Large Sample Estimators of the Stochastic Discount Factor*

Soohun Kim† Robert A. Korajczyk‡

November 14, 2018

Abstract

We propose estimators of the stochastic discount factor (SDF) using large cross-section of individual stocks. We suggest a correction for the small sample bias in a standard GMM estimator induced by having a finite time series and show how to use the correction in exploiting unbalanced panels of individual stock returns. Our estimators can utilize both a prespecified set of traded or non-traded factors implied by a specific asset pricing model and latent factors estimated by multivariate statistical methods. The estimators perform well in simulations designed to mimic the U.S. equity markets. We apply our SDF estimators to the 10,112 individual stock price dynamics in the U.S. over 50 years from 1976 to 2016, and identify which factors in popular asset pricing models command a price of risk.

JEL classification codes: G12

Keywords: asset pricing, factor structure, stochastic discount factor

*We thank Christian Brownlees, Gregory Connor, Ravi Jagannathan, Clark Liu, Andreas Nuehierl, Kuntara Pukthuanthong, Richard Roll, Olivier Scaillet, and participants at the Conference on Asia-Pacific Financial Markets, the Frontiers of Factor Investing Conference at Lancaster University, the ERFIN workshop at the Warsaw School of Economics, and seminar participants at Northwestern University, the University of Kentucky and the University of York for helpful comments.

†Scheller College of Business, Georgia Institute of Technology, Atlanta, GA 30308. E-mail: soohun.kim@scheller.gatech.edu.

‡Kellogg School of Management, Northwestern University, Evanston, IL 60208. E-mail: r-korajczyk@kellogg.northwestern.edu.

1 Introduction

In an economy without arbitrage opportunities, there exists a valid stochastic discount factor (SDF) such that the price of any security is obtained as the expected value of the discounted (by the SDF) future payoff. The SDF representation of asset pricing models has been widely used in the empirical finance literature. It is often the case that empirical studies using asset returns to estimate the SDF use a small number of portfolios.[1] This paper proposes alternative estimators of the stochastic discount factor so that empirical researchers can fully exploit useful information on asset prices from a large panel of individual stock data.

The intuition behind our SDF estimators builds on Hansen and Jagannathan (1997). Within a set of candidate SDFs, one can search for the one which minimizes the norm of pricing errors. Pukthuanthong and Roll (2017) propose an interesting "agnostic" candidate stochastic discount factor estimator (PR–SDF), which utilizes a time series of returns on a large collection of individual assets. The PR-SDF is agnostic in the sense that no assumptions are required about agents' utility functions or about the nature of systematic risk, such as the number of pervasive factors, in the economy. Since the approach picks a linear combination of asset returns that assigns the right prices to the underlying assets, the SDF-mimicking portfolio chooses weights based on explaining mean returns rather than the covariance matrix of returns. That is, since Pukthuanthong and Roll derive their SDF estimator to match prices in the cross section, the estimator favors pervasive factors that matter for pricing. Our estimators are designed to deal with several issues with the agnostic SDF estimator: it requires very large samples (larger than available in most empirical studies) to converge to the true SDF; it requires a balanced panel with its inherent survivorship biases; and it is biased for small time-series samples. In fact, the agnostic estimator is equivalent to the pricing error minimizing SDF when the SDF is a linear function of factors estimated by asymptotic principal components (Connor and Korajczyk (1986)) and the number of factors is equal to the number of time periods.

We find that imposing a finite factor structure with fewer factors improves the efficiency of our estimator appreciably. Our estimator can accommodate both prespecified factor models and those estimated by principal-component methods. Applying

[1] Jagannathan and Wang (1996) is an early example of papers estimating an SDF by imposing the pricing restrictions on a small number of portfolios.

statistical factor models in linear asset pricing models raises the issue of rotational indeterminacy of the estimated factors and factor loadings. The advantage of the SDF representation in this setting is that the estimated SDF is a function of the estimated factors and a projection vector. Rotating the factors changes the projection vector so that the estimated SDF is not affected.

Furthermore, we address the prevalence of short-lived individual stocks. While the top graph in Figure 1 shows that roughly thousands of individual stocks are traded in U.S. market at a specific time, the bottom graph reveals that a large portion of the individual stocks has substantial missing returns. The proportion of individual stocks with missing observations over the previous five (ten) years averages 36% (60%) over 1977-2016. Therefore, there are few individual assets that have a sufficiently long time series of returns to match the large T requirements of the agnostic estimator. In tackling this issue, we introduce a short block structure of unbalanced panel data with the mild assumption of balanced panel within each short block. We propose a short time-series bias correction in each short block, which allows us both to eliminate bias and to splice the estimator across multiple blocks, thus overcoming survivorship biases. Moreover, exploiting the multiple blocks, we develop a fully operational asymptotic theory for testing implications of no arbitrage pricing. We apply our novel tests to popular asset pricing models suggested in the literature (Sharpe, 1964, Fama and French, 1992, Hou et al. 2015, Fama and French, 2015, Pástor and Stambaugh, 2003, Barillas and Shanken, 2017) and identify which factors command the price of risk in a large cross-section of short-lived individual stocks.

This paper is not the first attempt to exploit the SDF representations of a large number of assets. Araújo and Issler (2012) show that the SDF can be summarized by a scaled inverse of the cross-sectional geometric average of returns. We allow multiple pervasive risks (possibly priced and non-priced factors) in an economy and let our estimators find the priced factors among those by observing the price dynamics of the large panel. As mentioned above, our paper is closely related to the agnostic estimator of Pukthuanthong and Roll (2017). Alternatively, Kozak et al. (2018) repackage individual stocks using a vast array of characteristics and suggest various methods to obtain robust SDF which achieves the bound by Hansen and Jagannathan (1991). In contrast, we directly use unbalanced panel of individual stocks and allow factors to be non-traded.

We also contribute to a broader literature of using individual stocks for the empirical studies of asset pricing models. The arbitrage pricing theory of Ross (1976) and Chamberlain and Rothschild (1983) provides a framework to dichotomize a large cross section of returns into pervasive factors and diversifiable risks. A long literature derives methods to extract pervasive factors from a large cross-sectional data (e.g., Connor and Korajczyk (1986, 1987, 1988), Stroyny (1992), Stock and Watson (1998, 2002), and Jones (2001)). In a recent effort to extract crucial factors for pricing, Kelly et al. (2018) propose to apply PCA to individual stock returns projected on dynamic characteristics and Lettau and Pelger (2018) suggest a novel method of estimating factors that can explain the covariance matrix as well as fit the pricing equation.[2] We contribute to this literature by providing a simple tool to select factors (priced in unbalanced panels of individual stocks) among the pervasive common factors extracted from a large panel of data. As pointed out in Merton (1973), Jagannathan and Wang (1996), Campbell and Vuolteenaho (2004), Kelly and Pruitt (2013), and Jagannathan and Marakani (2015), not all pervasive factors (i.e., those that explain common movements in asset returns) need be important for explaining the cross section of asset prices.

Alternatively, the pricing of a given pervasive factor can be examined with the beta pricing form. Exploiting a large cross section of assets in estimating beta pricing models, a series of papers have proposed risk premia estimators using large cross-sections (see Litzenberger and Ramaswamy (1979), Shanken (1992), and Jagannathan et al. (2010)). The recent papers by Gagliardini et al. (2016) and Kim and Skoulakis (2018a, 2018b) obtain the large panel asymptotic distribution of the risk premia estimator along with an estimator of its variance–covariance matrix. Our paper is differentiated in that we are using the SDF representation, not a beta pricing representation. This difference is particularly important when we use large-panel-data setting where the measurement errors in individual asset betas can severely bias estimated risk premia. Because of the advantage that we do not need to estimate individual stock betas in the SDF representation, we are able to develop a formal sampling theory exploiting a large cross-section of short-lived stocks. Although the equivalence between SDF form and beta form is well-known in the small-N/large-T setup (Jagannathan and Wang (2002)),

[2] Interestingly, we find that the agnostic estimator of Pukthuanthong and Roll (2017) is closely related to the first factor of Lettau and Pelger (2018) in the extreme version where a researcher puts an infinite weight on the cross-sectional pricing equation. See Proposition 2.1 for details. Lettau and Pelger (2018) tame the behavior of factors by imposing the additional restrictions in the second moments. In contrast, we use a prespecified set of a small number of factors implied by a specific asset pricing model or a finite number of latent factors estimated by multivariate statistical methods.

more work is required to understand the differences between the two approaches in the large panel data setting.

In Section 2, we introduce our large cross-sectional economy. Section 2.1 provides the economic intuition for our SDF estimator. We propose unbalanced panel estimators of the stochastic discount factor (SDF) for pricing risky assets and develop an inference framework in Section 2.2. In Section 3, we simulate an economy in which asset risks match those in the U.S. equity markets and examine the performance of our SDF estimators across various sample sizes. The estimators perform well and the imposition of a factor structure improves the estimators' performance relative to purely agnostic alternatives. The bias correction for unbalanced panels works. In Section 4, we apply our methodology to a large cross section of individual stock returns in U.S. equity markets and provide evidence that profitability and investment factors in Hou et al. (2015) or Fama and French (2015) are important for pricing individual stocks. Section 5 concludes. All proofs are in the Appendix.

2 Economy

We assume that the gross-return generating process of each individual security follows a K-factor model. In particular, the gross return of the i-th asset at time t, $R_{i,t}$, is expressed as

$$R_{i,t} = \alpha_i + \boldsymbol{\beta}_i'\mathbf{f}_t + e_{i,t}, \text{ for } i = 1, \cdots, N \text{ and } t = 1, \cdots, T, \tag{2.1}$$

where $\boldsymbol{\beta}_i$ is the $(K \times 1)$ vector of factor loadings of the i-th asset on the $(K \times 1)$ vector of factors, $\mathbf{f}_t$. As is standard, we assume $\mathbb{E}[e_{i,t}] = 0$ and $\mathbb{E}[\mathbf{f}_t e_{i,t}] = \mathbf{0}_K$, a $(K \times 1)$ vector of zeros. We allow the factor of $\mathbf{f}_t$ to be either traded excess returns, traded gross returns, latent, or nontraded factors.

With some mild assumptions on the cross-sectional dependency among residuals of $e_{i,t}$, Ross (1976) and Chamberlain and Rothschild (1983) show that in an economy without statistical arbitrage, there exist a scalar, λ_0, the gross return on the riskless asset, and a $(K \times 1)$ vector, $\boldsymbol{\lambda}_f$, such that

$$\mathbb{E}[R_{i,t}] \approx \lambda_0 + \boldsymbol{\beta}_i'\boldsymbol{\lambda}_f. \tag{2.2}$$

We assume that exact factor pricing holds, so that the equation (2.2) holds as an equality (as in Connor (1984)). Let the $(K \times 1)$ vector $\boldsymbol{\mu}_f$ be $\boldsymbol{\mu}_f = \mathbb{E}[\mathbf{f}_t]$. By combining the return generating process of (2.1) and the exact form of the pricing restriction of (2.2), we have

$$\mathbb{E}[R_{i,t}] = \alpha_i + \boldsymbol{\beta}_i' \boldsymbol{\mu}_f = \lambda_0 + \boldsymbol{\beta}_i' \boldsymbol{\lambda}_f,$$

implying that

$$\alpha_i = \lambda_0 + \boldsymbol{\beta}_i' (\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f).$$

Then, plugging the above expression into the process of (2.1) yields

$$R_{i,t} = \lambda_0 + \boldsymbol{\beta}_i' (\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f + \mathbf{f}_t) + e_{i,t}. \tag{2.3}$$

Equation (2.3) allows for many different specifications of the nature of the factor vector, $\mathbf{f}_t$. If $\mathbf{f}_t$ is an observed vector of portfolio excess returns, then $\boldsymbol{\mu}_f = \boldsymbol{\lambda}_f$. If $\mathbf{f}_t$ is an observed vector of portfolio gross returns, then $\boldsymbol{\mu}_f = \mathbf{1}_K \lambda_0 + \boldsymbol{\lambda}_f$ and spanning of the mean-variance frontier by the factors implies that (2.3) reduces to

$$R_{i,t} = \boldsymbol{\beta}_i' \mathbf{f}_t + e_{i,t}, \tag{2.4}$$

with the added constraint that $\boldsymbol{\beta}_i' \mathbf{1}_K = 1$ (see Huberman and Kandel (1987)). If $\mathbf{f}_t$ is an observed vector of pre-whitened macroeconomic variables, then $\boldsymbol{\mu}_f = \mathbf{0}_K$. In the literature, there are a number of papers that use a combination of traded excess returns and pre-whitened macroeconomic variables (e.g., Chen et al. (1986) or Shanken and Weinstein (2006)). In this case the expected value of the factors is the factor risk premium for the excess return factors and zero for the pre-whitened variables. Finally, if $\mathbf{f}_t$ is an unobserved vector of latent portfolio excess returns (as in Connor and Korajczyk (1986)) then $\boldsymbol{\mu}_f = \boldsymbol{\lambda}_f$, but the procedure requires a consistent estimator of the excess returns on factor-mimicking portfolios.

Next, we specify the stochastic discount factor (SDF) m_t in this economy such that

$$\mathbb{E}[R_{i,t} m_t] = 1 \text{ for } i = 1, \cdots, N.$$

The realized SDF is a linear function of the realization of the systematic factors:

$$m_t = \delta_0 + \mathbf{f}_t' \boldsymbol{\delta}_f, \tag{2.5}$$

which satisfies $\mathbb{E}\left[R_{i,t} m_t\right] = 1$ when the scalar δ_0 and the $(K \times 1)$ vector, $\boldsymbol{\delta}_f$, are given by

$$\delta_0 = \frac{1}{\lambda_0}\left(1 + \boldsymbol{\mu}_f' \Sigma_f^{-1} \boldsymbol{\lambda}_f\right) \tag{2.6}$$

$$\boldsymbol{\delta}_f = -\frac{1}{\lambda_0}\left(\Sigma_f^{-1} \boldsymbol{\lambda}_f\right), \tag{2.7}$$

where

$$\Sigma_f = \mathbb{E}\left[\left(\mathbf{f}_t - \boldsymbol{\mu}_f\right)\left(\mathbf{f}_t - \boldsymbol{\mu}_f\right)'\right].$$

The expected value of m_t is λ_0^{-1}.

So far, we describe an economy with N assets and specify the form of the stochastic discount factor as a linear function of systematic factors, which prices the gross returns of the N assets. In many cases, when a risk-free asset exists, empirical research studies the returns of the N assets in excess of the risk-free return. If there exists a risk-free asset, then the expression of (2.3) implies that the gross return of the risk-free asset is λ_0, since it has neither any exposure to the factor ($\boldsymbol{\beta}_i = \mathbf{0}_K$) nor residual risk ($e_{i,t} = 0$). Hence, from (2.3), the excess return of the i-th asset at time t can be written as

$$R_{e,i,t} = R_{i,t} - \lambda_0 = \boldsymbol{\beta}_i'\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f + \mathbf{f}_t\right) + e_{i,t}. \tag{2.8}$$

Now, we characterize a stochastic discount factor $m_{e,t}$, which prices the excess returns of the N assets, i.e.,

$$\mathbb{E}\left[R_{e,i,t} m_{e,t}\right] = 0 \text{ for } i = 1, \cdots, N.$$

It can be shown that we can construct a stochastic discount factor

$$m_{e,t} = a + \mathbf{f}_t' \boldsymbol{\delta}_e, \tag{2.9}$$

satisfying $\mathbb{E}\left[R_{e,i,t}m_{e,t}\right] = 0$, with the $(K \times 1)$ vector of $\boldsymbol{\delta}_e$, given by

$$\boldsymbol{\delta}_e = -a\left(\boldsymbol{\Sigma}_f + \boldsymbol{\lambda}_f\boldsymbol{\mu}_f'\right)^{-1}\boldsymbol{\lambda}_f. \tag{2.10}$$

We obtain an extra degree of freedom when pricing excess returns, rather than gross returns, since we do not require the SDF to pin down the mean of m or, equivalently, the riskless rate of return. Thus, the constant a is not identified and the SDF can be off in pricing gross returns by a constant that cancels when excess returns are analyzed. This necessitates a normalization and we choose to set a equal to 1 (see Cochrane (2009, section 6.3)).

From the expressions of (2.6)-(2.10), when factors are tradable or, equivalently, when $\boldsymbol{\lambda}_f = \boldsymbol{\mu}_f$, one can estimate the SDF directly utilizing the first two moments of the factors (see Kozak et al. (2018)). Hence, for the case of traded factors, a researcher can evaluate a model of interest by comparing the SDF constructed by using only factor data with our SDF using a large panel data – analogous to the common practice of comparing the average returns of traded factors with the cross-sectional coefficients on assets' associated beta from a Fama and MacBeth (1973) regression.

For the case of observable, but non-tradable, factors or latent factors, researchers need to construct factor mimicking portfolios in order to estimate $\boldsymbol{\lambda}_f$ before estimating (2.6), (2.7), or (2.10) or they can use our SDF estimators directly. We propose several alternative SDF estimators which are based on using large cross sections of individual assets or portfolios. We start with an estimator assuming a balanced panel of asset returns to provide intuition for the approach and motivate the need for small-T bias correction.

2.1 Motivation using Balanced Panel Data

For expositional purposes, we assume that we observe the gross returns of $R_{i,t}$ or the excess returns of $R_{i,t}^e$ for assets $i = 1, \cdots, N$ over the full time period $t = 1, \cdots, T$. The intuition behind the SDF estimator proposed in this subsection will be cast into a more realistic data structure later.

It is convenient to represent the gross-return generating process of (2.3) and the excess-return generating process of (2.8) in matrix form:

$$\mathbf{R} = \lambda_0 \mathbf{1}_N \mathbf{1}_T' + \mathbf{B}\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f\right)\mathbf{1}_T' + \mathbf{B}\mathbf{F}' + \mathbf{E}, \tag{2.11}$$

and

$$\mathbf{R}_e = \mathbf{B}\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f\right)\mathbf{1}_T' + \mathbf{B}\mathbf{F}' + \mathbf{E}, \tag{2.12}$$

where the (i,t) element of the $(N \times T)$ matrices of $\mathbf{R}$ and $\mathbf{R}_e$ are $R_{i,t}$ and $R_{e,i,t}$, respectively, $\mathbf{1}_N$ is the $(N \times 1)$ vector of ones, the i-th row of the $(N \times K)$ matrix of $\mathbf{B}$ is $\boldsymbol{\beta}_i'$, the t-th row of the $(T \times K)$ matrix of $\mathbf{F}$ is $\mathbf{f}_t'$, and the (i,t) element of the $(N \times T)$ matrix of $\mathbf{E}$ is $e_{i,t}$.

We make standard assumptions on the systematic factors and factor loadings.

Assumption 1. *As $N \to \infty$, $\frac{1}{N}\mathbf{B}'\mathbf{1}_N \to \boldsymbol{\mu}_\beta$ and $\frac{1}{N}\mathbf{B}'\mathbf{B} \to \mathbf{V}_\beta = \Sigma_\beta + \boldsymbol{\mu}_\beta\boldsymbol{\mu}_\beta'$, where Σ_β is a positive definite matrix. Also, as $T \to \infty$, $\frac{1}{T}\mathbf{F}'\mathbf{1}_T \xrightarrow{p} \boldsymbol{\mu}_f$ and $\frac{1}{T}\mathbf{F}'\mathbf{F} \xrightarrow{p} \mathbf{V}_f = \Sigma_f + \boldsymbol{\mu}_f\boldsymbol{\mu}_f'$, where Σ_f is a positive definite matrix.*

Assumption 1 specifies that loadings on each factor are pervasive across a large number of assets and that each factor is neither redundant nor nonstationary over time, which is reasonably acceptable for the return generating process. The assumption does not imply that all pervasive factors are priced, so that it allows for factors that explain common variation but are not deemed important (i.e., are not priced) by investors.

Next, we make assumptions on the distributional properties of the residual terms of $e_{i,t}$. We use $\mathbf{0}_{m \times n}$ to denote the $(m \times n)$ matrix of zeros.

Assumption 2. *As $N, T \to \infty$, $\frac{\mathbf{1}_N'\mathbf{E}\mathbf{1}_T}{NT} \xrightarrow{p} 0$, $\frac{\mathbf{F}'\mathbf{E}'\mathbf{1}_N}{NT} \xrightarrow{p} \mathbf{0}_K$, $\frac{\mathbf{B}'\mathbf{E}\mathbf{1}_T}{NT} \xrightarrow{p} \mathbf{0}_K$, and $\frac{\mathbf{B}'\mathbf{E}\mathbf{F}}{NT} \xrightarrow{p} \mathbf{0}_{K \times K}$. Also, there exists a positive constant $M_0 < \infty$ such that the maximum eigenvalue of $\frac{\mathbf{E}'\mathbf{E}}{N}$ is smaller than M_0 for all N, T.*

The first four conditions in Assumption 2 state that the average residual terms over the $(N \times T)$ panel data converge to zero even when the average is weighted by factor realizations (in the time-series dimension) or factor loadings (in cross-sectional dimension). The last condition in Assumption 2 provides a regularity condition on the behavior of residuals. The conventional assumptions that the probability limit of $\frac{1}{N}e_{i,t}^2$ is uniformly bounded and the probability limit of $\frac{1}{N}e_{i,t}e_{i,s}$ becomes negligible for any $t \neq s$ are sufficient to satisfy this regularity condition.

The following theorem establishes that we can recover the stochastic discount factor as a linear function of factors from a large panel of asset return data.

Theorem 2.1. *Under Assumptions 1 and 2, as* $N, T \to \infty$, $\widetilde{m}_t = \widetilde{\delta}_0 + \mathbf{f}_t' \widetilde{\boldsymbol{\delta}}_f$ *and* $\widetilde{m}_{e,t} = 1 + \mathbf{f}_t' \widetilde{\boldsymbol{\delta}}_e$ *(using the normalization a =1 in* (2.10)*) converge to* m_t *and* $m_{e,t}$ *given in* (2.5) *and* (2.9)*, respectively, when the* $((K+1) \times 1)$ *vector of* $\widetilde{\boldsymbol{\delta}} = \left[\widetilde{\delta}_0 \ \widetilde{\boldsymbol{\delta}}_f'\right]'$ *and the* $(K \times 1)$ *vector of* $\widetilde{\boldsymbol{\delta}}_e$ *are constructed by*

$$\widetilde{\boldsymbol{\delta}} = \widetilde{\mathbf{D}}^{-1} \widetilde{\mathbf{U}} \tag{2.13}$$

$$\widetilde{\boldsymbol{\delta}}_e = -\widetilde{\mathbf{D}}_e^{-1} \widetilde{\mathbf{U}}_e, \tag{2.14}$$

where $\mathbf{F}_\triangle = [\mathbf{1}_T \ \mathbf{F}]$ *and*

$$\widetilde{\mathbf{D}} = \frac{\mathbf{F}_\triangle' \mathbf{R}' \mathbf{R} \mathbf{F}_\triangle}{NT^2}, \ \widetilde{\mathbf{U}} = \frac{\mathbf{F}_\triangle' \mathbf{R}' \mathbf{1}_N}{NT}$$
$$\widetilde{\mathbf{D}}_e = \frac{\mathbf{F}' \mathbf{R}_e' \mathbf{R}_e \mathbf{F}}{NT^2}, \ \widetilde{\mathbf{U}}_e = \frac{\mathbf{F}' \mathbf{R}_e' \mathbf{R}_e \mathbf{1}_T}{NT^2}.$$

The estimator proposed in Theorem 2.1 can be intuitively understood as follows. By specifying the $(T \times 1)$ vector of the realized SDF, $[m_1 \cdots m_T]'$, as $\mathbf{m} = \mathbf{F}_\triangle \boldsymbol{\delta}$, the realized mispricing of the N assets' gross returns can be formulated by

$$\mathbf{1}_N - \frac{\mathbf{Rm}}{T} = \mathbf{1}_N - \frac{\mathbf{RF}_\triangle}{T} \boldsymbol{\delta},$$

and the estimator $\widetilde{\boldsymbol{\delta}}$ in (2.13) can be obtained as the solution of minimizing the squared pricing error:

$$\widetilde{\boldsymbol{\delta}} = \arg\min_{\boldsymbol{\delta}} \left(\mathbf{1}_N - \frac{\mathbf{RF}_\triangle}{T} \boldsymbol{\delta}\right)' \left(\mathbf{1}_N - \frac{\mathbf{RF}_\triangle}{T} \boldsymbol{\delta}\right). \tag{2.15}$$

Note that $\widetilde{\boldsymbol{\delta}}$ is the estimate of $\boldsymbol{\delta}$ from an OLS regression of $\mathbf{1}_N$ on $\frac{\mathbf{RF}_\triangle}{T}$, or the GMM estimator with an identity weighting matrix. Using the identity matrix for the weighting matrix allows us to accommodate the situation in which the cross-sectional sample is much larger than the time series sample, in which case the common efficient GMM weighting matrix in infeasible (the inverse of a singular matrix).

Similarly, given the $(T \times 1)$ vector of the realized SDF, $[m_{e,1} \cdots m_{e,t}]'$, denoted by $\mathbf{m}_e = \mathbf{1}_T + \mathbf{F} \boldsymbol{\delta}_e$, with the associated moment condition $\mathbb{E}\left[\frac{\mathbf{R}_e . \mathbf{m}_e}{T}\right] = \mathbb{E}\left[\frac{\mathbf{R}_e \mathbf{1}_T}{T} + \frac{\mathbf{R}_e \mathbf{F}}{T} \boldsymbol{\delta}_e\right] = 0$. The estimator $\widetilde{\boldsymbol{\delta}}_e$ in (2.14) can be interpreted as the solution of minimizing the sum of squared pricing errors:

$$\widetilde{\boldsymbol{\delta}}_e = \arg\min_{\boldsymbol{\delta}_e} \left(\frac{\mathbf{R}_e \mathbf{1}_T}{T} + \frac{\mathbf{R}_e \mathbf{F}}{T} \boldsymbol{\delta}_e \right)' \left(\frac{\mathbf{R}_e \mathbf{1}_T}{T} + \frac{\mathbf{R}_e \mathbf{F}}{T} \boldsymbol{\delta}_e \right). \tag{2.16}$$

The formation of $\widetilde{\boldsymbol{\delta}}$ and $\widetilde{\boldsymbol{\delta}}_e$ as in (2.15) and (2.16) implies that they are regression coefficients (regressing $\mathbf{1}_N$ on $\frac{\mathbf{R}\mathbf{F}_{\triangle}}{T}$ for gross returns and regressing $\frac{\mathbf{R}_e \mathbf{1}_T}{T}$ on $-\frac{\mathbf{R}_e \mathbf{F}}{T}$ for excess returns). The above interpretation of Theorem 2.1 confirms that the GMM estimator with identity weighting matrix can be utilized as an SDF estimator even when the number of moment conditions is very large.

For the estimators in Theorem 2.1, we need the true, but possibly mean-deficient (i.e., non-traded), factors. An alternative approach is to treat $\mathbf{F}$ as latent factors that are estimated through multivariate statistical techniques. For this case we do not directly observe factor realizations, $\mathbf{F}$, but estimate those with $\mathbf{F}^* = [\mathbf{f}_1^* \cdots \mathbf{f}_T^*]'$ with the following properties.

Assumption 3. *The factor estimator $\mathbf{F}^*$ satisfies that $\mathbf{f}_t^* \xrightarrow{p} \mathcal{O}'\mathbf{f}_t$ for each t and $\frac{1}{T}(\mathbf{F}^* - \mathbf{F}\mathcal{O})'(\mathbf{F}^* - \mathbf{F}\mathcal{O}) \xrightarrow{p} \mathbf{0}_{K\times K}$ for some rotation matrix of $\mathcal{O}$.*

The conventional PCA estimator using the return generating process given by (2.3) or (2.12) satisfies these properties under commonly used identification assumptions. See Stock and Watson (2002) for details. It turns out that the consistent estimation of the stochastic discount factor is still feasible as shown in the following corollary.

Corollary 2.1. *Under Assumptions 1, 2 and 3, it holds that $\widetilde{m}_t^* = \widetilde{\delta}_0^* + \mathbf{f}_t^{*\prime}\widetilde{\boldsymbol{\delta}}_f^*$ and $\widetilde{m}_{e,t}^* = 1 + \mathbf{f}_t^{*\prime}\widetilde{\boldsymbol{\delta}}_e^*$ converge to m_t and $m_{e,t}$ given by* (2.5) *and* (2.9)*, respectively, when the $((K+1)\times 1)$ vector of $\widetilde{\boldsymbol{\delta}}^* = \left[\widetilde{\delta}_0^* \ \widetilde{\boldsymbol{\delta}}_f^{*\prime}\right]'$ and the $(K\times 1)$ vector of $\widetilde{\boldsymbol{\delta}}_e^*$ are equivalent to $\widetilde{\boldsymbol{\delta}}$ and $\widetilde{\boldsymbol{\delta}}_e$ with $\mathbf{F}$ and $\mathbf{F}_{\triangle}$ are replaced by $\mathbf{F}^* = [\mathbf{f}_1^* \ \cdots \ \mathbf{f}_T^*]'$ and $\mathbf{F}_{\triangle}^* = [\mathbf{1}_T \ \mathbf{F}^*]$.*

Furthermore, even without taking a stand on a factor structure (or, equivalently, setting the number of factors equal to the number of time periods, T) we can consistently estimate the SDF for gross returns with some restrictions on the residual variances and the sequential asymptotics of $N \to \infty$ and then $T \to \infty$. It is worth emphasizing that the SDF estimator for the gross returns in this case is identical to the agnostic estimator proposed by Pukthuanthong and Roll (2017).

Proposition 2.1. *Let Assumptions 1 and 2 hold. Under the homoskedasticity condition of* $\frac{1}{N}\mathbf{E}'\mathbf{E} \xrightarrow{p} s\mathbf{I}_T$, *it holds that as* $N \to \infty$ *and then* $T \to \infty$, $\breve{m}_t$ *converges to* m_t *given by* (2.5), *when* $\breve{m}_t$ *is defined by*

$$\breve{m}_t = \iota_t' \left(\frac{\mathbf{R}'\mathbf{R}}{NT^2} \right)^{-1} \left(\frac{\mathbf{R}'\mathbf{1}_N}{NT} \right), \tag{2.17}$$

where ι_t *is the* $(T \times 1)$ *vector of zeros except the* t*-th element of one.*

The Pukthuanthong and Roll estimator is totally agnostic as to the number of pervasive factors. Our estimator of (2.5) in Theorem 2.1 is equivalent to that of Pukthuanthong and Roll when we let $K = T$.[3] Also, this estimator is closely related to the risk-premia PCA method proposed by Lettau and Pelger (2018). Lettau and Pelger (2018) suggest to put positive weight to the information of the realized risk premia in extracting systematic factors. In fact, the estimator by Pukthuanthong and Roll is a multiple of the first extracted factor by Lettau and Pelger (2018) when the infinite weight is put on the realized risk premia.

2.2 Unbalanced Panel Estimator

Since the estimators proposed in the previous subsection utilize the full balanced panel, they are appropriate for the case with a large number of portfolios over a long horizon. However, if empirical researchers wish to use individual stocks to construct the SDF, the estimators are problematic due to the survivorship biases induced by requiring a balanced panel of individual assets. Additionally, the estimators are biased for finite time series samples. To see this, note that $\frac{\mathbf{Rm}}{T} = \frac{\mathbf{RF}_\triangle}{T}\boldsymbol{\delta} = \mathbb{E}\left[\frac{\mathbf{Rm}}{T}\right] + u = \mathbf{1}_N + u$. However, $-u$ is the error in the regression of $\mathbf{1}_N$ on $\frac{\mathbf{RF}_\triangle}{T}$, so the error term is clearly correlated with the regressor. As T approaches to infinity, the pricing error u approaches zero as does the bias. In our simulation below the bias is substantial for time series sample sizes typically used in practice.

In this section, we propose estimators that deal with unbalanced panel data by

[3] Let $\widetilde{\mathbf{m}} = [\widetilde{m}_1 \ \cdots \ \widetilde{m}_T]$ and $\breve{\mathbf{m}} = [\breve{m}_1 \ \cdots \ \breve{m}_T]$. Then, when $K = T$, it holds that

$$\widetilde{\mathbf{m}} = \mathbf{F}_\triangle \widetilde{\boldsymbol{\delta}} = \mathbf{F}_\triangle \left(\frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{R}\mathbf{F}_\triangle}{NT^2} \right)^{-1} \left(\frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{1}_N}{NT} \right) = \mathbf{F}_\triangle \mathbf{F}_\triangle^{-1} \left(\frac{\mathbf{R}'\mathbf{R}}{NT^2} \right)^{-1} \left(\mathbf{F}_\triangle'\right)^{-1} \mathbf{F}_\triangle' \left(\frac{\mathbf{R}'\mathbf{1}_N}{NT} \right) = \breve{\mathbf{m}}.$$

estimating the SDF over non-overlapping time periods of length τ with the number of blocks increasing as T approaches infinity. Since the estimators proposed in Theorem 2.1 (and the estimator of Pukthuanthong and Roll (2017) in Proposition 2.1) are biased for finite values of T, we propose a bias correction for the SDF estimators.

We formalize the data structure of our unbalanced panel. We split the time period of length T into B non-overlapping time blocks of length τ such that $T = B\tau$. We fix τ. Hence, as T increases, B increases. We use $b = 1, \cdots, B$ as an index of time blocks. For example, the first block of $b = 1$ covers the time period $t = 1, \cdots, \tau$, and the second block of $b = 2$ covers the time period $t = \tau + 1, \cdots, 2\tau$. For a given block, b, we use all individual stocks with full return data over that time block, $t = (b-1)\tau + 1, \cdots, b\tau$. Although this restriction can be relaxed by assuming missing-at-random within a block (as in Connor and Korajczyk (1987) and Stock and Watson (1998)), we assume full time series returns in a single time block for simplicity. Hence, we require balanced panel within a specific block but allow unbalanced panel across blocks. We relabel stocks in block b with the index of $i_{[b]} = 1, \cdots, N_{[b]}$, where $N_{[b]}$ is the number of individual stocks with full returns over the b-th time block.

Next, we express the observed return generating process in the b-th time block similarly to the original full-panel representation of (2.11) and (2.12):

$$\mathbf{R}_{[b]} = \lambda_0 \mathbf{1}_{N_{[b]}} \mathbf{1}'_\tau + \mathbf{B}_{[b]} \left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f \right) \mathbf{1}'_\tau + \mathbf{B}_{[b]} \mathbf{F}'_{[b]} + \mathbf{E}_{[b]}, \tag{2.18}$$

and

$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]} \left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f \right) \mathbf{1}'_\tau + \mathbf{B}_{[b]} \mathbf{F}'_{[b]} + \mathbf{E}_{[b]}, \tag{2.19}$$

where the $\left(i_{[b]}, s\right)$ element of the $\left(N_{[b]} \times \tau\right)$ matrices of $\mathbf{R}_{[b]}$ and $\mathbf{R}_{e,[b]}$ are $R_{i_{[b]},(b-1)\tau+s}$ and $R^e_{i_{[b]},(b-1)\tau+s}$, respectively, $\mathbf{1}_m$ is the $(m \times 1)$ vector of ones, the $i_{[b]}$-th row of the $\left(N_{[b]} \times K\right)$ matrix of $\mathbf{B}_{[b]}$ is $\boldsymbol{\beta}'_{i_{[b]}}$, the s-th row of the $(\tau \times K)$ matrix of $\mathbf{F}_{[b]}$ is $\mathbf{f}'_{(b-1)\tau+s}$, and the $\left(i_{[b]}, s\right)$ element of the $\left(N_{[b]} \times \tau\right)$ matrix of $\mathbf{E}_{[b]}$ is $e_{i_{[b]},(b-1)\tau+s}$.

2.2.1 SDF Estimator

To construct a consistent SDF estimator using unbalanced panel data, we need the following conditions within and across blocks.

