomic data on the U.S. economy

First, we can visually appreciate nonstationarity in the data: investment grows considerably faster than the other variables. Second, we can see that with the assumption of the time-varying depreciation rate, the model (18)–(21) can reproduce the growth rates of all model's variables.

The main goal of this application is not to advocate the empirical relevance of the time-varying depreciation rate or some specific estimation and calibration techniques. Rather, we would like to illustrate how estimation and calibration of the parameters can be carried out in the context of a nested fixed-point problem without assuming stationarity and balanced growth. Similar to the depreciation rate, we could have made all other parameters time dependent, including the discount factor β, the share of capital in production α and the parameters of the process for the productivity level (21). Furthermore, our simple estimation-calibration technique can be replaced by more sophisticated econometric techniques such as maximum likelihood, simulated method of moments, etc.

8 Conclusion

A class of stationary Markov dynamic models is a dominant framework in the recent economic literature. The conventional assumption in this literature is that parameter shifts and drifts come at random and are unanticipated by the agents. With stationary environment, the optimal value and decision functions depend only on the economy's state but not on time. In the paper, we study a more flexible class of models – nonstationary Markov models – in which parameters are subject to both anticipated and unanticipated shifts and drifts. In our case, the optimal value and decision functions depend not just on state but also on time. We propose a simple and general EFP framework for analyzing time-dependent models which combines a familiar time iteration method with turnpike theory. Our analysis of time-dependent models complements the mainstream of the literature on state-dependent models and makes it possible to study many challenging nonstationary applications that are not studied in the literature yet.

The goal of the present paper is to introduce and illustrate the EFP methodology. Given this goal, we restrict our attention to a simple context of the optimal growth model. However, the EFP framework can be used for analyzing many other nonstationary applications that go far beyond the optimal growth model. In one ongoing project, we apply the EFP framework to analyze a general equilibrium unbalanced growth model with a CES production function in line with Krusell, Ohanian, Ríos-Rull and Violante (2000); in another project, we attempt to reproduce a historical sequence of events during the Great Recession as documented in Taylor (2012); and finally, in the other project, we augment a stylized new Keynesian model (see, e.g., Maliar and Maliar, 2015) to include anticipated regime switches and time-varying unconventional monetary policies. These are just three examples but many other interesting and empirically relevant questions can be addressed by using EFP.

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Appendices to "A Tractable Framework for Analyzing Nonstationary and Unbalanced Growth Models"

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Appendix A. Asymptotic convergence of T-period stationary economy to nonstationary economy

In this section, we introduce notation, provide several relevant definitions about random processes and elaborate the proof of Theorem 2 (turnpike theorem) formulated in Section 3.2, specifically, it shows that the optimal program of the T-period stationary economy converges to the optimal program of the nonstationary economy (1)–(3) as $T \to \infty$. The proof relies on three lemmas presented in Appendices A1-A3. In Appendix A1, we construct a limit program of a finite horizon economy with a terminal condition $k_T = 0$. In Appendix A2, we show that the optimal program of the T-period stationary economy, constructed in Section 3.1, converges to the same limit program as does the finite horizon economy with a zero terminal condition $k_T = 0$. In Appendix A3, we show that the limit program of the finite horizon economy with a zero terminal condition $k_T = 0$ is also an optimal program for the infinite horizon nonstationary economy (1)–(3). Finally, in Appendix A4, we combine the results of Appendices A1-A3 to establish the claim of Theorem 2. Our construction relies on mathematical tools developed in Majumdar and Zilcha (1987), Mitra and Nyarko (1991), Joshi (1997). We use the convention that equalities and inequalities hold almost everywhere (a.e.) except for a set of measure zero.

Appendix A0. Notation and definitions

Our exposition relies on standard measure theory notation; see, e.g., Stokey and Lucas with Prescott (1989), Santos (1999) and Stachurski (2009). Time is discrete and infinite, $t = 0, 1, ...$ Let $(\Omega, \mathcal{F}, P)$ be a probability space:

a) $\Omega = \Pi_{t=0}^{\infty} \Omega_t$ is a space of sequences $\varepsilon \equiv (\varepsilon_0, \varepsilon_1 ...)$ such that $\varepsilon_t \in \Omega_t$ for all t, where Ω_t is a compact metric space endowed with the Borel σ–field $\mathcal{E}_t$. Here, Ω_t is the set of all possible states of the environment at t and $\varepsilon_t \in \Omega_t$ is the state of the environment at t.

b) $\mathcal{F}$ is the σ–algebra on Ω generated by cylinder sets of the form $\Pi_{\tau=0}^{\infty} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all τ and $A_\tau = \Omega_\tau$ for all but finitely many τ.

c) P is the probability measure on $(\Omega, \mathcal{F})$.

We denote by $\{\mathcal{F}_t\}$ a filtration on Ω, where $\mathcal{F}_t$ is a sub σ–field of $\mathcal{F}$ induced by a partial history up of environment $h_t = (\varepsilon_0, ..., \varepsilon_t) \in \Pi_{\tau=0}^{t} \Omega_\tau$ up to period t, i.e., $\mathcal{F}_t$ is generated by cylinder sets of the form $\Pi_{\tau=0}^{t} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all $\tau \leq t$ and $A_\tau = \Omega_\tau$ for $\tau > t$. In particular, we have that $\mathcal{F}_0$ is the course σ–field $\{0, \Omega\}$, and that $\mathcal{F}_\infty = \mathcal{F}$. Furthermore, if Ω consists of either finite or countable states, ε is called a *discrete state process* or *chain*; otherwise, it is called a *continuous state process*. Our analysis focuses on continuous state processes, however, can be generalized to chains with minor modifications.

We provide some definitions that will be useful for characterizing random processes; these definitions are standard and closely follow Stokey and Lucas with Prescott (1989, Ch. 8.2).

Definition A1. *(Stochastic process).* A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ–algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$; a measurable space $(Z, \mathcal{Z})$; and a sequence of functions $z_t : \Omega \to Z$ for $t \geq 0$ such that each z_t is $\mathcal{F}_t$ measurable.

Stationarity is an assumption that is commonly used in economic literature.

Definition A2. *(Stationary process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary if the unconditional probability measure, given by

$$P_{t+1,...,t+n}(C) = P(\{\varepsilon \in \Omega : [z_{t+1}(\varepsilon), ..., z_{t+n}(\varepsilon)] \in C\}), \tag{22}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

A related notion is stationary (time-invariant) transition probabilities. Let us denote by $P_{t+1,...,t+n}(C | z_t = \overline{z}_t, ..., z_0 = \overline{z}_0)$ the probability of the event $\{\varepsilon \in \Omega : [z_{t+1}(\varepsilon), ..., z_{t+n}(\varepsilon)] \in C\}$, given that the event $\{\varepsilon \in \Omega : \overline{z}_t = z_t(\varepsilon), ..., \overline{z}_0 = z_0(\varepsilon)\}$ occurs.

Definition A3. *(Stationary transition probabilities).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is said to have stationary transition probabilities if the conditional probabilities

$$P_{t+1,...,t+n}(C | z_t = \overline{z}_t, ..., z_0 = \overline{z}_0) \tag{23}$$

are independent of t for all $C \in \mathcal{Z}^n$, $\varepsilon \in \Omega$, $t \geq 0$ and $n \geq 1$.

The assumption of stationary transition probabilities (23) implies the property of stationarity (22) provided that the corresponding unconditional probability measures exist. However, a process can be nonstationary even if transition probabilities are stationary, for example, a unit root process or explosive process is nonstationary; see Stokey and Lucas with Prescott (1989, Ch 8.2) for a related discussion. This kind of nonstationary processes is not studied explicitly in the present paper, i.e., we focus on nonstationarity that arises because transition probabilities change from one period to another.

