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Dynamic Escalation in Multiplayer Rivalries

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Many war-of-attrition-like races, such as online crowdsourcing challenges, online penny auctions, lobbying, R&D races, and political contests among and within parties involve more than two contenders. An important strategic decision in these settings is the timing of escalation. Anticipating the possible event in which another rival is to compete with the current frontrunner, a trailing contender may delay its own escalation effort, and thus avoid the instantaneous sunk cost, without necessarily conceding defeat. Such a free-rider effect, in the n-player dynamic war of attrition considered here, is shown to outweigh the opposite, competition effect intensified by having more rivals. Generalizing the dollar auction framework we construct subgame perfect equilibria where more than two players participate in escalation and, at critical junctures of the process, free-ride one another's escalation efforts. These equilibria generate larger total surplus for all rivals than the equilibrium where only two players participate in escalation.

Key words: dynamic escalation, multiplayer rivalry, subgame perfect equilibrium, markov games, auction theory

1. Introduction

In recent years crowdsourcing R&D through open online challenges has become a popular business innovation strategy. The first and most celebrated of such initiatives was the 2006 Netflix Prize, where the online movie streaming company offered one million dollars to the best performing movie recommendation algorithm.[1] Like an ascending bidding process, the online challenge openly updated the submissions and their performances ("bids") so that contenders could dynamically up their efforts to outperform each others and most importantly the frontrunner. A key element in the Netflix Prize—and most online challenges—was that Netflix retained exclusive rights to *all* submissions thereby becoming the beneficiary of all contenders' sunk cost efforts. Other examples of escalation and rivalry dynamics among multiple players includes conflicts/contests between nation-states (O'Neill 1986), political rivals (Dekel et al. 2009, Gul and Pesendorfer 2012), oligopolists (Fudenberg and Tirole 1986, Bulow and Klemperer 1999), species (Smith 1974), bargaining parties (Damiano et al. 2012, 2017), jurors (Meyer-ter-Vehn et al. 2017), college athletics departments

(Murphy 1996), and online penny auctions (Platt et al. 2013, Kakhbod 2013, Ødegaard and Anderson 2014, Augenblick 2016, Hinnosaar 2016).[2] The motivation of this paper is to analyze escalation dynamics with sunk costs in settings with multiple rivals.

Traditionally, the extant literature considers dynamic escalation in the realm of war-of-attrition stopping games (c.f. Hendricks et al. 1988) or all-pay clock auctions (c.f. Krishna and Morgan 1997), where players continuously make sunk-cost investments, as long as they stay in the game, and dropout decisions are irrevocable. While this framework suitably mimics escalation rivalry among two players—which much of the literature assumes—it is not innocuous when considering three or more rivals. There, the timing for a rival to make the next escalation investment becomes a strategic consideration. Whereas in the two-rival case not making the investment means immediate concession, with multiple rivals a contender may stay put without conceding to his rivals and instead free-ride the other rivals' escalation efforts for a while until the time comes for him to try leapfrogging to the lead. Thus, at certain junctures of the game the incentive for escalation might be lessened. That of course needs to be squared with the opposite effect of intensified competition caused by having more rivals. The question is: in escalation dynamics where rivals at equilibrium can strategize on the timing of the next escalation investment, can having more rivals make all rivals better-off than the equilibrium with only two rivals?

To answer this question, we consider an n-player dynamic game with complete information in which each player's strategic decision is whether to make his escalation effort immediately or to wait for a later period. The complete-information assumption is to sharpen the contrast between bilateral and multilateral rivalries: In a bilateral rivalry, with the value of the contested prize identical and commonly known, the total surplus for the two rivals is nearly zero. With more than two players, by contrast, this paper presents equilibria that not only generate larger total surplus but also Pareto dominate bilateral rivalry for all players.