Assumption 4. *As $N, T \to \infty$, the following hold:*

(i) $N_{[b]} \to \infty$,

(ii) $\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{1}_{N_{[b]}} \to \boldsymbol{\mu}_\beta$, $\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{B}_{[b]} \to \mathbf{V}_\beta = \Sigma_\beta + \boldsymbol{\mu}_\beta\boldsymbol{\mu}'_\beta$, $\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{1}_{N_{[b]}} \xrightarrow{p} \mathbf{0}_\tau$, $\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{B}_{[b]} \xrightarrow{p} \mathbf{0}_{\tau\times K}$ *and* $\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N} \xrightarrow{p} \mathbf{V}_{e,[b]}$, *where* $\mathbf{V}_{e,[b]}$ *is a* $(\tau \times \tau)$ *diagonal matrix. There exists a positive constant* $M_1 < \infty$ *such that the maximum eigenvalue of* $\mathbf{V}_{e,[b]}$ *is smaller than* M_1 *for all* b,

(iii) $\frac{1}{B}||\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{1}_{N_{[b]}} - \boldsymbol{\mu}_\beta|| \to 0$, $\frac{1}{B}||\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{B}_{[b]} - \mathbf{V}_\beta|| \to 0$, $\frac{1}{B}||\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{1}_{N_{[b]}}|| \xrightarrow{p} 0$, $\frac{1}{B}||\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{B}_{[b]}|| \xrightarrow{p} 0$, *and* $\frac{1}{B}||\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N} - \mathbf{V}_{e,[b]}|| \xrightarrow{p} 0$, *where* $||\mathbf{A}|| = trace\,(\mathbf{A}'\mathbf{A})$.

The condition (i) is for the availability of large cross-sectional data in each time block. It is worth highlighting that the first two limits of condition (ii) allow the beta at the individual-stock level to vary over time. We require that only the first two moments of the cross-sectional distribution factor loadings be stable over time. Also, note that from the last limit of $\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N} \xrightarrow{p} \mathbf{V}_{e,[b]}$ in condition (ii), the variance of residuals can vary within a block as well as across blocks, as in Jones (2001). The condition (iii) simply states that the block-wise averages of squared errors in the realized cross-sectional moments disappear with large N, T.

Besides, we need the following regularity condition on the behavior of factors in our short time-block structure.

Assumption 5. *Fix a continuous function of* $\mathbf{F}_{[b]}$*;* $g\ : \mathbb{R}^{\tau\times K} \to \mathbb{R}^m$. *Then, there exists a positive number* $M_g < \infty$ *such that* $\lim_{T\to\infty}\frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}\right) < M_g$.

Assumption 5 simply restricts the behavior of factor realizations to be stationary enough so that the block-wise averages do not explode as T increases. In fact, Assumptions 5 and 4(iii) guarantees that the block-wise average of the interaction between cross-sectional errors and a function of factors becomes negligible.

So far, we specified all the necessary assumptions to construct a stochastic discount factor utilizing unbalanced panel data. Before we present our main theorem, we need to introduce an estimator of $\mathbf{V}_{e,[b]}$, which will be an essential element of our small-τ bias correction. A bias correction in a short time series has been addressed in other papers (Litzenberger and Ramaswamy (1979), Shanken (1992)), and the relation of our correction to those papers is discussed below. We utilize the estimator of $\mathbf{V}_{e,[b]}$ proposed by Kim and Skoulakis (2018b).

Lemma 2.1. *Let Assumptions 1, 4, and 5 be in effect. Define* $\widehat{\mathbf{V}}_{e,[b]}$ *by*

$$\widehat{\mathbf{V}}_{e,[b]} = diag\left(\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}' vec\left(\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}\right)\right), \tag{2.20}$$

where

$$\begin{aligned} \mathbf{H}_{[b]} &= \mathbf{J}_\tau - \mathbf{J}_\tau \mathbf{F}_{[b]} \left(\mathbf{F}'_{[b]} \mathbf{J}_\tau \mathbf{F}_{[b]}\right)^{-1} \mathbf{F}'_{[b]} \mathbf{J}_\tau, \\ \mathbf{J}_\tau &= \mathbf{I}_\tau - \frac{1}{\tau} \mathbf{1}_{\tau\times\tau}, \end{aligned} \tag{2.21}$$

the operator $\odot$ *denotes the Hadamard product (entry-wise product), the* (i,j)*-th element of the* $(\tau^2 \times \tau)$ *selection matrix of* $\mathcal{S}$ *is given by* $\mathbf{1}\left(i = (j-1)\,\tau + j\right)$*, and the* $(N \times \tau)$ *matrix of* $\widehat{\mathbf{E}}_{[b]}$ *is defined, for the case of using the gross returns, by* $\widehat{\mathbf{E}}_{[b]} = \mathbf{R}_{[b]}\mathbf{H}_{[b]}$ *and, for the case of using excess returns, by* $\widehat{\mathbf{E}}_{[b]} = \mathbf{R}_{e,[b]}\mathbf{H}_{[b]}$*. Then, as* $N, T \to \infty$*,* $\widehat{\mathbf{V}}_{e,[b]} \xrightarrow{p} \mathbf{V}_{e,[b]}$ *for each* $b = 1, \cdots, B$*.*

The intuition of the estimator defined by (2.20) follows. Given the expressions of $\mathbf{R}_{[b]}$ and $\mathbf{H}_{[b]}$ in (2.18) and (2.21), respectively, it holds that $\widehat{\mathbf{E}}_{[b]} = \mathbf{R}_{[b]}\mathbf{H}_{[b]} = \mathbf{E}_{[b]}\mathbf{H}_{[b]}$. Hence, Assumption 4(ii) implies that $\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}} = \mathbf{H}_{[b]}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\mathbf{H}_{[b]} \xrightarrow{p} \mathbf{H}_{[b]}\mathbf{V}_{e,[b]}\mathbf{H}_{[b]}$. To extract the diagonal matrix of $\mathbf{V}_{e,[b]}$ in the probability limit of $\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}$, we manipulate the matrix of $\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}$ as in (2.20).

Lastly, the following theorem asserts that a consistent estimator of the SDF can be constructed with unbalanced panel data.

Theorem 2.2. *Under Assumptions 1, 4, and 5, as* $N, T \to \infty$*,* $\widehat{m}_t = \widehat{\delta}_0 + \mathbf{f}'_t\widehat{\boldsymbol{\delta}}_f$ *and* $\widehat{m}_{e,t} = 1 + \mathbf{f}'_t\widehat{\boldsymbol{\delta}}_e$ *converge to* m_t *and* $m_{e,t}$ *given in* (2.5) *and* (2.9)*, respectively, when the* $((K+1)\times 1)$ *vector of* $\widehat{\boldsymbol{\delta}} = \left[\widehat{\delta}_0 \ \widehat{\boldsymbol{\delta}}'_f\right]'$ *and the* $(K \times 1)$ *vector of* $\widehat{\boldsymbol{\delta}}_e$ *are constructed by*

$$\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}} \tag{2.22}$$

$$\widehat{\boldsymbol{\delta}}_e = -\widehat{\mathbf{D}}_e^{-1}\widehat{\mathbf{U}}_e, \tag{2.23}$$

where $\mathbf{F}_{\triangle,[b]} = \left[\mathbf{1}_\tau \ \mathbf{F}_{[b]}\right]$ *and*

$$
\begin{aligned}
\widehat{\mathbf{D}} &= \left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B} d_{[b]}, && (2.24)\\
d_{[b]} &= \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}\right),\\
\widehat{\mathbf{U}} &= \frac{1}{B}\sum_{b=1}^{B} u_{[b]}, && (2.25)\\
u_{[b]} &= \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau},\\
\widehat{\mathbf{D}}_e &= \left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B} d_{e,[b]}, && (2.26)\\
d_{e,[b]} &= \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{F}_{[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{[b]}}{\tau^2}\right),\\
\widehat{\mathbf{U}}_e &= \left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B} u_{e,[b]}, && (2.27)\\
u_{e,[b]} &= \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_{\tau}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{1}_{\tau}}{\tau^2}\right),
\end{aligned}
$$

and $\widehat{\mathbf{V}}_{e,[b]}$ *is given in* (2.20)*.*

The intuition behind Theorem 2.2 follows. We focus on the case of $\widehat{\boldsymbol{\delta}}$ given by (2.22) because the underlying intuition can be applied to $\widehat{\boldsymbol{\delta}}_e$ given by (2.23) in a similar manner. First, compare the expression of $\widehat{\boldsymbol{\delta}}$ in (2.22) with that of $\widetilde{\boldsymbol{\delta}}$ in (2.13). The matrices $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{U}}$ given by (2.24) and (2.25) are designed to mimic $\widetilde{\mathbf{D}} = \left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}\right)$ and $\widetilde{\mathbf{U}} = \left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}\right)$ in (2.13), respectively. Note that $\widehat{\mathbf{U}}$ is the average of the block-wise analogue of $\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}$ across blocks. In fact, it turns out that the probability limits of $\widehat{\mathbf{U}}$ and $\widetilde{\mathbf{U}}$ are identical under Assumptions 1-5. Next, we provide the intuition that $\widehat{\mathbf{D}}$ mimics $\widetilde{\mathbf{D}}$. For expositional simplicity, we consider the traded-factor case, i.e., $\boldsymbol{\lambda}_f = \boldsymbol{\mu}_f$. Then, the return generating process of (2.3) can be rewritten as follows:

$$
\mathbf{R}_{[b]} = \mathbf{X}\mathbf{F}'_{\triangle,[b]} + \mathbf{E}_{[b]},
$$

where $\mathbf{X} = \begin{bmatrix}\lambda_0\mathbf{1}_{N_{[b]}} & \mathbf{B}_{[b]}\end{bmatrix}$. Then, with the realized value of the linear SDF over the b-th block, denoted by the $(\tau \times 1)$ vector of $\mathbf{m}_{[b]} = \mathbf{F}_{\triangle,[b]}\boldsymbol{\delta}$, the realized mispricing can be written as

$$\mathbf{1}_{N,[b]} - \frac{\mathbf{R}_{[b]}\mathbf{m}_{[b]}}{\tau} = \mathbf{1}_{N,[b]} - \frac{\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\delta} = \mathbf{1}_{N,[b]} - \left(\mathbf{X}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} + \frac{\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)\boldsymbol{\delta}.$$

Hence, if we simply regress the true price of $\mathbf{1}_{N,[b]}$ on $\frac{\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}$ to estimate $\boldsymbol{\delta}$, a bias will be induced by the non-negligible term of $\frac{\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}$ with the finite τ in the regressor. That is, even though this term has zero expectation and disappears as τ approaches infinity, it is nonzero for any finite τ. This is why we need to deduct $\frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}$ (mimicking $\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{E}'_{[b]}\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2}$) from $\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2}$ as in (2.24), while we do not need such an adjustment for $\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}$ in (2.13). Furthermore, we need a slight adjustment of multiplying the inverse of the sample moment $\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)$ over the b-th block to properly average the adjusted values across blocks, and then we rescale the adjustment by multiplying the sample moment $\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)$ calculated over the whole time series. We incorporate these modifications to $\widehat{\mathbf{D}}$ given by (2.25) and find that $\widehat{\mathbf{D}}$ using unbalanced panel data recovers the probability limit of $\widetilde{\mathbf{D}} = \frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}$ using balanced panel data.

Recall that, in the previous subsection on the balanced panel estimator, we show that the stochastic discount factor can be consistently estimated even without observing the true factors (see Corollary 2.1) through PCA-based methods. We find that the unbalanced panel estimator also has such a desired property. In fact, we can recover SDF consistently with the estimated factors using the same SDF estimator for the unbalanced panel data. The following corollary confirms that the unbalanced panel estimators in Theorem 2.2 still recover the true stochastic discount factors.

Corollary 2.2. *Under Assumptions 1, 3, 4 and 5, it holds that* $\widehat{m}_t^* = \widehat{\delta}_0^* + \mathbf{f}_t^{*\prime}\widehat{\boldsymbol{\delta}}_f^*$ *and* $\widehat{m}_{e,t}^* = 1 + \mathbf{f}_t^{*\prime}\widehat{\boldsymbol{\delta}}_e^*$ *converge to* m_t *and* $m_{e,t}$ *given by* (2.5) *and* (2.9)*, respectively, when the* $((K+1)\times 1)$ *vector of* $\widehat{\boldsymbol{\delta}}^* = \begin{bmatrix}\widehat{\delta}_0^* & \widehat{\boldsymbol{\delta}}_f^{*\prime}\end{bmatrix}'$ *and the* $(K \times 1)$ *vector of* $\widehat{\boldsymbol{\delta}}_e^*$ *are constructed by*

$$\widehat{\boldsymbol{\delta}}^* = \left(\widehat{\mathbf{D}}^*\right)^{-1}\widehat{\mathbf{U}}^* \tag{2.28}$$

$$\widehat{\boldsymbol{\delta}}_e^* = -\left(\widehat{\mathbf{D}}_e^*\right)^{-1}\widehat{\mathbf{U}}_e^*. \tag{2.29}$$

The matrices of $\widehat{\mathbf{D}}^*$, $\widehat{\mathbf{U}}^*$, $\widehat{\mathbf{D}}_e^*$, *and* $\widehat{\mathbf{U}}_e^*$ *are the analogues of* $\widehat{\mathbf{D}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{D}}_e$, *and* $\widehat{\mathbf{U}}_e$ *where* $\mathbf{F}$, $\mathbf{F}_{\triangle}$, $\mathbf{F}_{[b]}$, *and* $\mathbf{F}_{\triangle,[b]}$ *are replaced by* $\mathbf{F}^*$, $\mathbf{F}_{\triangle}^* = \left[\mathbf{1}_T\ \mathbf{F}^*\right]$, $\mathbf{F}_{[b]}^*$, *and* $\mathbf{F}_{\triangle,[b]}^* = \left[\mathbf{1}_\tau\ \mathbf{F}_{[b]}^*\right]$, *respectively.*

2.2.2 Asymptotic Variance

We examine the asymptotic variance of our SDF estimator for unbalanced panel data. In particular, we analyze the asymptotic distribution of $\widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\delta}}_e$ defined in Theorem 2.2.

We require additional regularity conditions on the panel data structure and impose smoothness on the behavior of cross-sectional and time-series variables.

Assumption 6. *As $N, T \to \infty$, the following conditions hold:*
(i) there exists a small positive constant δ such that and $\frac{N^{1-\delta}}{T} \to \infty$ and $\frac{N_{[b]}}{N} > \delta$ for all $b = 1, \cdots, B$,
(ii) $\sqrt{N_{[b]}}\left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{1}_{N_{[b]}} - \boldsymbol{\mu}_\beta\right) \xrightarrow{d} N\left(0, \mathbf{V}_{1,[b]}\right)$, $\sqrt{N_{[b]}}\left(vec\left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{B}_{[b]} - \mathbf{V}_\beta\right)\right) \xrightarrow{d} N\left(0, \mathbf{V}_{2,[b]}\right)$, $\sqrt{N_{[b]}}\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{1}_{N_{[b]}}\right) \xrightarrow{d} N\left(0, \mathbf{V}_{3,[b]}\right)$, $\sqrt{N_{[b]}}\left(vec\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{B}_{[b]}\right)\right) \xrightarrow{d} N\left(0, \mathbf{V}_{4,[b]}\right)$, $\sqrt{N_{[b]}}\left(vec\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N} - \mathbf{V}_{e,[b]}\right)\right) \xrightarrow{d} N\left(0, \mathbf{V}_{5,[b]}\right)$, *and there exists a positive constant $M_1 < \infty$ such that the maximum eigenvalue of $\mathbf{V}_{i,[b]}$ is smaller than M_1 for all $i = 1, \cdots, 5$ and $b = 1, \cdots, B$,*
(iii) $\sqrt{T}vec\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle - \mathbf{V}_{\triangle,f}\right) \xrightarrow{d} N\left(0, \mathbf{V}_{f^2}\right)$ *where* $\lim_{T\to\infty}\frac{\mathbf{F}'_\triangle\mathbf{F}_\triangle}{T} = \mathbf{V}_{\triangle,f}$ *and $\mathbf{V}_{f^2}$ is a $(K+1)\times(K+1)$ positive semidefinite matrix.*

The condition (i) states that the cross-sectional size in each block does not grow at the slower rate than the total cross-sectional size and that the size of cross-section grows at a slightly faster rate than that of time-series, which are reasonable to describe the structure of the individual stock return data. The condition (ii) strengthens Assumption 4(ii) by constraining that various cross-sectional errors are distributed in a manner such that the central limit theorem is applied in the cross-sectional dimension. Similarly, the condition (iii) requires that the product of factors behaves smoothly enough so that the central limit theorem kicks in over time.

By adding the above assumption, we can identify the asymptotic distribution of $\widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\delta}}_e$. Noting that $\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}}$ and $\widehat{\boldsymbol{\delta}}_e = \widehat{\mathbf{D}}_e^{-1}\widehat{\mathbf{U}}_e$ as shown in Theorem 2.2, we see that the asymptotic distribution of $\widehat{\boldsymbol{\delta}}$ ($\widehat{\boldsymbol{\delta}}_e$) is determined by the asymptotic distribution of $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{U}}$ ($\widehat{\mathbf{D}}_e$ and $\widehat{\mathbf{U}}_e$). Denote $\mathbf{D}$, $\mathbf{U}$, $\mathbf{D}_e$ and $\mathbf{U}_e$ be the probability limits of $\widehat{\mathbf{D}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{D}}_e$ and $\widehat{\mathbf{U}}_e$, respectively. Then, after some algebra (see Lemmas A.31-A.34 in the appendix), we find that

$$
\begin{aligned}
\sqrt{T}\text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right) &= \mathbf{\Pi}_D\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),\\
\sqrt{T}\left(\widehat{\mathbf{U}}-\mathbf{U}\right) &= \mathbf{\Pi}_U\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),\\
\sqrt{T}\text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}_e\right) &= \mathbf{\Pi}_{D_e}\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),\\
\sqrt{T}\left(\widehat{\mathbf{U}}_e-\mathbf{U}_e\right) &= \mathbf{\Pi}_{U_e}\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),
\end{aligned}
$$

where the expressions of $\mathbf{\Pi}_D$, $\mathbf{\Pi}_U$, $\mathbf{\Pi}_{D_e}$, $\mathbf{\Pi}_{U_e}$ are given in Lemmas A.31, A.32, A.33, A.34, respectively.

Next, using the delta method and the above distributions of $\widehat{\mathbf{D}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{D}}_e$ and $\widehat{\mathbf{U}}_e$, we establish the asymptotic distribution of $\widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\delta}}_e$ in the following theorem.

Theorem 2.3. *Under Assumptions 1, 4, 5, 6, as $N,T\to\infty$,*

$$\sqrt{T}\left(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)\xrightarrow{d}N\left(0,\mathbf{\Sigma}_{\boldsymbol{\delta}}\right)$$

and

$$\sqrt{T}\left(\widehat{\boldsymbol{\delta}}_e-\boldsymbol{\delta}_e\right)\xrightarrow{d}N\left(0,\mathbf{\Sigma}_{\boldsymbol{\delta}_e}\right),$$

where

$$\mathbf{\Sigma}_{\boldsymbol{\delta}}=\mathbf{\Psi}\mathbf{\Pi}\mathbf{V}_{f^2}\mathbf{\Pi}'\mathbf{\Psi},\ \ \mathbf{\Psi}=\left[1\ \ -\boldsymbol{\delta}'\right]\otimes\mathbf{D}^{-1}\ \mathbf{\Pi}=\left[\mathbf{\Pi}_U'\ \mathbf{\Pi}_D'\right]'$$

and

$$\mathbf{\Sigma}_{\boldsymbol{\delta}_e}=\mathbf{\Psi}_e\mathbf{\Pi}_e\mathbf{V}_{f^2}\mathbf{\Pi}_e'\mathbf{\Psi}_e,\ \ \mathbf{\Psi}_e=-\left[1\ \boldsymbol{\delta}_e'\right]\otimes\mathbf{D}_e^{-1}\ \mathbf{\Pi}_e=\left[\mathbf{\Pi}_{U_e}'\ \mathbf{\Pi}_{D_e}'\right]'.$$

See Lemmas A.31, A.32, A.33, A.34 for the expressions of $\mathbf{\Pi}_D$, $\mathbf{\Pi}_U$, $\mathbf{\Pi}_{D_e}$, $\mathbf{\Pi}_{U_e}$ *and Lemmas A.15, A.16 for the expressions of* $\mathbf{D}$, $\mathbf{U}$, $\mathbf{D}_e$ *and* $\mathbf{U}_e$.

Note that the expressions of $\mathbf{\Psi}$ and $\mathbf{\Psi}_e$ are given by the delta method and that the middle part of $\mathbf{\Pi}\mathbf{V}_{f^2}\mathbf{\Pi}'$ ($\mathbf{\Pi}_e\mathbf{V}_{f^2}\mathbf{\Pi}_e'$), which is determined by the joint distribution of $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{U}}$ ($\widehat{\mathbf{D}}_e$ and $\widehat{\mathbf{U}}_e$).

Finally, we propose an estimator for the asymptotic variance. From the above theorem, we have the following equation:

$$\sqrt{T}\left(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right)=\boldsymbol{\Psi}\sqrt{T}\begin{bmatrix}\widehat{\mathbf{U}}-\mathbf{U}\\ \text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)\end{bmatrix}+o_p(1).$$

Because we can readily construct a consistent estimator for $\boldsymbol{\Psi}$ from Theorem A.5, we focus on the estimation of the asymptotic variance of $\sqrt{T}\begin{bmatrix}\widehat{\mathbf{U}}-\mathbf{U}\\ \text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)\end{bmatrix}$, or $\boldsymbol{\Pi}\mathbf{V}_{f^2}\boldsymbol{\Pi}'$. For the estimation for $\boldsymbol{\Pi}\mathbf{V}_{f^2}\boldsymbol{\Pi}'$, we assume the block-wise independence of factors as follows:[4]

Assumption 7. *As $T\to\infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}vec\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)vec\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)'\to\frac{1}{\tau}\mathbf{V}_{f^2}.$$

Note that $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{D}}$ in Theorem 2.2 are expressed as the summation across blocks. Mimicking the summation expressions with proper adjustments, we define a block-wise error term as

$$\boldsymbol{\eta}_{[b]}=\begin{bmatrix}u_{[b]}-\widehat{\mathbf{U}}\\ \text{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}d_{[b]}-\widehat{\mathbf{D}}\right)+\text{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{[b]}\right)-\widehat{\mathbf{D}}\right)\end{bmatrix}. \tag{2.30}$$

In fact, $\boldsymbol{\eta}_{[b]}\sim\boldsymbol{\Pi}\text{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)$ and it turns out that $\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{[b]}\boldsymbol{\eta}'_{[b]}\xrightarrow{p}\frac{1}{\tau}\boldsymbol{\Pi}\mathbf{V}_{f^2}\boldsymbol{\Pi}'$ (See Lemma A.40 for details). Similarly, for the excess return case, we define

$$\boldsymbol{\eta}_{e,[b]}=\begin{bmatrix}\left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]}-\widehat{\mathbf{U}}_e\right)+\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)-\widehat{\mathbf{U}}_e\right)\\ \text{vec}\left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}d_{e,[b]}-\widehat{\mathbf{D}}_e\right)+\text{vec}\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)-\widehat{\mathbf{D}}_e\right)\end{bmatrix}, \tag{2.31}$$

which is motivated by the expressions of $\widehat{\mathbf{U}}_e$ and $\widehat{\mathbf{D}}_e$ in Theorem 2.2. Using this reconstructed error term, we can achieve that $\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{e,[b]}\boldsymbol{\eta}'_{e,[b]}\xrightarrow{p}\frac{1}{\tau}\boldsymbol{\Pi}_e\mathbf{V}_{f^2}\boldsymbol{\Pi}'_e$.

Collecting the above results, we propose estimators for $\boldsymbol{\Sigma}_{\boldsymbol{\delta}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\delta}_e}$. The consistency of those estimator is established in the following theorem.

[4] We can relax this assumption to incorporate cross block dependency. However, we do not find a standard approach to handle time series dependency across time series blocks. Hence, we assume block-wise dependency. We can construct a consistent variance estimator by accordingly modifying the estimator to reflect the dependency across blocks.

Theorem 2.4. *Under Assumptions 1, 4-7, as $N, T \to \infty$,*

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}} = \tau \widehat{\boldsymbol{\Psi}} \left(\frac{1}{B} \sum_{b=1}^{B} \boldsymbol{\eta}_{[b]} \boldsymbol{\eta}_{[b]}' \right) \widehat{\boldsymbol{\Psi}}' \xrightarrow{p} \boldsymbol{\Sigma}_{\boldsymbol{\delta}}$$

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}_e} = \tau \widehat{\boldsymbol{\Psi}}_e \left(\frac{1}{B} \sum_{b=1}^{B} \boldsymbol{\eta}_{e,[b]} \boldsymbol{\eta}_{e,[b]}' \right) \widehat{\boldsymbol{\Psi}}_e' \xrightarrow{p} \boldsymbol{\Sigma}_{\boldsymbol{\delta}_e},$$

where $\widehat{\boldsymbol{\Psi}} = \begin{bmatrix} 1 & -\widehat{\boldsymbol{\delta}}' \end{bmatrix} \otimes \widehat{\mathbf{D}}^{-1}$, $\widehat{\boldsymbol{\Psi}}_e = -\begin{bmatrix} 1 & \widehat{\boldsymbol{\delta}}_e' \end{bmatrix} \otimes \widehat{\mathbf{D}}_e^{-1}$, and $\boldsymbol{\eta}_{[b]}$ and $\boldsymbol{\eta}_{e,[b]}$ are given by (2.30) and (2.31), respectively.

Finally, combining the results in Theorems 2.2, 2.3, 2.4, we develop a feasible test on whether the k-th factor of a given asset pricing model requires significant discounting in the large panel data or not. In particular, under the null that it does not require discounting, we can use the following distribution to test the null hypothesis:

$$\sqrt{T} \frac{\widehat{\boldsymbol{\delta}}\,(k+1)}{\sqrt{\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}\,(k+1, k+1)}}, \ \sqrt{T} \frac{\widehat{\boldsymbol{\delta}}_e\,(k)}{\sqrt{\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}_e}\,(k, k)}} \sim N\,(0, 1)\,,$$

where $\mathbf{x}\,(i)$ represents the i-th element in the vector $\mathbf{x}$ and $\mathbf{X}\,(i, j)$ represents the (i, j)-th element in the matrix $\mathbf{X}$.

3 Performance of the SDF estimators in a simulated economy

We provide the simulation evidence on the properties of our SDF estimators. We simulate economies that are constructed so that returns follow a strict K-factor model and compare the estimated SDF with the true SDF. The simulation design is similar to that in Chen et al. (2018).

3.1 Calibration

To simulate returns, we need to take a stance on the return generating process in (2.3). We consider three return generating processes implied by the CAPM, the Fama and French (1993) three-factor model (FF3), and the Fama and French (2015) five-factor model (FF5). For monthly factor returns of the three models as well as the risk-free return, we use data from Ken French's database.[5] In particular, we use the U.S. value-weighted stock market excess returns for all of the three models, SMB (small minus big) and HML (high minus low) factors for FF3 and FF5, and RMW (robust minus weak) and CMA (conservative minus aggressive) factors for FF5. We use the factor realizations over 600 months (January 1967 to December 2016) to estimate the first two moments of the factors: $\boldsymbol{\mu}_f = \frac{1}{600}\sum_{t=1}^{600}\mathbf{f}_t$ and $\boldsymbol{\Sigma}_f = \frac{1}{600-1}\sum_{t=1}^{600}\left(\mathbf{f}_t - \boldsymbol{\mu}_f\right)\left(\mathbf{f}_t - \boldsymbol{\mu}_f\right)'$. The riskless gross return is estimated as the average of the gross realized risk-free return over the same period: $\lambda_0 = \frac{1}{600}\sum_{t=1}^{600} R_{f,t}$.

To obtain the parameters for a large number of assets in the simulation, we exploit all available individual stock returns over 600 months (January 1967 to December 2016) from the CRSP monthly database. We estimate the factor betas ($\boldsymbol{\beta}_i$) and the variances of residual returns $\left(\sigma^2_{i,\varepsilon} = \mathbb{E}\left[\varepsilon^2_{i,t}\right]\right)$ of individual stocks by regressing the excess returns of $R_{i,t} - R_{f,t}$ on a constant and a vector of factor returns:

$$R_{i,t} - R_{f,t} = \alpha_i + \boldsymbol{\beta}_i'\mathbf{f}_t + e_{i,t}.$$

After this process, we have the estimated betas ($\boldsymbol{\beta}_i$) and the variance of residual returns $\left(\sigma^2_{i,e} = \mathbb{E}\left[e^2_{i,t}\right]\right)$ for each 14,277 individual stocks that have more than 60 observations over our sample period, January 1967 to December 2016.

3.2 Simulation Evidence

We simulate economies for the three asset pricing models of CAPM, FF3, and FF5 with N stocks over T periods, where N and T are set by N = 500, 1,000, 2,000, and 4,000 and T =60, 120, 240, and 480. The N stocks are randomly selected, with replacement, from 14,277 stocks available on CRSP over our sample period. If the j-th asset in the simulation is chosen to be asset i from CRSP, then it is assigned the beta vector($\boldsymbol{\beta}_i$) and the variance of residual returns $\left(\sigma^2_{i,e} = \mathbb{E}\left[e^2_{i,t}\right]\right)$ calibrated for the i-th stock in CRSP. We draw $\mathbf{f}_t \sim \mathcal{N}\left(\boldsymbol{\mu}_f, \boldsymbol{\Sigma}_f\right)$ and $e_{i,t} \sim \mathcal{N}\left(0, \sigma^2_{i,e}\right)$ for $t = 1, \cdots, T$ and $j = 1, \cdots, N$ in each repetition. With the calibrated $\boldsymbol{\beta}_i$ and λ_0 and the simulated $\mathbf{f}_t$ and $e_{i,t}$, the return process described in (2.3) can be generated. Note that $\boldsymbol{\lambda}_f = \boldsymbol{\mu}_f$ in this economy because $\mathbf{f}_t$ is traded for the three asset pricing models under consideration.

We examine the performance of our SDF estimator by comparing the estimated

[5]See `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

SDF with the true SDF given by (2.5) for gross returns and (2.9) for excess returns. In particular, since the estimated SDF, asymptotically, is the true SDF plus estimation error, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m:

$$\widehat{m}_t = a + b \cdot m_t + error_t.$$

If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We use these three statistics of R^2, a, and b as metrics for the performance of the SDF estimator. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Tables 1 (gross returns) and 2 (excess returns) report the SDF estimator performance in an economy where the RGP follows FF5. We repeat the same exercise for the CAPM as well as FF3 and report those results in Tables A1-A4, in the online Appendix. Panel A of each table shows the performance of the unbalanced panel estimators derived in Theorem 2.2. We set $\tau = 30$. Hence, T =60, 120, 240, and 480 corresponds to B =2, 4, 8, and 16. Under the infeasible assumption of observability of full panel data, Panel B of each table reports the results from estimators in Theorem 2.1 to highlight (i) the good performance of our unbalanced panel estimator (Panel A) over the balanced (hence infeasible at individual stock level) panel estimator (Panel B) and (ii) the importance of corrections in our unbalanced panel estimator for bias reduction. Note that the balanced panel estimators in Theorem 2.1 can be interpreted as special cases of the unbalanced panel estimators in Theorem 2.2 where there is only one block without any bias correction. Furthermore, to investigate the implication of Corollaries 2.1 and 2.2, stating that our SDF estimators are robust to the case of using estimated factors, we consider both cases of using true factors (Panels A-1 and B-1) and estimated factors (Panels A-2 and B-2). To estimate pervasive factors, we use the asymptotic principal components (APC) method of Connor and Korajczyk (1986) applied to the simulated returns. Also, to emphasize the importance of small T bias correction even in the full panel data (which can be practically applicable to large cross section of portfolios), we report the results of our SDF estimator with bias correction in Panel B-3 and B-4 by treating the full panel over T as a single block. Lastly, for comparison, we report the performance of Pukthuanthong and Roll (2017) estimator in Panel C of Table 1 for gross returns. Note that Pukthuanthong and Roll's (2017) estimator is not applicable to excess returns.

In terms of R^2, the performance using true factors and estimated factors becomes similar as N and T increase. For example, in Panel A of Table 2, when $N = 4000$ and $T = 480$, the average R^2 is 0.91 for the case using the true factors and 0.85 for the case using estimated factors. The bias for the intercept (a) and slope (b) in Panel B-1 of Table 1 is attenuated in Panel A-1 due to the correction terms in the unbalanced panel estimator when N and T are large. However, when N is small, the unbalanced panel estimator still suffers from the bias due to the errors in the estimated factors (Panels A-1 versus) A-2. Interestingly, the comparison between Panels B-1 and B-3 or Panels B-2 vs B-4 of Table 1 reveals that the bias correction is quite effective even in the balanced estimator. We report the performance of Pukthuanthong and Roll (2017) estimator in Panel C of Table 1. We first note that their estimator has much lower R^2 than ours. For example, for $N = 2000$ and $T = 480$, the average R^2 is only 13%. Additionally the average R^2 values actually decline as T increases.[6] This evidence shows that being slightly less agnostic by imposing a more restrictive factor structure on asset returns and using a small number of extracted factors leads to significant improvement in the performance of the estimated SDF, compared to the fully agnostic approach by Pukthuanthong and Roll (2017). Also, their estimator has bias in the intercept (a) and slope (b) especially with small T.

We investigate the SDF specification tests using our unbalanced panel estimator in Theorem 2.2 and the variance estimator in Theorem 2.4. In particular, we focus on the empirical rejection frequencies of t-statistics for each coefficient in SDF. Table 3 (4) reports the rejection frequency when the gross (excess) returns follows CAPM (Panel A), FF3 (Panel B) and FF5 (Panel C). We allow $N = 1000,\ 2000$ and $(T, \tau) = (450, 30),\ (750, 50),\ (600, 30),\ (1000, 50)$ to cover various cases of empirically relevant sample sizes. We consider three nominal levels of significance, 1%, 5%, and 10%, and compute the corresponding empirical rejection frequencies from 10,000 Monte Carlo repetitions. The reported figures show that our novel tests yield rejection frequencies reasonably close to the corresponding nominal levels of significance.

The simulation exercise shows that our SDF estimators have some desirable properties. As N and T increase to a size typical of financial panel data in developed markets, R^2 in the regression of the estimated SDF on the true SDF approaches 1. Furthermore,

[6] Some readers may find this result puzzling, given Proposition 2.1. However, untabulated simulation results show that increasing N to 1 million yields R^2 values for the Pukthuanthong and Roll (2017) estimator close to 1.

the intercept (slope) converges to 0 (1) with large, but empirically relevant, values of N and T. Also, we find that SDFs based on APC factor estimates perform similarly to those using known factors when N is large. The SDF estimators for the excess returns suffer less from the small T bias and show faster convergence to the true SDF than the estimators for the gross returns. Resorting to the superior performance of SDF estimators using excess returns relative to those using gross returns, we mainly focus on the estimators utilizing excess returns in our empirical results, reported below.

4 Empirical Application

In this section, we apply our SDF estimator to U.S. individual stock return data. Recall that our SDF estimator can be used either for a set of factors proposed by a specific asset pricing model (e.g., Sharpe's (1964) CAPM) or a set of statistical factors (e.g, Connor and Korajczyk (1986)) or non traded factors. Hence, we consider various cases.

In particular, we consider eight asset pricing models in total. The first set of six models are those with a specific set of factors and the second set of two models are with statistical factors. The list of models in the first set is as follows: CAPM, FF3, HXZ4 (Hou et al. 2015), FF5 (Fama and French, 2015), PS5 (Pástor and Stambaugh, 2003), BS6 (Barillas and Shanken, 2017) and two asset pricing models with statistical factors. As noted above, the CAPM is a model with a single factor of market excess return. FF3 considers two additional factors of size (SMB) and value (HML). HXZ4 augment the set of factors by adding profitability (ROE) and investment (I/A). However, they drop the value factor with the claim that the value factor becomes redundant with their two new factors. FF5 use different factors for profitability (RMW) and investment (CMA). Also, we specifically consider PS5 which contains a non-traded liquidity factor. They propose a five factor model which extends FF3 by including momentum (MOM) and non-traded liquidity (LIQ) factors. BS6 revive the value factor by using the monthly updated version (HML devil) in conjunction with the momentum (MOM) factor. Barillas and Shanken (2017) argue that the model with the six factors of market, size, value, momentum, profitability (ROE) and investment (I/A) performs the best relative to other potential combinations.[7]

[7]We obtain factors for CAPM, FF3, FF5 from French's database and those for HXZ4 from the authors of Hou et al. (2015). The non-traded liquidity factor (LIQ) is from Pástor's web site

Besides, we exploit Corollary 2.2, confirming that our SDF estimator is applicable to statistical factors, by using statistical factors from the methods in Connor and Korajczyk (1986, 1991). In particular, we extract factors from individual stock returns in a single block. Then, to overcome the rotational indeterminancy, we regress FF3 factors on the extracted factors to find a proper rotation and splice the rotated factors across blocks.