In general, $P_{t+1,...,t+n}(C)$ and $P_{t+1,...,t+n}(C|\cdot)$ depend on the entire history of the events up to t (i.e., the stochastic process z_t is measurable with respect to the sub σ–field $\mathcal{F}_t$). However, history-dependent processes are difficult to analyze in a general case. It is of interest to distinguish special cases in which the dependence on history has relatively simple and tractable form. A well-known case is a class of Markov processes.

Definition A4. *(Markov process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is (first-order) Markov if

$$P_{t+1,...,t+n}\left(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0\right) = P_{t+1,...,t+n}\left(C|z_t = \overline{z}_t\right), \tag{24}$$

for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

The key property of a Markov process is that it is memoryless, namely, all past history $(z_t, ..., z_0)$ is irrelevant for determining the future realizations except of the most recent past z_t.

Appendix A1. Limit program of finite horizon economy with a zero terminal capital

In this section, we consider a finite horizon version of the economy (1)–(3) with a given terminal condition for capital k_T. Specifically, we assume that the agent solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} E_0\left[\sum_{t=0}^{T} \beta^t u_t\left(c_t\right)\right] \tag{25}$$

$$\text{s.t. (2), (3),} \tag{26}$$

where initial condition (k_0, z_0) and terminal condition k_T are given. We first define feasible programs for the finite horizon economy.

Definition A5 *(Feasible programs in the finite horizon economy).* A feasible program in the finite horizon economy is a pair of adapted (i.e., $\mathcal{F}_t$ measurable for all t) processes $\{c_t, k_t\}_{t=0}^{T}$ such that given initial condition k_0 and any partial history $h_T = (\varepsilon_0, ..., \varepsilon_T)$, they reach a given terminal condition k_T at T, satisfy $c_t \geq 0$, $k_t \geq 0$ and (2), (3) for all $t = 1, ...T$.

In this section, we focus on a finite horizon economy that reaches a zero terminal condition, $k_T = 0$, at T. We denote by $\Im^{T,0}$ a set of all finite horizon feasible programs from given initial capital k_0 and any partial history $h_T \equiv (\varepsilon_0, ..., \varepsilon_T)$ that attain given $k_T = 0$ at T. We next introduce the concept of solution for the finite horizon model.

Definition A6 *(Optimal program in the finite horizon model).* A feasible finite horizon program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ is called optimal if

$$E_0\left[\sum_{t=0}^{T} \beta^t \left\{u_t(c_t^{T,0}) - u_t\left(c_t\right)\right\}\right] \geq 0 \tag{A1}$$

for every feasible process $\{c_t, k_t\}_{t=0}^{T} \in \Im^{T,0}$.

The existence result for the finite horizon version of the economy (25), (26) with a zero terminal condition is established in the literature. Namely, under Assumptions A1-A3, there exists an optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ and it is both interior and unique. The existence of the optimal program can be shown by using either a Bellman equation approach (see Mitra and

Nyarko (1991), Theorem 3.1) or an Euler equation approach (see Majumdar and Zilcha (1987), Theorems 1 and 2).

We next show that under terminal condition $k_T^{T,0} = k_T = 0$, the optimal program in the finite horizon economy (25), (26) has a well-defined limit.

Lemma 1. *A finite horizon optimal program* $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ *with a zero terminal condition* $k_T^{T,0} = 0$ *converges to a limit program* $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ *when* $T \to \infty$*, i.e.,*

$$k_t^{\lim} \equiv \lim_{T\to\infty} k_t^{T,0} \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^{T,0}, \quad \text{for } t = 0, 1, ... \tag{A2}$$

Proof. The existence of the limit program follows by three arguments (for any history):

i) Extending time horizon from T to $T+1$ increases T-period capital of the finite horizon optimal program, i.e., $k_T^{T+1,0} > k_T^{T,0}$. To see this, note that the model with time horizon T has zero (terminal) capital $k_T^{T,0} = 0$ at T. When time horizon is extended from T to $T+1$, the model has zero (terminal) capital $k_{T+1}^{T+1,0} = 0$ at $T+1$ but it has strictly positive capital $k_T^{T+1,0} > 0$ at T; this follows by the Inada conditions–Assumption A1d.

ii) The optimal program for the finite horizon economy has the following property of monotonicity with respect to the terminal condition: if $\{c'_t, k'_t\}_{t=0}^{T}$ and $\{c''_t, k''_t\}_{t=0}^{T}$ are two optimal programs for the finite horizon economy with terminal conditions $\kappa' < \kappa''$, then the respective optimal capital choices have the same ranking in each period, i.e., $k'_t \leq k''_t$ for all $t = 1, ...T$. This monotonicity result follows by either Bellman equation programming techniques (see Mitra and Nyarko (1991, Theorem 3.2 and Corollary 3.3)) or Euler equation programming techniques (see Majumdar and Zilcha (1987, Theorem 3)) or lattice programming techniques (see Hopenhayn and Prescott (1992)); see also Joshi (1997, Theorem 1) for generalizations of these results to nonconvex economies. Hence, the stochastic process $\left\{k_t^{T,0}\right\}_{t=0}^{T}$ shifts up (weakly) in a pointwise manner when T increases to $T+1$, i.e., $k_t^{T+1,0} \geq k_t^{T,0}$ for $t \geq 0$.

iii) By construction, the capital program from the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T}$ is bounded from above by the capital accumulation process $\{0, k_t^{\max}\}_{t=0}^{T}$ defined in (5), i.e., $k_t^{T,0} \leq k_t^{\max} < \infty$ for $t \geq 0$. A sequence that is bounded and monotone is known to have a well-defined limit. ■

Appendix A2. Limit program of the T-period stationary economy

We now show that the optimal program of the T-period stationary economy, introduced in Section 3.1, converges to the same limit program (A2) as the optimal program of the finite horizon economy (25), (26) with a zero terminal condition. We denote by $\Im^T$ a set of all feasible finite horizon programs that attains a terminal condition of the T-period stationary economy. (We assume the same initial capital (k_0, z_0) and the same partial history $h_T \equiv (\varepsilon_0, ..., \varepsilon_T)$ as those fixed for the finite horizon economy (25), (26)).

Lemma 2. *The optimal program of the* T*-period stationary economy* $\left\{c_t^T, k_t^T\right\}_{t=0}^{T} \in \Im^T$ *converges to a unique limit program* $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ *defined in (A2) as* $T \to \infty$ *i.e., for all*

$t \geq 0$

$$k_t^{\lim} \equiv \lim_{T\to\infty} k_t^T \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^T. \tag{A3}$$

Proof. The proof of the lemma follows by six arguments (for any history).

i). Observe that, by Assumptions A1 and A2, the optimal program of the T-period stationary economy has a positive capital stock $k_T^T > 0$ at T (since the terminal capital is generated by the capital decision function of a stationary version of the model), while for the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^T \in \Im^{T,0}$ of the finite horizon economy, it is zero by definition, $k_T^{T,0} = 0$.

ii). The property of monotonicity with respect to terminal condition implies that if $k_T^T > k_T^{T,0}$, then $k_t^T \geq k_t^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof to Lemma 1.

iii). Let us fix some $\tau \in \{1, ..., T\}$. We show that up to period τ, the optimal program $\left\{c_t^T, k_t^T\right\}_{t=0}^\tau$ does not give higher expected utility than $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^\tau$, i.e.,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t(c_t^{T,0})\right\}\right] \leq 0. \tag{A4}$$

Toward contradiction, assume that it does, i.e.,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t(c_t^{T,0})\right\}\right] > 0. \tag{A5}$$

