The Lattice of Envy-free Matchings

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Introduction

- A stable matching is a matching that has no blocking pair.
- Envy-freeness is a relaxation of stability. Informally speaking, an envy-free matching allows blocking pairs between doctors and empty positions of hospitals.
- Suppose we start with a stable matching. When some doctors retire or new hospital positions are created, this matching may become unstable, but it remains envy-free.
- In such a market, if hospitals with empty positions make offers to the doctors they like, and doctors accept offers that are better than their current hospitals, then we will see a lot of so-called vacancy chains.

More On Vacancy Chains

- Imagine a professor at Harvard retires, his position may be filled by a professor from MIT. Now MIT has a vacant position, which may in turn attract a Stanford professor. Then Stanford would want to hire a new professor, and so on. This is a "vacancy chain".
- Suppose we start with an envy-free matching. If each hospital with empty positions makes an offer to its favorite blocking doctor, and each doctor accepts his most preferred offer received, then a new envy-free matching is formed and the first round of vacancy chains is completed.
- If this process repeats until all vacancy chains have ended, in the end we reach a stable matching; and until then, we will be observing envy-free matchings.

A Many-To-One Matching Model

- Standard many-to-one matching model with strict responsive preferences.
- There is a finite set of hospitals **H** and a finite set of doctors **D**.
- Each doctor d has strict preferences $\succ_d$ over the set of hospitals and being unmatched, denoted by $\emptyset$.
- Each hospital h: (1) has a capacity q_h; (2) has strict preferences $\succ_h$ over subsets of doctors and being unmatched; (3) its preference is **responsive**: any two groups of doctors that differ in a single doctor are preference ordered by the preference for individual doctors.

Stability

- A matching μ is **individually rational** if: (1) $\forall d \in \mathbf{D}$, $\mu(d) \succsim_d \emptyset$; (2) $\forall h \in \mathbf{H}$, $d \in \mu(h)$, we have $d \succsim_h \emptyset$.
- A doctor-hospital pair (d, h) **blocks** μ if $h \succ_d \mu(d)$ and at least one of the following situations happen: (1) $\exists d' \in \mu(h)$ such that $d \succ_h d'$; (2) $|\mu(h)| < q_h$ and $d \succ_h \emptyset$.
- A matching μ is **stable** if and only if it is individually rational and there is no blocking pair.
- Type (1) blocking pair is often called "justified envy"; and type (2) blocking pair is often called "wastefulness".

Envy-Freeness

- Given a matching μ, a doctor d has **justified envy** toward d' who is assigned to hospital h, if (i) $h \succ_d \mu(d)$ and (ii) $d \succ_h d'$.
- A matching μ is **envy-free** if it is individually rational and no doctor has justified envy.
- In other words, we allow blocking pairs in envy-free matchings, but only between doctors and empty positions of hospitals.
- Some examples of envy-free matchings: all stable matchings are envy-free; the empty matching in which everyone is unmatched is envy-free; after each round of a hospital proposing deferred acceptance algorithm, the temporary matching is envy-free.

Conway's Lattice Theorem

- Recall that a partially ordered set P is called a **join-semilattice** if any two elements in P have a least upper bound (called join, denoted by $\vee$); and a **meet-semilattice** if any two elements in P have a greatest lower bound (called meet, denoted by $\wedge$). A partially ordered set P is a **lattice** if it is both a join-semilattice and meet-semilattice.
- We know the set of stable matchings forms a lattice under the common preferences of doctors: a natural candidate for $\mu \vee \mu'$ is a "matching" λ that matches each doctor d to his more preferred hospital between $\mu(d)$ and $\mu'(d)$; similarly a "matching" ν that matches each doctor d to his less preferred hospital between $\mu(d)$ and $\mu'(d)$ is a candidate for $\mu \wedge \mu'$. Conway proved that λ and ν are indeed stable matchings, and serve and the join and meet respectively.

Hasse Diagram

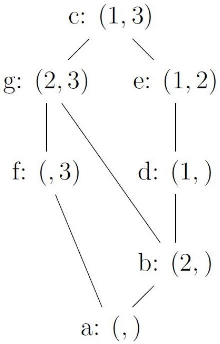
d_1: $h_1 \succ_{d_1} h_2$

d_2: $h_3 \succ_{d_2} h_2$

h_1: d_1

h_2: $d_1 \succ_{h_2} d_2$

h_3: d_2



We see that ν is not necessarily an envy-free matching: look at (e) and (g), both d_1 and d_2 like h_2 less. On the other hand, λ defines an envy-free matching.

The Lattice of Envy-free Matchings

- Lemma: the matching $\lambda = \mu \vee \mu'$ is always envy-free and therefore the set of envy-free matchings $\mathcal{L}$ is a join-semilattice under $\succsim_D$ (common preferences of doctors).
- Technical lemma: A finite join-semilattice with a minimum is a lattice.
- Theorem: the set of envy-free matchings $\mathcal{L}$ is a lattice under $\succsim_D$. (The empty matching is the smallest element)
- We don't know much about the meet from the (non-constructive) proof of the technical lemma.

Properties of the Lattice

- The maximum: the doctor optimal stable matching. (Sotomayor)
- The minimum: the empty matching.
- It is non-distributive: every maximal chain in a finite distributive lattice has the same length, not the case in the example.
- Join: Conway-style.

  We show:

- The join of a stable matching and an envy-free matching is a stable matching.
- The Conway-style meet of a stable matching and an envy-free matching is an envy-free matching.
- Let μ be any envy-free matching. If $\mu \gtrsim_D \mu_H$, then μ is stable.

Tarski's Fixed Point Theorem

- A lattice is called complete if every subset (and not just every pair of elements) has a join and a meet.
- (Tarski 1955) Let $(\mathcal{L}, \leq)$ be a complete lattice and $T : \mathcal{L} \to \mathcal{L}$ be isotone, i.e. $\forall x, y \in \mathcal{L}$, $x \leq y \Rightarrow T(x) \leq T(y)$, then the set of fixed points of T is nonempty and forms a complete lattice with respect to $\leq$.
- Use Tarski's fixed point theorem to study stable matchings: Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Ostrovsky (2008). Also envy-free matchings: Kamada and Kojima (2017).

Tarski's (Vacancy Chain) Operator T

- Operates on the set of envy-free matchings.
- If the matching is already stable, do nothing.
- Otherwise, let all the hospitals send offers to their favorite blocking doctors.
- Doctors accept their favorite offers and move to the corresponding hospitals.
- We have a new matching, denote it by $T(\mu)$.

The Fixed Points

- One can check that (1) $T(\mu)$ is an envy-free matching; and (2) T is isotone, therefore Tarski's fixed point theorem applies.
- The fixed points of T are stable matchings.
- Also notice $T(\mu) \succsim_D \mu$.
- Theorem: let μ be an envy-free matching, denote the fixed point of T starting from μ by $F(\mu)$. Then $F(\mu) = \mu \vee \mu_H$.
- If $\mu = \emptyset$, then $F(\mu) = \mu_H$, we recover a version of the hospital proposing deferred-acceptance algorithm.

Summary

- Blocking pairs are allowed in envy-free matchings, but only between doctors and empty positions of hospitals.
- The set of envy-free matchings forms a lattice with a point-wise join, but non-point-wise meet.
- There is a Tarski's operator on this lattice that can be interpreted as the dynamics of vacancy chains.
- Markets with vacancy chains eventually converge to stable matchings. $(\mu \vee \mu_H)$
- This process might take time and we observe envy-free matchings along the way.