We explain the filters that we apply to individual stock return data. We consider all individual stocks which were traded in the three main exchanges of NYSE, AMEX, and NASDAQ over our sample period of 50 years from January 1967 to December 2016. The share code is required to be 10 or 11 so that only common stocks are included in our sample. We apply price filter of five dollars at the beginning of each month. After applying the three filters, we obtain 10,112 individual stocks. Note that in applying our SDF estimator in Theorem 2.2 and the associated tests in using the variance estimator in Theorem 2.4, we need to specify the block structure to the data. We split the total 600 months data from January 1967 to December 2016 into 20 blocks with equal length of 30 months. The first block is from January 1967 to June 1969 and the last block is from July 2014 to December 2016.[8] The number of stocks are ranged from 1578 to 3443 with the average of 2455 over 20 blocks.

Table 5 reports the estimates $\widehat{\boldsymbol{\delta}}_e$ and the associated t-statistics under the null that each factor does not enter the SDF. Recall that for the case of traded factors, the coefficients of $\boldsymbol{\delta}_e$ can be interpreted as the price of risk when the risk is measured by second moments (See (2.10) with $\boldsymbol{\lambda}_f = \boldsymbol{\mu}_f$). For example, in a CAPM world, if the market risk premium is 8%/year and the annual standard deviation of market return is 15%, the coefficient on the market excess returns in the SDF is roughly $-3.5 \left(\simeq -\frac{0.08}{0.15^2+0.08^2}\right)$. For models only with traded factors, we report the alternative SDF coefficients, the maximum Sharpe ratio portfolio weights $-\left(\widehat{\boldsymbol{\Sigma}}_f + \widehat{\boldsymbol{\mu}}_f \widehat{\boldsymbol{\mu}}_f'\right)^{-1} \widehat{\boldsymbol{\mu}}_f$ below t-statistics. In Panel A, the behavior of the estimated stochastic discount factor tend to align with intuition. Interestingly, across all models, the coefficient on MKT is significant with the expected negative sign. This finding is consistent with Gagliardini et al. (2016), which find significant market risk premium in various models, and which concludes that the cross-sectional ex-post risk premia on the market are similar to the time series average

`https://faculty.chicagobooth.edu/lubos.pastor/research/liq_data_1962_2017.txt`. Lastly, HML devil factor is available at `https://www.aqr.com/library/data-sets`.

[8]We also try different block sizes of 45 months and 60 months. The results are qualitatively and quantitatively similar.

of factor realizations over short horizons. We find that HML becomes almost negligible in FF3 and the sign of coefficients on HML becomes flipped in FF5, which might be a reflection of redundancy of HML as Hou et al. (2015) claim. However, it is interesting to observe that HML or HML(devil) becomes significant along with MOM in PS5 or BS6, respectively. In HXZ4, every factor is significant in the SDF at least 10% significance level. From the results on PS5, we find that the non-traded liquidity factor does not seem to be significantly priced in the cross-section of individual stocks. All factors in BS6 except I/A appear to be important for pricing individual stocks. Lastly, for overall comparison, we allow all factors, except HML to avoid multicollinearity, to be in SDF and find that significantly priced factors agree with the results in BS6.[9] We turn to Panel B of Table 5. As explained before, we extract statistical factors using the methods in Connor and Korajczyk (1986, 1991). PC1, PC2 and PC3 are statistical factors, rotated to mimic MKT, SMB, HML factors, respectively. We find that PC1 is significantly priced in both one-factor and three-factor settings. However, PC2 and PC3 do not appear to be significantly priced in the cross-section of individual stocks. To show the robustness of our results to the choice of τ, we repeat the same exercise with $\tau = 60$ and report the results in Table A5, which are mostly consistent to Table 5.

5 Conclusion

While a large panel of unbalanced individual stock return data are available, the empirical asset pricing literature has tended to utilize small numbers of portfolios in the cross section to examine asset pricing models. Inspired by the agnostic SDF estimator of Pukthuanthong and Roll (2017), we propose novel estimators of the stochastic discount factor which are built on the intuition of minimizing the sum of squared pricing errors across a very large cross section of assets. Our estimators can be applied to prespecified factor models with either traded or non-traded factors as well as statistical factor models, such as Asymptotic Principal Components.

Our estimators are designed to extend the agnostic estimator of Pukthuanthong and Roll in several dimensions. We find that imposing a factor structure, rather than letting the number of factors equal the number of time periods, as in the agnostic estimator, leads to significant improvements in the precision of the SDF estimator. We propose

[9] We also try all factors without HML(devil) and find that HML is not significantly priced.

a bias correction that allows us to estimate the SDF over shorter time intervals, thus reducing selection biases from requiring a balanced panel of data. The SDF approach helps overcome this issue since the estimated SDF is the product of the statistical factors and a projection vector. Any rotation in the statistical factors is undone by the projection vector chosen to minimize pricing errors. Simulation evidence shows that our SDF estimators provide more precise and less biased estimates of the SDF in an economy with asset risk matching that of the U.S. equity market. The bias correction for unbalanced panels works well in eliminating the bias associated with small time-series samples. When applied to actual return data, the relation between the estimated SDF and the pervasive factors tend to be in line with long-run estimates of risk premia and factor risks. We find that the market factor commands a significant premium when pricing individual stocks across various models. The recently proposed factors in Hou et al. (2015) and Fama and French (2015) command a significant risk premia. The HML factor appears to be significant along with MOM factor as suggested by Barillas and Shanken (2017).

References

Araújo, Fabio, and João Victor Issler, 2012, A stochastic discount factor approach to asset pricing using panel data asymptotics, Unpublished Manuscript, Princeton University.

Barillas, Francisco, and Jay Shanken, 2017, Which alpha?, *The Review of Financial Studies* 30, 1316–1338.

Campbell, John Y, and Tuomo Vuolteenaho, 2004, Bad beta, good beta, *American Economic Review* 94, 1249–1275.

Chamberlain, Gary, and Michael Rothschild, 1983, Arbitrage, factor structure, and mean-variance analysis on large asset markets, *Econometrica* 1281–1304.

Chen, Nai-Fu, Richard Roll, and Stephen A Ross, 1986, Economic forces and the stock market, *Journal of business* 383–403.

Chen, Zhuo, Gregory Connor, and Robert A Korajczyk, 2018, A performance comparison of large-n factor estimators, *The Review of Asset Pricing Studies* 8, 153–182.

Cochrane, John H, 2009, *Asset pricing: Revised edition* (Princeton university press).

Connor, Gregory, 1984, A unified beta pricing theory, *Journal of Economic Theory* 34, 13–31.

Connor, Gregory, and Robert A Korajczyk, 1986, Performance measurement with the arbitrage pricing theory: A new framework for analysis, *Journal of Financial Economics* 15, 373–394.

Connor, Gregory, and Robert A Korajczyk, 1987, Estimating pervasive economic factors with missing observations, Unpublished Manuscript, Northwestern University.

Connor, Gregory, and Robert A Korajczyk, 1988, Risk and return in an equilibrium apt: Application of a new test methodology, *Journal of Financial Economics* 21, 255–289.

Connor, Gregory, and Robert A Korajczyk, 1991, The attributes, behavior, and performance of us mutual funds, *Review of Quantitative Finance and Accounting* 1, 5–26.

Fama, Eugene F, and Kenneth R French, 1992, The cross-section of expected stock returns, *Journal of Finance* 47, 427–465.

Fama, Eugene F, and Kenneth R French, 2015, A five-factor asset pricing model, *Journal of Financial Economics* 116, 1–22.

Fama, Eugene F, and James D MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *Journal of Political Economy* 81, 607–636.

Gagliardini, Patrick, Elisa Ossola, and Olivier Scaillet, 2016, Time-varying risk premium in large cross-sectional equity data sets, *Econometrica* 84, 985–1046.

Hansen, Lars Peter, and Ravi Jagannathan, 1991, Implications of security market data for models of dynamic economies, *Journal of Political Economy* 99, 225–262.

Hansen, Lars Peter, and Ravi Jagannathan, 1997, Assessing specification errors in stochastic discount factor models, *The Journal of Finance* 52, 557–590.

Hou, Kewei, Chen Xue, and Lu Zhang, 2015, Digesting anomalies: An investment approach, *Review of Financial Studies* 28, 660–705.

Huberman, Gur, and Shmuel Kandel, 1987, Mean-variance spanning, *The Journal of Finance* 42, 873–888.

Jagannathan, Ravi, and Srikant Marakani, 2015, Price-dividend ratio factor proxies for long-run risks, *The Review of Asset Pricing Studies* 5, 1–47.

Jagannathan, Ravi, Georgios Skoulakis, and Zhenyu Wang, 2010, The analysis of the cross section of security returns, *Handbook of financial econometrics* 2, 73–134.

Jagannathan, Ravi, and Zhenyu Wang, 1996, The conditional capm and the cross-section of expected returns, *The Journal of finance* 51, 3–53.

Jagannathan, Ravi, and Zhenyu Wang, 2002, Empirical evaluation of asset-pricing models: A comparison of the sdf and beta methods, *The Journal of Finance* 57, 2337–2367.

Jones, Christopher S, 2001, Extracting factors from heteroskedastic asset returns, *Journal of Financial economics* 62, 293–325.

Kelly, Bryan, and Seth Pruitt, 2013, Market expectations in the cross-section of present values, *The Journal of Finance* 68, 1721–1756.

Kelly, Bryan, Seth Pruitt, and Yinan Su, 2018, Characteristics are covariances: A unified model of risk and return, *Journal of Financial Economics* forthcoming.

Kim, Soohun, and Georgios Skoulakis, 2018a, Ex-post risk premia estimation and asset pricing tests using large cross sections: The regression-calibration approach, *Journal of Econometrics* 204, 159–188.

Kim, Soohun, and Georgios Skoulakis, 2018b, Testing ex-post implications of asset pricing models using individual stocks over short horizons, Working Paper.

Kozak, Serhiy, Stefan Nagel, and Shrihari Santosh, 2018, Shrinking the cross section, Unpublished Manuscript, University of Chicago.

Lettau, Martin, and Markus Pelger, 2018, Estimating latent asset-pricing factors, Unpublished Manuscript, University of California at Berkeley.

Litzenberger, Robert H, and Krishna Ramaswamy, 1979, The effect of personal taxes and dividends on capital asset prices: Theory and empirical evidence, *Journal of Financial Economics* 7, 163–195.

Merton, Robert C., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–887.

Pástor, L'uboš, and Robert F Stambaugh, 2003, Liquidity risk and expected stock returns, *Journal of Political Economy* 111, 642–685.

Pukthuanthong, Kuntara, and Richard Roll, 2017, An agnostic and practically useful estimator of the stochastic discount factor, Unpublished Manuscript, Caltech.

Ross, Stephen A, 1976, The arbitrage theory of capital asset pricing, *Journal of Economic Theory* 13, 341–360.

Shanken, Jay, 1992, On the estimation of beta-pricing models, *The review of financial studies* 5, 1–33.

Shanken, Jay, and Mark I Weinstein, 2006, Economic forces and the stock market revisited, *Journal of Empirical Finance* 13, 129–144.

Sharpe, William F, 1964, Capital asset prices: A theory of market equilibrium under conditions of risk, *The Journal of Finance* 19, 425–442.

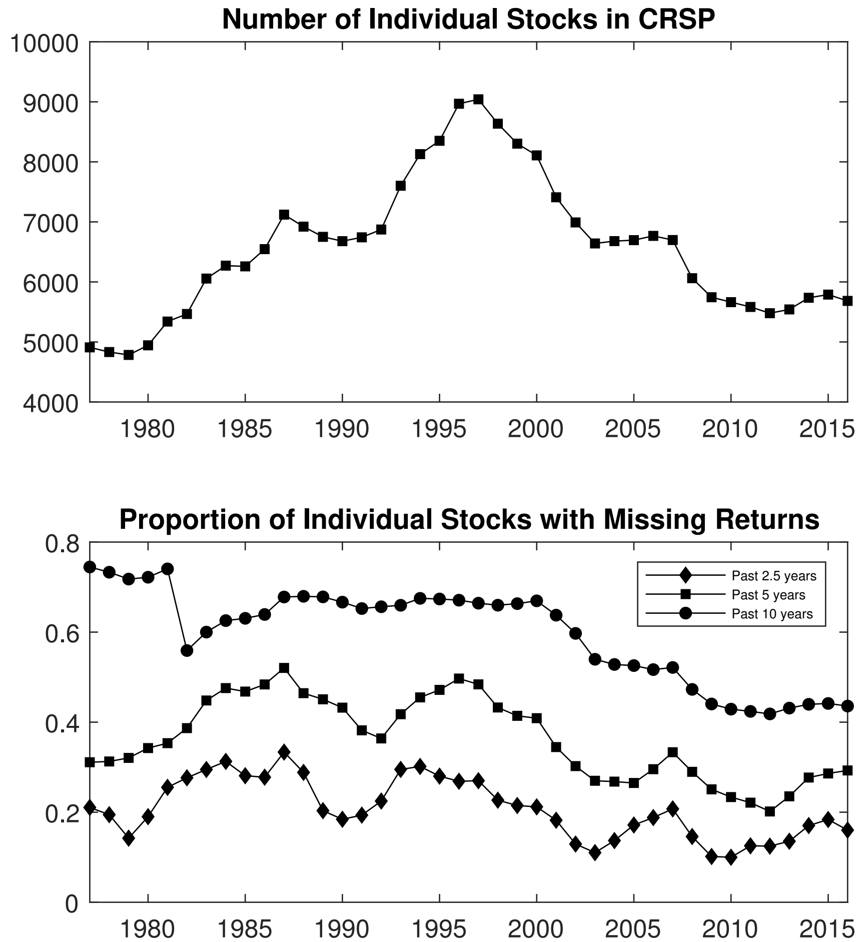
Stock, James H., and Mark W. Watson, 1998, Diffusion indexes, NBER Working paper 6702.

Stock, James H, and Mark W Watson, 2002, Forecasting using principal components from a large number of predictors, *Journal of the American Statistical Association* 97, 1167–1179.

Stroyny, Alvin L, 1992, Still more on em factor analysis, Unpublished Manuscript, University of Wisconsin.

A Figures and Tables

Figure 1: Unbalanced Panel of CRSP Data



The top graph shows the total number of individual stocks in the CRSP NYSE/NASDAQ/AMEX database for at the beginning of each year: 1977-2016. The bottom graph shows the proportion of individual stocks with missing returns over the past 2.5, 5 or 10 years.

Table 1: SDF Estimator Performance when Gross Returns Follow FF5

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.19	0.20	0.23	0.27	-0.74	1.97	-0.30	-0.05	1.75	-1.10	1.30	1.06
1000	0.21	0.23	0.28	0.37	-0.46	0.10	-0.06	-0.01	1.44	0.91	1.06	1.02
2000	0.26	0.30	0.37	0.49	-0.12	-0.07	-0.02	-0.01	1.14	1.08	1.03	1.01
4000	0.31	0.38	0.49	0.63	-0.10	-0.03	-0.02	-0.01	1.12	1.04	1.02	1.02
A-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.19	0.22	0.27	0.36	0.21	0.30	0.30	0.30	0.80	0.70	0.70	0.70
1000	0.23	0.27	0.36	0.47	0.15	0.20	0.20	0.21	0.86	0.80	0.80	0.79
2000	0.26	0.34	0.45	0.59	0.13	0.12	0.13	0.13	0.88	0.88	0.87	0.87
4000	0.32	0.42	0.56	0.71	0.03	0.06	0.07	0.06	0.98	0.95	0.93	0.94
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.49	0.67	0.81	0.90	0.45	0.28	0.16	0.09	0.56	0.72	0.84	0.91
1000	0.53	0.69	0.83	0.91	0.44	0.28	0.16	0.09	0.56	0.72	0.85	0.92
2000	0.55	0.72	0.84	0.91	0.44	0.28	0.16	0.09	0.56	0.73	0.84	0.91
4000	0.56	0.73	0.84	0.92	0.44	0.27	0.16	0.08	0.56	0.73	0.84	0.92
B-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.32	0.45	0.56	0.64	0.57	0.47	0.39	0.35	0.43	0.53	0.61	0.65
1000	0.40	0.53	0.65	0.72	0.53	0.41	0.32	0.27	0.47	0.59	0.68	0.73
2000	0.46	0.61	0.72	0.79	0.50	0.36	0.26	0.20	0.51	0.64	0.74	0.80
4000	0.51	0.67	0.78	0.86	0.47	0.32	0.21	0.14	0.53	0.68	0.79	0.86
B-3: With Observed Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.46	0.65	0.80	0.90	-0.13	-0.05	-0.02	-0.01	1.15	1.05	1.03	1.01
1000	0.51	0.69	0.83	0.91	-0.08	-0.03	-0.02	-0.01	1.10	1.03	1.03	1.01
2000	0.53	0.71	0.83	0.91	-0.05	-0.02	-0.01	0.00	1.07	1.03	1.01	1.01
4000	0.55	0.72	0.84	0.91	-0.04	-0.02	-0.01	-0.01	1.06	1.03	1.01	1.01
B-4: With Estimated Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.33	0.45	0.57	0.64	0.33	0.31	0.30	0.30	0.68	0.69	0.70	0.70
1000	0.40	0.54	0.65	0.72	0.22	0.21	0.21	0.21	0.79	0.79	0.80	0.79
2000	0.45	0.61	0.72	0.79	0.14	0.14	0.14	0.14	0.88	0.87	0.86	0.87
4000	0.50	0.66	0.78	0.86	0.07	0.07	0.07	0.06	0.94	0.94	0.93	0.94
Panel C: Pukthuanthong and Roll's (2017) Estimator												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.11	0.09	0.05	0.00	0.46	0.29	0.17	0.09	0.55	0.71	0.83	0.92
1000	0.17	0.16	0.11	0.06	0.45	0.29	0.16	0.09	0.55	0.71	0.84	0.91
2000	0.24	0.24	0.20	0.13	0.45	0.28	0.17	0.09	0.55	0.72	0.83	0.91
4000	0.32	0.35	0.31	0.23	0.45	0.28	0.17	0.09	0.56	0.72	0.84	0.91

This table summarizes the performance of various SDF estimators for gross returns when the true return generating process of each individual asset follows FF5. We consider different levels of $N =$ 500, 1000, 2000, and 4000 and $T =$ 60, 120, 240, and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table 2: SDF Estimator Performance when Excess Returns Follow FF5

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.38	0.55	0.71	0.83	-0.08	-1.18	-0.01	0.00	0.92	1.97	0.90	0.90
1000	0.46	0.63	0.77	0.87	-0.02	0.00	-0.01	0.00	0.87	0.88	0.90	0.90
2000	0.51	0.67	0.80	0.89	0.00	0.00	0.00	0.00	0.85	0.88	0.89	0.90
4000	0.54	0.70	0.82	0.91	0.00	0.00	0.00	0.00	0.84	0.88	0.89	0.90
A-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.33	0.45	0.56	0.63	0.28	0.27	0.26	0.26	0.59	0.63	0.65	0.66
1000	0.40	0.53	0.64	0.72	0.20	0.19	0.18	0.18	0.67	0.71	0.73	0.74
2000	0.45	0.60	0.71	0.78	0.13	0.12	0.12	0.12	0.72	0.77	0.78	0.79
4000	0.50	0.65	0.77	0.85	0.08	0.07	0.06	0.06	0.77	0.82	0.83	0.85
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.55	0.70	0.82	0.90	0.00	0.00	0.00	0.00	0.85	0.88	0.89	0.90
1000	0.56	0.71	0.84	0.91	0.00	0.00	-0.01	0.00	0.85	0.88	0.90	0.90
2000	0.56	0.72	0.84	0.91	0.00	0.00	0.00	0.00	0.84	0.88	0.89	0.90
4000	0.57	0.73	0.84	0.92	0.00	0.00	0.00	0.00	0.84	0.88	0.89	0.90
B-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.34	0.47	0.57	0.64	0.28	0.27	0.26	0.26	0.59	0.63	0.65	0.67
1000	0.41	0.54	0.66	0.73	0.19	0.19	0.18	0.18	0.67	0.71	0.73	0.74
2000	0.46	0.61	0.72	0.79	0.13	0.12	0.12	0.12	0.72	0.77	0.78	0.79
4000	0.50	0.66	0.78	0.86	0.08	0.07	0.06	0.06	0.77	0.82	0.83	0.85
B-3: With Observed Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.47	0.66	0.81	0.90	0.00	0.00	0.00	0.00	0.85	0.88	0.89	0.90
1000	0.52	0.69	0.83	0.91	0.00	0.00	-0.01	0.00	0.85	0.87	0.90	0.90
2000	0.54	0.71	0.83	0.91	0.00	0.00	0.00	0.00	0.84	0.88	0.89	0.90
4000	0.55	0.72	0.84	0.92	0.00	0.00	0.00	0.00	0.84	0.88	0.89	0.90
B-4: With Estimated Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.34	0.47	0.57	0.64	0.28	0.27	0.26	0.26	0.59	0.63	0.65	0.67
1000	0.41	0.54	0.66	0.73	0.19	0.19	0.18	0.18	0.67	0.71	0.73	0.74
2000	0.46	0.61	0.72	0.79	0.13	0.12	0.12	0.12	0.72	0.77	0.78	0.79
4000	0.50	0.66	0.78	0.86	0.08	0.07	0.06	0.06	0.77	0.82	0.83	0.85

This table summarizes the performance of various SDF estimators for excess returns when the true return generating process of each individual asset follows FF5. We consider different levels of $N =$ 500, 1000, 2000, and 4000 and $T =$ 60, 120, 240, and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table 3: Empirical Rejection Frequencies of SDF Specification Tests using Gross Returns

N	1000												2000											
(T,τ)	(450,30)			(750,50)			(600,30)			(1000,50)			(450,30)			(750,50)			(600,30)			(1000,50)		
Nominal Size	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0
Panel A: CAPM																								
$\delta_{\mathtt{MKT}}$	2.1	7.6	13.7	2.6	8.4	14.1	1.6	6.7	12.4	2.1	7.2	13.3	2.4	8.3	14.1	2.7	8.5	14.9	2.1	7.2	13.3	2.2	7.8	13.6
Panel B: FF3																								
$\delta_{\mathtt{MKT}}$	2.1	7.2	13.5	2.6	7.7	13.6	1.5	6.2	11.9	1.8	7.0	12.4	2.3	7.2	13.0	2.4	8.1	14.1	1.4	6.5	12.3	1.9	7.2	13.1
$\delta_{\mathtt{SMB}}$	2.1	6.9	12.9	2.2	8.2	14.2	1.6	6.2	12.0	2.0	7.3	12.7	2.3	7.6	13.4	2.3	7.8	13.4	1.5	6.3	12.1	1.9	7.2	13.0
$\delta_{\mathtt{HML}}$	1.7	6.8	12.6	2.5	8.2	13.9	1.8	7.0	12.8	1.8	6.9	12.5	2.0	7.2	13.1	2.5	8.5	14.4	1.8	7.1	12.5	2.1	7.1	12.9
Panel C: FF5																								
$\delta_{\mathtt{MKT}}$	1.5	6.0	11.7	2.1	7.4	13.1	1.4	5.7	10.5	1.7	7.2	12.9	1.9	7.4	13.2	2.2	8.0	14.0	1.6	6.2	11.4	2.2	7.4	13.2
$\delta_{\mathtt{SMB}}$	1.5	6.2	11.5	2.3	7.6	13.5	1.5	5.8	11.0	1.9	7.0	12.3	1.7	6.9	12.6	2.3	7.8	13.4	1.6	6.5	12.0	1.9	7.0	12.9
$\delta_{\mathtt{HML}}$	1.6	6.4	12.0	2.2	7.4	13.2	1.5	6.0	11.3	2.0	6.6	12.4	1.8	7.2	13.2	2.7	8.1	14.0	1.9	6.7	12.2	1.9	7.4	13.2
$\delta_{\mathtt{RMW}}$	1.5	6.5	11.7	2.3	8.2	13.8	1.3	6.0	11.6	2.1	7.2	13.0	2.0	7.1	13.1	2.4	8.0	13.6	1.5	6.4	11.9	2.0	7.2	13.1
$\delta_{\mathtt{CMA}}$	1.7	6.2	11.7	2.1	7.3	13.0	1.4	6.0	11.4	1.7	6.7	12.2	1.8	7.2	13.1	2.5	7.9	13.7	1.7	6.2	12.2	2.0	6.9	12.5

This table presents simulation results on the rejection frequency (size) of the SDF specification tests based on our unbalanced panel estimator when the gross returns follows CAPM (Panel A), FF3 (Panel B) and FF5 (Panel C). We consider $N = 1000,\ 2000$ and $(T,\tau) = (450,30)\,,\ (750,50)\,,\ (600,30)\,,\ (1000,50)\,$. The reported figures are based on 10,000 Monte Carlo repetitions.

Table 4: Empirical Rejection Frequencies of SDF Specification Tests using Excess Returns

N	1000												2000											
(T,τ)	(450,30)			(750,50)			(600,30)			(1000,50)			(450,30)			(750,50)			(600,30)			(1000,50)		
Nominal Size	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0
Panel A: CAPM																								
$\delta_{\texttt{MKT}}$	2.4	7.4	12.3	2.6	7.3	12.8	1.7	6.4	11.8	2.0	6.8	11.7	2.5	7.4	12.7	2.0	6.9	12.1	2.0	7.0	12.1	2.0	6.6	11.7
Panel B: FF3																								
$\delta_{\texttt{MKT}}$	2.3	7.7	13.1	2.6	7.7	12.7	1.6	6.0	11.2	1.9	6.7	12.0	2.5	7.5	12.8	2.3	7.0	12.1	1.9	6.5	12.1	1.8	6.4	11.6
$\delta_{\texttt{SMB}}$	2.6	7.8	12.9	2.3	7.3	12.4	2.2	6.5	11.7	2.3	7.1	12.2	2.3	7.1	12.5	2.5	7.0	12.3	1.8	6.5	11.5	1.9	6.3	11.7
$\delta_{\texttt{HML}}$	2.4	7.5	12.3	2.5	7.3	13.0	2.1	6.7	11.8	2.0	6.7	11.9	2.3	7.5	12.3	2.1	7.1	12.7	1.8	6.6	11.9	2.1	6.6	12.0
Panel C: FF5																								
$\delta_{\texttt{MKT}}$	2.2	6.9	11.9	2.3	6.8	12.2	1.8	6.5	11.8	2.1	6.3	11.6	2.2	7.0	12.2	2.1	6.8	12.0	1.7	6.5	11.5	1.7	6.2	11.2
$\delta_{\texttt{SMB}}$	2.2	7.2	12.6	2.4	7.3	12.7	1.6	6.0	10.9	1.9	6.9	12.0	2.2	7.4	12.5	2.3	6.9	12.3	1.8	6.8	12.0	2.1	7.0	12.4
$\delta_{\texttt{HML}}$	2.3	7.4	12.9	2.2	7.3	12.9	1.7	6.5	11.8	2.1	6.4	11.6	2.3	7.4	12.7	2.3	7.3	12.3	1.7	6.4	11.5	1.8	6.5	11.4
$\delta_{\texttt{RMW}}$	2.5	7.6	13.0	2.5	6.9	12.1	2.1	6.6	11.6	2.2	6.7	12.0	2.3	7.1	12.4	2.4	7.4	12.6	2.0	6.7	11.7	1.9	6.5	11.8
$\delta_{\texttt{CMA}}$	2.3	7.6	12.7	2.3	7.5	12.7	1.8	6.6	12.1	1.8	6.4	11.6	2.4	7.5	12.6	2.5	7.3	12.6	1.9	6.6	11.6	1.9	7.0	12.1

This table presents simulation results on the rejection frequency (size) of the SDF specification tests based on our unbalanced panel estimator when the excess returns follows CAPM (Panel A), FF3 (Panel B) and FF5 (Panel C). We consider $N = 1000,\ 2000$ and $(T,\tau) = (450,30)\,,\ (750,50)\,,\ (600,30)\,,\ (1000,50)\,$. The reported figures are based on 10,000 Monte Carlo repetitions.

Table 5: SDF Estimation Results

	MKT	SMB	HML	I/A	ROE	CMW	RMW	MOM	LIQ	HML(devil)
Panel A: Specific Asset Pricing Models										
CAPM	-4.46									
	(-4.38)									
	-2.51									
FF3	-3.42	-2.69	-0.94							
	(-3.49)	(-1.39)	(-0.42)							
	-3.08	-1.90	-5.97							
HXZ4	-5.04	-5.81		-7.85	-12.73					
	(-4.05)	(-4.59)		(-1.73)	(-6.19)					
	-4.79	-4.84		-14.90	-10.32					
FF5	-5.45	-3.31	4.62			-8.48	-11.57			
	(-3.96)	(-1.70)	(1.48)			(-2.38)	(-2.05)			
	-4.83	-3.55	0.40			-8.74	-12.50			
PS5	-5.54	-2.60	-5.37					-8.59	0.19	
	(-6.54)	(-2.71)	(-3.08)					(-6.06)	(0.22)	
BS6	-5.41	-5.73		-5.03	-8.29			-7.43		-5.16
	(-4.28)	(-4.69)		(-1.59)	(-2.86)			(-3.85)		(-2.66)
	-4.90	-5.99		-5.55	-13.25			-5.22		-10.60
All	-5.50	-4.95		-5.79	-9.23	1.91	1.68	-7.96	-0.15	-5.92
	(-4.23)	(-4.12)		(-0.94)	(-2.17)	(0.56)	(0.24)	(-4.05)	(-0.17)	(-2.60)
Panel B: Statistical Factor Model										
	PC1	PC2	PC3							
Single-Factor	-4.58									
	(-4.38)									
Three-Factor	-3.42	-0.96	-2.75							
	(-2.08)	(-0.25)	(-0.68)							

This table reports the estimated values of $\boldsymbol{\delta}_e$ using our estimator $\widehat{\boldsymbol{\delta}}_e$ in Theorem 2.2. In panel A, we consider six asset pricing models: CAPM (Sharpe, 1964), FF3 (Fama and French, 1992), HXZ4 (Hou et al. 2015), FF5 (Fama and French, 2015), PS5 (Pástor and Stambaugh, 2003), BS6 (Barillas and Shanken, 2017). For models only with traded factors, we report the alternative SDF coefficients, $-\left(\widehat{\boldsymbol{\Sigma}}_f + \widehat{\boldsymbol{\mu}}_f \widehat{\boldsymbol{\mu}}_f'\right)^{-1} \widehat{\boldsymbol{\mu}}_f$ below t-statistics. In Panel B, we examine statistical factors computed by methods in Connor and Korajczyk (1986, 1991). The t-statistics are computed by our asymptotic variance estimator in Theorem 2.4. The sample periods are 600 months over the sample period January 1976 to December 2016. We set $\tau = 30$.

B Proofs

Proof of Theorem 2.1 First, we show that $\widetilde{\boldsymbol{\delta}} \xrightarrow{p} \boldsymbol{\delta}$, implying $\widetilde{m}_t \xrightarrow{p} m_t$. From Lemma A.2, we have that

$$\widetilde{\boldsymbol{\delta}} = \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{U}} \xrightarrow{p} \left(\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_\triangle}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}\right)^{-1}\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right]' = \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}^{-1}\mathbf{V}_{\beta_\triangle}^{-1}\left[1\ \boldsymbol{\mu}_\beta'\right]'. \tag{B.1}$$

Because

$$\begin{aligned}
\mathbf{V}_{\triangle,f} &= \begin{bmatrix} \left(1+\boldsymbol{\mu}_f'\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\mu}_f\right) & -\boldsymbol{\mu}_f'\boldsymbol{\Sigma}_f^{-1} \\ -\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\mu}_f & \boldsymbol{\Sigma}_f^{-1} \end{bmatrix} \\
\boldsymbol{\Lambda}^{-1} &= \begin{bmatrix} \frac{1}{\lambda_0} & \mathbf{0}_K' \\ \frac{1}{\lambda_0}\left(\boldsymbol{\mu}_f-\boldsymbol{\lambda}\right) & \mathbf{I}_K \end{bmatrix} \\
\mathbf{V}_{\beta_\triangle}^{-1} &= \begin{bmatrix} \left(1+\boldsymbol{\mu}_\beta'\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta\right) & -\boldsymbol{\mu}_\beta'\boldsymbol{\Sigma}_\beta^{-1} \\ -\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta & \boldsymbol{\Sigma}_\beta^{-1} \end{bmatrix},
\end{aligned}$$

it follows that

$$\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}^{-1}\mathbf{V}_{\beta_\triangle}^{-1}\left[1\ \boldsymbol{\mu}_\beta'\right]' = \frac{1}{\lambda_0}\begin{bmatrix} \left(1+\boldsymbol{\mu}_f'\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\lambda}_f\right) \\ -\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\lambda}_f \end{bmatrix} = \boldsymbol{\delta}. \tag{B.2}$$

Combining (B.1) and (B.2), we prove the first claim in the theorem.