Then, consider a new process $\{c_t', k_t'\}_{t=0}^\tau$ that follows $\left\{c_t^T, k_t^T\right\}_{t=0}^T \in \Im^T$ up to period $\tau - 1$ and that drops down at τ to match $k_\tau^{T,0}$ of the finite horizon program $\left\{c_t^T, k_t^T\right\}_{t=0}^T \in \Im^{T,0}$, i.e., $\{c_t', k_t'\}_{t=0}^\tau \equiv \left\{c_t^T, k_t^T\right\}_{t=0}^{\tau-1} \cup \left\{c_\tau^T + k_\tau^T - k_\tau^{T,0}, k_\tau^{T,0}\right\}$. By monotonicity ii). we have $k_\tau^T - k_\tau^{T,0} \geq 0$, so that

$$\begin{aligned} E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t'\right) - u_t\left(c_t^T\right)\right\}\right] &= \\ &= E_0\left[\beta^\tau \left\{u_t\left(c_\tau^T + k_\tau^T - k_\tau^{T,0}\right) - u_t\left(c_\tau^T\right)\right\}\right] \geq 0, \end{aligned} \tag{A6}$$

where the last inequality follows by Assumption A1b of strictly increasing u_t.

iv). By construction $\{c_t', k_t'\}_{t=0}^\tau$ and $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^\tau$ reach the same capital $k_\tau^{T,0}$ at τ. Let us extend the program $\{c_t', k_t'\}_{t=0}^\tau$ to T by assuming that it follows the process $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^T$ from the period $\tau + 1$ up to T, i.e., $\{c_t', k_t'\}_{t=\tau+1}^T \equiv \left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=\tau+1}^T$. Then, we have

$$\begin{aligned} E_0\left[\sum_{t=0}^{T} \beta^t \left\{u_t\left(c_t'\right) - u_t(c_t^{T,0})\right\}\right] &= E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t'\right) - u_t(c_t^{T,0})\right\}\right] \\ &\geq E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^T\right) - u_t\left(c_t^{T,0}\right)\right\}\right] > 0, \end{aligned} \tag{A7}$$

where the last two inequalities follow by result (A6) and assumption (A5), respectively. Thus, we obtain a contradiction: The constructed program $\{c'_t, k'_t\}_{t=0}^{T} \in \Im^{T,0}$ is feasible in the finite horizon economy with a zero terminal condition, $k'_T = 0$, and it gives strictly higher expected utility than the optimal program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ in that economy.

v). Holding τ fixed, we compute the limit of (A4) by letting T go to infinity:

$$\lim_{T\to\infty} E_0 \left[\sum_{t=0}^{\tau} \beta^t \left\{ u_t\left(c_t^T\right) - u_t(c_t^{T,0}) \right\} \right] =$$

$$\lim_{T\to\infty} E_0 \left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^T\right) \right] - E_0 \left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^{\lim}\right) \right] \leq 0. \quad \text{(A8)}$$

vi). The last inequality implies that for any $\tau \geq 1$, the limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ of the finite horizon economy $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ with a zero terminal condition $k_T^{T,0} = 0$ gives at least as high expected utility as the optimal limit program $\left\{c_t^T, k_t^T\right\}_{t=0}^{T} \in \Im^T$ of the T-period stationary economy. Since Assumptions A1 and A2 imply that the optimal program is unique, we conclude that $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ defined in (A2) is a unique limit of the optimal program $\left\{c_t^T, k_t^T\right\}_{t=0}^{T} \in \Im^T$ of the T-period stationary economy. ■

Appendix A3. Convergence of finite horizon economy to infinite horizon economy

We now show a connection between the optimal programs of the finite horizon and infinite horizon economies. Namely, we show that the finite horizon economy (25), (26) with a zero terminal condition $k_T^{T,0} = 0$ converges to the nonstationary infinite horizon economy (1)–(3) as $T \to \infty$ provided that we fix the same initial condition k_0 and partial history $h_T = (\varepsilon_0, ..., \varepsilon_T)$ for both economies.

Lemma 3. *The limit program* $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ *is a unique optimal program* $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ *in the infinite horizon nonstationary economy (1)–(3).*

Proof. We prove this lemma by contradiction. We use the arguments that are similar to those used in the proof of Lemma 2.

i). Toward contradiction, assume that $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ is not an optimal program of the infinite horizon economy $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$. By definition of limit, there exists a real number $\epsilon > 0$ and a subsequence of natural numbers $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$ such that $\{c_t^{\infty}, k_t^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ gives strictly higher expected utility than the limit program of the finite horizon economy $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$, i.e.,

$$E_0 \left[\sum_{t=0}^{T_n} \beta^t \left\{ u_t\left(c_t^{\infty}\right) - u_t(c_t^{\lim}) \right\} \right] > \epsilon \text{ for all } T_n \in \{T_1, T_2, ...\}. \quad \text{(A9)}$$

ii). Let us fix some $T_n \in \{T_1, T_2, ...\}$ and consider any finite $T \geq T_n$. Assumptions A1 and A2 imply that $k_T^{\infty} > 0$ while $k_T^{T,0} = 0$ by definition of the finite horizon economy with a zero terminal condition. The monotonicity of the optimal program with respect to a terminal condition implies that if $k_T^\infty > k_T^{T,0}$, then $k_t^\infty \geq k_t^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof of Lemma 1.

iii). Following the arguments in iii). of the proof of Lemma 2, we can show that up to period T_n, the optimal program $\{c_t^\infty, k_t^\infty\}_{t=0}^{T_n}$ does not give higher expected utility than $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T_n}$, i.e.,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^\infty\right) - u_t(c_t^{T,0})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A10}$$

iv). Holding T_n fixed, we compute the limit of (A10) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^\infty\right) - u_t(c_t^{T,0})\right\}\right]$$
$$= E_0\left[\sum_{t=0}^{T_n} \beta^t u_t\left(c_t^\infty\right) - \beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0 \text{ for all } T_n. \tag{A11}$$

However, result (A11) contradicts to our assumption in (A9).

v). We conclude that for any subsequence $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$, we have

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^\infty\right) - u_t(c_t^{\lim})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A12}$$

However, under Assumptions A1 and A2, the optimal program $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty \in \Im^\infty$ is unique, and hence, it must be that $\{c_t^\infty, k_t^\infty\}_{t=0}^\infty$ coincides with $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^\infty$ for all $t \geq 0$. ■

Appendix A4. Proof to the turnpike theorem

We now combine the results of Lemmas 1-3 together into a turnpike-style theorem to show the convergence of the optimal program of the T-period stationary economy to that of the infinite horizon nonstationary economy. To be specific, Lemma 1 shows that the optimal program of the finite horizon economy with a zero terminal condition $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^T \in \Im^{T,0}$ converges to the limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^\infty$. Lemma 2 shows that the optimal program of the T-period stationary economy $\left\{c_t^T, k_t^T\right\}_{t=0}^T$ also converges to the same limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^\infty$. Finally, Lemma 3 shows that the limit program of the finite horizon economies $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^\infty$ is optimal in the nonstationary infinite horizon economy. Then, it must be the case that the limit optimal program of the T-period stationary economy $\left\{c_t^T, k_t^T\right\}_{t=0}^T$ is optimal in the infinite horizon nonstationary economy. This argument is formalized below.

Proof to Theorem 2 (turnpike theorem). The proof follows by definition of limit and Lemmas 1-3. Let us fix a real number $\varepsilon > 0$ and a natural number τ such that $1 \leq \tau < \infty$ and consider a possible partial history $h_T = (\varepsilon_0, ..., \varepsilon_T)$.

i). Lemma 1 shows that $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to a limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ as $T \to \infty$. Then, definition of limit implies that there exists $T_1(h_T) > 0$ such that $\left|k_t^{T,0} - k_t^{\lim}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

ii). Lemma 2 implies that the finite horizon problem of the T-period stationary economy $\left\{c_t^T, k_t^T\right\}_{t=0}^{T}$ also converges to limit program $\left\{c_t^{\lim}, k_t^{\lim}\right\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_2(h_T) > 0$ such that $\left|k_t^{\lim} - k_t^T\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iii). Lemma 3 implies the program $\left\{c_t^{T,0}, k_t^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to the infinite horizon optimal program $\left\{c_t^{\infty}, k_t^{\infty}\right\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_3(h_T) > 0$ such that $\left|k_t^{T,0} - k_t^{\infty}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iv). Then, the triangular inequality implies

$$\begin{aligned}\left|k_t^T - k_t^{\infty}\right| &= \left|k_t^T - k_t^{\lim} + k_t^{\lim} - k_t^{T,0} + k_t^{T,0} - k_t^{\infty}\right| \\ &\leq \left|k_t^T - k_t^{\lim}\right| + \left|k_t^{\lim} - k_t^{T,0}\right| + \left|k_t^{T,0} - k_t^{\infty}\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for $T(h_T) \geq \max\left\{T_1(h_T), T_2(h_T), T_3(h_T)\right\}$.

v). Finally, consider all possible partial histories $\{h_T\}$ and define $T(\varepsilon, \tau) \equiv \max_{\{h_T\}} T(h_T)$. By construction, for any $T > T(\varepsilon, \tau)$, the inequality (11) holds. ■

Remark A1. Our proof of the turnpike theorem addresses a technical issue that does not arise in the literature that focuses on finite horizon economies with a zero terminal condition; see, e.g., Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997). Their construction relies on the fact that the optimal program of the finite horizon economy is always pointwise below the optimal program of the infinite horizon economy, i.e., $k_t^T \leq k_t^{\infty}$, for $t = 1, ..., \tau$, and it gives strictly higher expected utility up to T than does the infinite horizon optimal program (because excess capital can be consumed at terminal period T). This argument does not directly applies to our T-period stationary economy: our finite horizon program can be either below or above the infinite horizon program depending on a specific T-period terminal condition; see the experiments with terminal conditions κ' and κ'' in Figure 1, respectively. Our proof addresses this issue by constructing in Lemma 2 a separate limit program for the T-period stationary economy.

Remark A2. We also proved a similar turnpike theorem for a more general version of the economy (1)–(3). First, we relax the assumption of Markov structure of the stochastic process (3) (i.e., we consider a general stochastic environment that satisfies only a weak assumptions of measurability); and second, we relax the assumption that the terminal condition comes from the T-period stationary economy (i.e., we consider an arbitrary terminal condition k_T). To save on space, we do not include this more general turnpike theorem in the paper but limit ourselves to the nonstationary Markov setup that is actually studied in our numerical experiments.

Appendix B. Implementation of EFP

In this section, we describe the implementation of the EFP method used to produce the numerical results in the main text.

Algorithm 1 (implementation). Extended function path.

The goal of EFP.

EFP is aimed at approximating a solution of a nonstationary model during the first τ periods, i.e., it finds approximating functions $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ such that $\widehat{K}_t \approx K_t$ for $t = 1, ...\tau$, where K_t and $\widehat{K}_t$ are a t-period true capital function and its parametric approximation, respectively.

Step 0. Initialization.

a. Choose time horizon $T \gg \tau$ for constructing T-period stationary economy.

b. Construct a deterministic path $\{z_t^*\}_{t=0}^T$ for exogenous state variable $\{z_t\}_{t=0}^T$ satisfying
$z_{t+1}^* = \varphi_t\left(z_t^*, E_t\left[\varepsilon_{t+1}\right]\right)$ for $t = 0, .., T$.

c. Construct a deterministic path $\{k_t^*\}_{t=0}^T$ for endogenous state variable $\{k_t\}_{t=0}^T$ satisfying
$u_t'(c_t^*) = \beta u_t'(c_{t+1}^*)(1 - \delta + f_{t+1}'\left(k_{t+1}^*, z_{t+1}^*\right))$.
$c_t^* + k_{t+1}^* = (1-\delta)\, k_t^* + f_t\left(k_t^*, z_t^*\right)$ for $t = 0, .., T$.

d. For $t = 0, ..., T$:
Construct a grid $\{(k_{m,t}, z_{m,t})\}_{m=1}^M$ centered at (k_t^*, z_t^*).
Choose integration nodes, $\varepsilon_{j,t}$, and weights, $\omega_{j,t}$ for $j = 1, ..., J$.
Construct future shocks $z'_{m,j,t} = \varphi_t\left(z_{m,t}, \varepsilon_{j,t}\right)$.

e. Write a t-period discretized system of the optimality conditions:

i). $u_t'(c_{m,t}) = \beta \sum\limits_{j=1}^{J} \omega_{j,t}\left[u_t'(c'_{m,j,t})\left\{1 - \delta + f_{t+1}\left(k'_{m,t}, z'_{m,j,t}\right)\right\}\right]$

ii). $c_{m,t} + k'_{m,t} = (1-\delta)\, k_{m,t} + f_t\left(k_{m,t}, z_{m,t}\right)$

iii). $c'_{m,j,t} + k''_{m,j,t} = (1-\delta)\, k'_{m,t} + f_{t+1}\left(k'_{m,t}, z'_{m,j,t}\right)$

iv). $k'_{m,t} = \widehat{K}_t\left(k_{m,t}, z_{m,t}\right)$ and $k''_{m,j,t} = \widehat{K}_{t+1}\left(k'_{m,t}, z'_{m,j,t}\right)$.

d. Assume that the model becomes stationary at T.

Step 1. Solving the T-period stationary model.

Find $\widehat{K}_T = \widehat{K}_{T+1}$ that approximately solves the system i).-iv). on the grid for the T-period stationary economy $f_{T+1} = f_T$, $u_{T+1} = u_T$, $\varphi_{T+1} = \varphi_T$.

Step 2. Solving for a function path for $t = 0, 1, ..., T-1$.

a. Construct the function path $\left(\widehat{K}_0, ..., \widehat{K}_{T-1}, \widehat{K}_T\right)$ that approximately solves the system i).-iv) for each $t = 0, ..., T$ and that matches the given terminal function $\widehat{K}_T$ constructed in Step 1.

The EFP solution:

Use $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ as an approximation to $(K_0, ..., K_\tau)$ and discard the remaining $T - \tau$ functions.

The EFP method is more expensive than conventional solution methods for stationary models because decision functions must be constructed not just once but for T periods. We implement EFP in the way that keeps its cost relatively low: First, to approximate decision functions, we use a version of the Smolyak (sparse) grid technique. Specifically, we use a version of the Smolyak method that combines a Smolyak grid with ordinary polynomials for approximating functions off the grid. This method is described in Maliar, Maliar and Judd (2011) who find it to be sufficiently accurate in the context of a similar growth model, namely, unit-free residuals in the model's equations do not exceed 0.01% on a stochastic simulation of 10,000 observations). For this version of the Smolyak method, the polynomial coefficients are overdetermined, for example, in a 2-dimensional case, we have 13 points in a second-level Smolyak grid, and we have only six coefficients in second-degree ordinary polynomial. Hence, we identify the coefficients using a least-squares regression; we use an SVD decomposition, to enhance numerical stability; see Judd, Maliar and Maliar (2011) for a discussion of this and other numerically stable approximation methods. We do not construct the Smolyak grid within a hypercube normalized to $[-1,1]^2$, like do Smolyak methods that rely on Chebyshev polynomials used in, e.g., Krueger and Kubler (2004) and Judd, Maliar, Maliar and Valero (2014). Instead, we construct a sequence of Smolyak grids around actual steady state and thus, the hypercube, in which the Smolyak grid is constructed, grows over time as shown in Figure 1.