Next, in a similar manner, we show that $\widetilde{\boldsymbol{\delta}}^e \xrightarrow{p} \boldsymbol{\delta}_e$, implying $\widetilde{m}_t^e \xrightarrow{p} m_{e,t}$. From Lemma A.3, we have that

$$\begin{aligned}
\widetilde{\boldsymbol{\delta}}^e = \widetilde{\mathbf{D}}_e^{-1}\widetilde{\mathbf{U}}_e &\xrightarrow{p} -\left(\left[\boldsymbol{\mu}_f\ \mathbf{V}_f\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[\boldsymbol{\mu}_f\ \mathbf{V}_f\right]'\right)^{-1}\left[\boldsymbol{\mu}_f\ \mathbf{V}_f\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[\begin{array}{cc} 1 & \boldsymbol{\mu}_f' \end{array}\right]' \\
&= -\left(\boldsymbol{\Lambda}_e\left[\boldsymbol{\mu}_f\ \mathbf{V}_f\right]'\right)^{-1}\boldsymbol{\Lambda}_e\left[\begin{array}{cc} 1 & \boldsymbol{\mu}_f' \end{array}\right]' = -\left(\left(\boldsymbol{\lambda}_f-\boldsymbol{\mu}_f\right)\boldsymbol{\mu}_f'+\mathbf{V}_f\right)^{-1}\left(\boldsymbol{\lambda}_f-\boldsymbol{\mu}_f+\boldsymbol{\mu}_f\right) \\
&= -\left(\boldsymbol{\lambda}_f\boldsymbol{\mu}_f'+\boldsymbol{\Sigma}_f\right)^{-1}\boldsymbol{\lambda}_f = \boldsymbol{\delta}_e. \qquad (\text{B.3})
\end{aligned}$$

The limits of (B.2) and (B.3) complete the proof of the theorem. □

Proof of Theorem 2.2 We will show that $\widehat{\boldsymbol{\delta}} \xrightarrow{p} \boldsymbol{\delta}$ and $\widehat{\boldsymbol{\delta}}_e \xrightarrow{p} \boldsymbol{\delta}_e$, implying $\widehat{m}_t \xrightarrow{p} m_t$ and $\widehat{m}_{e,t} \xrightarrow{p} m_{e,t}$, respectively. From Lemma A.15, we have that

$$\begin{aligned}\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}} &\xrightarrow{p} \left(\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_\triangle}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}\right)^{-1}\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right]' \\ &= \frac{1}{\lambda_0}\left[\begin{array}{c}\left(1+\boldsymbol{\mu}_f\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\lambda}_f\right) \\ -\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\lambda}_f\end{array}\right] = \boldsymbol{\delta},\end{aligned}$$

where the next to the last equality is from (B.2). Also, from Lemma A.16, we have that

$$\begin{aligned}\widehat{\boldsymbol{\delta}}_e &= -\left(\widehat{\mathbf{D}}_e\right)^{-1}\widehat{\mathbf{U}}_e \\ &\xrightarrow{p} -\left(\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'\right)^{-1}\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[1\ \boldsymbol{\mu}_f'\right]' \\ &= -\left(\boldsymbol{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'\right)^{-1}\boldsymbol{\Lambda}_e\left[1\ \boldsymbol{\mu}_f'\right]' = -\left(\left(\boldsymbol{\lambda}_f-\boldsymbol{\mu}_f\right)\boldsymbol{\mu}_f'+\mathbf{V}_f\right)^{-1}\left(\boldsymbol{\lambda}_f-\boldsymbol{\mu}_f+\boldsymbol{\mu}_f\right) \\ &= -\left(\boldsymbol{\lambda}_f\boldsymbol{\mu}_f+\boldsymbol{\Sigma}_f\right)^{-1}\boldsymbol{\lambda}_f = \boldsymbol{\delta}_e.\end{aligned}$$

The above two limits complete the proof of the theorem. □

Proof of Theorem 2.3 Let $\mathbf{D}$ and $\mathbf{U}$ denote the probability limit of $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{U}}$, respectively. Note that

Then,

$$\begin{aligned}\sqrt{T}\left(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right) &= \sqrt{T}\left(\widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}}-(\mathbf{D})^{-1}\mathbf{U}\right) \\ &= \widehat{\mathbf{D}}^{-1}\sqrt{T}\left(\widehat{\mathbf{U}}-\mathbf{U}\right)+\sqrt{T}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right)\mathbf{U} \\ &= \mathbf{D}^{-1}\sqrt{T}\left(\widehat{\mathbf{U}}-\mathbf{U}\right)+\sqrt{T}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right)\mathbf{U}+o_p(1) \\ &= \mathbf{D}^{-1}\boldsymbol{\Pi}_U\sqrt{T}\mathrm{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_{\triangle,f}\right)+\sqrt{T}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right)\mathbf{U}+o_p(1), \end{aligned}\tag{B.4}$$

where the fourth equality is from Lemmas A.31 and A.32. From (A.5),

$$\sqrt{T}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right)\mathbf{U} = \left(\mathbf{U}'\otimes\mathbf{I}_{K+1}\right)\sqrt{T}\mathrm{vec}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right). \tag{B.5}$$

Furthermore, using delta method,

$$\sqrt{T}\mathrm{vec}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right) = -\left(\mathbf{D}^{-1}\otimes\mathbf{D}^{-1}\right)\sqrt{T}\mathrm{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)+o_p(1). \tag{B.6}$$

Plugging (B.6) into (B.5), we have that

$$\begin{aligned}
\sqrt{T}\left(\widehat{\mathbf{D}}^{-1}-\mathbf{D}^{-1}\right)\mathbf{U} &= -\left(\mathbf{U}'\otimes\mathbf{I}_{K+1}\right)\left(\mathbf{D}^{-1}\otimes\mathbf{D}^{-1}\right)\sqrt{T}\text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)+o_p\left(1\right)\\
&= -\left(\left(\mathbf{U}'\mathbf{D}^{-1}\right)\otimes\mathbf{D}^{-1}\right)\sqrt{T}\text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)+o_p\left(1\right)\\
&= -\left(\boldsymbol{\delta}'\otimes\mathbf{D}^{-1}\right)\sqrt{T}\text{vec}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)+o_p\left(1\right)\\
&= -\left(\boldsymbol{\delta}'\otimes\mathbf{D}^{-1}\right)\mathbf{\Pi}_D\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle-\mathbf{V}_{\triangle,f}\right)+o_p\left(1\right),
\end{aligned}\tag{B.7}$$

where the third equality is from $\boldsymbol{\delta}'=\mathbf{U}'\mathbf{D}^{-1}$ and the last equality is from Lemma A.31. Finally, plugging (B.7) to (B.4) yields that

$$\begin{aligned}
\sqrt{T}\left(\widehat{\boldsymbol{\delta}}-\boldsymbol{\delta}\right) &= \left(\mathbf{D}^{-1}\mathbf{\Pi}_U+\left(\boldsymbol{\delta}'\otimes\mathbf{D}^{-1}\right)\mathbf{\Pi}_D\right)\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle-\mathbf{V}_{\triangle,f}\right)+o_p\left(1\right).\\
&= \mathbf{\Psi}\mathbf{\Pi}\text{vec}\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle-\mathbf{V}_{\triangle,f}\right)+o_p\left(1\right),
\end{aligned}\tag{B.8}$$

where

$$\begin{aligned}
\mathbf{\Psi} &= \left[1\ \ -\boldsymbol{\delta}'\right]\otimes\mathbf{D}^{-1}\\
\mathbf{\Pi} &= \left[\mathbf{\Pi}'_U\ \mathbf{\Pi}'_D\right]'.
\end{aligned}$$

Repeating the above procedures while replacing $\mathbf{D}$, $\mathbf{U}$, $\widehat{\mathbf{D}}$, $\widehat{\mathbf{U}}$ with $\mathbf{D}_e$, $-\mathbf{U}_e$, $\widehat{\mathbf{D}}_e$, $-\widehat{\mathbf{U}}_e$ yields that

$$\begin{aligned}
\sqrt{T}\left(\widehat{\boldsymbol{\delta}}_e-\boldsymbol{\delta}_e\right) &= \left(\mathbf{D}_e^{-1}\mathbf{\Pi}_{U_e}+\left(\boldsymbol{\delta}'_e\otimes\mathbf{D}_e^{-1}\right)\mathbf{\Pi}_{D_e}\right)\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle-\mathbf{V}_{\triangle,f}\right)+o_p\left(1\right).\\
&= \mathbf{\Psi}_e\mathbf{\Pi}_e\text{vec}\left(\frac{1}{T}\mathbf{F}'_\triangle\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),
\end{aligned}\tag{B.9}$$

where

$$\begin{aligned}
\mathbf{\Psi}_e &= -\left[\mathbf{D}_e^{-1}\ \ \left(\boldsymbol{\delta}'_e\otimes\mathbf{D}_e^{-1}\right)\right]\\
\mathbf{\Pi}_e &= \left[\mathbf{\Pi}'_{U_e}\ \mathbf{\Pi}'_{D_e}\right]'.
\end{aligned}$$

The equations of (B.8) and (B.9) complete the proof of the theorem. □

Proof of Theorem 2.4 Note that Lemmas A.15 and A.16 imply $\widehat{\mathbf{\Pi}}\xrightarrow{p}\mathbf{\Pi}$ and $\widehat{\mathbf{\Pi}}_e\xrightarrow{p}\mathbf{\Pi}_e$. Then, the desired results directly follow from Lemmas A.40 and A.41. This completes the proof of the theorem. □

Online Appendix for Large Sample Estimators of the Stochastic Discount Factor

Supplementary Proofs

Let $\lambda_{\max}(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ denote the maximum eigenvalue and the trace of a square matrix $\mathbf{A}$, respectively. The following properties of eigenvalues and trace operator, vectorize operator are useful for the proof of the lemmas:

1. Consider a $(L \times 1)$ vector of $\mathbf{x}$ and a $(L \times L)$ symmetric positive semidefinite matrix of $\mathbf{A}$. Then, it holds that
$$\mathbf{x}'\mathbf{A}\mathbf{x} \leq \lambda_{\max}(\mathbf{A})\,\mathbf{x}'\mathbf{x}. \tag{A.1}$$

2. Consider a $(L \times M)$ matrix of $\mathbf{A}$ and $(M \times L)$ matrix of $\mathbf{B}$. Then, it holds that
$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}). \tag{A.2}$$

3. Consider $(L \times L)$ positive semidefinite matrices of $\mathbf{A}$, $\mathbf{B}$. Then, it holds that
$$\mathrm{tr}(\mathbf{AB}) \leq \lambda_{\max}(\mathbf{A})\,\mathrm{tr}(\mathbf{B}). \tag{A.3}$$

4. Consider $(L \times M)$ matrices of $\mathbf{A}$ and $\mathbf{B}$. Then, it holds that
$$\mathrm{vec}(\mathbf{A})'\,\mathrm{vec}(\mathbf{B}) = \mathrm{tr}(\mathbf{A}'\mathbf{B}). \tag{A.4}$$

5. Consider $(L \times M)$, $(M \times N)$, $(M \times O)$ matrices of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Then, it holds that
$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B}). \tag{A.5}$$

We introduce the following notations:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_0 & \mathbf{0}'_K \\ (\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f) & \mathbf{I}_K \end{bmatrix} \tag{A.6}$$

$$\boldsymbol{\Lambda}_e = \begin{bmatrix} (\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f) & \mathbf{I}_K \end{bmatrix} \tag{A.7}$$

$$\mathbf{B}_\triangle = \begin{bmatrix} \mathbf{1}_N & \mathbf{B} \end{bmatrix} \tag{A.8}$$

$$\mathbf{F}_\triangle = \begin{bmatrix} \mathbf{1}_T & \mathbf{F} \end{bmatrix}. \tag{A.9}$$

Lemma A.1. *Under Assumption 1, it holds that as N, T increases,*

$$\frac{\mathbf{B}'_\triangle \mathbf{B}_\triangle}{N} \to \begin{bmatrix} 1 & \boldsymbol{\mu}'_\beta \\ \boldsymbol{\mu}_\beta & \mathbf{V}_\beta \end{bmatrix} = \mathbf{V}_{\beta_\triangle} \tag{A.10}$$

$$\frac{\mathbf{F}'_\triangle \mathbf{F}_\triangle}{T} \xrightarrow{p} \begin{bmatrix} 1 & \boldsymbol{\mu}'_f \\ \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix} = \mathbf{V}_{\triangle,f}. \tag{A.11}$$

Under Assumption 2, it holds that as N, T increases,

$$\frac{\mathbf{B}'_\triangle \mathbf{E} \mathbf{F}_\triangle}{NT} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)} \tag{A.12}$$

$$\frac{\mathbf{F}'_\triangle \mathbf{E}' \mathbf{E} \mathbf{F}_\triangle}{NT^2} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)}. \tag{A.13}$$

Proof Assumption 1 implies that

$$\frac{\mathbf{B}'_\triangle \mathbf{B}_\triangle}{N} = \begin{bmatrix} 1 & \frac{\mathbf{1}'_N \mathbf{B}}{N} \\ \frac{\mathbf{B}'\mathbf{1}_N}{N} & \frac{\mathbf{B}'\mathbf{B}}{N} \end{bmatrix} \to \begin{bmatrix} 1 & \boldsymbol{\mu}'_\beta \\ \boldsymbol{\mu}_\beta & \mathbf{V}_\beta \end{bmatrix}$$

$$\frac{\mathbf{F}'_\triangle \mathbf{F}_\triangle}{T} = \begin{bmatrix} 1 & \frac{\mathbf{1}'_T \mathbf{F}}{T} \\ \frac{\mathbf{F}'\mathbf{1}_T}{T} & \frac{\mathbf{F}'\mathbf{F}}{T} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} 1 & \boldsymbol{\mu}'_f \\ \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix},$$

verifying (A.10) and (A.11). Assumption 2 implies that

$$\frac{\mathbf{B}'_\triangle \mathbf{E} \mathbf{F}_\triangle}{NT} = \begin{bmatrix} \frac{\mathbf{1}'_N \mathbf{E} \mathbf{1}_T}{NT} & \frac{\mathbf{1}'_N \mathbf{E}\mathbf{F}}{NT} \\ \frac{\mathbf{B}'\mathbf{E}\mathbf{1}_T}{NT} & \frac{\mathbf{B}'\mathbf{E}\mathbf{F}}{NT} \end{bmatrix} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)},$$

showing (A.12).

We turn to (A.13). Note that $\frac{\mathbf{F}'_\triangle \mathbf{E}'\mathbf{E}\mathbf{F}_\triangle}{NT^2}$ is a positive semidefinite matrix. Hence, for

(A.13), it suffices to show that the trace of $\frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{E}\mathbf{F}_\triangle}{NT^2}$ converges to zero, which is followed by

$$\begin{aligned}
\operatorname{tr}\left(\frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{E}\mathbf{F}_\triangle}{NT^2}\right) &= \frac{1}{T}\operatorname{tr}\left(\frac{\mathbf{E}'\mathbf{E}}{N}\frac{\mathbf{F}_\triangle\mathbf{F}_\triangle'}{T}\right) \\
&< \frac{M_0}{T}\operatorname{tr}\left(\frac{\mathbf{F}_\triangle\mathbf{F}_\triangle'}{T}\right) \xrightarrow{p} 0\cdot\operatorname{tr}\left(\mathbf{V}_{\triangle,f}\right) = 0,
\end{aligned}$$

where the equality is from (A.2) and the inequality is from Assumption 2 and (A.3). This completes the proof of the lemma. □

Lemma A.2. *Under Assumptions 1 and 2, it holds that as $N, T \to \infty$,*

$$\begin{aligned}
\widetilde{\mathbf{D}} &= \frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{R}\mathbf{F}_\triangle}{NT^2} \xrightarrow{p} \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_\triangle}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f} \\
\widetilde{\mathbf{U}} &= \frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{1}_N}{NT} \xrightarrow{p} \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right].
\end{aligned}$$

Proof Rewrite the return generating process of $\mathbf{R}$ in (2.11) as

$$\mathbf{R} = \mathbf{1}_N\lambda_0\mathbf{1}_T' + \mathbf{B}\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f\right)\mathbf{1}_T' + \mathbf{B}\mathbf{F}' + \mathbf{E} = \mathbf{B}_\triangle\boldsymbol{\Lambda}\mathbf{F}_\triangle' + \mathbf{E},$$

where $\boldsymbol{\Lambda}$, $\mathbf{B}_\triangle$ and $\mathbf{F}_\triangle$ are given by (A.6), (A.8) and (A.9), respectively.

From Lemma A.1, we have that

$$\begin{aligned}
\frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{R}\mathbf{F}_\triangle}{NT^2} &= \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'\frac{\mathbf{B}_\triangle'\mathbf{B}_\triangle}{N}\boldsymbol{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T} + \frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{B}_\triangle}{NT}\boldsymbol{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T} + \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'\frac{\mathbf{B}_\triangle'\mathbf{E}\mathbf{F}_\triangle}{NT} \\
&\quad + \frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{E}\mathbf{F}_\triangle}{NT^2} \xrightarrow{p} \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_\triangle}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}
\end{aligned}$$

and that

$$\begin{aligned}
\frac{\mathbf{F}_\triangle'\mathbf{R}'\mathbf{1}_N}{NT} &= \frac{\mathbf{F}_\triangle'\left(\mathbf{F}_\triangle\boldsymbol{\Lambda}'\mathbf{B}_\triangle' + \mathbf{E}'\right)\mathbf{1}_N}{NT} \\
&= \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'\frac{\mathbf{B}_\triangle'\mathbf{1}_N}{N} + \frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{1}_N}{NT} \xrightarrow{p} \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right]'.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma A.3. *Under Assumptions 1 and 2, it holds that as $N, T \to \infty$,*

$$\widetilde{\mathbf{D}}_e = \frac{\mathbf{F}'_\triangle \mathbf{R}'\mathbf{R}\mathbf{F}_\triangle}{NT^2} \xrightarrow{p} \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix} \boldsymbol{\Lambda}'_e \mathbf{V}_\beta \boldsymbol{\Lambda}_e \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix}'$$
$$\widetilde{\mathbf{U}}_e = \frac{\mathbf{F}'_\triangle \mathbf{R}'\mathbf{1}_N}{NT} \xrightarrow{p} \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix} \boldsymbol{\Lambda}'_e \mathbf{V}_\beta \boldsymbol{\Lambda}_e \begin{bmatrix} 1 & \boldsymbol{\mu}'_f \end{bmatrix}'.$$

Proof The return generating process of $\mathbf{R}_e$ in (2.12) is rewritten as

$$\mathbf{R}_e = \mathbf{B}\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f\right)\mathbf{1}'_T + \mathbf{B}\mathbf{F}' + \mathbf{E} = \mathbf{B}\boldsymbol{\Lambda}_e\mathbf{F}'_\triangle + \mathbf{E},$$

where $\boldsymbol{\Lambda}_e$ and $\mathbf{F}_\triangle$ are given by (A.7) (A.9). From Assumption 1 and Lemma A.1, we have that

$$\frac{\mathbf{F}'\mathbf{R}'_e\mathbf{R}_e\mathbf{F}}{NT^2} = \frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'\mathbf{B}}{N}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_\triangle\mathbf{F}}{T} + \frac{\mathbf{F}'\mathbf{E}'\mathbf{B}}{NT}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_\triangle\mathbf{F}}{T} + \frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'\mathbf{E}\mathbf{F}}{NT}$$
$$+ \frac{\mathbf{F}'\mathbf{E}'\mathbf{E}\mathbf{F}}{NT^2} \xrightarrow{p} \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix} \boldsymbol{\Lambda}'_e \mathbf{V}_\beta \boldsymbol{\Lambda}_e \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix}'$$

and that

$$\frac{\mathbf{F}'\mathbf{R}'_e\mathbf{R}_e\mathbf{1}_T}{NT^2} = \frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'\mathbf{B}}{N}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_\triangle\mathbf{1}_T}{T} + \frac{\mathbf{F}'\mathbf{E}'\mathbf{B}}{NT}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_\triangle\mathbf{1}_T}{T} + \frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'\mathbf{E}\mathbf{1}_T}{NT}$$
$$+ \frac{\mathbf{F}'\mathbf{E}'\mathbf{E}\mathbf{1}_T}{NT^2} \xrightarrow{p} \begin{bmatrix} \boldsymbol{\mu}_f & \mathbf{V}_f \end{bmatrix} \boldsymbol{\Lambda}'_e \mathbf{V}_\beta \boldsymbol{\Lambda}_e \begin{bmatrix} 1 & \boldsymbol{\mu}'_f \end{bmatrix}'.$$

This completes the proof of the lemma. □

We need Lemmas A.4-A.8 for Corollary 2.1. We introduce the following notation:

$$\mathcal{O}_\triangle = \begin{bmatrix} 1 & \mathbf{0}'_K \\ \mathbf{0}_K & \mathcal{O} \end{bmatrix}. \tag{A.14}$$

Lemma A.4. *Under Assumptions 1-3, there exist positive constants $C_1, C_2 < \infty$ such that as $N, T \to \infty$, it holds that $\lambda_{\max}\left(\frac{\mathbf{R}'\mathbf{R}}{NT}\right) < C_1$ and $\lambda_{\max}\left(\frac{\mathbf{R}'_e\mathbf{R}_e}{NT}\right) < C_2$.*

Proof First, we show the existence of C_1 such that $\lambda_{\max}\left(\frac{\mathbf{R}'\mathbf{R}}{NT}\right) < C_1$. Because $\frac{\mathbf{R}'\mathbf{R}}{NT}$ is positive semidefinite, it suffices to show the existence of C_1 such that $\mathrm{tr}\left(\frac{\mathbf{R}'\mathbf{R}}{NT}\right) < C_1$. Recall that

$$\mathbf{R} = \mathbf{B}_\triangle\boldsymbol{\Lambda}\mathbf{F}'_\triangle + \mathbf{E},$$

where $\mathbf{\Lambda}$, $\mathbf{B}_\triangle$ and $\mathbf{F}_\triangle$ are given by (A.6), (A.8) and (A.9), respectively. Hence,

$$\frac{\mathbf{R}\mathbf{R}'}{NT} = l_1 + l_2 + l_2' + l_3,$$

where

$$l_1 = \frac{\mathbf{B}_\triangle \mathbf{\Lambda} \mathbf{F}_\triangle' \mathbf{F}_\triangle \mathbf{\Lambda}' \mathbf{B}_\triangle'}{NT},\ l_2 = \frac{\mathbf{B}_\triangle \mathbf{\Lambda} \mathbf{F}_\triangle' \mathbf{E}'}{NT},\ l_3 = \frac{\mathbf{E}\mathbf{E}'}{NT}.$$

Note that

$$\begin{aligned}\operatorname{tr}(l_1) &= \operatorname{tr}\left(\frac{\mathbf{B}_\triangle \mathbf{\Lambda} \mathbf{F}_\triangle' \mathbf{F}_\triangle \mathbf{\Lambda}' \mathbf{B}_\triangle}{NT}\right) = \operatorname{tr}\left(\frac{\mathbf{\Lambda} \mathbf{F}_\triangle' \mathbf{F}_\triangle \mathbf{\Lambda}' \mathbf{B}_\triangle' \mathbf{B}_\triangle}{NT}\right) = \operatorname{tr}\left(\mathbf{\Lambda}\left(\frac{\mathbf{F}_\triangle' \mathbf{F}_\triangle}{T}\right)\mathbf{\Lambda}'\left(\frac{\mathbf{B}_\triangle' \mathbf{B}_\triangle}{N}\right)\right) \\ &\xrightarrow{p} \operatorname{tr}\left(\mathbf{\Lambda} \mathbf{V}_{\triangle,f} \mathbf{\Lambda}' \mathbf{V}_{\beta_\triangle}\right), \qquad \text{(A.15)}\end{aligned}$$

where the second equality is from (A.2) and the last limit is from Lemma A.1, and that

$$\begin{aligned}\operatorname{tr}(l_2) &= \operatorname{tr}\left(\frac{\mathbf{B}_\triangle \mathbf{\Lambda} \mathbf{F}_\triangle' \mathbf{E}'}{NT}\right) = \operatorname{tr}\left(\mathbf{\Lambda}\frac{\mathbf{F}_\triangle' \mathbf{E}' \mathbf{B}_\triangle}{NT}\right) \\ &\xrightarrow{p} \operatorname{tr}\left(\mathbf{\Lambda} \mathbf{0}_{(K+1)\times(K+1)}\right) = 0, \qquad \text{(A.16)}\end{aligned}$$

where the second equality is from (A.2) and the limit is from Lemma A.1, and that

$$\operatorname{tr}(l_3) = \operatorname{tr}\left(\frac{\mathbf{E}\mathbf{E}'}{NT}\right) = \frac{1}{T}\operatorname{tr}\left(\frac{\mathbf{E}'\mathbf{E}}{N}\right) \le \frac{1}{T} T \cdot \lambda_{\max}\left(\frac{\mathbf{E}'\mathbf{E}}{N}\right) \le \frac{1}{T} T \cdot M_0 = M_0, \qquad \text{(A.17)}$$

where the second equality is from (A.2), the first inequality is from the positivity of $\frac{\mathbf{E}'\mathbf{E}}{N}$ and the second inequality is from Assumption 2.

Lastly, from (A.15)-(A.17), we obtain that with large N, T

$$\operatorname{tr}\left(\frac{\mathbf{R}'\mathbf{R}}{NT}\right) = \operatorname{tr}\left(\frac{\mathbf{R}\mathbf{R}'}{NT}\right) < \operatorname{tr}\left(\mathbf{\Lambda} \mathbf{V}_{\triangle,f} \mathbf{\Lambda}' \mathbf{V}_{\beta_\triangle}\right) + M_0 + 1.$$

Hence, the first statement of the lemma holds by setting $C_1 = \operatorname{tr}\left(\mathbf{\Lambda} \mathbf{V}_{\triangle,f} \mathbf{\Lambda}' \mathbf{V}_{\beta_\triangle}\right) + M_0 + 1$.

Next, we turn to the existence of C_2 such that $\lambda_{\max}\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right) < C_2$. As before, we find C_2 such that $\operatorname{tr}\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right) < C_2$. Recall that

$$\mathbf{R}_e = \mathbf{B}\mathbf{\Lambda}_e \mathbf{F}_\triangle' + \mathbf{E},$$

where $\mathbf{\Lambda}_e$ and $\mathbf{F}_\triangle$ are given by (A.7) and (A.9), respectively. Hence,

$$\frac{\mathbf{R}_e\mathbf{R}_e'}{NT} = m_1 + m_2 + m_2' + m_3,$$

where

$$m_1 = \frac{\mathbf{B}\mathbf{\Lambda}_e\mathbf{F}_\triangle'\mathbf{F}_\triangle\mathbf{\Lambda}_e'\mathbf{B}'}{NT},\ m_2 = \frac{\mathbf{B}\mathbf{\Lambda}_e\mathbf{F}_\triangle'\mathbf{E}'}{NT},\ m_3 = \frac{\mathbf{E}\mathbf{E}'}{NT}.$$

Note that

$$\begin{aligned}\operatorname{tr}(m_1) = \operatorname{tr}\left(\frac{\mathbf{B}\mathbf{\Lambda}_e\mathbf{F}_\triangle'\mathbf{F}_\triangle\mathbf{\Lambda}_e'\mathbf{B}'}{NT}\right) &= \operatorname{tr}\left(\frac{\mathbf{\Lambda}_e\mathbf{F}_\triangle'\mathbf{F}_\triangle\mathbf{\Lambda}_e'\mathbf{B}'\mathbf{B}}{NT}\right) = \operatorname{tr}\left(\mathbf{\Lambda}_e\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\mathbf{\Lambda}_e'\left(\frac{\mathbf{B}'\mathbf{B}}{N}\right)\right) \\ &\xrightarrow{p} \operatorname{tr}\left(\mathbf{\Lambda}_e\mathbf{V}_{\triangle,f}\mathbf{\Lambda}_e'\mathbf{V}_\beta\right),\end{aligned} \tag{A.18}$$

where the second equality is from (A.2) and the last limit is from Lemma A.1 and Assumption 1, and that

$$\begin{aligned}\operatorname{tr}(m_2) = \operatorname{tr}\left(\frac{\mathbf{B}\mathbf{\Lambda}_e\mathbf{F}_\triangle'\mathbf{E}'}{NT}\right) &= \operatorname{tr}\left(\mathbf{\Lambda}_e\frac{\mathbf{F}_\triangle'\mathbf{E}'\mathbf{B}}{NT}\right) \\ &\xrightarrow{p} \operatorname{tr}\left(\mathbf{\Lambda}\mathbf{0}_{(K+1)\times K}\right) = 0,\end{aligned} \tag{A.19}$$

where the second equality is from (A.2) and the limit is from Lemma A.1, and that

$$\operatorname{tr}(m_3) = \operatorname{tr}\left(\frac{\mathbf{E}\mathbf{E}'}{NT}\right) = \frac{1}{T}\operatorname{tr}\left(\frac{\mathbf{E}'\mathbf{E}}{N}\right) \le M_0, \tag{A.20}$$

where the second equality is from (A.2) and the inequality is from Assumption 2.

Lastly, from (A.18)-(A.20), it holds that with large N, T

$$\operatorname{tr}\left(\frac{\mathbf{R}_e\mathbf{R}_e'}{NT}\right) = \operatorname{tr}\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right) < \operatorname{tr}\left(\mathbf{\Lambda}_e\mathbf{V}_{\triangle,f}\mathbf{\Lambda}_e'\mathbf{V}_\beta\right) + M_0 + 1.$$

Hence, the second statement holds with $C_2 = \operatorname{tr}\left(\mathbf{\Lambda}_e\mathbf{V}_{\triangle,f}\mathbf{\Lambda}_e'\mathbf{V}_\beta\right) + M_0 + 1$. This completes the proof of the lemma $\square$

Lemma A.5. *Under Assumptions 1-3, there exist positive numbers $C_3, C_4 < \infty$ such that as $N, T \to \infty$, it holds that $\frac{\mathbf{1}_N'\mathbf{R}\mathbf{R}'\mathbf{1}_N}{N^2T} < C_3$ and $\frac{\mathbf{1}_T'\mathbf{R}_e'\mathbf{R}_e\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{N^2T^3} < C_4$.*

Proof First, we find C_3. Note that

$$\frac{\mathbf{1}_N'\mathbf{R}\mathbf{R}'\mathbf{1}_N}{N^2T} = \left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)'\left(\frac{\mathbf{R}\mathbf{R}'}{NT}\right)\left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)$$
$$\leq \lambda_{\max}\left(\frac{\mathbf{R}\mathbf{R}'}{NT}\right)\left(\left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)'\left(\frac{1}{\sqrt{N}}\mathbf{1}_N\right)\right) = \lambda_{\max}\left(\frac{\mathbf{R}\mathbf{R}'}{NT}\right),$$

where the inequality is from (A.1). Hence, the inequality of $\frac{\mathbf{1}_N'\mathbf{R}\mathbf{R}'\mathbf{1}_N}{N^2T} < C_3$ holds by setting $C_3 = C_1$ given by Lemma A.4.

Next, we find C_4. Note that

$$\frac{\mathbf{1}_T'\mathbf{R}_e'\mathbf{R}_e\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{N^2T^3} = \left(\frac{1}{\sqrt{T}}\mathbf{1}_T\right)'\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right)^2\left(\frac{1}{\sqrt{T}}\mathbf{1}_T\right)$$
$$\leq \lambda_{\max}\left(\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right)^2\right)\left(\left(\frac{1}{\sqrt{T}}\mathbf{1}_T\right)'\left(\frac{1}{\sqrt{T}}\mathbf{1}_T\right)\right) = \left(\lambda_{\max}\left(\frac{\mathbf{R}_e\mathbf{R}_e'}{NT}\right)\right)^2,$$

where the inequality is from (A.1). Hence, the inequality of $\frac{\mathbf{1}_N'\mathbf{R}\mathbf{R}'\mathbf{1}_N}{N^2T} < C_4$ holds by setting $C_4 = C_2^2$, where C_2 is given by Lemma A.4. This completes the proof of the lemma. $\square$

Lemma A.6. *Let Assumption 3 be in effect. Consider a $(T \times L)$ matrix $\mathbf{X}$, where L is fixed. If there exists a positive constant $C < \infty$ such that as $N, T \to \infty$, $\lambda_{\max}\left(\frac{\mathbf{X}'\mathbf{X}}{T}\right) < C$, then, the probability limit of $\frac{\mathbf{F}^{*\prime}\mathbf{X}}{T}$ is identical to that of $\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}}{T}$.*

Proof Note that $\frac{\mathbf{F}^{*\prime}\mathbf{X}}{T} = \frac{(\mathbf{F}\mathcal{O}+\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}}{T} = \mathcal{O}'\frac{\mathbf{F}'\mathbf{X}}{T} + \frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}}{T}$. Hence, it suffices to show that $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}}{T}$ becomes negligible. Define $\iota_{l,L}$ as the $(L \times 1)$ vector of zeros except the l-th element of one and $\iota_{k,K}$ as the $(K \times 1)$ vector of zeros except the k-th element of one. Then, it follows that

$$\iota_{k,K}'\frac{(\mathbf{F}^* - \mathbf{F}\mathcal{O})'\mathbf{X}}{T}\iota_{l,L} = \left(\frac{(\mathbf{F}^* - \mathbf{F}\mathcal{O})\iota_{k,K}}{\sqrt{T}}\right)'\left(\frac{\mathbf{X}\iota_{l,L}}{\sqrt{T}}\right)$$
$$\leq \sqrt{\iota_{k,K}'\left(\frac{(\mathbf{F}^* - \mathbf{F}\mathcal{O})'(\mathbf{F}^* - \mathbf{F}\mathcal{O})}{T}\right)\iota_{k,K}} \cdot \sqrt{\iota_{l,L}'\left(\frac{\mathbf{X}'\mathbf{X}}{T}\right)\iota_{l,L}}$$
$$\leq \sqrt{\iota_{k,K}'\left(\frac{(\mathbf{F}^* - \mathbf{F}\mathcal{O})'(\mathbf{F}^* - \mathbf{F}\mathcal{O})}{T}\right)\iota_{k,K}} \cdot \sqrt{\lambda_{\max}\left(\frac{\mathbf{X}'\mathbf{X}}{T}\right)\iota_{l,L}'\iota_{l,L}}$$
$$\xrightarrow{p} \iota_{k,K}'\mathbf{0}_{K\times K}\iota_{k,K} \cdot C = 0,$$

where the first inequality is from Cauchy-Schwarz inequality, the second inequality is from (A.1) and the last limit is from the assumption of $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T} \xrightarrow{p} \mathbf{0}_{K\times K}$ and $\lambda_{\max}\left(\frac{\mathbf{X}'\mathbf{X}}{T}\right) <$

C with large N,T. Repeating this exercise for each $k=1,\cdots,K$ and $l=1,\cdots,L$ completes the proof of the lemma. $\square$

Lemma A.7. *Let Assumption 3 be in effect. Consider a $(T\times T)$ symmetric positive semidefinite matrix $\mathbf{X}$. If there exists a positive constant $C<\infty$ such that as $N,T\to\infty$, $\lambda_{\max}(\mathbf{X})<C$, then, the probability limit of $\frac{\mathbf{F}^{*\prime}\mathbf{X}\mathbf{F}^*}{T}$ is identical to that of $\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}\mathbf{F}}{T}\mathcal{O}$.*

Proof Note that

$$\begin{aligned}\frac{\mathbf{F}^{*\prime}\mathbf{X}\mathbf{F}^*}{T}&=\frac{(\mathbf{F}\mathcal{O}+\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}\mathcal{O}+\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}\\&=\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}\mathbf{F}}{T}\mathcal{O}+\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}\mathbf{F}}{T}\mathcal{O}+\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}+\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}.\end{aligned}\tag{A.21}$$

We prove the lemma by showing that $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}\mathbf{F}}{T}$ and $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}$ becomes negligible with large T.

First, we verify $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}\mathbf{F}}{T}\xrightarrow{p}\mathbf{0}_{K\times K}$. Because $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}\mathbf{F}}{T}=\frac{\mathbf{F}^{*\prime}\mathbf{X}\mathbf{F}}{T}-\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}\mathbf{F}}{T}$, from Lemma A.6, it suffices to show that the boundedness of $\lambda_{\max}\left(\frac{\mathbf{X}\mathbf{F}\mathbf{F}'\mathbf{X}'}{T}\right)$. In fact, we show that the boundedness of $\operatorname{tr}\left(\frac{\mathbf{X}\mathbf{F}\mathbf{F}'\mathbf{X}'}{T}\right)$, implying the boundedness of $\lambda_{\max}\left(\frac{\mathbf{X}\mathbf{F}\mathbf{F}'\mathbf{X}'}{T}\right)$. Note that

$$\operatorname{tr}\left(\frac{\mathbf{X}\mathbf{F}\mathbf{F}'\mathbf{X}'}{T}\right)=\operatorname{tr}\left(\mathbf{X}'\mathbf{X}\frac{\mathbf{F}\mathbf{F}'}{T}\right)\le(\lambda_{\max}(\mathbf{X}))^2\operatorname{tr}\left(\frac{\mathbf{F}\mathbf{F}'}{T}\right),$$

where the equality is from (A.2) and the inequality is from (A.3). Because $\lambda_{\max}(\mathbf{X})<C$ and $\operatorname{tr}\left(\frac{\mathbf{F}\mathbf{F}'}{T}\right)\xrightarrow{p}\operatorname{tr}(\mathbf{\Sigma}_f)$, it follows that $\lambda_{\max}\left(\frac{\mathbf{X}\mathbf{F}\mathbf{F}'\mathbf{X}'}{T}\right)$ is bounded when N,T are large. Hence, from Lemma A.6, we have that

$$\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}\mathbf{F}}{T}\xrightarrow{p}\mathbf{0}_{K\times K}.\tag{A.22}$$

Second, we verify $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}\xrightarrow{p}\mathbf{0}_{K\times K}$. Since $\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}$ is positive semidefinite, it suffices to show that $\operatorname{tr}\left(\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}\right)$ converges to zero. Note that

$$\begin{aligned}\operatorname{tr}\left(\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})'\mathbf{X}(\mathbf{F}^*-\mathbf{F}\mathcal{O})}{T}\right)&=\operatorname{tr}\left(\mathbf{X}\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})(\mathbf{F}^*-\mathbf{F}\mathcal{O})'}{T}\right)\\&\le\lambda_{\max}(\mathbf{X})\operatorname{tr}\left(\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})(\mathbf{F}^*-\mathbf{F}\mathcal{O})'}{T}\right),\end{aligned}$$

where the equality is from (A.2) and the inequality is from (A.3). Because $\lambda_{\max}(\mathbf{X})<C$ and

$\operatorname{tr}\left(\frac{(\mathbf{F}^*-\mathbf{F}\mathcal{O})(\mathbf{F}^*-\mathbf{F}\mathcal{O})'}{T}\right) \xrightarrow{p} \operatorname{tr}\left(\mathbf{0}_{K\times K}\right)$, it follows that

$$\frac{(\mathbf{F}^* - \mathbf{F}\mathcal{O})' \mathbf{X} (\mathbf{F}^* - \mathbf{F}\mathcal{O})}{T} \xrightarrow{p} \mathbf{0}_{K\times K}. \tag{A.23}$$

Lastly, plugging (A.22) and (A.23) into (A.21) yields that the limit of $\frac{\mathbf{F}^{*\prime}\mathbf{X}\mathbf{F}^*}{T}$ converges to the limit of $\mathcal{O}'\frac{\mathbf{F}'\mathbf{X}\mathbf{F}}{T}\mathcal{O}$. This completes the proof of the lemma. □

Lemma A.8. *Under Assumptions 1-3, the probability limits of* $\frac{\mathbf{F}^{*\prime}_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}^*_{\triangle}}{NT^2}$, $\frac{\mathbf{F}^{*\prime}_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}$, $\frac{\mathbf{F}^{*\prime}\mathbf{R}_e'\mathbf{R}_e\mathbf{F}^*}{NT^2}$, $\frac{\mathbf{F}^{*\prime}\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{NT^2}$ *are identical to those of* $\mathcal{O}_{\triangle}'\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}\mathcal{O}_{\triangle}$, $\mathcal{O}_{\triangle}'\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}$, $\mathcal{O}'\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{F}}{NT^2}\mathcal{O}$, $\mathcal{O}'\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{NT^2}$, *respectively, where* $\mathcal{O}_{\triangle}$ *is given by (A.14).*

Proof Note that when N, T are large, $\lambda_{\max}\left(\frac{\mathbf{R}'\mathbf{R}}{NT}\right)$, $\lambda_{\max}\left(\frac{\mathbf{R}_e'\mathbf{R}_e}{NT}\right)$, $\frac{\mathbf{1}_N'\mathbf{R}\mathbf{R}'\mathbf{1}_N}{N^2T}$, $\frac{\mathbf{1}_T'\mathbf{R}_e'\mathbf{R}_e\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{N^2T^3}$ are bounded from Lemmas A.4 and A.5. Then, Lemmas A.6 and A.7 guarantess the stated results. This completes the proof of the lemma. □

Using the above lemmas, we prove Corollary 2.1.