Second, to approximate expectation functions, we use Gauss-Hermite quadrature rule with 10 integration nodes. However, a comparison analysis in Judd, Maliar and Maliar (2011) shows that for models with smooth decision functions like ours, the number of integration nodes plays only a minor role in the properties of the solution, for example, the results will be the same up to six digits of precision if instead of ten integration nodes we use just two nodes or a simple linear monomial rule (a two-node Gauss-Hermite quadrature rule is equivalent to a linear monomial integration rule for the two-dimensional case). However, simulation-based Monte-Carlo-style integration methods produce very inaccurate approximations for integrals and are not considered in this paper; see Judd, Maliar and Maliar (2011) for discussion.

Third, to solve for the coefficients of decision functions, we use a simple derivative-free fixed-point iteration method in line with Gauss-Jacobi iteration. Let us re-write the Euler equation i). constructed in the initialization step of the algorithm by pre-multiplying both sides by t-period capital

$$\widehat{k}'_{m,t} = \beta \sum_{j=1}^{J} \varepsilon_{j,t} \left[\frac{u'_t(c'_{m,j,t})}{u'_t(c_{m,t})} \left\{ 1 - \delta + f_{t+1}\left(k'_{m,t}k^*_{t+1}, z'_{m,j,t}z^*_{t+1}\right) \right\} \right] k'_{m,t}. \tag{27}$$

We use different notation, $k'_{m,t}$ and $\widehat{k}'_{m,t}$, for t-period capital in the left and right side of (27), respectively, in order to describe our fixed-point iteration method. Namely, we substitute $k'_{m,t}$ in the right side of (27) and in the constraints ii). and iii). in the initialization step to compute $c_{m,t}$ and $c'_{m,j,t}$, respectively, and we obtain a new set of values of the capital function on the grid $\widehat{k}'_{m,t}$ in the left side. We iterate on these steps until convergence.

Our approximation functions $\widehat{K}_t$ are ordinary polynomial functions characterized by a time-dependent vector of parameters b_t, i.e., $\widehat{K}_t = \widehat{K}(\cdot; b_t)$. So, operationally, the iteration is performed not on the grid values $k'_{m,t}$ and $\widehat{k}'_{m,t}$ but on the coefficients of the approximation functions. The iteration procedure differs in Steps 1 and 2.

In Step 1, we construct a solution to T-period stationary economy. For iteration i, we fix some initial vector of coefficients b, compute $k'_{m,T+1} = \widehat{K}(k_{m,T}, z_{m,T}; b)$, find $c_{m,T}$ and $c'_{m,j,T}$

to satisfy constraints ii) and iii), respectively and find $\widehat{k}'_{m,T+1}$ from the Euler equation i). We run a regression of $\widehat{k}'_{m,T+1}$ on $\widehat{K}\left(k_{m,T}, z_{m,T}; \cdot\right)$ in order to re-estimate the coefficients $\widehat{b}$ and we compute the coefficients for iteration $i+1$ as a weighted average, i.e., $b^{(i+1)} = (1-\xi)\, b^{(i)} + \xi \widehat{b}^{(i)}$, where $\xi \in (0,1)$ is a damping parameter (typically, $\xi = 0.05$). We use partial updating instead of full updating $\xi = 1$ because fixed-point iteration can be numerically unstable and using partial updating enhances numerical stability; see Maliar, Maliar and Judd (2011). This kind of fixed-point iterations are used by numerical methods that solve for equilibrium in conventional stationary Markov economies; see e.g., Judd, Maliar and Maliar (2011), Judd, Maliar, Maliar and Valero (2014).

In Step 2, we iterate on the path for the polynomial coefficients using Gauss-Jacobi style iterations in line with Fair and Taylor (1983). Specifically, on iteration j, we take a path for the coefficients vectors $\left\{b_1^{(j)}, ..., b_T^{(j)}\right\}$, compute the corresponding path for capital quantities using $k'_{m,t} = \widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t^{(j)}\right)$, and find a path for consumption quantities $c_{m,t}$ and $c'_{m,j,t}$ from constraints ii) and iii), respectively, for $t = 0, ..., T$. Substitute these quantities in the right side of a sequence of Euler equations for $t = 0, ..., T$ to obtain a new path for capital quantities in the left side of the Euler equation $\widehat{k}'_{m,t}$ for $t = 0, ..., T-1$. Run $T-1$ regressions of $\widehat{k}'_{m,t}$ on polynomial functional forms $\widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t\right)$ for $t = 0, ..., T-1$ to construct a new path for the coefficients $\left\{\widehat{b}_0^{(j)}, ..., \widehat{b}_{T-1}^{(j)}\right\}$. Compute the path of the coefficients for iteration $j+1$ as a weighted average, i.e., $b_t^{(j+1)} = (1-\xi)\, b_t^{(j)} + \xi \widehat{b}_t^{(j)}$, $t = 0, ..., T-1$, where $\xi \in (0,1)$ is a damping parameter which we again typically set at $\xi = 0.05$. (Observe that this iteration procedure changes all the coefficients on the path except of the last one $b_T^{(j)} \equiv b$, which is a given terminal conditions that we computed in Step 1 from T-period stationary economy).

In fact, the problem of constructing a path for function coefficients is similar to the problem of constructing a path for variables: in both cases, we need to solve a large system of nonlinear equations. The difference is that under EFP, the arguments of this system are not variables but parameters of the approximating functions. Instead of Gauss-Jacobi style iteration on path, we can use Gauss-Siedel fixed-point iteration (shooting), Newton-style solvers or any other technique that can solve a system of nonlinear equations; see Lipton, Poterba, Sachs and Summers (1980), Atolia and Buffie (2009a,b), Heer and Maußner (2010), and Grüne, Semmler and Stieler (2013) for examples of such techniques.

Appendix C. Path-solving methods for nonstationary models

We first describe the shooting method of Lipton, Poterba, Sachs and Summers (1980) for a nonstationary deterministic economy, and we then elaborate the extended path (EP) of Fair and Taylor (1983) for a nonstationary economy with uncertainty.

Shooting methods To illustrate the class of shooting methods, let us substitute c_t and c_{t+1} from (2) into the Euler equation of (1)–(3) to obtain a second-order difference equation,

$$\begin{aligned} &u_t'((1-\delta)\,k_t + f_t\,(k_t, z_t) - k_{t+1}) \\ &\quad = \beta E_t \left[u_{t+1}'((1-\delta)\,k_{t+1} + f_{t+1}\,(k_{t+1}, z_{t+1}) - k_{t+2})(1-\delta + f_{t+1}'\,(k_{t+1}, z_{t+1})) \right]. \end{aligned} \quad (28)$$

Initial condition (k_0, z_0) is given. Let us abstract from uncertainty by assuming that $z_t = 1$ for all t, choose a sufficiently large T and fix some terminal condition k_{T+1} (typically, the literature assumes that the economy arrives in the steady state $k_{T+1} = k^*$).[10] To approximate the optimal path, we must solve numerically a system of T nonlinear equations (28) with respect to T unknowns $\{k_1, ..., k_T\}$. It is possible to solve the system (28) by using a Newton-style or any other numerical solver. However, a more efficient alternative could be numerical methods that exploit the recursive structure of the system (28) such as shooting methods (Gauss-Siedel iteration). There are two types of shooting methods: a forward shooting and a backward shooting. A typical forward shooting method expresses k_{t+2} in terms of k_t and k_{t+1} using (28) and constructs a forward path $(k_1, ..., k_{T+1})$; it iterates on k_1 until the path reaches a given terminal condition $k_{T+1} = k^*$. In turn, a typical reverse shooting method expresses k_t in terms of k_{t+1} and k_{t+2} and constructs a backward path $\{k_T, ..., k_0\}$; it iterates on k_T until the path reaches a given initial condition k_0. A shortcoming of shooting methods is that they tend to produce explosive paths, in particular, forward shooting methods; see Atolia and Buffie (2009 a, b) for a careful discussion and possible treatments of this problem.

Fair and Taylor (1984) method The EP method of Fair and Taylor (1983) allows us to solve nonstationary economic models with uncertainty by approximating expectation functions under the assumption of certainty equivalence. To see how this method works, consider the system (28) with uncertainty and as an example, assume that z_{t+1} follows a possibly nonstationary Markov process $\ln(z_{t+1}) = \rho_t \ln(z_t) + \sigma_t \varepsilon_{t+1}$, where the sequences $(\rho_0, \rho_1, ...)$ and $(\sigma_0, \sigma_1, ...)$ are deterministically given at $t = 0$ and $\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$. Again, let us choose a sufficiently large T and fix some terminal condition such as $k_{T+1} = k^*$, so that the turnpike argument applies. Fair and Taylor (1983) propose to construct a solution path to (28) by setting all future innovations to their expected values, $\varepsilon_1 = \varepsilon_2 = ... = 0$. This produces a path on which technology evolves as $\ln(z_{t+1}) = \rho_t \ln(z_t)$ gradually converging to $z^* = 1$ and the models's variables gradually converge to the steady state. Note that only the first entry k_1 of the constructed path $(k_1, ..., k_T)$ is meaningful; the remaining entries $(k_2, ..., k_T)$ are obtained under a supplementary assumption of zero future innovations and they are only needed to accurately construct k_1. Thus, k_1 is stored and the rest of the sequence is discarded. By applying the same procedure to next state (k_1, z_1), we produce k_2, and so on until the path of desired length τ is constructed.

However, certainty equivalence approximation of Fair and Taylor (1983) has its limitations. It is exact for linear and linearized models, and it can be sufficiently accurate for models that are close to linear; see Cagnon and Taylor (1990), and Love (2010). However, it becomes highly inaccurate when either volatility and/or the degrees of nonlinearity increase; see our accuracy evaluations in Section 5.

[10] The turnpike theorem implies that in initial τ periods, the optimal path is insensitive to a specific terminal condition used if $\tau \ll T$.

Another novelty of the EP method relative to shooting methods is that it iterates on the economy's path at once using Gauss-Jacobi iteration. This type of iteration is more stable than Gauss-Siedel and allows us to avoid explosive behavior. To be specific, it guesses the economy's path $(k_1, ..., k_{T+1})$, substitute the quantities for $t = 1, ...T + 1$ it in the right side of T Euler equations (28), respectively, and obtains a new path $(k_0, ..., k_T)$ in the left side of (28); and it iterates on the path until the convergence is achieved. Finally, Fair and Taylor (1983) propose a simple procedure for determining T that insures that a specific terminal condition used does not affect the quality of approximation, namely, they suggested to increase T (i.e., extend the path) until the solution in the initial period(s) becomes insensitive to further increases in T.

We now elaborate the description of the version of Fair and Taylor's (1983) method used to produce the results in the main text. We use a slightly different representation of the optimality conditions of the model (1)–(3) (we assume $\delta = 1$ and $u(c) = \ln(c)$ for expository convenience). The Euler equation and budget constraint, respectively, are:

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} (1 - \delta + z_{t+1} f'(k_{t+1})) \right],$$
$$c_t + k_{t+1} = (1 - \delta) k_t + z_t f(k_t).$$

We combine the above two conditions to get

$$k_{t+1} = z_t f(k_t) - \left[E_t \left(\frac{\beta z_{t+1} f'(k_{t+1}))}{z_{t+1} f(k_{t+1}) - k_{t+2}} \right) \right]^{-1} \approx$$
$$z_t f(k_t) - \frac{z^e_{t+1} f(k_{t+1}) - k_{t+2}}{\beta z^e_{t+1} f'(k_{t+1}))}, \tag{29}$$

where the path for $z^e_{t+1}$is constructed under the certainty equivalence assumption that $\varepsilon_{t+1} = 0$ for all $t \geq 0$. Under the conventional AR(1) process for productivity levels (4), this means that $\ln z^e_{t+1} = \rho \ln z^e_t$ for all $t \geq 0$, or equivalently $z^e_{t+1} = (z^e_t)^\rho$, where $z^e_0 = z_0$. To solve for the path of variables, we use derivative-free iteration in line with Gauss-Jacobi method as in Fair and Taylor (1983):

Algorithm 2. Extended path (EP) framework by Fair and Taylor (1983).

The goal of EP framework of Fair and Taylor (1983).

EFP is aimed at approximating a path for variables satisfying the model's equations during the first τ periods, i.e., it finds $\widehat{k}_0, ..., \widehat{k}_\tau$ such that $\left\| k_t - \widehat{k}_t \right\| < \varepsilon$ for $t = 1, ...\tau$, where $\varepsilon > 0$ is target accuracy, $\|\cdot\|$ is an absolute value, and k_t and $\widehat{k}_t$ are the t-period true capital stocks and their approximation, respectively.

Step 0. Initialization.

a. Fix $t = 0$ period state (k_0, z_0).

b. Choose time horizon $T \gg \tau$ and terminal condition $\widehat{k}_{T+1}$.

c. Construct and fix $\left\{z_{t+1}^e\right\}_{t=0,...,T}$ such that $z_{t+1}^e = (z_t^e)^\rho$ for all t, where $z_0^e = z_0$.

d. Guess an equilibrium path $\left\{\widehat{k}_t^{(1)}\right\}_{t=1,...,T'}$ for iteration $j = 1$.

e. Write a t-period system of the optimality conditions in the form:

$$\widehat{k}_{t+1} = z_t^e f\left(\widehat{k}_t\right) - \frac{z_{t+1}^e f\left(\widehat{k}_{t+1}\right) - \widehat{k}_{t+2}}{\beta z_{t+1}^e f'(\widehat{k}_{t+1}))},$$

where $\widehat{k}_0 = k_0$.

Step 1. Solving for a path using Gauss-Jacobi method.

a. Substitute a path $\left\{\widehat{k}_t^{(j)}\right\}_{t=1,...,T'}$ into the right side of (29) to find

$$\widehat{k}_{t+1}^{(j+1)} = z_t^e f\left(\widehat{k}_t^{(j)}\right) - \frac{z_{t+1}^e f\left(\widehat{k}_{t+1}^{(j)}\right) - \widehat{k}_{t+2}^{(j)}}{\beta z_{t+1}^e f'(\widehat{k}_{t+1}^{(j)}))}, \qquad t = 1, ..., T$$

b. End iteration if the convergence is achieved $\left|\widehat{k}_{t+1}^{(j+1)} - \widehat{k}_{t+1}^{(j)}\right| <$ tolerance level.
Otherwise, increase j by 1 and repeat Step 1.

The EP solution:

Use the first entry $\widehat{k}_1$ of the constructed path $\widehat{k}_1, ..., \widehat{k}_T$ as an approximation to the true solution k_1 n period $t = 0$ and discard the remaining $k_2, ..., k_T$ values.