Proof of Corollary 2.1 First, we show that $\widetilde{m}_t^* \xrightarrow{p} m_t$. Because $\mathbf{f}_t^* \xrightarrow{p} \mathcal{O}'\mathbf{f}_t$, it suffices to establish that $\widetilde{\boldsymbol{\delta}}^* \xrightarrow{p} \begin{bmatrix} 1 & \mathbf{0}_K' \\ \mathbf{0}_K & \mathcal{O}' \end{bmatrix} \boldsymbol{\delta}$. From the following expression of $\widetilde{\boldsymbol{\delta}}^* = \left(\frac{\mathbf{F}^{*\prime}_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}^*_{\triangle}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}\right)$, Lemma A.8 shows that $\widetilde{\boldsymbol{\delta}}^*$ converges to the limit of $\mathcal{O}_{\triangle}'\left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}\right)$. Using $\left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{R}'\mathbf{1}_N}{NT}\right) \xrightarrow{p} \mathcal{O}_{\triangle}'\boldsymbol{\delta}$ given in the proof of Theorem 2.1, we have that $\widetilde{\boldsymbol{\delta}}^* \xrightarrow{p} \mathcal{O}_{\triangle}'\boldsymbol{\delta}$, which in conjunction with $\mathbf{f}_t^* \xrightarrow{p} \mathcal{O}'\mathbf{f}_t$ implies

$$\widetilde{m}_t^* \xrightarrow{p} m_t. \tag{A.24}$$

Next, we turn to $\widetilde{m}_{e,t}^* \xrightarrow{p} m_{e,t}$. In a similar manner, we show that $\widetilde{\boldsymbol{\delta}}_e^* \xrightarrow{p} \mathcal{O}'\boldsymbol{\delta}_e$. From the expression of $\widetilde{\boldsymbol{\delta}}^* = -\left(\frac{\mathbf{F}^{*\prime}\mathbf{R}_e'\mathbf{R}_e\mathbf{F}^*}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{NT^2}\right)$, Lemma A.8 shows that $\widetilde{\boldsymbol{\delta}}_e^*$ converges to the limit of $\mathcal{O}'\left(\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{F}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{NT^2}\right)$. Using $\left(\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{F}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}'\mathbf{R}_e'\mathbf{R}_e\mathbf{1}_T}{NT^2}\right) \xrightarrow{p} \boldsymbol{\delta}_e$ given in the proof of Theorem 2.1, we have that $\widetilde{\boldsymbol{\delta}}_e^* \xrightarrow{p} \mathcal{O}'\boldsymbol{\delta}_e$, which in conjunction with $\mathbf{f}_t^* \xrightarrow{p} \mathcal{O}'\mathbf{f}_t$ implies

$$\widetilde{m}_{e,t}^* \xrightarrow{p} m_{e,t}. \tag{A.25}$$

The limits of (A.24) and (A.25) complete the proof of the corollary. □

Proof of Proposition 2.1 Note that from Assumptions 1, 2, and the limits in Lemma A.1 and the homoskedasticity condition, as $N \to \infty$,

$$
\begin{aligned}
\frac{\mathbf{R}'\mathbf{R}}{N} &= \mathbf{F}_\triangle \boldsymbol{\Lambda}' \frac{\mathbf{B}'_\triangle \mathbf{B}_\triangle}{N} \boldsymbol{\Lambda} \mathbf{F}'_\triangle + \frac{\mathbf{E}'\mathbf{E}}{N} + \frac{\mathbf{E}'\mathbf{B}_\triangle}{N} \boldsymbol{\Lambda} \mathbf{F}'_\triangle + \mathbf{F}_\triangle \boldsymbol{\Lambda}' \frac{\mathbf{B}'_\triangle \mathbf{E}}{N} \\
&\xrightarrow{p} \mathbf{F}_\triangle \boldsymbol{\Lambda}' \mathbf{V}_{\beta_\triangle} \boldsymbol{\Lambda} \mathbf{F}'_\triangle + s\mathbf{I}_T \text{ and} \qquad (A.26) \\
\frac{\mathbf{R}'\mathbf{1}_N}{N} &= \mathbf{F}_\triangle \boldsymbol{\Lambda}' \frac{\mathbf{B}'_\triangle \mathbf{1}_N}{N} + \frac{\mathbf{E}'\mathbf{1}_N}{N} \xrightarrow{p} \mathbf{F}_\triangle \boldsymbol{\Lambda}' \begin{bmatrix} 1 & \boldsymbol{\mu}'_\beta \end{bmatrix}'. \qquad (A.27)
\end{aligned}
$$

From the N-limits of (A.26) and (A.27), after some algebras, we find that as $N \to \infty$,

$$
\begin{aligned}
\breve{m}_t =& \iota'_t \left(\frac{\mathbf{R}'\mathbf{R}}{NT} \right)^{-1} \left(\frac{\mathbf{R}'\mathbf{1}_N}{N} \right) \\
\xrightarrow{p}& T \iota'_t \left(\left[\mathbf{F}_\triangle \boldsymbol{\Lambda}' \mathbf{V}_{\beta_\triangle} \boldsymbol{\Lambda} \mathbf{F}'_\triangle \right] + s\mathbf{I}_T \right)^{-1} \left(\mathbf{F}_\triangle \boldsymbol{\Lambda}' \left[1 \ \boldsymbol{\mu}'_\beta \right]' \right) \\
=& \iota'_t \left(\mathbf{F}_\triangle \boldsymbol{\Lambda}' \mathbf{V}^{1/2}_{\beta_\triangle} \left(\frac{s}{T} \mathbf{I}_{K+1} + \mathbf{V}^{1/2}_{\beta_\triangle} \boldsymbol{\Lambda} \frac{\mathbf{F}'_\triangle \mathbf{F}_\triangle}{T} \boldsymbol{\Lambda}' \mathbf{V}^{1/2}_{\beta_\triangle} \right)^{-1} \right) \left(\mathbf{V}^{-1/2}_{\beta_\triangle} \left[1 \ \boldsymbol{\mu}'_\beta \right]' \right) \\
=& \left[1 \ \ \mathbf{f}'_t \right] \boldsymbol{\Lambda}' \mathbf{V}^{1/2}_{\beta_\triangle} \left(\frac{s}{T} \mathbf{I}_{K+1} + \mathbf{V}^{1/2}_{\beta_\triangle} \boldsymbol{\Lambda} \frac{\mathbf{F}'_\triangle \mathbf{F}_\triangle}{T} \boldsymbol{\Lambda}' \mathbf{V}^{1/2}_{\beta_\triangle} \right)^{-1} \mathbf{V}^{-1/2}_{\beta_\triangle} \left[1 \ \boldsymbol{\mu}'_\beta \right]'.
\end{aligned}
$$

Hence, as $N \to \infty$ and then $T \to \infty$,

$$
\begin{aligned}
\breve{m}_t &= \iota'_t \left(\frac{\mathbf{R}'\mathbf{R}}{NT^2} \right)^{-1} \left(\frac{\mathbf{R}'\mathbf{1}_N}{NT} \right) \\
&\xrightarrow{p} \left[1 \ \ \mathbf{f}'_t \right] \left(\mathbf{V}_{\triangle,f} \boldsymbol{\Lambda}^{-1} \right) \left(\mathbf{V}^{-1}_{\beta_\triangle} \left[1 \ \boldsymbol{\mu}'_\beta \right]' \right) = \left[1 \ \ \mathbf{f}'_t \right] \boldsymbol{\delta} = m_t,
\end{aligned}
$$

where the next to the last equality is from (B.2).

This completes the proof of the proposition. □

We define $\mathcal{S}$ as the $\left(\tau^2 \times \tau\right)$ selection matrix such that the $(\tau(s-1)+1, s)$ element of $\mathcal{S}$ is 1, for $s = 1, \cdots, \tau$ and all other elements are zero.

Proof of Lemma 2.1 Define the $(\tau \times 1)$ vector of $\mathbf{v}_{e,[b]}$ such that $\mathbf{V}_{e,[b]} = diag\left(\mathbf{v}_{e,[b]}\right)$. From the expression of $\widehat{\mathbf{V}}_{e,[b]}$ given in (2.20), we have that $\widehat{\mathbf{V}}_{e,[b]} = \operatorname{diag}\left(\widehat{\mathbf{v}}_{e,[b]}\right)$, where

$$
\widehat{\mathbf{v}}_{e,[b]} = \left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]} \right)^{-1} \mathcal{S}' \operatorname{vec} \left(\frac{\widehat{\mathbf{E}}'_{[b]} \widehat{\mathbf{E}}_{[b]}}{N_{[b]}} \right).
$$

Hence, it suffices to show $\widehat{\mathbf{v}}_{e,[b]} \xrightarrow{p} \mathbf{v}_{e,[b]}$. The invertibility of $\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)$ is discussed in footnote 7 of Kim and Skoulakis (2018b).

First, we verify the N-limit of vec $\left(\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}\right)$. Since $\mathbf{1}'_\tau \mathbf{H}_{[b]} = \mathbf{0}'_\tau$ and $\mathbf{F}'_{[b]}\mathbf{H}_{[b]} = \mathbf{0}_{K\times\tau}$, for both the gross returns case of (2.18) and the excess return case of (2.19), it holds that

$$\widehat{\mathbf{E}}_{[b]} = \mathbf{E}_{[b]}\mathbf{H}_{[b]}.$$

Using (A.5), we have that

$$\begin{aligned}\mathrm{vec}\left(\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}\right) &= \mathrm{vec}\left(\mathbf{H}_{[b]}\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\mathbf{H}_{[b]}\right) = \left(\mathbf{H}_{[b]}\otimes\mathbf{H}_{[b]}\right)\mathrm{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\\ &\xrightarrow{p} \left(\mathbf{H}_{[b]}\otimes\mathbf{H}_{[b]}\right)\mathrm{vec}\left(\mathbf{V}_{e,[b]}\right),\end{aligned}$$

where the last limit is from Assumption 4(ii).

Hence, from the above limit and the properties of selection matrix of $\mathcal{S}$ such that vec $\left(\mathbf{V}_{e,[b]}\right) = \mathcal{S}\mathbf{v}_{e,[b]}$ and that $\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]} = \mathcal{S}'\left(\mathbf{H}_{[b]}\otimes\mathbf{H}_{[b]}\right)\mathcal{S}$, we have that

$$\begin{aligned}\widehat{\mathbf{v}}_{e,[b]} &= \left(\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]}\right)^{-1}\mathcal{S}'\mathrm{vec}\left(\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}\right)\\ &\xrightarrow{p} \left(\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]}\right)^{-1}\mathcal{S}'\left(\mathbf{H}_{[b]}\otimes\mathbf{H}_{[b]}\right)\mathrm{vec}\left(\mathbf{V}_{e,[b]}\right)\\ &= \left(\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]}\right)^{-1}\mathcal{S}'\left(\mathbf{H}_{[b]}\otimes\mathbf{H}_{[b]}\right)\mathcal{S}\mathbf{v}_{e,[b]}\\ &= \left(\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]}\right)^{-1}\left(\mathbf{H}_{[b]}\odot\mathbf{H}_{[b]}\right)\mathbf{v}_{e,[b]} = \mathbf{v}_{e,[b]},\end{aligned}$$

which completes the proof of the lemma. □

We provide lemmas to prove Theorem 2.2. Define $e_{k,[b]}$ for $k = 1, \cdots, 5$ as follows:

$$\begin{aligned}&e_{1,[b]} = \left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{1}_{N_{[b]}} - \boldsymbol{\mu}_\beta\right),\ e_{2,[b]} = \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{B}_{[b]} - \mathbf{V}_\beta\right),\\ &e_{3,[b]} = \frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{1}_{N_{[b]}},\ e_{4,[b]} = \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{B}_{[b]}\right),\ e_{5,[b]} = \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{E}_{[b]} - \mathbf{V}_{e,[b]}\right).\end{aligned} \tag{A.28}$$

Lemma A.9. *Let Assumptions 4 and 5 be in effect. Consider any set of continuous functions of* $\mathbf{F}_{[b]}$*;* $f_1 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^K$, $f_2 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{K^2}$, $f_3 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^\tau$, $f_4 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{K\tau}$, *and*

$f_5 : \mathbb{R}^{\tau \times K} \to \mathbb{R}^{\tau^2}$. *For* $e_{k,[b]}$ *defined by (A.28), it holds that as* $N, T \to \infty$,

$$\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0$$

for $k = 1, \cdots, 5$.

Proof Note that from Assumption 4(iii), it holds that

$$\frac{1}{B}\sum_{b=1}^{B} e'_{k,[b]} e_{k,[b]} \xrightarrow{p} 0 \tag{A.29}$$

for $k = 1, \cdots, 5$.

Fix k. Let f_k be the corresponding function. As N, T increases, it follows that

$$\begin{aligned}
\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} &\leq \frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{f_k\left(\mathbf{F}_{[b]}\right)' f_k\left(\mathbf{F}_{[b]}\right)}\sqrt{e'_{k,[b]} e_{k,[b]}}\right) \\
&= \frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{g_k\left(\mathbf{F}_{[b]}\right)}\sqrt{e'_{k,[b]} e_{k,[b]}}\right) \\
&\leq \left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} g_k\left(\mathbf{F}_{[b]}\right)}\right)\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} e'_{k,[b]} e_{k,[b]}}\right) \\
&\leq \sqrt{M_{g_k}} \cdot \sqrt{\frac{1}{B}\sum_{b=1}^{B} e'_{k,[b]} e_{k,[b]}} \xrightarrow{p} 0,
\end{aligned} \tag{A.30}$$

where the first equality holds by defining $g_k\left(\mathbf{F}_{[b]}\right) = \sqrt{f_k\left(\mathbf{F}_{[b]}\right)' f_k\left(\mathbf{F}_{[b]}\right)}$, the first and second inequalities are from the Cauchy-Schwarz inequality, the third inequality is from Assumption 5, and the last limit is from (A.29). In a similar manner, we can show that as N, T increases,

$$-\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{[b]} \leq \sqrt{M_{g_k}} \cdot \sqrt{\frac{1}{B}\sum_{b=1}^{B} e'_{k,[b]} e_{k,[b]}} \xrightarrow{p} 0. \tag{A.31}$$

Lastly, combining (A.30) and (A.31) in conjunction with the squeeze theorem, we have that as $N, T \to \infty$,

$$\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0.$$

Repeating this exercise for $k = 1, \cdots, 5$ completes the proof of the lemma. $\square$

Lemma A.10. *It holds that*

$$vec\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}\right) = \mathbf{K}_{[b]} vec\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right),$$

where

$$\mathbf{K}_{[b]} = \left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right)\right),$$

and $\widehat{\mathbf{V}}_{e,[b]}$ *and* $\mathbf{H}_{[b]}$ *are given by* (2.20) *and* (2.21).

Proof From (2.20),

$$\mathrm{vec}\left(diag\left(\widehat{\mathbf{v}}_{e,[b]}\right)\right) = \mathcal{S}\widehat{\mathbf{v}}_{e,[b]} = \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right) \mathrm{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right).$$

Hence,

$$\begin{aligned}
&\mathrm{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - diag\left(\widehat{\mathbf{v}}_{e,[b]}\right)\right)\\
=&\left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right)\right) \mathrm{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\\
=&\left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right)\right) \mathrm{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right),
\end{aligned}$$

where the last equality is from

$$\begin{aligned}
&\left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right)\right) \mathrm{vec}\left(\mathbf{V}_{e,[b]}\right)\\
=&\left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right)\right) \mathcal{S}\mathbf{v}_{e,[b]}\\
=&\mathcal{S}\mathbf{v}_{e,[b]} - \mathcal{S}\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}_{[b]} \otimes \mathbf{H}_{[b]}\right) \mathcal{S}\mathbf{v}_{e,[b]} = \mathbf{0}_{\tau^2}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Lemma A.11. *The* $d_{[b]}$ *given in Theorem 2.2 can be expressed as*

$$d_{[b]} = \mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} + \mathcal{E}_{D,[b]},$$

where $vec\left(\mathcal{E}_{D,[b]}\right)$ *is given by* (A.35). *Under Assumptions 1, 4 and 5, it holds that as* $N, T \to$

∞,

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\xrightarrow{p}\mathbf{0}_{(K+1)\times(K+1)}.$$

Proof Rewrite $\mathbf{R}_{[b]}$ in (2.18) as

$$\mathbf{R}_{[b]}=\mathbf{B}_{\triangle,[b]}\boldsymbol{\Lambda}\mathbf{F}'_{\triangle,[b]}+\mathbf{E}_{[b]}, \tag{A.32}$$

where $\mathbf{B}_{\triangle,[b]}=\left[\mathbf{1}_N\ \mathbf{B}_{[b]}\right]$ and $\boldsymbol{\Lambda}$ is given in (A.6). Plugging the expression of (A.32), we have

$$\begin{aligned}
&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2}-\frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}\\
=&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\boldsymbol{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}+\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}-\widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{F}_{\triangle,[b]}}{\tau}\\
+&\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}+\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}_{\triangle,[b]}}{\tau},
\end{aligned}$$

yielding

$$d_{[b]}=\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2}-\frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}\right) \tag{A.33}$$

$$=\boldsymbol{\Lambda}'\mathbf{V}_{\triangle,\beta}\boldsymbol{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}+\mathcal{E}_{D,[b]}, \tag{A.34}$$

where

$$\begin{aligned}
\mathcal{E}_{D,[b]}=&\boldsymbol{\Lambda}'\left(\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}-\mathbf{V}_{\triangle,\beta}\right)\boldsymbol{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\\
&+\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}-\widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{F}_{\triangle,[b]}}{\tau}\\
&+\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}+\boldsymbol{\Lambda}'\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}_{\triangle,[b]}}{\tau}.
\end{aligned}$$

Using the property of (A.5) and Lemma A.10 for $\widehat{\mathbf{V}}_{e,[b]}$, we have

$$\begin{aligned}\operatorname{vec}\left(\mathcal{E}_{D,[b]}\right) &= \left(\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}'\right)\otimes\mathbf{\Lambda}'\right)\operatorname{vec}\left(\frac{\mathbf{B}_{\triangle,[b]}'\mathbf{B}_{\triangle,[b]}}{N_{[b]}}-\mathbf{V}_{\triangle,\beta}\right)\\ &+\left(\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\otimes\left(\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\right)\right)\mathbf{K}_{[b]}\operatorname{vec}\left(\frac{\mathbf{E}_{[b]}'\mathbf{E}_{[b]}}{N_{[b]}}-\mathbf{V}_{e,[b]}\right)\\ &+\left(\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}'\right)\otimes\left(\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\right)\right)\operatorname{vec}\left(\frac{\mathbf{E}_{[b]}'\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\\ &+\left(\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\otimes\mathbf{\Lambda}'\right)\operatorname{vec}\left(\left(\frac{\mathbf{E}_{[b]}'\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\right).\end{aligned}\tag{A.35}$$

This verifies the first claim of the lemma.

Applying Lemma A.9 to the expression of (A.35), we have that $\operatorname{vec}\left(\mathcal{E}_{D,[b]}\right)\xrightarrow{p}\mathbf{0}_{(K+1)^2}$, which in turn implies that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\xrightarrow{p}\mathbf{0}_{(K+1)\times(K+1)},$$

verifying the second claim of the lemma. This completes the proof of the lemma. □

Lemma A.12. *The $u_{[b]}$ given in Theorem 2.2 can be expressed as*

$$u_{[b]}=\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)\mathbf{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right]+\mathcal{E}_{U,[b]},$$

where $\mathcal{E}_{U,[b]}$ is given by (A.38)*. Under Assumptions 1, 4 and 5, it holds that as $N,T\to\infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]}\xrightarrow{p}\mathbf{0}_{K+1}.$$

Proof Rewrite $\mathbf{R}_{[b]}$ in (2.18) as

$$\mathbf{R}_{[b]}=\mathbf{B}_{\triangle,[b]}\mathbf{\Lambda}\mathbf{F}_{\triangle,[b]}'+\mathbf{E}_{[b]},\tag{A.36}$$

where $\mathbf{B}_{\triangle,[b]}=\left[\mathbf{1}_N\ \mathbf{B}_{[b]}\right]$ and $\mathbf{\Lambda}$ is given in (A.6). Hence, it holds that

$$\begin{aligned}\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{R}_{[b]}'\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau} &= \left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}\mathbf{\Lambda}'}{\tau^2}\right)\left(\frac{\mathbf{B}_{\triangle,[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)+\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{E}_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau^2}\\ &= \left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)\mathbf{\Lambda}'\left[1\ \boldsymbol{\mu}_\beta'\right]+\mathcal{E}_{U,[b]},\end{aligned}\tag{A.37}$$

where

$$\mathcal{E}_{U,[b]} = \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}\boldsymbol{\Lambda}'}{\tau^2}\right)\left(\frac{\mathbf{B}_{\triangle,[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}} - \left[1\ \boldsymbol{\mu}'_\beta\right]'\right) + \left(\frac{\mathbf{F}'_{\triangle,[b]}}{\tau^2}\right)\left(\frac{\mathbf{E}_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right). \tag{A.38}$$

Hence, the first claim of the lemma holds.

Applying Lemma A.9 to the expression of (A.38), we have that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)},$$

verifying the second claim of the lemma. This completes the proof of the lemma. □

Lemma A.13. *The $d_{e,[b]}$ given in Theorem 2.2 can be expressed as*

$$d_{e,[b]} = \boldsymbol{\Lambda}'_e\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} + \mathcal{E}_{D_e,[b]},$$

where vec $\left(\mathcal{E}_{D_e,[b]}\right)$ is given by (A.42)*. Under Assumptions 1, 4 and 5, it holds that as $N, T \to \infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)}.$$

Proof Rewrite $\mathbf{R}_{e,[b]}$ in (2.19) as

$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]}\boldsymbol{\Lambda}_e\mathbf{F}'_{\triangle,[b]} + \mathbf{E}_{[b]}. \tag{A.39}$$

Hence, it follows that

$$\begin{aligned}
&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{[b]}\mathbf{F}_{[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{[b]}}{\tau^2} \\
=&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} + \frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{F}_{[b]}}{\tau} \\
+&\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} + \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'_e\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}_{[b]}}{\tau},
\end{aligned}$$

yielding

$$d_{e,[b]} = \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{F}_{[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{[b]}}{\tau^2}\right) \tag{A.40}$$

$$= \mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} + \mathcal{E}_{D_{e,[b]}}, \tag{A.41}$$

where $\mathcal{E}_{D_{e,[b]}}$ is given by

$$\begin{aligned}
\mathcal{E}_{D_{e,[b]}} &= \mathbf{\Lambda}'_e\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right)\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} \\
&+ \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{F}_{[b]}}{\tau} \\
&+ \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{[b]}}{\tau} + \mathbf{\Lambda}'_e\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}_{[b]}}{\tau}.
\end{aligned}$$

Using the property of (A.5) and Lemma A.10 for $\widehat{\mathbf{V}}_{e,[b]}$, we have

$$\begin{aligned}
\operatorname{vec}\left(\mathcal{E}_{D_{e,[b]}}\right) &= \left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}'_e \otimes \mathbf{\Lambda}'_e\right)\operatorname{vec}\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right) \\
&+ \left(\frac{\mathbf{F}'_{[b]}}{\tau} \otimes \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\right)\mathbf{K}_{[b]}\operatorname{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right) \\
&+ \left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}'_e \otimes \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\right)\operatorname{vec}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right) \\
&+ \left(\frac{\mathbf{F}'_{[b]}}{\tau} \otimes \mathbf{\Lambda}'_e\right)\operatorname{vec}\left(\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\right).
\end{aligned} \tag{A.42}$$

This verifies the first claim of the lemma.

Applying Lemma A.9 to the expression of A.42, we have that $\operatorname{vec}\left(\mathcal{E}_{D_{e,[b]}}\right) \xrightarrow{p} \mathbf{0}_{(K+1)^2}$, which in turn implies that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_{e,[b]}} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)},$$

verifying the last claim of the lemma. This completes the proof of the lemma. □

Lemma A.14. *The $u_{e,[b]}$ given in Theorem 2.2 can be expressed as*

$$u_{e,[b]} = \mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \mathcal{E}_{U_{e,[b]}},$$

where $\mathcal{E}_{U_e,[b]}$ is given by (A.45). *Under Assumptions 1, 4 and 5, it holds that as $N, T \to \infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]} \xrightarrow{p} \mathbf{0}_{K+1}.$$

Proof Rewrite $\mathbf{R}_{e,[b]}$ in (2.19) as

$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]}\boldsymbol{\Lambda}_e\mathbf{F}'_{\triangle,[b]} + \mathbf{E}_{[b]}. \tag{A.43}$$

Hence, it holds that

$$\begin{aligned}
&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{1}_\tau}{\tau^2}\\
=&\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'_e\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{1}_\tau}{\tau}\\
+&\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\boldsymbol{\Lambda}'_e\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{1}_\tau}{\tau},
\end{aligned}$$

yielding

$$\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{1}_\tau}{\tau^2}\right) = \boldsymbol{\Lambda}'_e\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \mathcal{E}_{U_e,[b]}, \tag{A.44}$$

where

$$\begin{aligned}
\mathcal{E}_{U_e,[b]} =& \boldsymbol{\Lambda}'_e\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right)\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau}\\
&+\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}\right)\frac{\mathbf{1}_\tau}{\tau}\\
&+\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}'_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \boldsymbol{\Lambda}'_e\left(\frac{\mathbf{E}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{1}_\tau}{\tau}.
\end{aligned}$$

Using the property of (A.5) and Lemma A.10 for $\widehat{\mathbf{V}}_{e,[b]}$, we have

$$\begin{aligned}\mathcal{E}_{U_e,[b]} &= \left(\frac{\mathbf{1}_\tau'\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}_e' \otimes \mathbf{\Lambda}_e'\right)\mathrm{vec}\left(\frac{\mathbf{B}_{[b]}'\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right) \\ &+ \left(\frac{\mathbf{1}_\tau'}{\tau} \otimes \left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\right)\mathbf{K}_{[b]}\mathrm{vec}\left(\frac{\mathbf{E}_{[b]}'\mathbf{E}_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right) \\ &+ \left(\frac{\mathbf{1}_\tau'\mathbf{F}_{\triangle,[b]}}{\tau}\mathbf{\Lambda}_e' \otimes \left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}_{\triangle,[b]}'}{\tau}\right)\mathrm{vec}\left(\frac{\mathbf{E}_{[b]}'\mathbf{B}_{[b]}}{N_{[b]}}\right) \\ &+ \left(\frac{\mathbf{1}_\tau'}{\tau} \otimes \mathbf{\Lambda}_e'\right)\mathrm{vec}\left(\left(\frac{\mathbf{E}_{[b]}'\mathbf{B}_{[b]}}{N_{[b]}}\right)'\right). \end{aligned} \tag{A.45}$$

This verifies the first claim of the lemma.

Applying Lemma A.9 to the expression of A.45, we have that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]} \xrightarrow{p} \mathbf{0}_{K+1}$$

verifying the last claim of the lemma. This completes the proof of the lemma. □

Lemma A.15. *Under Assumptions 1, 4 and 5, as $N, T \to \infty$, it holds that*

$$\begin{aligned}\widehat{\mathbf{D}} &= \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\frac{1}{B}\sum_{b=1}^{B} d_{[b]} \xrightarrow{p} \mathbf{D} = \mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_\triangle}\mathbf{\Lambda}\mathbf{V}_{\triangle,f} \\ \widehat{\mathbf{U}} &= \frac{1}{B}\sum_{b=1}^{B} u_{[b]} \xrightarrow{p} \mathbf{U} = \mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\begin{bmatrix}1 & \boldsymbol{\mu}_\beta'\end{bmatrix},\end{aligned}$$

where $d_{[b]}$ and $u_{[b]}$ are given in Lemmas A.11 and A.12, respectively.

Proof From Lemmas A.1 and A.11, it follows that

$$\begin{aligned}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\frac{1}{B}\sum_{b=1}^{B} d_{[b]} &= \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T} + \frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]} \\ &\xrightarrow{p} \mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_\triangle}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}.\end{aligned}$$

Also, from Lemmas A.1 and A.12, it follows that

$$\begin{aligned}\frac{1}{B}\sum_{b=1}^{B}u_{[b]} &= \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right] + \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]} \\ &\xrightarrow{p} \mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}.\end{aligned}$$

This completes the proof of the lemma. □

Lemma A.16. *Under Assumptions 1, 4 and 5, as $N, T \to \infty$, it holds that*

$$\begin{aligned}\widehat{\mathbf{D}}_e &= \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{e,[b]} \xrightarrow{p} \left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]' \\ \widehat{\mathbf{U}}_e &= \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\sum_{b=1}^{B}u_{e,[b]} \xrightarrow{p} \left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\left[\begin{array}{cc}1 & \boldsymbol{\mu}_f'\end{array}\right]',\end{aligned}$$

where $d_{e,[b]}$ and $u_{e,[b]}$ are given in Lemmas A.13 and A.14, respectively.

Proof From Lemmas A.1 and A.13, it follows that

$$\begin{aligned}\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{e,[b]} &= \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}}{T} + \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]} \\ &\xrightarrow{p} \left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'.\end{aligned}$$

Also, from Lemmas A.1 and A.14, it follows that

$$\begin{aligned}\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\sum_{b=1}^{B}u_{e,[b]} &= \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{1}_{\tau}}{\tau} + \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]} \\ &\xrightarrow{p} \left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\mathbf{\Lambda}_e'\mathbf{V}_{\beta}\mathbf{\Lambda}_e\left[\begin{array}{cc}1 & \boldsymbol{\mu}_f'\end{array}\right]'.\end{aligned}$$

This completes the proof of the lemma. □

We introduce notations. Let $\widehat{\mathbf{D}}^*$, $\widehat{\mathbf{U}}^*$, $\widehat{\mathbf{D}}_e^*$ and $\widehat{\mathbf{D}}_e^*$ denote the followings:

$$\widehat{\mathbf{D}}^* = \left(\frac{\mathbf{F}_\triangle^{*\prime}\mathbf{F}_\triangle^*}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\left[\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{F}_{\triangle,[b]}^*}{\tau}\right)^{-1}\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{R}_{[b]}^{\prime}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}^*}{N_{[b]}\tau^2} - \frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\widehat{\mathbf{V}}_{e,[b]}^*\mathbf{F}_{\triangle,[b]}^*}{\tau^2}\right)\right], \tag{A.46}$$

$$\widehat{\mathbf{U}}^* = \frac{1}{B}\sum_{b=1}^{B}\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{R}_{[b]}^{\prime}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau}, \tag{A.47}$$

$$\widehat{\mathbf{D}}_e^* = \left(\frac{\mathbf{F}^{*\prime}\mathbf{F}_\triangle^*}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\left[\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{F}_{\triangle,[b]}^*}{\tau}\right)^{-1}\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{R}_{e,[b]}^{\prime}\mathbf{R}_{e,[b]}\mathbf{F}_{[b]}^*}{N_{[b]}\tau^2} - \frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\widehat{\mathbf{V}}_{e,[b]}^*\mathbf{F}_{[b]}^*}{\tau^2}\right)\right], \tag{A.48}$$

$$\widehat{\mathbf{U}}_e^* = \left(\frac{\mathbf{F}^{*\prime}\mathbf{F}_\triangle^*}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\left[\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{F}_{\triangle,[b]}^*}{\tau}\right)^{-1}\left(\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{R}_{e,[b]}^{\prime}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\widehat{\mathbf{V}}_{e,[b]}^*\mathbf{1}_\tau}{\tau^2}\right)\right], \tag{A.49}$$

where

$$\widehat{\mathbf{V}}_{e,[b]}^* = diag\left(\widehat{\mathbf{v}}_{e,[b]}^*\right)$$
$$\widehat{\mathbf{v}}_{e,[b]}^* = \left(\mathbf{H}_{[b]}^* \odot \mathbf{H}_{[b]}^*\right)^{-1}\mathcal{S}^\prime\text{vec}\left(\frac{\widehat{\mathbf{E}}_{[b]}^{*\prime}\widehat{\mathbf{E}}_{[b]}^*}{N_{[b]}}\right)$$

and $\widehat{\mathbf{E}}_{[b]}^*$ is defined by for the case of using the gross returns $\widehat{\mathbf{E}}_{[b]}^* = \mathbf{R}_{[b]}\mathbf{H}_{[b]}^*$ and for the case of using excess returns $\widehat{\mathbf{E}}_{[b]} = \mathbf{R}_{e,[b]}\mathbf{H}_{[b]}^*$and

$$\mathbf{H}_{[b]}^* = \mathbf{J}_\tau - \mathbf{J}_\tau\mathbf{F}_{[b]}^*\left(\mathbf{F}_{[b]}^{*\prime}\mathbf{J}_\tau\mathbf{F}_{[b]}^*\right)^{-1}\mathbf{F}_{[b]}^{*\prime}\mathbf{J}_\tau \tag{A.50}$$
$$\mathbf{J}_\tau = \mathbf{I}_\tau - \frac{1}{\tau}\mathbf{1}_{\tau\times\tau}.$$

Also, we express the return process as

$$\mathbf{R}_{[b]} = \mathbf{B}_{\triangle,[b]}\mathbf{\Lambda}\mathcal{O}_\triangle\mathbf{F}_{\triangle,[b]}^{*\prime} + \mathbf{E}_{[b]}^* \tag{A.51}$$
$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]}\mathbf{\Lambda}_e\mathcal{O}_\triangle^\prime\mathbf{F}_{\triangle,[b]}^{*\prime} + \mathbf{E}_{e,[b]}^*, \tag{A.52}$$

where

$$\mathbf{E}_{[b]}^* = \mathbf{B}_{\triangle,[b]}\mathbf{\Lambda}\left(\mathbf{F}_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}^*\mathcal{O}_\triangle^\prime\right)^\prime + \mathbf{E}_{[b]}$$
$$\mathbf{E}_{e,[b]}^* = \mathbf{B}_{[b]}\mathbf{\Lambda}_e\left(\mathbf{F}_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}^*\mathcal{O}_\triangle^\prime\right)^\prime + \mathbf{E}_{[b]}.$$

We need the following lemmas for Corollary 2.2.