In terms of our notations, Fair and Taylor (1983) use $\tau = 1$, i.e., they keep only the first element $\widehat{k}_1$ from the constructed path $\left(\widehat{k}_1, ..., \widehat{k}_T\right)$ and disregard the rest of the path; then, they draw a next period shock z_1 and solve for a new path $\left(\widehat{k}_1, ..., \widehat{k}_{T+1}\right)$ starting from $\widehat{k}_1$ and ending in a given $\widehat{k}_{T+1}$ and store $\widehat{k}_2$, again disregarding the rest of the path; and they advance forward until the path of the given length τ is constructed. T is chosen so that further its extensions do not affect the solution in the initial period of the path. For instance, to find a solution $\widehat{k}_1$, Fair and Taylor (1983) solve the model several times under $T + 1$, $T + 2$, $T + 3, ...$ and check that $\widehat{k}_1$ remains the same (up to a given degree of precision).

As is typical for fixed-point-iteration style methods, Gauss-Jacobi iteration may fail to converge. To deal with this issue, Fair and Taylor (1983) use damping, namely, they update the path over iteration only by a small amount $k_{t+1}^{(j+1)} = \xi k_{t+1}^{(j+1)} + (1 - \xi)\, k_{t+1}^{(j)}$ where $\xi \in (0, 1)$ is a small number close to zero (e.g., 0.01).

Steps 1a and *1b* of Fair and Taylor's (1983) method are called Type I and Type II iterations and are analogous to *Step 2* of the EFP method when the sequence of the decision functions is constructed. The extension of path is called Type III iteration and gives the name to Fair and Taylor (1983) method.

In our examples, we implement Fair and Taylor's (1983) method using a conventional Newton style numerical solver instead of Gauss-Jacobi iteration; a similar implementation is used in Heer and Maußner (2010). The cost of Fair and Taylor's (1983) method can depend considerably on a specific solver used and can be very high (as we need to solve a system of equations with hundreds of unknowns numerically). In our simple examples, a Newton-style solver was sufficiently fast and reliable. In more complicated models, we are typically unable to derive closed-form laws of motion for the state variables, and derivative-free fixed-point iteration advocated in Fair and Taylor (1983) can be a better alternative.

Appendix D. Solving the test model using the associated stationary model

We first convert the nonstationary model (1)–(3), (12) with labor augmenting technological progress into a stationary model using the standard change of variables $\widehat{c}_t = c_t/A_t$ and $\widehat{k}_t = k_t/A_t$. This leads us to the following model

$$\max_{\left\{\widehat{k}_{t+1},\widehat{c}_t\right\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} (\beta^*)^t \frac{\widehat{c}_t^{\,1-\gamma}}{1-\gamma} \tag{30}$$

$$\text{s.t. } \widehat{c}_t + g_A \widehat{k}_{t+1} = (1-\delta)\,\widehat{k}_t + z_t \widehat{k}_t^{\alpha}, \tag{31}$$

$$\ln z_{t+1} = \rho_t \ln z_t + \sigma_t \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \tag{32}$$

where $\beta^* \equiv \beta g_A^{1-\gamma}$. We solve this stationary model by using the same version of the Smolyak method that is used within EFP to find a solution to T-period stationary economy.

After a solution to the stationary model (30)–(32) is constructed, a solution for nonstationary variables can be recovered by using an inverse transformation $c_t = \widehat{c}_t A_t$ and $k_t = \widehat{k}_t A_t$.

For the sake of our comparison, we also need to recover the path of nonstationary decision functions in terms of their parameters. Let us show how this can be done under polynomial approximation of decision functions. Let us assume that a capital policy function of the stationary model is approximated by complete polynomial of degree L, namely, $\hat{k}_{t+1} = \sum_{l=0}^{L} \sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1} \hat{k}_t^m z_t^{l-m}$, where b_i is a polynomial coefficient, $i = 0, ..., L+\frac{(L-1)(L+2)}{2}+1$. Given that the stationary and nonstationary solutions are related by $\hat{k}_{t+1} = k_{t+1}/\left(A_0 g_A^{t+1}\right)$, we have

$$k_{t+1} = A_0 g_A^{t+1} \hat{k}_{t+1} = A_0 g_A^{t+1} \sum_{l=0}^{L} \sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1} \hat{k}_t^m z_t^{l-m} =$$

$$A_0 \sum_{l=0}^{L} \sum_{m=0}^{l} {g_A}^{1-(m-1)t} b_{m+\frac{(l-1)(l+2)}{2}+1} k_t^m z_t^{l-m}. \tag{33}$$

For example, for first-degree polynomial $L = 1$, we construct the coefficients vector of the nonstationary model by premultiplying the coefficient vector $b \equiv (b_0, b_1, b_2)$ of the stationary model by a vector $\left(A_0 g_A^{t+1}, A_0 g_A, A_0 g_A^{t+1}\right)^\top$, which yields $b_{t+1} \equiv \left(b_0 A_0 g_A^{t+1}, b_1 A_0 g_A, b_2 A_0 g_A^{t+1}\right)$, $t = 0, ..., T$, where T is time horizon (length of simulation in the solution procedure). Note that a similar relation will hold even if the growth rate g_A is time variable.

Appendix E. Sensitivity results for the model with labor augmenting technological progress

In this appendix, we provide Table 2 which contains the results on accuracy and cost of the version of the EFP method studied in Section 5. We use $\tau = 200$ and $T = 400$ and consider several alternative parameterizations for $\{\gamma, \sigma_\varepsilon, g_A\}$.

Table 2: Sensitivity analysis for the EFP method.

Parameters	Model 1	Model 2	Model 3	Model 4	Model 5	Models 6	Model 7
γ	5	5	5	5	0.1	1	10
σ_ϵ	0.03	0.03	0.03	0.01	0.01	0.01	0.01
g_A	1.01	1.00	1.05	1.01	1.01	1.01	1.01
Mean errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-7.01	-6.67	-7.34	-7.03	-7.03	-6.61	-7.30
$t \in [0, 100]$	-6.82	-6.44	-7.25	-6.84	6.92	-6.48	-7.08
$t \in [0, 150]$	-6.73	-6.33	-7.22	-6.76	-6.89	-6.43	-6.98
$t \in [0, 175]$	-6.70	-6.29	-7.22	-6.74	-6.87	-6.41	-6.95
$t \in [0, 200]$	-6.68	-6.26	-7.21	-6.72	-6.87	-6.37	-6.93
Maximum errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-6.42	-6.31	-7.13	-6.66	-6.08	-6.24	-6.81
$t \in [0, 100]$	-5.99	-6.12	-7.05	-6.54	-5.97	-6.18	-6.36
$t \in [0, 150]$	-5.98	-6.04	-7.05	-6.52	-5.97	-6.18	-6.35
$t \in [0, 175]$	-5.98	-6.01	-7.05	-6.52	-5.97	-6.13	-6.33
$t \in [0, 200]$	-5.92	-5.99	-7.05	-6.51	-5.96	-5.88	-6.24
Running time, in seconds							
Solution	225.9	150.0	193.0	216.98	836.5	300.7	245.9
Simulation	5.6	5.7	5.8	5.66	5.6	5.6	5.7
Total	231.6	155.7	198.8	222.64	842.1	306.3	251.6

Notes: "Mean errors" and "Maximum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by EFP under the parameterization in the column. The difference between the solutions is computed across 100 simulations. The time horizon is T=400, and the terminal condition is constructed by using the T-period stationary economy in all experiments.

Appendix F. Additional figures

In Figure 10, we plot the simulated solution to the model with both deterministic technology switches and stochastic productivity shocks following an AR(1) process (32); this corresponds to a version of Application 3 with a productivity drift.