Lemma A.17. *Under Assumption 3, as $T \to \infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right) \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)}$$

and

$$tr\left(\frac{1}{B}\sum_{b=1}^{B}\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right) \xrightarrow{p} 0.$$

Proof Note that

$$\left(\mathbf{F}^*_{\triangle} - \mathbf{F}_{\triangle}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle} - \mathbf{F}_{\triangle}\mathcal{O}\right) = \sum_{b=1}^{B}\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right).$$

Hence,

$$\begin{aligned}\frac{1}{B}\sum_{b=1}^{B}\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right) &= \tau\frac{1}{T}\left(\mathbf{F}^*_{\triangle} - \mathbf{F}_{\triangle}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle} - \mathbf{F}_{\triangle}\mathcal{O}\right) \\ &\xrightarrow{p} \tau\mathbf{0}_{(K+1)\times(K+1)} = \mathbf{0}_{(K+1)\times(K+1)},\end{aligned}$$

where the limit is from the second condition in Assumption 3. Furthermore, note that the first condition in Assumption 3 yields that

$$\mathbf{F}^*_{\triangle,[b]} \xrightarrow{p} \mathbf{F}_{\triangle,[b]}\mathcal{O}.$$

Hence, it holds that when N, T are large,

$$\begin{aligned}&\operatorname{tr}\left(\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right) \\ >&\left(\operatorname{tr}\left(\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right)\right)^2 \\ >&\operatorname{tr}\left(\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right),\end{aligned}$$

which in turn implies that

$$\begin{aligned}&\frac{1}{B}\sum_{b=1}^{B}\operatorname{tr}\left(\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right) \\ >&\frac{1}{B}\sum_{b=1}^{B}\operatorname{tr}\left(\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)'\left(\mathbf{F}^*_{\triangle,[b]} - \mathbf{F}_{\triangle,[b]}\mathcal{O}\right)\right).\end{aligned}$$

Noting that the LHS of the above inequality converges to zero and the RHS is non-negative, we conclude that the second claim of the lemma holds. This completes the proof of the lemma. □

Lemma A.18. *Let Assumption 3 be in effect. As* $T \to \infty$,

$$\frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T} = \frac{1}{B}\sum_{b=1}^{B}\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{F}_{\triangle,[b]}^{*}}{\tau} \xrightarrow{p} \mathcal{O}_{\triangle}^{\prime}\mathbf{V}_{\triangle,f}\mathcal{O}_{\triangle}.$$

Proof This follows from Lemma A.7. □

Lemma A.19. *Let Assumption 3 and 5 be in effect. Consider a continuous function of* $\mathbf{F}_{[b]}^{*}$*;* $g\ : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{m}$*. Then, there exists a positive number* $M_g^* < \infty$ *such that*

$$\lim_{T\to\infty}\frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}^{*}\right) < M_g^*.$$

Proof Due to the continuity of g, Assumption 3 yields that

$$g\left(\mathbf{F}_{[b]}^{*}\right) < g\left(\mathbf{F}_{[b]}\mathcal{O}\right) + 1.$$

Hence,

$$\frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}^{*}\right) < \frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}\mathcal{O}\right) + 1.$$

From Assumption 5, $\frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}\mathcal{O}\right)$ is bounded. Hence, we can always find M_g^* such that $\lim_{T\to\infty}\frac{1}{B}\sum_{b=1}^{B} g\left(\mathbf{F}_{[b]}^{*}\right) < M_g^*$. This proves the lemma. □

Lemma A.20. *It holds that*

$$vec\left(\frac{\mathbf{E}_{[b]}^{*\prime}\mathbf{E}_{[b]}^{*}}{N_{[b]}} - \widehat{\mathbf{V}}_{e,[b]}^{*}\right) = \mathbf{K}_{[b]}^{*}vec\left(\frac{\mathbf{E}_{[b]}^{*\prime}\mathbf{E}_{[b]}^{*}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right),$$

where

$$
\begin{aligned}
\widehat{\mathbf{V}}^*_{e,[b]} &= diag\left(\left(\mathbf{H}^*_{[b]} \odot \mathbf{H}^*_{[b]}\right)^{-1} \mathcal{S}' vec\left(\frac{\widehat{\mathbf{E}}^{*\prime}_{[b]}\widehat{\mathbf{E}}^*_{[b]}}{N_{[b]}}\right)\right) \\
\widehat{\mathbf{E}}^*_{[b]} &= \mathbf{R}_{[b]}\mathbf{H}^*_{[b]} \ \ or\ \widehat{\mathbf{E}}^*_{[b]} = \mathbf{R}_{e,[b]}\mathbf{H}^*_{[b]} \\
\mathbf{K}^*_{[b]} &= \left(\mathbf{I}_{\tau^2} - \mathcal{S}\left(\mathbf{H}^*_{[b]} \odot \mathbf{H}^*_{[b]}\right)^{-1} \mathcal{S}'\left(\mathbf{H}^*_{[b]} \otimes \mathbf{H}^*_{[b]}\right)\right) \\
\mathbf{H}^*_{[b]} &= \mathbf{J}_\tau - \mathbf{J}_\tau \mathbf{F}^*_{[b]}\left(\mathbf{F}^{*\prime}_{[b]}\mathbf{J}_\tau\mathbf{F}^*_{[b]}\right)^{-1}\mathbf{F}^{*\prime}_{[b]}\mathbf{J}_\tau \\
\mathbf{J}_\tau &= \mathbf{I}_\tau - \frac{1}{\tau}\mathbf{1}_{\tau\times\tau}.
\end{aligned}
$$

Proof The proof is identical to that of Lemma A.10 where $\widehat{\mathbf{V}}_{e,[b]}$, $\mathbf{H}_{[b]}$ and $\mathbf{E}_{[b]}$ are replaced by $\widehat{\mathbf{V}}^*_{e,[b]}$, $\mathbf{H}^*_{[b]}$ and $\mathbf{E}^*_{[b]}$, respectively. $\square$

Lemma A.21. *Let Assumptions 1, 3-5 be in effect. Consider any set of continuous functions of $\mathbf{F}^*_{[b]}$; $f_1 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^K$, $f_2 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{K^2}$, $f_3 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^\tau$, $f_4 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{K\tau}$, and $f_4 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{\tau^2}$. Then, as $N, T \to \infty$, it holds that*

$$
\begin{aligned}
\frac{1}{B}\sum_{b=1}^{B} f_1\left(\mathbf{F}^*_{[b]}\right)'\left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{1}_{N_{[b]}} - \boldsymbol{\mu}_\beta\right) &\xrightarrow{p} 0 \\
\frac{1}{B}\sum_{b=1}^{B} f_2\left(\mathbf{F}^*_{[b]}\right)' vec\left(\frac{1}{N_{[b]}}\mathbf{B}'_{[b]}\mathbf{B}_{[b]} - \mathbf{V}_\beta\right) &\xrightarrow{p} 0 \\
\frac{1}{B}\sum_{b=1}^{B} f_3\left(\mathbf{F}^*_{[b]}\right)'\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{1}_{N_{[b]}}\right) &\xrightarrow{p} 0 \\
\frac{1}{B}\sum_{b=1}^{B} f_4\left(\mathbf{F}^*_{[b]}\right)' vec\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{B}_{[b]}\right) &\xrightarrow{p} 0 \\
\frac{1}{B}\sum_{b=1}^{B} f_5\left(\mathbf{F}^*_{[b]}\right)' vec\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{U}^*_{[b]} - \mathbf{V}_{e,[b]}\right) &\xrightarrow{p} 0,
\end{aligned}
$$

where $\mathbf{U}^*_{[b]}$ is either $\mathbf{E}^*_{[b]}$ or $\mathbf{E}^*_{e,[b]}$.

Proof The lemma is proved by going through the following steps. We define $e_{[b]}$ locally as $e_{[b]} = \mathbf{F}_{\triangle,[b]} - \mathbf{F}^*_{\triangle,[b]}\mathcal{O}'_\triangle$.

Step 1. Let $\mathbf{X}_{[b]}$ either $\mathbf{B}_{\triangle,[b]}\boldsymbol{\Lambda}$ or $\mathbf{B}_{[b]}\boldsymbol{\Lambda}_e$. Then, there exits a positive constant $c_0 < \infty$ such that $\lambda_{\max}\left(\frac{\mathbf{X}'_{[b]}\mathbf{X}_{[b]}}{N_{[b]}}\right) < c_0$: This directly follows from $\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}} \to \mathbf{V}_{\triangle,\beta}$.

Step 2. It holds that $\mathbf{U}^*_{[b]} = \mathbf{X}_{[b]}\left(\mathbf{F}_{\triangle,[b]} - \mathbf{F}^*_{\triangle,[b]}\mathcal{O}'_\triangle\right)' + \mathbf{E}_{[b]}$ for $\mathbf{X}_{[b]}$ in Step 1: Recall that $\mathbf{E}^*_{[b]} = \mathbf{B}_{\triangle,[b]}\boldsymbol{\Lambda}\left(\mathbf{F}_{\triangle,[b]} - \mathbf{F}^*_{\triangle,[b]}\mathcal{O}'_\triangle\right)' + \mathbf{E}_{[b]}$ and $\mathbf{E}^*_{e,[b]} = \mathbf{B}_{[b]}\boldsymbol{\Lambda}_e\left(\mathbf{F}_{\triangle,[b]} - \mathbf{F}^*_{\triangle,[b]}\mathcal{O}'_\triangle\right)' + \mathbf{E}_{[b]}$.

Step 3. There is a positive constant $c_1 < \infty$ such that $\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)\right) < c_1$: Note that

$$
\begin{aligned}
&\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)\right) < \mathrm{tr}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)\right) \\
=&\frac{1}{N_{[b]}}\mathrm{tr}\left(\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\mathbf{1}_{N_{[b]}}\mathbf{1}'_{N_{[b]}}\right) < \lambda_{\max}\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right) = \lambda_{\max}\left(\frac{\mathbf{X}'_{[b]}\mathbf{X}_{[b]}}{N_{[b]}}\right) < c_0 \equiv c_1,
\end{aligned}
$$

where the first inequality is due to the positivity of the matrix, the first and third equalities are from (A.2), the second equality is from (A.3), the last inequality is from Step 2.

Step 4. There is a positive constant $c_2 < \infty$ such that $\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\right) < c_2$: Note that

$$
\begin{aligned}
&\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\right) < \mathrm{tr}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\right) \\
=&\mathrm{tr}\left(\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\left(\frac{\mathbf{B}_{[b]}\mathbf{B}'_{[b]}}{N_{[b]}}\right)\right) < \lambda_{\max}\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\mathrm{tr}\left(\frac{\mathbf{B}_{[b]}\mathbf{B}'_{[b]}}{N_{[b]}}\right) \\
<&\lambda_{\max}\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)K\lambda_{\max}\left(\frac{\mathbf{B}_{[b]}\mathbf{B}'_{[b]}}{N_{[b]}}\right) < Kc_0\lambda_{\max}\left(\mathbf{V}_\beta\right) + 1 \equiv c_2, \qquad \text{(A.53)}
\end{aligned}
$$

where the first inequality is due to the positivity of the matrix, the first equality is from (A.2), the second inequality is from (A.3), the last inequality is from Step 2.

Step 5. There is a positive constant $c_3 < \infty$ such that $\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\right) < c_3$: Note that

$$
\begin{aligned}
&\lambda_{\max}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\right) < \mathrm{tr}\left(\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)'\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\right) \\
=&\mathrm{tr}\left(\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\left(\frac{\mathbf{E}_{[b]}\mathbf{E}'_{[b]}}{N_{[b]}}\right)\right) < \mathrm{tr}\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\lambda_{\max}\left(\frac{\mathbf{E}_{[b]}\mathbf{E}'_{[b]}}{N_{[b]}}\right) \\
<&K\lambda_{\max}\left(\frac{\mathbf{X}_{[b]}\mathbf{X}'_{[b]}}{N_{[b]}}\right)\lambda_{\max}\left(\frac{\mathbf{E}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right) < Kc_1\left(M_1 + 1\right) \equiv c_3. \qquad \text{(A.54)}
\end{aligned}
$$

where the first and third inequalities are due to the positivity of the matrix, the second inequality from (A.2), the last inequality is from Step 2 and Assumption 4(ii).

Step 6. Let $a_{1,[b]} = e_{[b]}\left(\frac{\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right)$. It holds that $a'_{1,[b]}a_{1,[b]} < c_1\mathrm{tr}\left(e'_{[b]}e_{[b]}\right)$: Note that

$$
\begin{aligned}
a'_{1,[b]}a_{1,[b]} &= \left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}\right)' e'_{[b]}e_{[b]}\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}\right) \\
&= \mathrm{tr}\left(\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}\right)\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{1}_{N_{[b]}}\right)' e'_{[b]}e_{[b]}\right) \\
&< c_1\mathrm{tr}\left(e'_{[b]}e_{[b]}\right),
\end{aligned}
$$

where the second equality is from (A.2) and the inequality is from (A.3) and Step 3.

Step 7. Let $a_{2,[b]} = \mathrm{vec}\left(e_{[b]}\left(\frac{\mathbf{X}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\right)$. It holds that $a'_{2,[b]}a_{2,[b]} < c_2\mathrm{tr}\left(e'_{[b]}e_{[b]}\right)$: Note that

$$
\begin{aligned}
a'_{2,[b]}a_{2,[b]} &= \mathrm{tr}\left(\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{B}_{[b]}\right)' e'_{[b]}e_{[b]}\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{B}_{[b]}\right)\right) \\
&= \mathrm{tr}\left(\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{B}_{[b]}\right)\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{B}_{[b]}\right)' e'_{[b]}e_{[b]}\right) \\
&< c_2\mathrm{tr}\left(e'_{[b]}e_{[b]}\right),
\end{aligned}
$$

where the second equality is from (A.2) and the inequality is from (A.3) and Step 4.

Step 8. Let $a_{3,[b]} = \mathrm{vec}\left(e_{[b]}\left(\frac{\mathbf{X}'_{[b]}\mathbf{E}_{[b]}}{N_{[b]}}\right)\right)$. It holds that $a'_{3,[b]}a_{3,[b]} < c_3\mathrm{tr}\left(e'_{[b]}e_{[b]}\right)$: Note that

$$
\begin{aligned}
a'_{3,[b]}a_{3,[b]} &= \mathrm{tr}\left(\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{E}_{[b]}\right)' e'_{[b]}e_{[b]}\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{E}_{[b]}\right)\right) \\
&= \mathrm{tr}\left(\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{E}_{[b]}\right)\left(\frac{1}{N_{[b]}}\mathbf{X}'_{[b]}\mathbf{E}_{[b]}\right)' e'_{[b]}e_{[b]}\right) \\
&< c_3\mathrm{tr}\left(e'_{[b]}e_{[b]}\right),
\end{aligned}
$$

where the second equality is from (A.2) and the inequality is from (A.3) and Step 5.

Step 9. Let $a_{4,[b]} = \mathrm{vec}\left(e_{[b]}\left(\frac{\mathbf{X}'_{[b]}\mathbf{X}_{[b]}}{N_{[b]}}\right)e'_{[b]}\right)$. It holds that $a'_{4,[b]}a_{4,[b]} < c_0^2\mathrm{tr}\left(e'_{[b]}e_{[b]}e'_{[b]}e_{[b]}\right)$:

Note that

$$
\begin{aligned}
a_{4,[b]}^{\prime} a_{4,[b]} &= \operatorname{tr}\left(e_{[b]}\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime} e_{[b]}\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime}\right) \\
&= \operatorname{tr}\left(\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime} e_{[b]}\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime} e_{[b]}\right) \\
&< c_0 \operatorname{tr}\left(e_{[b]}^{\prime} e_{[b]}\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime} e_{[b]}\right) = c_0 \operatorname{tr}\left(\left(\frac{\mathbf{X}_{[b]}^{\prime} \mathbf{X}_{[b]}}{N_{[b]}}\right) e_{[b]}^{\prime} e_{[b]} e_{[b]}^{\prime} e_{[b]}\right) \\
&< c_0^2 \operatorname{tr}\left(e_{[b]}^{\prime} e_{[b]} e_{[b]}^{\prime} e_{[b]}\right),
\end{aligned}
$$

where the first equality is from (A.4), the second and third equalities are from (A.2) and the inequalities are from (A.3) and Step 1.

Step 10. $\frac{1}{B} \sum_{b=1}^{B} f_3\left(\mathbf{F}_{[b]}^*\right)^{\prime}\left(\frac{1}{N_{[b]}} \mathbf{U}_{[b]}^{*\prime} \mathbf{1}_{N_{[b]}}-\frac{1}{N_{[b]}} \mathbf{E}_{[b]}^{\prime} \mathbf{1}_{N_{[b]}}\right)=\frac{1}{B} \sum_{b=1}^{B} f_3\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{1,[b]}$: This is trivial.

Step 11. $\frac{1}{B} \sum_{b=1}^{B} f_4\left(\mathbf{F}_{[b]}^*\right)^{\prime}\left(\operatorname{vec}\left(\frac{1}{N_{[b]}} \mathbf{U}_{[b]}^{*\prime} \mathbf{B}_{[b]}\right)-\operatorname{vec}\left(\frac{1}{N_{[b]}} \mathbf{E}_{[b]}^{\prime} \mathbf{B}_{[b]}\right)\right)=\frac{1}{B} \sum_{b=1}^{B} f_4\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{2,[b]}$: This is trivial.

Step 12. $\frac{1}{B} \sum_{b=1}^{B} f_5\left(\mathbf{F}_{[b]}^*\right)^{\prime}\left(\operatorname{vec}\left(\frac{1}{N_{[b]}} \mathbf{U}_{[b]}^{*\prime} \mathbf{U}_{[b]}^*\right)-\operatorname{vec}\left(\frac{1}{N_{[b]}} \mathbf{E}_{[b]}^{\prime} \mathbf{E}_{[b]}\right)\right)=2 \frac{1}{B} \sum_{b=1}^{B} f_5\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{3,[b]}+$ $\frac{1}{B} \sum_{b=1}^{B} f_3\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{4,[b]}$: This is trivial.

Step 13. $\frac{1}{B} \sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{l,[b]} \xrightarrow{p} 0$ for $(k, l)=(3,1),(4,2),(5,3)$: Note that

$$
\begin{aligned}
& \frac{1}{B} \sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}^*\right)^{\prime} a_{l,[b]} \leq \frac{1}{B} \sum_{b=1}^{B} \sqrt{f_k\left(\mathbf{F}_{[b]}^*\right)^{\prime} f_k\left(\mathbf{F}_{[b]}^*\right)} \sqrt{a_{l,[b]}^{\prime} a_{l,[b]}} \\
\leq & \frac{1}{B} \sum_{b=1}^{B} \sqrt{f_k\left(\mathbf{F}_{[b]}^*\right)^{\prime} f_k\left(\mathbf{F}_{[b]}^*\right)} \sqrt{c_l \operatorname{tr}\left(e_{[b]}^{\prime} e_{[b]}\right)} \\
\leq & \sqrt{\frac{1}{B} \sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}^*\right)^{\prime} f_k\left(\mathbf{F}_{[b]}^*\right)} \sqrt{c_l \operatorname{tr}\left(\frac{1}{B} \sum_{b=1}^{B} e_{[b]}^{\prime} e_{[b]}\right)},
\end{aligned}
$$

where the first and third inequalities are from Cauchy-Schwarz inequility and the second inequality is from Steps 6-8. Lastly, Assumption 3 confirms that the upper bound goes to zero. Similarly, we can show that the lower bound goes to zero.

Step 14. $\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}^*_{[b]}\right)' a_{l,[b]} \xrightarrow{p} 0$ for $(k,l)=(5,4)$: Note that

$$
\begin{aligned}
\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}^*_{[b]}\right)' a_{l,[b]} \leq & \frac{1}{B}\sum_{b=1}^{B}\sqrt{f_k\left(\mathbf{F}^*_{[b]}\right)' f_k\left(\mathbf{F}^*_{[b]}\right)}\sqrt{a'_{l,[b]}a_{l,[b]}} \\
\leq & \frac{1}{B}\sum_{b=1}^{B}\sqrt{f_k\left(\mathbf{F}^*_{[b]}\right)' f_k\left(\mathbf{F}^*_{[b]}\right)}\sqrt{c_0^2 \mathrm{tr}\left(e'_{[b]}e_{[b]}e'_{[b]}e_{[b]}\right)} \\
\leq & \sqrt{\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}^*_{[b]}\right)' f_3\left(\mathbf{F}^*_{[b]}\right)}\sqrt{c_k^2 \mathrm{tr}\left(\frac{1}{B}\sum_{b=1}^{B} e'_{[b]}e_{[b]}e'_{[b]}e_{[b]}\right)},
\end{aligned}
$$

where the first and third inequalities are from Cauchy-Schwarz inequility and the second inequality is from Step 9. Lemma A.17 confirms that the upper bound goes to zero. We can show that the lower bound goes to zero in a similar manner.

Step 15. The following equalities hold:

$$
\begin{aligned}
\frac{1}{B}\sum_{b=1}^{B} f_3\left(\mathbf{F}^*_{[b]}\right)'\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{1}_{N_{[b]}}\right) &= \frac{1}{B}\sum_{b=1}^{B} f_3\left(\mathbf{F}^*_{[b]}\right)'\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{1}_{N_{[b]}}\right) + o_p(1) \\
\frac{1}{B}\sum_{b=1}^{B} f_4\left(\mathbf{F}^*_{[b]}\right)' \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{B}_{[b]}\right) &= \frac{1}{B}\sum_{b=1}^{B} f_4\left(\mathbf{F}^*_{[b]}\right)' \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{B}_{[b]}\right) + o_p(1) \\
\frac{1}{B}\sum_{b=1}^{B} f_5\left(\mathbf{F}^*_{[b]}\right)' \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{U}^{*\prime}_{[b]}\mathbf{U}^*_{[b]} - \mathbf{V}_{e,[b]}\right) &= \frac{1}{B}\sum_{b=1}^{B} f_5\left(\mathbf{F}^*_{[b]}\right)' \mathrm{vec}\left(\frac{1}{N_{[b]}}\mathbf{E}'_{[b]}\mathbf{E}_{[b]} - \mathbf{V}_{e,[b]}\right) + o_p(1).
\end{aligned}
$$

The follows from applying the results in Steps 13-14 to the expressions in Steps 10-12.

Step 16. The claim holds: From Step 15, we can use Lemma A.9 while replacing $\mathbf{F}_{[b]}$ with $\mathbf{F}^*_{[b]}$ and substituting Assumption 5 with Lemma A.19. This completes the proof of the lemma. □

Lemma A.22. *Define $d^*_{[b]}$ as*

$$
d^*_{[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}^*_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^*_{e,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau^2}\right).
$$

Then, it holds that

$$
d^*_{[b]} = \mathcal{O}'_{\triangle}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau} + \mathcal{E}^*_{D,[b]},
$$

where vec$\left(\mathcal{E}^*_{D,[b]}\right)$ is given by (A.58). *Under Assumptions 1, 3, 4 and 5, it holds that as $N,T\to\infty$,*

$$
\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}^*_{D,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)}.
$$

Proof We rewrite (A.51) here:

$$\mathbf{R}_{[b]} = \mathbf{B}_{\triangle,[b]}\boldsymbol{\Lambda}\mathcal{O}'_{\triangle}\mathbf{F}^{*\prime}_{\triangle,[b]} + \mathbf{E}^{*}_{[b]}. \tag{A.55}$$

Using the expression of (A.55), we have

$$\begin{aligned}
&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}^{*}_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^{*}_{e,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau^2} \\
=&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\left(\mathcal{O}'_{\triangle}\boldsymbol{\Lambda}'\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\right)\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^{*}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^{*}_{e,[b]}\right)\frac{\mathbf{F}^{*}_{\triangle,[b]}}{\tau} \\
&+\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\mathcal{O}'_{\triangle}\boldsymbol{\Lambda}'\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}^{*}_{\triangle,[b]}}{\tau},
\end{aligned}$$

yielding

$$d^{*}_{[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}^{*}_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^{*}_{e,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau^2}\right) \tag{A.56}$$

$$= \mathcal{O}'_{\triangle}\boldsymbol{\Lambda}'\mathbf{V}_{\triangle,\beta}\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau} + \mathcal{E}^{*}_{D,[b]}, \tag{A.57}$$

where

$$\begin{aligned}
\mathcal{E}^{*}_{D,[b]} =& \mathcal{O}'_{\triangle}\boldsymbol{\Lambda}'\left(\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}} - \mathbf{V}_{\triangle,\beta}\right)\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau} \\
&+\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^{*}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^{*}_{e,[b]}\right)\frac{\mathbf{F}^{*}_{\triangle,[b]}}{\tau} \\
&+\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau} + \mathcal{O}'_{\triangle}\boldsymbol{\Lambda}'\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}^{*}_{\triangle,[b]}}{\tau}.
\end{aligned}$$

Using the property of (A.5) and Lemma A.20 for $\widehat{\mathbf{V}}^*_{e,[b]}$, we have

$$\begin{aligned}\mathrm{vec}\left(\mathcal{E}^*_{D,[b]}\right) =& \left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\left(\boldsymbol{\Lambda}\mathcal{O}_\triangle\right)'\right)\otimes\left(\boldsymbol{\Lambda}\mathcal{O}_\triangle\right)'\right)\mathrm{vec}\left(\frac{\mathbf{B}'_{\triangle,[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}-\mathbf{V}_{\triangle,\beta}\right)\\ &+\left(\frac{\mathbf{F}^*_{\triangle,[b]}}{\tau}\otimes\left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\right)\mathbf{K}^*_{[b]}\mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^*_{[b]}}{N_{[b]}}-\mathbf{V}_{e,[b]}\right)\\ &+\left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\left(\boldsymbol{\Lambda}\mathcal{O}_\triangle\right)'\right)\otimes\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)\\ &+\left(\frac{\mathbf{F}^*_{\triangle,[b]}}{\tau}\otimes\left(\boldsymbol{\Lambda}\mathcal{O}_\triangle\right)'\right)\mathrm{vec}\left(\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{\triangle,[b]}}{N_{[b]}}\right)'\right). \end{aligned}\tag{A.58}$$

Hence, the first claim of the lemma holds.

From Lemma A.21 and the expressions of (A.58), we have that $\mathrm{vec}\left(\mathcal{E}^*_{D,[b]}\right)\xrightarrow{p}\mathbf{0}_{(K+1)^2}$, which in turn implies that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}^*_{D,[b]}\xrightarrow{p}\mathbf{0}_{(K+1)\times(K+1)},$$

verifying the last claim of the lemma. This completes the proof of the lemma. □

Lemma A.23. *Define $u^*_{[b]}$ as*

$$u^*_{[b]}=\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau}.$$

Then, it holds that

$$u^*_{[b]}=\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)\left(\boldsymbol{\Lambda}\mathcal{O}_\triangle\right)'\left[1\ \boldsymbol{\mu}'_\beta\right]+\mathcal{E}^*_{U,[b]},$$

where $\mathcal{E}_{U,[b]}$ is given by (A.60). *Under Assumptions 1, 3, 4 and 5, it holds that as $N,T\to\infty$,*

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}^*_{U,[b]}\xrightarrow{p}\mathbf{0}_{(K+1)}.$$

Proof Recall the expression of (A.51):

$$\mathbf{R}_{[b]}=\mathbf{B}_{\triangle,[b]}\boldsymbol{\Lambda}\mathcal{O}_\triangle\mathbf{F}^{*\prime}_{\triangle,[b]}+\mathbf{E}^*_{[b]}.$$

Hence, it holds that

$$\begin{aligned}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}^{\prime}_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau} &= \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\left(\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\right)^{\prime}\right)\left(\frac{\mathbf{B}_{\triangle,[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right) + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{E}^{*}_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau} \\ &= \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)\left(\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\right)^{\prime}\boldsymbol{\mu}_{\beta} + \mathcal{E}^{*}_{U,[b]},\end{aligned} \tag{A.59}$$

where

$$\mathcal{E}^{*}_{U,[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\right)^{\prime}\right)\left(\frac{\mathbf{B}_{\triangle,[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}} - \boldsymbol{\mu}_{\beta}\right) + \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\left(\frac{\mathbf{E}^{*}_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}}\right). \tag{A.60}$$

Hence, the first claim of the lemma holds.

Applying Lemma A.21 to the expressions of (A.60), we have that

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}^{*}_{U,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)},$$

verifying the last claim of the lemma. This completes the proof of the lemma. $\square$

Lemma A.24. *Define* $d^{*}_{e,[b]}$ *as*

$$d^{*}_{e,[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}^{\prime}_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{N_{[b]}\tau^{2}} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^{*}_{e,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau^{2}}\right).$$

Then, it holds that

$$d^{*}_{e,[b]} = \mathcal{O}_{\triangle}\boldsymbol{\Lambda}^{\prime}_{e}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\mathcal{O}^{\prime}_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} + \mathcal{E}^{*}_{D_e,[b]},$$

where vec$\left(\mathcal{E}^{*}_{D_e,[b]}\right)$ *is given by* (A.64). *Under Assumptions 1, 3, 4 and 5, it holds that as* $N,T\to\infty$,

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}^{*}_{D_e,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)}.$$

Proof Rewrite $\mathbf{R}_{e,[b]}$ in (A.52) as

$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]}\boldsymbol{\Lambda}_{e}\mathcal{O}^{\prime}_{\triangle}\mathbf{F}^{*\prime}_{\triangle,[b]} + \mathbf{E}^{*}_{[b]}. \tag{A.61}$$

Hence, it follows that

$$\begin{aligned}
&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{[b]}\mathbf{F}^{*}_{[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^{*}_{e,[b]}\mathbf{F}^{*}_{[b]}}{\tau^2} \\
=&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\mathbf{\Lambda}_e\mathcal{O}'_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^{*}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^{*}_{e,[b]}\right)\frac{\mathbf{F}^{*}_{[b]}}{\tau} \\
+&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\mathbf{\Lambda}_e\mathcal{O}'_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}^{*}_{[b]}}{\tau},
\end{aligned}$$

yielding

$$d^{*}_{e,[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{[b]}\mathbf{F}^{*}_{[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\widehat{\mathbf{V}}^{*}_{e,[b]}\mathbf{F}^{*}_{[b]}}{\tau^2}\right) \tag{A.62}$$

$$= \mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\mathbf{V}_{\beta}\mathbf{\Lambda}_e\mathcal{O}'_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} + \mathcal{E}^{*}_{D_e,[b]}, \tag{A.63}$$

where

$$\begin{aligned}
\mathcal{E}^{*}_{D_e,[b]} =& \mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_{\beta}\right)\mathbf{\Lambda}_e\mathcal{O}'_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} \\
&+ \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^{*}_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^{*}_{e,[b]}\right)\frac{\mathbf{F}^{*}_{[b]}}{\tau} \\
&+ \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\mathbf{\Lambda}_e\mathcal{O}'_{\triangle}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau} + \mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{F}^{*}_{[b]}}{\tau}.
\end{aligned}$$

Using the property of (A.5) and Lemma A.21 for $\widehat{\mathbf{V}}^{*}_{e,[b]}$, we have

$$\begin{aligned}
\mathrm{vec}\left(\mathcal{E}^{*}_{D_e,[b]}\right) =& \left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau}\left(\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\right)\right)\otimes\left(\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\right)\right)\mathrm{vec}\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_{\beta}\right) \\
&+ \left(\frac{\mathbf{F}^{*\prime}_{[b]}}{\tau}\otimes\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\mathbf{K}_{[b]}\mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^{*}_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right) \\
&+ \left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{[b]}}{\tau}\left(\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\right)\right)\otimes\left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^{*}_{\triangle,[b]}}{\tau}\right)^{-1}\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\right)\mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right) \\
&+ \left(\frac{\mathbf{F}^{*\prime}_{[b]}}{\tau}\otimes\left(\mathcal{O}_{\triangle}\mathbf{\Lambda}'_e\right)\right)\mathrm{vec}\left(\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\right).
\end{aligned} \tag{A.64}$$

Hence, the first claim of the lemma holds.

Applying Lemma A.21 to the expressions of (A.64), we have that vec $\left(\mathcal{E}^*_{D_e,[b]}\right) \xrightarrow{p} \mathbf{0}_{K^2}$, which in turn implies that

$$\frac{1}{B}\sum_{b=1}^{B} \mathcal{E}^*_{D_e,[b]} \xrightarrow{p} \mathbf{0}_{K\times K},$$

verifying the last claim of the lemma. This completes the proof of the lemma. $\square$

Lemma A.25. *Define* $u^*_{e,[b]}$ *as*

$$u^*_{e,[b]} = \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}^*_{e,[b]}\mathbf{1}_\tau}{\tau^2}\right).$$

Then, it holds that

$$u^*_{e,[b]} = \mathcal{O}_\triangle\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\mathcal{O}'_\triangle\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \mathcal{E}^*_{U_e,[b]},$$

where $\mathcal{E}^*_{U_e,[b]}$ *is given by* (A.66). *Under Assumptions 1, 3, 4 and 5, it holds that as* $N, T \to \infty$,

$$\frac{1}{B}\sum_{b=1}^{B} \mathcal{E}^*_{U_e,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)}.$$

Proof Rewrite $\mathbf{R}_{e,[b]}$ of (A.52):

$$\mathbf{R}_{e,[b]} = \mathbf{B}_{[b]}\mathbf{\Lambda}_e\mathcal{O}_\triangle\mathbf{F}^{*\prime}_{\triangle,[b]} + \mathbf{E}^*_{[b]}.$$

Hence, it holds that

$$\begin{aligned}
&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{1}_\tau}{\tau^2}\\
=&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\left(\mathbf{\Lambda}_e\mathcal{O}'_\triangle\right)'\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\mathbf{\Lambda}_e\mathcal{O}'_\triangle\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^*_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^*_{e,[b]}\right)\frac{\mathbf{1}_\tau}{\tau}\\
+&\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)\mathbf{\Lambda}_e\mathcal{O}'_\triangle\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\left(\mathbf{\Lambda}_e\mathcal{O}'_\triangle\right)'\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\frac{\mathbf{1}_\tau}{\tau},
\end{aligned}$$

yielding

$$\begin{aligned}
&\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1}\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{R}'_{e,[b]}\mathbf{R}_{e,[b]}\mathbf{1}_\tau}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}^*_{e,[b]}\mathbf{1}_\tau}{\tau^2}\right) \qquad \text{(A.65)}\\
=&\mathcal{O}_\triangle\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\mathcal{O}'_\triangle\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \mathcal{E}^*_{U_e,[b]},
\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}^*_{U_e,[b]} &= \mathcal{O}_\triangle \mathbf{\Lambda}'_e \left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right) \mathbf{\Lambda}_e \mathcal{O}'_\triangle \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} \\ &+ \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1} \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau} \left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^*_{[b]}}{N_{[b]}} - \widehat{\mathbf{V}}^*_{e,[b]}\right) \frac{\mathbf{1}_\tau}{\tau} \\ &+ \left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1} \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau} \left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right) \mathbf{\Lambda}_e \mathcal{O}'_\triangle \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \mathcal{O}_\triangle \mathbf{\Lambda}'_e \left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)' \frac{\mathbf{1}_\tau}{\tau}.\end{aligned} \tag{A.66}$$

Hence, the first claim of the lemma holds.