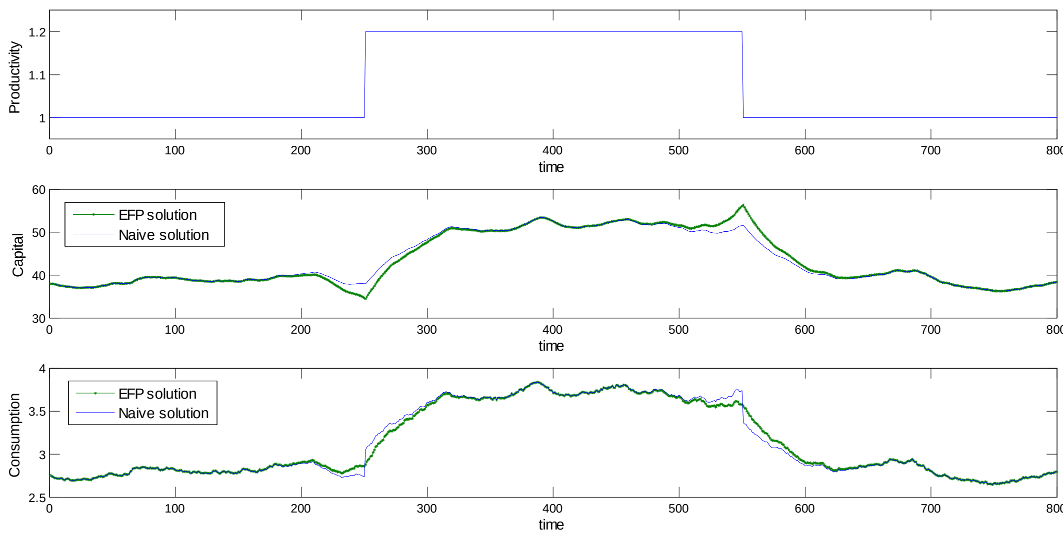

Figure 10. Deterministic technology switches and stochastic productivity shocks

In Figure 11, we provide a plot of simulated solution with both productivity drift and stochastic productivity shocks.

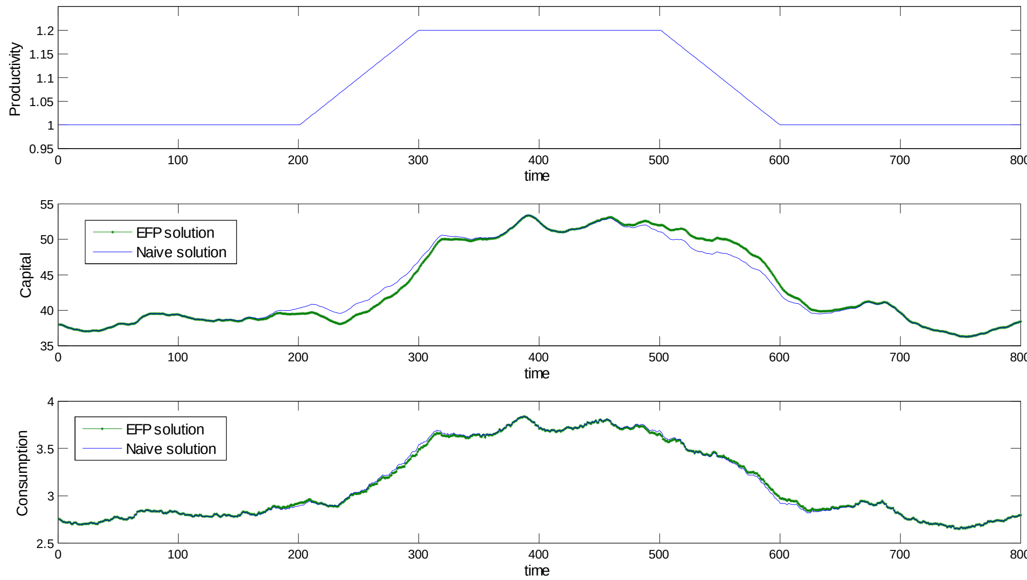

Figure 11. Productivity shifts and stochastic productivity shocks

Appendix G. Implementation details of the calibration-estimation procedure

In this appendix, we describe implementation details of the calibration-simulation procedure of Section 7.4.

Time series to match

Our macroeconomic data on the U.S. economy come from the webpages of the Bureau of Economic Analysis and the Federal Reserve Bank of St. Louis (namely, the data on capital and investment come from the former data base, while the data on the remaining time series, as well as that on the implicit price deflator, come from the latter data base); the sample spans over the period 1964:Q1 - 2011:Q4. Investment is defined as nonresidential and residential private fixed investment. Consumption is defined as a sum of nondurables and services. Capital is given by a sum of fixed assets and durables; capital series are annual (in contrast to the other series which are quarterly); we interpolate annual series of capital to get quarterly series using spline interpolation. Output is obtained as a sum of consumption and investment. We deflate the constructed variables with the corresponding implicit price deflator and we convert them in per capita terms by dividing them by the series of the total population.

Calibration and estimation of the model's parameters

To identify the model's parameters, we formulate the following set of restrictions:

$$A_t z_t = \frac{y_t}{k_t^{\alpha}}, \tag{34}$$

$$d_t \delta_t = \frac{i_t}{k_t} - \frac{k_{t+1} - k_t}{k_t}, \tag{35}$$

$$\frac{1}{\beta} = \frac{1}{T} \sum_{t=1}^{T} \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[1 - d_{t+1}\delta_{t+1} + \alpha A_{t+1} z_{t+1} k_{t+1}^{\alpha-1}\right]. \tag{36}$$

We set $\gamma = 1$, and we search for α that matches best the growth rates of variables in the data. First, given some α, we construct $A_t z_t$ using (34) , and we estimate the parameters ρ_z, $\sigma_{\varepsilon_z}^2$, g_A in the process for productivity $z_t = z_{t-1}^{\rho_z} \exp(\varepsilon_{z,t})$ using a linear regression method. To identify a growing and cycle components, A_t and z_t, respectively, we assume $z_0 = 1$. Second, we construct the data on $d_t \delta_t$ using (35), and we estimate the parameters ρ_δ, $\sigma_{\varepsilon_\delta}^2$, g_d in the process for productivity $\delta_t = \delta_{t-1}^{\rho_\delta} \exp(\varepsilon_{\delta,t})$ using a linear regression. Again, to separate growth and cycles, d_t and δ_t, respectively, we assume $\delta_0 = 1$. Finally, we calibrate the discount factor by using the Euler equation (36).

Our estimation-calibration procedure gives the following values of the parameters: $\beta = 0.9013$, $\rho_z = 0.9890$, $\sigma_{\varepsilon_z} = 0.0054$, $g_A = 1.002$, $\rho_\delta = 0.9538$, $\sigma_{\varepsilon_\delta} = 0.0381$ and $g_d = 1.002$. We observe a considerable positive growth rate in the depreciation rate $g_d = 1.002$. Furthermore, we find that the best fit of our criteria for the growth rate is obtained under $\alpha = 0.7$. This value for the capital share in output is larger than is typically used in the business cycle literature, however, it is roughly in line with the recent finding of Karabarbounis and Neiman (2014) that labor shares gradually declined over time; the implied gross capital shares reach 0.55.

We know that on the tail, the EFP solution will depend on a specific terminal condition used and may be insufficiently accurate. To deal with this issue, we extrapolate the data for 80 periods forward, using the growth rates that we estimate from the data on consumption, capital, output, and investment under the assumption of exponential growth. We implement EFP to match the initial and terminal conditions in the extrapolated data, i.e., we use $T = \tau + 80$. To identify the growth path in our unbalanced growth model, we use assumption (16). We construct a sequence of growing Smolyak grids. There are three state variables (k_t, z_t, δ_t) in this application and the corresponding second-level Smolyak grid consists of 25 multidimensional grid points. After we compute the EFP solution, we simulate the model using the sequence of shocks reconstructed from the data.