Applying Lemma A.21 to the expressions of (A.66), we have

$$\begin{aligned}\mathcal{E}^*_{U_e,[b]} &= \left(\left(\mathbf{\Lambda}_e \mathcal{O}'_\triangle \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau}\right)' \otimes (\mathcal{O}_\triangle \mathbf{\Lambda}'_e)\right) \mathrm{vec}\left(\frac{\mathbf{B}'_{[b]}\mathbf{B}_{[b]}}{N_{[b]}} - \mathbf{V}_\beta\right) \\ &+ \left(\frac{\mathbf{1}'_\tau}{\tau} \otimes \left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1} \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\right) \mathbf{K}^*_{[b]} \mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{E}^*_{[b]}}{N_{[b]}} - \mathbf{V}_{e,[b]}\right) \\ &+ \left(\left(\mathbf{\Lambda}_e \mathcal{O}'_\triangle \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{1}_\tau}{\tau}\right)' \otimes \left(\left(\frac{\mathbf{F}^{*\prime}_{\triangle,[b]}\mathbf{F}^*_{\triangle,[b]}}{\tau}\right)^{-1} \frac{\mathbf{F}^{*\prime}_{\triangle,[b]}}{\tau}\right)\right) \mathrm{vec}\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right) \\ &+ \left(\frac{\mathbf{1}'_\tau}{\tau} \otimes (\mathcal{O}_\triangle \mathbf{\Lambda}'_e)'\right) \mathrm{vec}\left(\left(\frac{\mathbf{E}^{*\prime}_{[b]}\mathbf{B}_{[b]}}{N_{[b]}}\right)'\right),\end{aligned}$$

implying that

$$\frac{1}{B}\sum_{b=1}^{B} \mathcal{E}^*_{U_e,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)}$$

verifying the last claim of the lemma. This completes the proof of the lemma. □

Lemma A.26. *Under Assumptions 1, 3, 4 and 5, as $N, T \to \infty$, it holds that*

$$\widehat{\mathbf{D}}^* = \frac{\mathbf{F}^{*\prime}_\triangle \mathbf{F}^*_\triangle}{T} \frac{1}{B} \sum_{b=1}^{B} d^*_{[b]} \xrightarrow{p} \mathcal{O}'_\triangle \mathbf{V}_{\triangle,f} \mathbf{\Lambda}' \mathbf{V}_{\beta\triangle} \mathbf{\Lambda} \mathbf{V}_{\triangle,f} \mathcal{O}_\triangle$$

$$\widehat{\mathbf{U}}^* = \frac{1}{B}\sum_{b=1}^{B} u^*_{[b]} \xrightarrow{p} \mathcal{O}'_\triangle \mathbf{V}_{\triangle,f} \mathbf{\Lambda}' \begin{bmatrix} 1 & \boldsymbol{\mu}'_\beta \end{bmatrix}',$$

where $d^*_{[b]}$ and $u^*_{[b]}$ are given in Lemmas A.22 and A.23, respectively.

Proof From Lemmas A.17 and A.22, it follows that

$$\frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{[b]} = \frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\mathcal{O}_{\triangle}^{\prime}\boldsymbol{\Lambda}^{\prime}\mathbf{V}_{\triangle,\beta}\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\frac{\mathbf{F}_{\triangle,[b]}^{*\prime}\mathbf{F}_{\triangle,[b]}^{*}}{\tau} + \frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}^{*}$$
$$\xrightarrow{p} \mathcal{O}_{\triangle}^{\prime}\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}^{\prime}\mathbf{V}_{\beta\triangle}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}\mathcal{O}_{\triangle}.$$

Also, from Lemmas A.17 and A.12, it follows that

$$\frac{1}{B}\sum_{b=1}^{B}u_{[b]}^{*} = \frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\left(\boldsymbol{\Lambda}\mathcal{O}_{\triangle}\right)^{\prime}\left[1\ \boldsymbol{\mu}_{\beta}^{\prime}\right] + \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}^{*}$$
$$\xrightarrow{p} \mathcal{O}_{\triangle}^{\prime}\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}^{\prime}\left[1\ \boldsymbol{\mu}_{\beta}^{\prime}\right]^{\prime}.$$

This completes the proof of the lemma. □

Lemma A.27. *Under Assumptions 1, 3, 4 and 5, as $N, T \to \infty$, it holds that*

$$\widehat{\mathbf{D}}_{e}^{*} = \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{e,[b]}^{*} \xrightarrow{p} \mathcal{O}^{\prime}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]^{\prime}\mathcal{O}$$
$$\widehat{\mathbf{U}}_{e}^{*} = \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}u_{e,[b]}^{*} \xrightarrow{p} \mathcal{O}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[1\ \boldsymbol{\mu}_{f}^{\prime}\right]^{\prime},$$

where $d_{e,[b]}^{*}$ and $u_{e,[b]}^{*}$ are given in Lemmas A.24 and A.25, respectively.

Proof From Lemmas A.1 and A.13, it follows that

$$\frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{e,[b]}^{*} = \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\mathcal{O}_{\triangle}\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\mathcal{O}_{\triangle}^{\prime}\frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{F}^{*}}{\tau} + \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_{e},[b]}^{*}$$
$$\xrightarrow{p} \mathcal{O}^{\prime}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]^{\prime}\mathcal{O}$$

Also, from Lemmas A.1 and A.14, it follows that

$$\frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}u_{e,[b]}^{*} = \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\frac{\mathbf{F}_{\triangle}^{*\prime}\mathbf{1}_{T}}{T} + \frac{\mathbf{F}^{*\prime}\mathbf{F}_{\triangle}^{*}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_{e},[b]}^{*}$$
$$\xrightarrow{p} \mathcal{O}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}^{\prime}\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[1\ \boldsymbol{\mu}_{f}^{\prime}\right]^{\prime}.$$

This completes the proof of the lemma. □

Proof of Corollary 2.2 Note that from Lemma A.26

$$\left(\widehat{\mathbf{D}}^{*}\right)^{-1}\widehat{\mathbf{U}}^{*}\xrightarrow{p}\mathcal{O}_{\triangle}'\left(\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}\right)^{-1}\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]'=\mathcal{O}_{\triangle}'\boldsymbol{\delta}$$

and that from Lemma A.27

$$\begin{aligned}&\left(\mathbf{D}_{e}^{*}\right)^{-1}\mathbf{U}_{e}^{*}\\ &\xrightarrow{p}-\mathcal{O}'\left(\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]'\right)^{-1}\left[\boldsymbol{\mu}_{f}\ \mathbf{V}_{f}\right]\boldsymbol{\Lambda}_{e}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_{e}\left[1\ \boldsymbol{\mu}_{f}'\right]'\\ &=-\mathcal{O}'\left(\boldsymbol{\Lambda}_{e}\left[\begin{array}{cc}\boldsymbol{\mu}_{f} & \mathbf{V}_{f}\end{array}\right]'\right)^{-1}\boldsymbol{\Lambda}_{e}\left[1\ \boldsymbol{\mu}_{f}'\right]'=-\mathcal{O}'\boldsymbol{\delta}_{e}.\end{aligned}$$

Hence, given the consistency of factor estimators in Assumption 3, the claim of the corollary is true. □

Lemma A.28. *Under Assumption 6, for $e_{k,[b]}$ defined by (A.28), it holds that as $N,T\to\infty$,*

$$\sum_{b=1}^{B}e_{k,[b]}'e_{k,[b]}\xrightarrow{p}0$$

for $k=1,\cdots,5$.

Proof Fix k. Pick any $c>0$. From Assumption 6(ii), when $N_{[b]}$ is large, it holds that

$$N_{[b]}^{-\delta}\left(N_{[b]}e_{k,[b]}'e_{k,[b]}\right)<c \tag{A.67}$$

for all b with δ given in Assumption 6(i). Then, it follows that

$$\begin{aligned}\sum_{b=1}^{B}e_{k,[b]}'e_{k,[b]}&=\frac{1}{N^{1-\delta}}\sum_{b=1}^{B}\left(\frac{N}{N_{[b]}}\right)^{1-\delta}N_{[b]}^{-\delta}\left(N_{[b]}e_{k,[b]}'e_{k,[b]}\right)\\ &<\frac{1}{N^{1-\delta}}B\sum_{b=1}^{B}\left(\frac{1}{\delta}\right)^{1-\delta}c\\ &=\frac{c}{\tau}\frac{T}{N^{1-\delta}}\left(\frac{1}{\delta}\right)^{1-\delta}\xrightarrow{p}0,\end{aligned}$$

where the first inequality is from Assumption 6(i) and (A.67) and the last limit is from Assumption 6(i).

Repeating this exercise for $k=1,\cdots,5$ completes the proof of the lemma. □

Lemma A.29. *Let Assumptions 1, 4, 5 and 6 be in effect. Consider any set of continuous functions of $\mathbf{F}_{[b]}$; $f_1:\mathbb{R}^{\tau\times K}\to\mathbb{R}^{K}$, $f_2:\mathbb{R}^{\tau\times K}\to\mathbb{R}^{K^2}$, $f_3:\mathbb{R}^{\tau\times K}\to\mathbb{R}^{\tau}$, $f_4:\mathbb{R}^{\tau\times K}\to\mathbb{R}^{K\tau}$,* *and* $f_4 : \mathbb{R}^{\tau\times K} \to \mathbb{R}^{\tau^2}$*. For* $e_{k,[b]}$ *defined by (A.28), it holds that as* $N, T \to \infty$*,*

$$\sqrt{T}\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0$$

for $k = 1, \cdots, 5$*.*

Proof Fix k. Let f_k be the corresponding function. As N, T increases, it follows that

$$\begin{aligned}
\sqrt{T}\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} &\leq \sqrt{T}\frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{f_k\left(\mathbf{F}_{[b]}\right)' f_k\left(\mathbf{F}_{[b]}\right)}\sqrt{e'_{k,[b]}e_{k,[b]}}\right) \\
&= \sqrt{T}\frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{g_k\left(\mathbf{F}_{[b]}\right)}\sqrt{e'_{k,[b]}e_{k,[b]}}\right) \\
&\leq \sqrt{T}\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} g_k\left(\mathbf{F}_{[b]}\right)}\right)\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} e'_{k,[b]}e_{k,[b]}}\right) \\
&\leq \sqrt{M_{g_k}} \cdot \sqrt{\tau\sum_{b=1}^{B} e'_{k,[b]}e_{k,[b]}} \xrightarrow{p} 0, \qquad \text{(A.68)}
\end{aligned}$$

where the first equality holds by defining $g_k\left(\mathbf{F}_{[b]}\right) = \sqrt{f_k\left(\mathbf{F}_{[b]}\right)' f_k\left(\mathbf{F}_{[b]}\right)}$, the first and second inequalities are from the Cauchy-Schwarz inequality, the third inequality is from Assumption 5, and the last limit is from Lemma A.28. In a similar manner, we can show that as N, T increases,

$$-\sqrt{T}\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{[b]} \leq \sqrt{M_{g_k}} \cdot \sqrt{\tau\sum_{b=1}^{B} e'_{k,[b]}e_{k,[b]}} \xrightarrow{p} 0. \qquad \text{(A.69)}$$

Lastly, combining (A.68) and (A.69) in conjunction with the squeeze theorem, we have that as $N, T \to \infty$,

$$\sqrt{T}\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0,$$

Repeating this exercise for $k = 1, \cdots, 5$ completes the proof of the lemma. □

Lemma A.30. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]},\ \frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]} \xrightarrow{p} \mathbf{0}_{(K+1)\times(K+1)},\ \frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]},\ \frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]} \xrightarrow{p} \mathbf{0}_{K+1},$$

where $vec\left(\mathcal{E}_{D,[b]}\right)$*,* $vec\left(\mathcal{E}_{D_e,[b]}\right)$*,* $\mathcal{E}_{U,[b]}$*,* $\mathcal{E}_{U_e,[b]}$ *are given by (A.35), (A.42), (A.38) and (A.45), respectively.*

Proof From the expressions of vec $\left(\mathcal{E}_{D,[b]}\right)$, vec $\left(\mathcal{E}_{D_e,[b]}\right)$, $\mathcal{E}_{U,[b]}$, $\mathcal{E}_{U_e,[b]}$ are given by (A.35), (A.42), (A.38) and (A.45), Lemma A.29 implies the desired result. $\square$

Lemma A.31. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\sqrt{T}\left(vec\left(\widehat{\mathbf{D}}\right)-vec\left(\mathbf{D}\right)\right)=\mathbf{\Pi}_D\sqrt{T}vec\left(\frac{1}{T}\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}-\mathbf{V}_f\right)+o_p\left(1\right),$$

where $\mathbf{D}$ *is the probability limit of* $\widehat{\mathbf{D}}$ *and*

$$\mathbf{\Pi}_D=\left(\mathbf{I}_{K+1}\otimes\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\right)+\left(\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\otimes\mathbf{I}_{K+1}\right).$$

Proof From the expression of $\widehat{\mathbf{D}}$ in Theorem 2.2, the expressions of $d_{[b]}$ in Lemma A.11 and the limit of $\widehat{\mathbf{D}}$ in Lemma A.15,

$$\begin{aligned}\widehat{\mathbf{D}}-\mathbf{D}&=\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{[b]}-\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}\\&=\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}+\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}-\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}.\end{aligned}$$

From Lemma A.30, we have that

$$\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}=o_p\left(1\right),$$

implying that

$$\sqrt{T}\left(\widehat{\mathbf{D}}-\mathbf{D}\right)=\sqrt{T}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}\right)+o_p\left(1\right).\qquad\text{(A.70)}$$

Besides, noting that

$$\begin{aligned}&\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}\\=&\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)+\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\beta\triangle}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}+\\&+\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right),\end{aligned}$$

we use (A.5) to obtain

$$
\begin{aligned}
&\sqrt{T}\text{vec}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}'\mathbf{V}_{\triangle,\beta}\boldsymbol{\Lambda}\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta_{\triangle}}\boldsymbol{\Lambda}\mathbf{V}_{\triangle,f}\right)\\
&= \left(\left(\mathbf{I}\otimes\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}\right) + \left(\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}\otimes\mathbf{I}\right)\right)\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle} - \mathbf{V}_{f}\right)\\
&\quad + \left(T^{0.25}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\otimes T^{0.25}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\right)\text{vec}\left(\boldsymbol{\Lambda}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}\right)\\
&= \left(\left(\mathbf{I}\otimes\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}\right) + \left(\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\mathbf{V}_{\beta}\boldsymbol{\Lambda}\otimes\mathbf{I}\right)\right)\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle} - \mathbf{V}_{f}\right) + o_p(1), \qquad \text{(A.71)}
\end{aligned}
$$

where the last equality holds from Assumption 6(iii). Plugging (A.71) into (A.70) completes the proof of the lemma. □

Lemma A.32. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\sqrt{T}\left(\widehat{\mathbf{U}} - \mathbf{U}\right) = \boldsymbol{\Pi}_U\sqrt{T}vec\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle} - \mathbf{V}_f\right) + o_p(1),$$

where $\mathbf{U}$ *is the probability limit of* $\widehat{\mathbf{U}}$ *and*

$$\boldsymbol{\Pi}_U = \left(\boldsymbol{\Lambda}'\boldsymbol{\mu}_{\beta}\right)'\otimes\mathbf{I}_{K+1}.$$

Proof From the expression of $\widehat{\mathbf{U}}$ in Theorem 2.2, the expressions of $u_{[b]}$ in Lemma A.12 and the limit of $\widehat{\mathbf{U}}$ in Lemma A.15,

$$
\begin{aligned}
\widehat{\mathbf{U}} - \mathbf{U} &= \frac{1}{B}\sum_{b=1}^{B}u_{[b]} - \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]\\
&= \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right] + \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]} - \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right].
\end{aligned}
$$

From Lemma A.30, we have that

$$\frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]} = o_p(1),$$

implying that

$$\sqrt{T}\left(\widehat{\mathbf{U}} - \mathbf{U}\right) = \sqrt{T}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right] - \mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]\right) + o_p(1). \qquad \text{(A.72)}$$

Besides, noting that

$$\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]$$
$$=\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right],$$

we use (A.5) to obtain

$$\sqrt{T}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}'\left[1\ \boldsymbol{\mu}_{\beta}'\right]\right)$$
$$=\left(\left(\boldsymbol{\Lambda}'\boldsymbol{\mu}_{\beta}\right)'\otimes\mathbf{I}_{K+1}\right)\sqrt{T}\mathrm{vec}\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}-\mathbf{V}_{f}\right). \tag{A.73}$$

Plugging (A.73) into (A.72) completes the proof of the lemma. □

Lemma A.33. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\sqrt{T}\left(vec\left(\widehat{\mathbf{D}}_{e}\right)-vec\left(\mathbf{D}_{e}\right)\right)=\boldsymbol{\Pi}_{D_e}\sqrt{T}vec\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}-\mathbf{V}_{f}\right)+o_p\left(1\right),$$

where $\mathbf{D}_e$ *is the probability limit of* $\widehat{\mathbf{D}}_e$ *and*

$$\boldsymbol{\Pi}_{D_e}=\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\right)+\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]\right).$$

Proof From the expression of $\widehat{\mathbf{D}}_e$ in Theorem 2.2, the expressions of $d_{e,[b]}$ in Lemma A.13 and the limit of $\widehat{\mathbf{D}}_e$ in Lemma A.16,

$$\widehat{\mathbf{D}}_e-\mathbf{D}_e=\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}d_{e,[b]}-\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'$$
$$=\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}}{T}+\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}-\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'.$$

From Lemma A.30, we have that

$$\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}=o_p\left(1\right),$$

implying that

$$\sqrt{T}\left(\widehat{\mathbf{D}}_e-\mathbf{D}_e\right)=\sqrt{T}\left[\mathbf{0}_K\ \mathbf{I}_K\right]\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_{\beta}\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\right)\left[\mathbf{0}_K\ \mathbf{I}_K\right]'+o_p\left(1\right). \tag{A.74}$$

Besides, noting that

$$
\begin{aligned}
&\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\\
=&\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)+\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}+\\
&+\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right),
\end{aligned}
$$

we use (A.5) to obtain

$$
\begin{aligned}
&\sqrt{T}\text{vec}\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\right)\left[\mathbf{0}_K\ \mathbf{I}_K\right]'\right)\\
=&\boldsymbol{\Pi}_{D_e}\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}-\mathbf{V}_f\right)\\
&+\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]T^{0.25}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]T^{0.25}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\right)\text{vec}\left(\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\\
=&\boldsymbol{\Pi}_{D_e}\sqrt{T}\text{vec}\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}-\mathbf{V}_f\right)+o_p\left(1\right),
\end{aligned}
\tag{A.75}
$$

where the last equality holds from Assumption 6(iii). Plugging (A.75) into (A.74) completes the proof of the lemma. □

Lemma A.34. *Under Assumptions 1, 4, 5, 6, it holds that*

$$
\sqrt{T}\left(\widehat{\mathbf{U}}_e-\mathbf{U}_e\right)=\boldsymbol{\Pi}_{U_e}\sqrt{T}vec\left(\frac{1}{T}\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}-\mathbf{V}_f\right)+o_p\left(1\right),
$$

where $\mathbf{U}_e$ *is the probability limit of* $\widehat{\mathbf{U}}_e$ *and*

$$
\boldsymbol{\Pi}_{U_e}=\left(\left[1\ \mathbf{0}_K'\right]\otimes\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\right)+\left(\left(\left[1\ \mathbf{0}_K'\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]\right).
$$

Proof From the expression of $\widehat{\mathbf{U}}_e$ in Theorem 2.2, the expressions of $u_{e,[b]}$ in Lemma A.14 and the limit of $\widehat{\mathbf{U}}_e$ in Lemma A.16,

$$
\begin{aligned}
\widehat{\mathbf{U}}_e-\mathbf{U}_e&=\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\sum_{b=1}^{B}u_{e,[b]}-\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[\begin{array}{cc}1 & \boldsymbol{\mu}_f'\end{array}\right]'\\
&=\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{1}_T}{T}+\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}-\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left[\begin{array}{cc}1 & \boldsymbol{\mu}_f'\end{array}\right]'.
\end{aligned}
$$

From Lemmas A.1 and A.30, we have that

$$\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\frac{\sqrt{T}}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}=o_p\left(1\right).$$

Hence, using the above equality with the following identities $\mathbf{F}=\mathbf{F}_\triangle\left[\mathbf{0}_K\ \mathbf{I}_K\right]'$, $\mathbf{1}_T=\mathbf{F}_\triangle\left[1\ \mathbf{0}_K'\right]'$, $\left[\begin{array}{cc}\boldsymbol{\mu}_f & \mathbf{V}_f\end{array}\right]'=\mathbf{V}_{\triangle,f}\left[\mathbf{0}_K\ \mathbf{I}_K\right]'$, $\left[\begin{array}{cc}1 & \boldsymbol{\mu}_f'\end{array}\right]'=\mathbf{V}_{\triangle,f}\left[1\ \mathbf{0}_K'\right]'$, we have that

$$\sqrt{T}\left(\widehat{\mathbf{U}}_e-\mathbf{U}_e\right)=\sqrt{T}\left[\mathbf{0}_K\ \mathbf{I}_K\right]\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\right)\left[1\ \mathbf{0}_K'\right]'+o_p\left(1\right). \tag{A.76}$$

Besides, noting that

$$\begin{aligned}&\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\\ =&\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)+\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}+\\ &+\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right),\end{aligned}$$

we use (A.5) to obtain

$$\begin{aligned}&\sqrt{T}\mathrm{vec}\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\mathbf{V}_{\triangle,f}\right)\left[1\ \mathbf{0}_K'\right]'\right)\\ =&\boldsymbol{\Pi}_{U_e}\sqrt{T}\mathrm{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)\\ &+\left(\left[1\ \mathbf{0}_K'\right]T^{0.25}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]T^{0.25}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\right)\mathrm{vec}\left(\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\\ =&\boldsymbol{\Pi}_{U_e}\sqrt{T}\mathrm{vec}\left(\frac{1}{T}\mathbf{F}_\triangle'\mathbf{F}_\triangle-\mathbf{V}_f\right)+o_p\left(1\right),\end{aligned} \tag{A.77}$$

where the expression of $\boldsymbol{\Pi}_{U_e}$ is given in the statement of the lemma, the last equality holds from Assumption 6(iii). Plugging (A.77) into (A.76) completes the proof of the lemma. □

Lemma A.35. *Let Assumptions 1, 5 be in effect. Consider a continuous function of* $\mathbf{F}_{[b]}$ *and* $\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}$*;* $h\ :\left(\mathbb{R}^{\tau\times K}\times\mathbb{R}^{(K+1)\times(K+1)}\right)\to\mathbb{R}$*. Then, there exists a positive number* $M_h<\infty$ *such that*

$$\lim_{T\to\infty}\frac{1}{B}\sum_{b=1}^{B}h\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)<M_h.$$

Proof Due to the continuity of h, Lemma A.1 yields that

$$h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right) < h\left(\mathbf{F}_{[b]}, \mathbf{V}_{\triangle,f}\right) + 1,$$

implying that

$$\frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right) < \frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \mathbf{V}_{\triangle,f}\right) + 1.$$

From Assumption 5, $\frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \mathbf{V}_{\triangle,f}\right)$ is bounded. Hence, we can always find M_h such that $\lim_{T\to\infty} \frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \widehat{\mathbf{V}}_{f\triangle}\right) < M_h$. This proves the lemma. $\square$

Lemma A.36. *Let Assumptions 4 and 5 be in effect. Consider any set of continuous functions of* $\mathbf{F}_{[b]}$ *and* $\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}$*;* $h_1 : \mathbb{R}^{\tau\times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to \mathbb{R}^{K}$*,* $h_2 : \mathbb{R}^{\tau\times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to \mathbb{R}^{K^2}$*,* $h_3 : \mathbb{R}^{\tau\times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to \mathbb{R}^{\tau}$*,* $h_4 : \mathbb{R}^{\tau\times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to \mathbb{R}^{K\tau}$*, and* $h_5 : \mathbb{R}^{\tau\times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to \mathbb{R}^{\tau^2}$*. For* $e_{k,[b]}$ *defined by (A.28), it holds that as* $N, T \to \infty$*,*

$$\frac{1}{B}\sum_{b=1}^{B} h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' e_{k,[b]} \xrightarrow{p} 0$$

for $k = 1, \cdots, 5$*.*

Proof Note that from Assumption 4(iii), it holds that

$$\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}' e_{k,[b]} \xrightarrow{p} 0 \tag{A.78}$$

for $k = 1, \cdots, 5$.

Fix k. Let f_k be the corresponding function. As N, T increases, it follows that

$$\frac{1}{B}\sum_{b=1}^{B} h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' e_{k,[b]} \leq \frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)}\sqrt{e_{k,[b]}' e_{k,[b]}}\right)$$

$$= \frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{g_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)}\sqrt{e_{k,[b]}' e_{k,[b]}}\right) \tag{A.79}$$

$$\leq \left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} g_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)}\right)\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}' e_{k,[b]}}\right)$$

$$\leq \sqrt{M_{g_k}} \cdot \sqrt{\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}' e_{k,[b]}} \xrightarrow{p} 0, \tag{A.80}$$

where the first equality holds by defining $g_k\left(\mathbf{F}_{[b]}\right) = \sqrt{h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' h_k\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)}$, the first and second inequalities are from the Cauchy-Schwarz inequality, the third inequality is from Lemma A.35, and the last limit is from (A.78). In a similar manner, we can show that as N, T increases,

$$-\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{[b]} \leq \sqrt{M_{g_k}} \cdot \sqrt{\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}' e_{k,[b]}} \xrightarrow{p} 0. \tag{A.81}$$

Lastly, combining (A.80) and (A.81) in conjunction with the squeeze theorem, we have that as $N, T \to \infty$,

$$\frac{1}{B}\sum_{b=1}^{B} f_k\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0,$$

Repeating this exercise for $k = 1, \cdots, 5$ completes the proof of the lemma. $\square$

We introduce the $(K+1)(K+2) \times 1$ vectors of $\mathcal{E}_{[b]}$ and $\mathcal{E}_{e,[b]}$ which are defined by

$$\mathcal{E}_{[b]} = \left[\mathcal{E}_{U,[b]}' \ \text{vec}\left(\mathcal{E}_{D,[b]}\right)'\right] \tag{A.82}$$

$$\mathcal{E}_{e,[b]} = \left[\mathcal{E}_{U_e,[b]}' \ \text{vec}\left(\mathcal{E}_{D_e,[b]}\right)'\right], \tag{A.83}$$

where $\mathcal{E}_{U,[b]}$, $\text{vec}\left(\mathcal{E}_{D,[b]}\right)$, $\mathcal{E}_{U_e,[b]}$, $\text{vec}\left(\mathcal{E}_{D_e,[b]}\right)$ are given by (A.38), (A.35), (A.45) and (A.42), respectively.

Lemma A.37. *Consider a continuous function of* $\mathbf{F}_{[b]}$ *and* $\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}$; $h : \mathbb{R}^{\tau \times K} \times \mathbb{R}^{(K+1)\times(K+1)} \to$

$\mathbb{R}^{(K+1)(K+2)}$. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' \mathcal{E}_{[b]},\ \frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' \mathcal{E}_{e,[b]} \xrightarrow{p} 0.$$

Proof Note that $\mathcal{E}_{[b]}$ and $\mathcal{E}_{e,[b]}$ can be expressed as $\sum_{k=1}^{5}\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)e_{k,[b]}$ for some matrix valued continuous functions of $\mathbf{H}_k$ for $k=1,\cdots,5$ from the definitions $e_{k,[b]}$ given by (A.28) for $k=1,\cdots,5$. Then, it follows that

$$\begin{aligned}
&\frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' \sum_{k=1}^{5}\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)e_{k,[b]}\\
=&\sum_{k=1}^{5}\left(\frac{1}{B}\sum_{b=1}^{B} h\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)' \mathbf{H}_k\left(\mathbf{F}_{[b]}\right)e_{k,[b]}\right) \xrightarrow{p} 0,
\end{aligned}$$

where the last limit is from Lemma A.36. □

Lemma A.38. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}_{[b]}',\ \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{e,[b]}\mathcal{E}_{e,[b]}' \xrightarrow{p} \mathbf{0}_{(K+1)(K+2)\times(K+1)(K+2)}.$$

Proof We prove the lemma in the following steps.

Step 1. For $e_{k,[b]}$ defined by (A.28) for $k=1,\cdots,5$, it holds that $\sum_{b=1}^{B}\frac{\left(e_{k,[b]}'e_{k,[b]}\right)^2}{B} \xrightarrow{p} 0$: Because $e_{k,[b]}$ is small from Assumption 4(ii), $\left(e_{k,[b]}'e_{k,[b]}\right)^2 < e_{k,[b]}'e_{k,[b]}$, implying that $\sum_{b=1}^{B}\frac{\left(e_{k,[b]}'e_{k,[b]}\right)^2}{B} < \sum_{b=1}^{B}\frac{e_{k,[b]}'e_{k,[b]}}{B}$. Also, Assumption 4(iii) states that $\sum_{b=1}^{B}\frac{e_{k,[b]}'e_{k,[b]}}{B} \xrightarrow{p} 0$. Hence, it follows that $\sum_{b=1}^{B}\frac{\left(e_{k,[b]}'e_{k,[b]}\right)^2}{B} \xrightarrow{p} 0$.

Step 2. Pick any $k,k'=1,\cdots,5$. Consider a conformable matrix valued continuous function $\mathbf{G}(\cdot)$ such $e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)e_{k',[b]}$ is well defined. Then, it holds that $\sum_{b=1}^{B}\frac{e_{k,[b]}'\mathbf{G}(\mathbf{F}_{[b]})e_{k',[b]}}{B} \xrightarrow{p} 0$: Note that

$$\begin{aligned}
&\frac{1}{B}\sum_{b=1}^{B}\left(e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)e_{k',[b]}\right)\\
\leq&\frac{1}{B}\sum_{b=1}^{B}\left(\sqrt{e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'e_{k,[b]}}\sqrt{e_{k',[b]}'e_{k',[b]}}\right)\\
\leq&\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B}e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'e_{k,[b]}}\right)\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B}e_{k',[b]}'e_{k',[b]}}\right),
\end{aligned} \tag{A.84}$$

where inequalities are from the Cauchy-Schwartz inequality. Assumption 4(iii) implies that

$$\frac{1}{B}\sum_{b=1}^{B} e_{k',[b]}' e_{k',[b]} \xrightarrow{p} 0. \tag{A.85}$$

Also, we have that

$$\begin{aligned} e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} &= \left(e_{k,[b]} \otimes e_{k,[b]}\right)' \operatorname{vec}\left(\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'\right) \\ &< \sqrt{\left(e_{k,[b]} \otimes e_{k,[b]}\right)'\left(e_{k,[b]} \otimes e_{k,[b]}\right)}\sqrt{\operatorname{tr}\left(\left(\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'\right)^2\right)} \\ &= \sqrt{\left(e_{k,[b]}' e_{k,[b]}\right)^2}\sqrt{\operatorname{tr}\left(\left(\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'\right)^2\right)}, \end{aligned} \tag{A.86}$$

where the first equality is from (A.5), the inequality is from the Cauchy-Schwartz inequality and (A.4). From (A.86) and the Cauchy-Schwartz inequality, we have that

$$\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} < \left(\sqrt{\sum_{b=1}^{B}\frac{\left(e_{k,[b]}' e_{k,[b]}\right)^2}{B}}\right)\left(\sqrt{\frac{1}{B}\sum_{b=1}^{B}\operatorname{tr}\left(\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'\right)^2}\right).$$

Because $\frac{1}{B}\sum_{b=1}^{B}\operatorname{tr}\left(\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)'\right)^2$ is bounded from Lemma 5, and $\sum_{b=1}^{B}\frac{\left(e_{k,[b]}' e_{k,[b]}\right)^2}{B} \xrightarrow{p} 0$ from Step 1, we have that

$$\frac{1}{B}\sum_{b=1}^{B} e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)\mathbf{G}\left(\mathbf{F}_{[b]}\right)' e_{k,[b]} \xrightarrow{p} 0. \tag{A.87}$$

By plugging (A.85) and (A.87) into (A.84), we have that $\sum_{b=1}^{B}\frac{e_{k,[b]}'\mathbf{G}\left(\mathbf{F}_{[b]}\right)e_{k',[b]}}{B} \xrightarrow{p} 0$.

Step 3. $\mathcal{E}_{[b]} = \sum_{k=1}^{5}\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)e_{k,[b]}$ for some matrix valued continuous functions of $\mathbf{H}_k$ for $k = 1, \cdots, 5$: This follows from the definitions $e_{k,[b]}$ given by (A.28) for $k = 1, \cdots, 5$, and the expressions of $\mathcal{E}_{[b]} = \left[\mathcal{E}_{U,[b]}' \operatorname{vec}\left(\mathcal{E}_{D,[b]}\right)'\right]$ given by (A.38), (A.35), respectively.

Step 4. $\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}'\mathcal{E}_{[b]} \xrightarrow{p} 0$: Note that

$$\begin{aligned} \mathcal{E}_{[b]}'\mathcal{E}_{[b]} &= \left(\sum_{k=1}^{5} e_{k,[b]}'\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)'\right)\left(\sum_{k'=1}^{5}\mathbf{H}_{k'}\left(\mathbf{F}_{[b]}\right)e_{k,[b]}\right) \\ &= \sum_{k,k'=1}^{5} e_{k,[b]}'\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)'\mathbf{H}_{k'}\left(\mathbf{F}_{[b]}\right)e_{k',[b]}. \end{aligned}$$

Hence,

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}'_{[b]}\mathcal{E}_{[b]}=\sum_{k,k'=1}^{5}\frac{1}{B}\sum_{b=1}^{B}\left(e'_{k,[b]}\mathbf{H}_k\left(\mathbf{F}_{[b]}\right)'\mathbf{H}_{k'}\left(\mathbf{F}_{[b]}\right)e_{k',[b]}\right)\xrightarrow{p}0$$

from Step 2.

Step 5. $\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}'_{[b]}\xrightarrow{p}\mathbf{0}_{(K+1)(K+2)\times(K+1)(K+2)}$: Note that $\operatorname{tr}\left(\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}'_{[b]}\right)=\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}'_{[b]}\mathcal{E}_{[b]}\xrightarrow{p}0$ from Step 4. Because $\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}'_{[b]}$ is positive semidefinite, $\operatorname{tr}\left(\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}'_{[b]}\right)\xrightarrow{p}0$ implies $\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}\mathcal{E}'_{[b]}\xrightarrow{p}\mathbf{0}_{(K+1)(K+2)\times(K+1)(K+2)}$. The result on the limit of $\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{e,[b]}\mathcal{E}'_{e,[b]}$ similarly follows by repeating the above steps with proper adjustments. This completes the proof of the lemma. □

Lemma A.39. *Define $\boldsymbol{\zeta}_{[b]}$ define as*

$$\boldsymbol{\zeta}_{[b]}=\boldsymbol{\zeta}_{[b],1}+\boldsymbol{\zeta}_{[b],2}+\boldsymbol{\zeta}_{[b],3},$$

where

$$\begin{aligned}\boldsymbol{\zeta}_{[b],1}&=\boldsymbol{\Xi}vec\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\\ \boldsymbol{\zeta}_{[b],2}&=\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)vec\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\\ \boldsymbol{\zeta}_{[b],3}&=\mathbf{G}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathcal{Z}_{[b]}+\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{Z}_{[b]}\end{aligned}$$

and $\boldsymbol{\Xi}$ is a conformable constant matrix, $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{L}$ are some conformable matrix valued continuous functions, $\mathcal{Z}_{[b]}$ is either $\mathcal{E}_{[b]}$ or $\mathcal{E}_{e,[b]}$, given by (A.82) and (A.83), respectively. Under Assumptions 1, 4, 5 and 6, it holds that

$$\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b]}\boldsymbol{\zeta}'_{[b]}\xrightarrow{p}\frac{1}{\tau}\boldsymbol{\Xi}\mathbf{V}_{f^2}\boldsymbol{\Xi}'.$$

Proof The following property of $\mathcal{Z}_{[b]}$ is useful:

$$\frac{1}{B}\sum_{b=1}^{B}\mathcal{Z}_{[b]}=o_p\left(1\right),\tag{A.88}$$

which directly follows from Lemmas A.11, A.12, A.13, A.14. For simplicity, we locally define a $(K+1)\times(K+1)$ vector $\mathbf{x}$ by $\mathbf{x}=\operatorname{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)$. Note that

$$\mathbf{x}=o_p\left(1\right),\tag{A.89}$$

from Lemma A.1.

Let $(L \times 1)$ be the size of $\boldsymbol{\zeta}_{[b]}$. Note that $\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b]}\boldsymbol{\zeta}_{[b]}' = \sum_{i,j=1}^{3}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],i}\boldsymbol{\zeta}_{[b],j}'\right)$. Hence, we examine $\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],i}\boldsymbol{\zeta}_{[b],j}'$ for $i, j = 1, 2, 3$. We find that

$$\begin{aligned}\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],1}\boldsymbol{\zeta}_{[b],1}' &= \boldsymbol{\Xi}\frac{1}{B}\sum_{b=1}^{B}\left(\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)'\right)\boldsymbol{\Xi}' \\ &\xrightarrow{p} \frac{1}{\tau}\boldsymbol{\Xi}\mathbf{V}_{f^2}\boldsymbol{\Xi}',\end{aligned} \tag{A.90}$$

where the last limit is from Assumption 7 and that

$$\begin{aligned}&\text{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],1}\boldsymbol{\zeta}_{[b],2}'\right) \\ =&\text{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\left(\boldsymbol{\Xi}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\mathbf{x}'\mathbf{H}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\right) \\ =&\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\left(\boldsymbol{\Xi}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\right)\right)\mathbf{x} \\ \xrightarrow{p}&\mathbf{0}_{L^2},\end{aligned} \tag{A.91}$$

where the second equality is from (A.5) and the limit follows from the boundedness of $\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)$ (Lemma A.35) and (A.89) and that

$$\begin{aligned}&\text{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],1}\boldsymbol{\zeta}_{[b],3}'\right) \\ =&\text{vec}\left(\boldsymbol{\Xi}\left(\frac{1}{B}\sum_{b=1}^{B}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\mathcal{Z}_{[b]}'\right)\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right) \\ &+\text{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\Xi}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)'\left(\mathbf{L}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\right)'\right) \\ =&\text{vec}\left(\boldsymbol{\Xi}\left(\frac{1}{B}\sum_{b=1}^{B}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\mathcal{Z}_{[b]}'\right)\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right) \\ &+\left(\frac{1}{B}\sum_{b=1}^{B}\left(\mathbf{L}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\boldsymbol{\Xi}\text{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\right)\right)\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right) \\ \xrightarrow{p}&\mathbf{0}_{L^2},\end{aligned} \tag{A.92}$$

where the second equality is from (A.5), the first object in RHS shrinks by Lemma A.37 and the second object in RHS evaporates due to Lemma A.35 (for the boundedness of

$\frac{1}{B}\sum_{b=1}^{B}\left(\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\mathbf{\Xi}\mathrm{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\right)$) and (A.88) and that

$$\begin{aligned}
&\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],2}\boldsymbol{\zeta}_{[b],2}'\right)\\
=&\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{x}\mathbf{x}'\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\\
=&\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\right)\mathrm{vec}\left(\mathbf{x}\mathbf{x}'\right)\\
\xrightarrow{p}&\mathbf{0}_{L\times L},
\end{aligned}\tag{A.93}$$

due to the boundedness of $\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)$ (Lemma A.35) and (A.89) and that

$$\begin{aligned}
&\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],2}\boldsymbol{\zeta}_{[b],3}'\right)\\
=&\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{x}\mathcal{Z}_{[b]}'\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right) &\text{(A.94)}\\
&+\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{x}\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)'\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\\
=&\mathrm{vec}\left(\left(\mathbf{x}'\otimes\mathbf{I}_L\right)\left(\frac{1}{B}\sum_{b=1}^{B}\left(\mathrm{vec}\left(\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\right)\mathcal{Z}_{[b]}'\right)\right)\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\\
&+\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\left(\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{x}\right)\right)\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)\\
\xrightarrow{p}&\mathbf{0}_{L^2}, &\text{(A.95)}
\end{aligned}$$

where the second equality is from (A.5), the first object in RHS shrinks because of Lemma A.37 and the second object in RHS evaporates due to Lemma A.35 (for the boundedness of $\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\left(\mathbf{H}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{x}\right)$) and (A.88) and that

$$\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\left(\frac{1}{B}\sum_{b=1}^{B}\mathcal{Z}_{[b]}\mathcal{Z}_{[b]}'\right)\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\xrightarrow{p}\mathbf{0}_{L\times L}\tag{A.96}$$

from Lemma A.38 and that

$$\begin{aligned}
&\text{vec}\left(\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathcal{Z}_{[b]}\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)'\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\right)\\
&=\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\left(\mathbf{G}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathcal{Z}_{[b]}\right)\right)\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)\xrightarrow{p}\mathbf{0}_{L^2},
\end{aligned}\tag{A.97}$$

where the equality is from (A.5), the limit is from Lemma A.37 and (A.88) that

$$\begin{aligned}
&\text{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)\left(\frac{1}{B}\sum_{b''=1}^{B}\mathcal{Z}_{[b'']}\right)'\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)'\right)\\
&=\left(\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\right)\text{vec}\left(\left(\frac{1}{B}\sum_{b'=1}^{B}\mathcal{Z}_{[b']}\right)\left(\frac{1}{B}\sum_{b''=1}^{B}\mathcal{Z}_{[b'']}\right)'\right)\xrightarrow{p}\mathbf{0}_{L^2}
\end{aligned}\tag{A.98}$$

from Lemma A.35 (for the boundedness of $\frac{1}{B}\sum_{b=1}^{B}\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\otimes\mathbf{L}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)$) and (A.88) and that

$$\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b],3}\boldsymbol{\zeta}_{[b],3}'\xrightarrow{p}\mathbf{0}_{(K+1)(K+2)\times(K+1)(K+2)},\tag{A.99}$$

from (A.96), (A.97) and (A.98).

Lastly combining (A.90)-(A.99), we have that

$$\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\zeta}_{[b]}\boldsymbol{\zeta}_{[b]}'\xrightarrow{p}\frac{1}{\tau}\boldsymbol{\Xi}\mathbf{V}_{f^2}\boldsymbol{\Xi}',$$

which completes the proof of the lemma. □

Lemma A.40. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{[b]}\boldsymbol{\eta}_{[b]}'\xrightarrow{p}\frac{1}{\tau}\boldsymbol{\Pi}\mathbf{V}_{f^2}\boldsymbol{\Pi}',$$

where

$$\boldsymbol{\eta}_{[b]}=\begin{bmatrix}u_{[b]}-\widehat{\mathbf{U}}\\ vec\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}d_{[b]}-\widehat{\mathbf{D}}\right)+vec\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{[b]}\right)-\widehat{\mathbf{D}}\right)\end{bmatrix},$$

and $\mathbf{\Pi} = [\mathbf{\Pi}'_U\ \mathbf{\Pi}'_D]'$, $\mathbf{\Pi}_D$ *and* $\mathbf{\Pi}_U$ *are given in Lemmas A.31 and A.32.*

Proof First, we show that

$$\boldsymbol{\eta}_{[b]} = \boldsymbol{\eta}_{[b],1} + \boldsymbol{\eta}_{[b],2} + \boldsymbol{\eta}_{[b],3}, \tag{A.100}$$

where

$$\begin{aligned}
\boldsymbol{\eta}_{[b],1} &= \mathbf{\Pi}\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\\
\boldsymbol{\eta}_{[b],2} &= \mathbf{H}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\\
\boldsymbol{\eta}_{[b],3} &= \mathbf{G}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathcal{E}_{[b]} + \mathbf{L}\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}
\end{aligned}$$

and $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{L}$ are some conformable matrix valued continuous functions.

We examine $u_{[b]} - \widehat{\mathbf{U}}$. From Lemmas A.12 and A.15, we have that

$$\begin{aligned}
u_{[b]} - \widehat{\mathbf{U}} &= \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\right)\mathbf{\Lambda}'\left[1\ \boldsymbol{\mu}'_{\beta}\right] + \mathcal{E}_{U,[b]} - \left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}'\left[1\ \boldsymbol{\mu}'_{\beta}\right] - \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]}\\
&= \left((\mathbf{\Lambda}'\boldsymbol{\mu}_{\beta})' \otimes \mathbf{I}_{K+1}\right)\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right) +\\
&\quad - \left((\mathbf{\Lambda}'\boldsymbol{\mu}_{\beta})' \otimes \mathbf{I}_{K+1}\right)\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right) + \mathcal{E}_{U,[b]} - \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]}\\
&= \mathbf{\Pi}_U\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right) - \mathbf{\Pi}_U\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right) + \mathcal{E}_{U,[b]} - \frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U,[b]}.
\end{aligned} \tag{A.101}$$

We turn to $\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}d_{[b]}-\widehat{\mathbf{D}}$. From Lemmas A.11 and A.15, we have that

$$\begin{aligned}
&\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}d_{[b]}-\widehat{\mathbf{D}}\\
=&\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}+\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\left(\mathcal{E}_{D,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right)\\
=&\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)+\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\\
&-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)+\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\left(\mathcal{E}_{D,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right),
\end{aligned}$$

which in turn yields

$$\begin{aligned}
\operatorname{vec}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}d_{[b]}-\widehat{\mathbf{D}}\right)=&\left(\mathbf{I}_{K+1}\otimes\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\right)\operatorname{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)+\\
&+\mathbf{H}_0\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\operatorname{vec}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\\
&+\left(\mathbf{I}_{K+1}\otimes\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\operatorname{vec}\left(\mathcal{E}_{D,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right),
\end{aligned}\tag{A.102}$$

where $\mathbf{H}_0$ is a conformable matrix valued continuous function.

We inspect $\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{[b]}\right)-\widehat{\mathbf{D}}$. From Lemmas A.11 and A.15, after some algebra, we find that

$$\begin{aligned}
&\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{[b]}\right)-\widehat{\mathbf{D}}=\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\frac{1}{B}\left(\sum_{b=1}^{B}d_{[b]}\right)\\
=&\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\left(\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}+\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right)\\
=&\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\mathbf{V}_{\triangle,f}+\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\\
&-\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'\mathbf{V}_{\triangle,\beta}\mathbf{\Lambda}\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}+\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]},
\end{aligned}$$

which implies

$$
\begin{aligned}
&\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B} d_{[b]}\right)-\widehat{\mathbf{D}}\right)\\
&=\left(\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'\mathbf{V}_{\beta}\mathbf{\Lambda}\otimes\mathbf{I}_{K+1}\right)\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\\
&\quad+\mathbf{H}_1\left(\mathbf{F}_{[b]},\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathrm{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\\
&\quad+\left(\mathbf{I}_{K+1}\otimes\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\right)\mathrm{vec}\left(\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right),
\end{aligned}
\tag{A.103}
$$

where $\mathbf{H}_1$ is a conformable matrix valued continuous function. Combining (A.101), (A.102) and (A.103) yields the expression of (A.100).

Next, note that given the expression of $\boldsymbol{\eta}_{[b]}$ in (A.100), Lemma A.39 implies that

$$
\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{[b]}\boldsymbol{\eta}'_{[b]}\xrightarrow{p}\frac{1}{\tau}\mathbf{\Pi}\mathbf{V}_{f^2}\mathbf{\Pi}'.
$$

This completes the proof of the lemma. □

Lemma A.41. *Under Assumptions 1, 4, 5 and 6, it holds that*

$$
\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{e,[b]}\boldsymbol{\eta}'_{e,[b]}\xrightarrow{p}\frac{1}{\tau}\mathbf{\Pi}_e\mathbf{V}_{f^2}\mathbf{\Pi}'_e,
$$

where

$$
\boldsymbol{\eta}_{e,[b]}=\begin{bmatrix}\left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]}-\widehat{\mathbf{U}}_e\right)+\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)-\widehat{\mathbf{U}}_e\right)\\ vec\left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}d_{e,[b]}-\widehat{\mathbf{D}}_e\right)+vec\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)-\widehat{\mathbf{D}}_e\right)\end{bmatrix},
$$

and $\mathbf{\Pi}_e=\left[\mathbf{\Pi}'_{U_e}\ \mathbf{\Pi}'_{D_e}\right]'$, $\mathbf{\Pi}_{D_e}$ *and* $\mathbf{\Pi}_{U_e}$ *are given in Lemmas A.33 and A.34.*

Proof First, we show that

$$
\boldsymbol{\eta}_{e,[b]}=\boldsymbol{\eta}_{e,[b],1}+\boldsymbol{\eta}_{e,[b],2}+\boldsymbol{\eta}_{e,[b],3},
\tag{A.104}
$$

where

$$\boldsymbol{\eta}_{e,[b],1} = \mathbf{\Pi}_e \text{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)$$
$$\boldsymbol{\eta}_{e,[b],2} = \mathbf{H}_e\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right) \text{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)$$
$$\boldsymbol{\eta}_{e,[b],3} = \mathbf{G}_e\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathcal{E}_{e,[b]} + \mathbf{L}_e\left(\mathbf{F}_{[b]}, \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{[b]}$$

and $\mathbf{H}_e$, $\mathbf{G}_e$ and $\mathbf{L}_e$ are some conformable matrix valued continuous functions.

The following identities are useful:

$$\begin{aligned}
&\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\\
=&\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right) - \mathbf{V}_{\triangle,f}\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\\
&+\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)
\end{aligned}\tag{A.105}$$

and

$$\begin{aligned}
&\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\\
=&\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\mathbf{V}_{\triangle,f} - \left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right)\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\mathbf{V}_{\triangle,f}\\
&+\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} - \frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} - \mathbf{V}_{\triangle,f}\right).
\end{aligned}\tag{A.106}$$

We examine $\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]} - \widehat{\mathbf{U}}_e$. From Lemmas A.14 and A.16, we have that

$$\begin{aligned}
\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]} - \widehat{\mathbf{U}}_e =& \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{1}_\tau}{\tau} + \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\mathcal{E}_{U_e,[b]}\\
&- \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}'_e\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}'\mathbf{1}_\tau}{T} - \frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]},
\end{aligned}$$

which in turn yields

$$\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]}-\widehat{\mathbf{U}}_e=[\mathbf{0}_K\ \mathbf{I}_K]\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\mathbf{\Lambda}_e'\mathbf{V}_\beta\mathbf{\Lambda}_e\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\right)[1\ \mathbf{0}_K']'$$
$$+\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\left(\mathcal{E}_{U_e,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}\right).$$

Plugging (A.105) to the above and using (A.5), we have that

$$\begin{aligned}\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]}-\widehat{\mathbf{U}}_e&=\operatorname{vec}\left(\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}u_{e,[b]}-\widehat{\mathbf{U}}_e\right)\\&=\left([1\ \mathbf{0}_K']\otimes[\mathbf{0}_K\ \mathbf{I}_K]\,\mathbf{V}_{\triangle,f}\mathbf{\Lambda}_e'\mathbf{V}_\beta\mathbf{\Lambda}_e\right)\operatorname{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\\&+\mathbf{H}_{U_e,0}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\operatorname{vec}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)\\&+\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\left(\mathcal{E}_{U_e,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}\right),\end{aligned}\tag{A.107}$$

where $\mathbf{H}_{U_e,0}$ is a conformable matrix valued continuous function.

We turn to $\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)-\widehat{\mathbf{U}}_e$. From Lemmas A.14 and A.16,

$$\begin{aligned}&\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)-\widehat{\mathbf{U}}_e\\&=\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)\\&=\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}_e'\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{1}_\tau}{\tau}+\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}\\&=[\mathbf{0}_K\ \mathbf{I}_K]\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}_e'\mathbf{V}_\beta\mathbf{\Lambda}_e\frac{\mathbf{F}_{\triangle}'\mathbf{F}_{\triangle}}{T}[1\ \mathbf{0}_K']'+\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}.\end{aligned}$$

Plugging (A.106) to the above and using (A.5), we get

$$\begin{aligned}
&\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}u_{e,[b]}\right)-\widehat{\mathbf{U}}_e\\
=&\left(\left(\left[1\ \mathbf{0}_K'\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]\right)\operatorname{vec}\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)\\
&+\mathbf{H}_{U_e,1}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\operatorname{vec}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\\
&+\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\right)\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{U_e,[b]}, \qquad \text{(A.108)}
\end{aligned}$$

where $\mathbf{H}_{U_e,1}$ is a conformable matrix valued continuous function.

Next, we examine $\operatorname{vec}\left(\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}d_{e,[b]}\right)-\operatorname{vec}\left(\widehat{\mathbf{D}}_e\right)$. From Lemmas A.13 and A.16, we have that

$$\begin{aligned}
&\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}d_{e,[b]}-\widehat{\mathbf{D}}_e\\
=&\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\frac{\mathbf{F}_\triangle'\mathbf{F}}{T}+\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\left(\mathcal{E}_{D,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right),
\end{aligned}$$

which in turn yields

$$\begin{aligned}
\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}d_{e,[b]}-\widehat{\mathbf{D}}_e=&\left[\mathbf{0}_K\ \mathbf{I}_K\right]\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\left(\frac{\mathbf{F}_{\triangle,[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\right)\left[\mathbf{0}_K\ \mathbf{I}_K\right]'\\
&+\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\left(\mathcal{E}_{D,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D,[b]}\right).
\end{aligned}$$

Plugging (A.105) to the above and using (A.5), we have that

$$\begin{aligned}
\operatorname{vec}\left(\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}d_{e,[b]}-\widehat{\mathbf{D}}_e\right)=&\left(\left[\mathbf{0}_K\ \mathbf{I}_K\right]\otimes\left[\mathbf{0}_K\ \mathbf{I}_K\right]\mathbf{V}_{\triangle,f}\boldsymbol{\Lambda}_e'\mathbf{V}_\beta\boldsymbol{\Lambda}_e\right)\operatorname{vec}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\\
&+\mathbf{H}_{D_e,0}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}\right)\operatorname{vec}\left(\frac{\mathbf{F}_\triangle'\mathbf{F}_\triangle}{T}-\mathbf{V}_{\triangle,f}\right)\\
&+\left(\mathbf{I}_{K+1}\otimes\frac{\mathbf{F}'\mathbf{F}_\triangle}{T}\right)\operatorname{vec}\left(\mathcal{E}_{D_e,[b]}-\frac{1}{B}\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}\right), \qquad \text{(A.109)}
\end{aligned}$$

where $\mathbf{H}_{D_e,0}$ is a conformable matrix valued continuous function.

Lastly, we check $\operatorname{vec}\left(\frac{\mathbf{F}_{[b]}'\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)-\widehat{\mathbf{D}}_e\right)$. From Lemmas A.13 and A.16, we have that

$$\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)-\widehat{\mathbf{D}}_e=\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)$$
$$=\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\left(\mathbf{\Lambda}'_e\mathbf{V}_{\beta}\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle}\mathbf{F}}{T}+\frac{1}{B}\left(\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}\right)\right)$$
$$=[\mathbf{0}_K\ \mathbf{I}_K]\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\mathbf{\Lambda}'_e\mathbf{V}_{\beta}\mathbf{\Lambda}_e\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}[\mathbf{0}_K\ \mathbf{I}_K]'$$
$$+\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\left(\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}\right).$$

Plugging (A.105) to the above and using (A.5), we have that

$$\operatorname{vec}\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}\frac{1}{B}\left(\sum_{b=1}^{B}d_{e,[b]}\right)-\widehat{\mathbf{D}}_e\right)$$
$$=\left([\mathbf{0}_K\ \mathbf{I}_K]\otimes\mathbf{V}_{\triangle,f}\mathbf{\Lambda}'_e\mathbf{V}_{\beta}\mathbf{\Lambda}_e\right)\operatorname{vec}\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\mathbf{V}_{\triangle,f}\right)+\mathbf{H}_{D_e,1}\left(\mathbf{F}_{[b]},\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\operatorname{vec}\left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}-\mathbf{V}_{\triangle,f}\right)$$
$$+\left(\frac{\mathbf{F}'_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau}-\frac{\mathbf{F}'\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\left(\sum_{b=1}^{B}\mathcal{E}_{D_e,[b]}\right),\tag{A.110}$$

where $\mathbf{H}_{D_e,1}$ is a conformable matrix valued continuous function. Combining (A.107), (A.108), (A.109) and (A.110) yields the expression of (A.104).

Next, note that given the expression of$\boldsymbol{\eta}_{e,[b]}$ in (A.104), Lemma A.39 implies that

$$\frac{1}{B}\sum_{b=1}^{B}\boldsymbol{\eta}_{e,[b]}\boldsymbol{\eta}'_{e,[b]}\xrightarrow{p}\frac{1}{\tau}\mathbf{\Pi}_e\mathbf{V}_{f^2}\mathbf{\Pi}'_e.$$

This completes the proof of the lemma. □

Supplementary Tables

Table A1: SDF Estimator Performance when Gross Returns Follow CAPM

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					-0.05	-0.06	0.00	0.01	1.07	1.06	1.01	0.99
1000	N.A.				-0.12	-0.02	0.00	0.00	1.14	1.03	1.01	1.00
2000					-0.06	-0.02	0.00	0.01	1.08	1.03	1.01	1.00
4000					-0.07	-0.03	-0.01	0.00	1.09	1.04	1.01	1.00
A-2: With Estimated Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.14	0.08	0.12	0.12	0.87	0.93	0.88	0.88
1000	0.96	0.96	0.97	0.98	-0.02	-0.03	0.02	0.03	1.04	1.04	0.98	0.97
2000	0.99	0.99	0.99	0.99	-0.04	0.01	0.01	0.01	1.06	1.00	1.00	0.99
4000	0.99	1.00	1.00	1.00	-0.03	-0.02	-0.01	0.00	1.05	1.03	1.01	1.00
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					0.58	0.36	0.21	0.11	0.43	0.64	0.79	0.89
1000	N.A.				0.58	0.37	0.21	0.11	0.42	0.64	0.79	0.89
2000					0.57	0.36	0.21	0.11	0.43	0.64	0.79	0.89
4000					0.58	0.35	0.21	0.11	0.43	0.65	0.79	0.89
B-2: With Estimated Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.60	0.43	0.30	0.21	0.40	0.57	0.70	0.79
1000	0.96	0.96	0.97	0.98	0.59	0.39	0.24	0.15	0.41	0.61	0.76	0.85
2000	0.99	0.99	0.99	0.99	0.57	0.37	0.22	0.12	0.43	0.63	0.78	0.88
4000	0.99	1.00	1.00	1.00	0.58	0.36	0.21	0.12	0.43	0.65	0.79	0.88
B-3: With Observed Factors + Bias Correction												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					-0.06	-0.02	0.00	-0.01	1.08	1.03	1.00	1.01
1000	N.A.				-0.05	-0.02	-0.01	0.00	1.07	1.02	1.01	1.01
2000					-0.05	-0.02	0.00	0.00	1.07	1.02	1.01	1.00
4000					-0.05	-0.03	0.00	0.00	1.07	1.04	1.01	1.00
B-4: With Estimated Factors + Bias Correction												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.13	0.13	0.13	0.12	0.89	0.88	0.87	0.88
1000	0.96	0.96	0.97	0.98	0.04	0.05	0.05	0.04	0.98	0.95	0.95	0.96
2000	0.99	0.99	0.99	0.99	-0.02	0.01	0.02	0.01	1.04	1.00	0.99	0.99
4000	0.99	1.00	1.00	1.00	-0.03	-0.02	0.00	0.00	1.05	1.03	1.00	1.00
Panel C: Pukthuanthong and Roll's (2017) Estimator												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.04	0.02	0.01	0.00	0.55	0.32	0.17	0.07	0.46	0.68	0.83	0.93
1000	0.06	0.04	0.02	0.01	0.56	0.34	0.18	0.09	0.44	0.66	0.82	0.92
2000	0.09	0.06	0.04	0.02	0.56	0.35	0.20	0.10	0.44	0.65	0.81	0.90
4000	0.14	0.11	0.07	0.04	0.57	0.34	0.20	0.10	0.43	0.66	0.80	0.90

This table summarizes the performance of various SDF estimators for gross returns when the true return generating process of each individual asset follows CAPM. We consider different levels of $N = 500,\ 1000,\ 2000,$ and 4000 and $T = 60,\ 120,\ 240,$ and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table A2: SDF Estimator Performance when Excess Returns Follow CAPM

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					-0.01	0.00	0.01	0.00	1.00	0.99	0.98	0.99
1000		N.A.			-0.01	0.01	0.00	0.00	1.00	0.98	0.99	0.99
2000					-0.01	0.01	0.01	0.00	1.00	0.99	0.98	0.99
4000					-0.01	-0.01	0.01	0.00	1.00	1.00	0.99	0.99
A-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.09	0.09	0.10	0.08	0.90	0.90	0.90	0.91
1000	0.96	0.96	0.97	0.98	0.04	0.04	0.03	0.02	0.95	0.95	0.96	0.97
2000	0.99	0.99	0.99	0.99	0.00	0.02	0.02	0.01	0.99	0.98	0.97	0.98
4000	0.99	1.00	1.00	1.00	0.00	-0.01	0.01	0.01	0.99	1.00	0.98	0.99
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					-0.01	0.00	0.01	0.00	1.00	0.99	0.98	0.99
1000		N.A.			-0.01	0.01	0.00	0.00	1.00	0.98	0.99	0.99
2000					-0.01	0.00	0.01	0.00	1.00	0.99	0.98	0.99
4000					-0.01	-0.01	0.01	0.00	1.00	1.00	0.99	0.99
B-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.09	0.09	0.10	0.08	0.90	0.90	0.90	0.91
1000	0.96	0.96	0.97	0.98	0.04	0.04	0.03	0.02	0.95	0.95	0.96	0.97
2000	0.99	0.99	0.99	0.99	0.00	0.02	0.02	0.01	0.99	0.98	0.97	0.98
4000	0.99	1.00	1.00	1.00	0.00	-0.01	0.01	0.01	0.99	1.00	0.98	0.99
B-3: With Observed Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500					-0.01	0.00	0.01	0.00	1.00	0.99	0.98	0.99
1000		N.A.			-0.01	0.01	0.00	0.00	1.00	0.98	0.99	0.99
2000					-0.01	0.00	0.01	0.00	1.00	0.99	0.98	0.99
4000					-0.01	-0.01	0.01	0.00	1.00	1.00	0.99	0.99
B-4: With Estimated Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.90	0.91	0.92	0.92	0.09	0.09	0.10	0.08	0.90	0.90	0.90	0.91
1000	0.96	0.96	0.97	0.98	0.04	0.04	0.03	0.02	0.95	0.95	0.96	0.97
2000	0.99	0.99	0.99	0.99	0.00	0.01	0.02	0.01	0.99	0.98	0.97	0.98
4000	0.99	1.00	1.00	1.00	0.00	-0.01	0.01	0.01	0.99	1.00	0.98	0.99

This table summarizes the performance of various SDF estimators for excess returns when the true return generating process of each individual asset follows CAPM. We consider different levels of $N =$ 500, 1000, 2000, and 4000 and $T =$ 60, 120, 240, and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table A3: SDF Estimator Performance when Gross Returns Follow FF3

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.36	0.38	0.40	0.44	-1.12	-1.26	-0.14	-0.03	2.09	2.29	1.15	1.03
1000	0.39	0.40	0.44	0.50	-0.32	-0.03	-0.03	-0.01	1.34	1.05	1.03	1.01
2000	0.41	0.44	0.50	0.59	-0.29	-0.07	-0.03	-0.01	1.31	1.08	1.03	1.01
4000	0.44	0.50	0.58	0.69	-0.04	-0.03	-0.02	-0.01	1.06	1.04	1.02	1.01
A-2: With Estimated Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.28	0.31	0.35	0.41	0.35	0.33	0.32	0.33	0.66	0.68	0.68	0.67
1000	0.33	0.38	0.45	0.55	0.21	0.21	0.19	0.18	0.80	0.79	0.81	0.82
2000	0.39	0.45	0.55	0.66	0.11	0.11	0.10	0.10	0.91	0.90	0.90	0.90
4000	0.44	0.53	0.65	0.77	0.04	0.03	0.03	0.03	0.98	0.98	0.97	0.97
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.47	0.62	0.77	0.88	0.52	0.33	0.19	0.10	0.49	0.67	0.81	0.90
1000	0.48	0.63	0.79	0.89	0.52	0.34	0.19	0.10	0.48	0.67	0.81	0.90
2000	0.50	0.66	0.80	0.90	0.52	0.32	0.19	0.10	0.48	0.68	0.81	0.90
4000	0.51	0.66	0.81	0.90	0.51	0.32	0.19	0.10	0.49	0.68	0.81	0.90
B-2: With Estimated Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.29	0.39	0.51	0.60	0.66	0.54	0.44	0.39	0.34	0.46	0.56	0.61
1000	0.36	0.49	0.63	0.73	0.60	0.46	0.34	0.27	0.40	0.54	0.66	0.73
2000	0.42	0.57	0.71	0.81	0.57	0.39	0.27	0.19	0.44	0.61	0.73	0.80
4000	0.47	0.63	0.77	0.87	0.53	0.35	0.22	0.13	0.47	0.65	0.78	0.87
B-3: With Observed Factors + Bias Correction												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.51	0.65	0.79	0.88	-0.10	-0.03	-0.02	-0.01	1.12	1.04	1.03	1.01
1000	0.53	0.66	0.80	0.89	-0.06	-0.02	-0.01	0.00	1.07	1.02	1.02	1.01
2000	0.55	0.69	0.81	0.90	-0.05	-0.03	-0.01	0.00	1.07	1.03	1.01	1.01
4000	0.56	0.69	0.82	0.91	-0.05	-0.02	-0.01	-0.01	1.07	1.03	1.02	1.01
B-4: With Estimated Factors + Bias Correction												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.33	0.42	0.53	0.61	0.37	0.36	0.33	0.33	0.64	0.65	0.67	0.67
1000	0.40	0.52	0.65	0.73	0.24	0.22	0.20	0.20	0.77	0.78	0.80	0.80
2000	0.47	0.60	0.73	0.81	0.13	0.11	0.11	0.11	0.89	0.89	0.89	0.90
4000	0.52	0.66	0.79	0.87	0.04	0.04	0.03	0.03	0.98	0.97	0.97	0.97
Panel C: Pukthuanthong and Roll's (2017) Estimator												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.06	0.04	0.02	0.00	0.51	0.32	0.17	0.08	0.49	0.68	0.83	0.92
1000	0.09	0.08	0.05	0.03	0.52	0.33	0.18	0.09	0.49	0.67	0.82	0.91
2000	0.14	0.13	0.10	0.06	0.52	0.32	0.19	0.10	0.49	0.68	0.82	0.90
4000	0.20	0.20	0.17	0.12	0.51	0.32	0.19	0.10	0.49	0.68	0.82	0.90

This table summarizes the performance of various SDF estimators for gross returns when the true return generating process of each individual asset follows FF3. We consider different levels of $N =$ 500, 1000, 2000, and 4000 and $T = 60$, 120, 240, and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table A4: SDF Estimator Performance when Excess Returns Follow FF3

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.47	0.59	0.73	0.84	-0.01	0.00	-0.01	0.00	0.94	0.94	0.96	0.96
1000	0.50	0.63	0.77	0.87	0.01	0.01	0.00	0.00	0.92	0.94	0.95	0.96
2000	0.53	0.67	0.79	0.89	0.00	0.00	0.00	0.00	0.93	0.95	0.95	0.95
4000	0.55	0.68	0.81	0.90	-0.01	0.00	0.00	0.00	0.93	0.95	0.96	0.96
A-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.33	0.43	0.53	0.61	0.32	0.30	0.28	0.28	0.62	0.65	0.69	0.69
1000	0.41	0.52	0.65	0.73	0.21	0.19	0.17	0.16	0.72	0.76	0.79	0.80
2000	0.47	0.60	0.72	0.81	0.12	0.10	0.09	0.09	0.81	0.85	0.86	0.87
4000	0.52	0.66	0.78	0.87	0.05	0.04	0.03	0.03	0.88	0.91	0.92	0.93
Panel B: (Infeasible) Balanced Panel Estimator												
B-1: With Observed Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.55	0.68	0.80	0.89	0.00	0.00	0.00	0.00	0.93	0.94	0.96	0.96
1000	0.56	0.68	0.81	0.90	0.00	0.01	0.00	0.00	0.92	0.94	0.95	0.96
2000	0.56	0.70	0.82	0.90	0.00	0.00	0.00	0.00	0.93	0.95	0.95	0.96
4000	0.57	0.70	0.82	0.91	-0.01	0.00	0.00	0.00	0.93	0.95	0.96	0.96
B-2: With Estimated Factors												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.34	0.44	0.55	0.63	0.32	0.30	0.28	0.28	0.62	0.65	0.68	0.69
1000	0.42	0.53	0.66	0.74	0.21	0.19	0.17	0.16	0.72	0.76	0.79	0.80
2000	0.48	0.61	0.73	0.82	0.12	0.10	0.09	0.09	0.81	0.85	0.86	0.87
4000	0.53	0.66	0.79	0.88	0.05	0.04	0.03	0.03	0.88	0.91	0.92	0.93
B-3: With Observed Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.51	0.65	0.79	0.89	0.00	0.00	0.00	0.00	0.93	0.94	0.96	0.96
1000	0.53	0.67	0.80	0.89	0.01	0.01	0.00	0.00	0.92	0.94	0.95	0.96
2000	0.55	0.69	0.82	0.90	0.00	0.00	0.00	0.00	0.93	0.95	0.95	0.96
4000	0.56	0.70	0.82	0.91	-0.01	0.00	0.00	0.00	0.94	0.95	0.96	0.96
B-4: With Estimated Factors + Bias Correction												
$N\backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.34	0.44	0.55	0.63	0.32	0.30	0.28	0.28	0.62	0.65	0.68	0.69
1000	0.42	0.53	0.66	0.74	0.21	0.19	0.17	0.16	0.72	0.76	0.79	0.80
2000	0.48	0.61	0.73	0.82	0.12	0.10	0.09	0.09	0.81	0.85	0.86	0.87
4000	0.53	0.66	0.79	0.88	0.05	0.04	0.03	0.03	0.88	0.91	0.92	0.93

This table summarizes the performance of various SDF estimators for excess returns when the true return generating process of each individual asset follows FF3. We consider different levels of $N =$ 500, 1000, 2000, and 4000 and $T =$ 60, 120, 240, and 480. We set $\tau = 30$. After obtaining a time series of estimates $\widehat{m}_t$ for $t = 1, \cdots, T$, we regress the estimated SDF $\widehat{m}$ on a constant and the true SDF m: $\widehat{m}_t = a + b \cdot m_t + error_t$. If the fit to the true SDF is perfect, R^2 is 1, the intercept (a) is zero, and the coefficient on the true SDF (b) is 1. We report the mean of the estimated R^2, a, and b across 10,000 repetitions.

Table A5: SDF Estimation Results with $\tau = 60$

Panel A: Specific Asset Pricing Models										
	MKT	SMB	HML	I/A	ROE	CMW	RMW	MOM	LIQ	HML(devil)
CAPM	-4.47									
	(-4.87)									
	-2.51									
FF3	-3.69	-2.51	-1.14							
	(-3.72)	(-1.50)	(-0.45)							
	-3.08	-1.90	-5.97							
HXZ4	-4.95	-6.34		-7.50	-14.83					
	(-3.52)	(-6.15)		(-1.24)	(-5.37)					
	-4.79	-4.84		-14.90	-10.32					
FF5	-5.01	-4.28	3.94			-10.66	-8.51			
	(-3.61)	(-2.90)	(1.30)			(-2.71)	(-2.56)			
	-4.83	-3.55	0.40			-8.74	-12.50			
PS5	-5.90	-2.22	-6.36					-9.01	-0.13	
	(-5.11)	(-2.06)	(-3.23)					(-4.50)	(-0.16)	
BS6	-5.08	-6.89		-1.54	-14.68			-5.96		-6.77
	(-3.34)	(-5.08)		(-0.42)	(-3.21)			(-2.40)		(-2.85)
	-4.90	-5.99		-5.55	-13.25			-5.22		-10.60
All	-5.40	-5.52		1.68	-1.92	-15.06	2.90	0.86	-7.96	-0.15
	(-4.04)	(-4.97)		(0.37)	(-0.17)	(-2.09)	(0.59)	(0.08)	(-2.67)	(-0.17)
Panel B: Statistical Factor Model										
	PC1	PC2	PC3							
Single-Factor	-4.60									
	(-4.80)									
Three-Factor	-3.94	-1.74	-0.22							
	(-3.73)	(-0.63)	(-0.07)							

This table reports the estimated values of $\boldsymbol{\delta}_e$ using our estimator $\widehat{\boldsymbol{\delta}}_e$ in Theorem 2.2. In panel A, we consider six asset pricing models: CAPM (Sharpe, 1964), FF3 (Fama and French, 1992), HXZ4 (Hou et al. 2015), FF5 (Fama and French, 2015), PS5 (Pástor and Stambaugh, 2003), BS6 (Barillas and Shanken, 2017).For models only with traded factors, we report the alternative SDF coefficients, $-\left(\widehat{\boldsymbol{\Sigma}}_f + \widehat{\boldsymbol{\mu}}_f \widehat{\boldsymbol{\mu}}_f'\right)^{-1} \widehat{\boldsymbol{\mu}}_f$ below t-statistics. In Panel B, we examine statistical factors computed by methods in Connor and Korajczyk (1986, 1991).The t-statistics are computed by our asymptotic variance estimator in Theorem 2.4. The sample periods are 600 months over the sample period January 1976 to December 2016. We set $\tau = 60$.