In this dynamic game of n-player, each round starts with a ranked order of the players. Those positioned behind the frontrunner choose whether to escalate or stay put. If no one escalates then the frontrunner wins the contested prize, without making any further payment, and the game ends. Otherwise, one of those who escalate is randomly selected to be the next frontrunner through bearing a sunk cost proportional to his distance from the current frontrunner; and then the game continues to the next round, with the ranked positioning updated so that the new frontrunner is just one step ahead the previous frontrunner and all other players are one step further behind the lead (the new frontrunner). Such one-step restriction, as explained later, is to rule out artificial shortcuts whereby rivals avoid detrimental escalation.

To focus on the stochastic, recursive dynamics of escalation, our solution concept is subgame perfect equilibrium (SPE) subject to three conditions: Markov, symmetry and independence of non-participants. Given Markov and symmetry, a player's equilibrium action in each subgame depends not on his name or identity but rather on the positioning of the players, and his own location in the positioning, at the start of the subgame. Given independence of nonparticipants, a player's equilibrium strategy does not vary with the increasing distance between the frontrunner and those who no longer escalate to compete for the prize.

Our analysis compares bilateral equilibria, where only two players escalate along the path, with multiple-rivalry equilibria, where more than two players do so. Capturing the detrimental and surplus-dissipating feature of escalation is the bilateral equilibrium, where only the player immediately following the frontrunner escalates. Given any positioning on the players, the equilibrium exists and is unique; the players' total surplus it generates is merely 2δ, the sunk cost that the follower needs to pay in order to top the frontrunner (Section 2.1). By contrast, when a cohort of ranked players are positioned close to one another, there is an m-rivalry equilibrium, where the top m players endogenously take turn to escalate. A striking feature of this equilibrium is that the players' total surplus it generates is $m\delta$, which goes up to just a δ shy of the entire value of the prize when m is sufficiently large (Proposition 1). Such a large surplus is achieved through a free-rider effect, suggested earlier, which gets accumulated by the large number of rivals in closely ranked positions.

The above contrast, stark as it is, requires that the game start with a consecutive positioning of at least the top three players, one immediately following the other. Without this condition, say the third-ranked player is more than one step behind the second-ranked player (called *follower*, who is always one step behind the frontrunner), then the m-rivalry equilibrium coincides with the detrimental, bilateral equilibrium. Thus, we investigate the possibility of improving upon the bilateral equilibrium by a multilateral equilibrium that is *robust* in the sense that the third-ranked player is willing to escalate despite his at least two steps behind the follower. We prove existence of robust, trilateral equilibria, where the top three players escalate against one another despite the third-ranked player's often lagging further behind (Theorem 2). Furthermore, not only does any such trilateral equilibrium generate larger total surplus, but it also makes the top three players each better-off, than the bilateral one (Theorems 3). Interestingly, each robust trilateral equilibrium corresponds to an even number, measuring the minimal lag of the third-ranked player—called *underdog*—at which he no longer escalates in the equilibrium (Theorem 1).

Such normative advantage of multilateral over bilateral rivalries has policy implications. For instance, in de facto two-party systems such as the United States, and contrary to the massive criticism against the Independence Party due to the Year-2000 United States Presidential Election, having a vibrant third political party could mitigate the escalating partisan conflict between the two sides.[3] On the flip side, applied to industrial organization situations where oligopolists struggle to survive in a market (c.f. Fudenberg and Tirole 1986), the policy implication is that having more than two competing firms is more conducive to their collusion than having only two firms. Additional real-world examples of sunk cost dynamics among multi-player rivalries include lobbying by cities to host large sporting events, like the Olympics and World Cup, or facility locations for large corporations. For instance, the 2017 open call by online retail behemoth Amazon for cities to "bid" on becoming their second North American headquarter initially involved 238 cities and subsequently reduced to 20 selected finalists (Wingfield 2018, Streitfeld 2018). Finally leapfrogging in the form of market-leadership rotation is also observed, albeit in a longer time frame, in R&D-intensive industries, e.g., the cold war era "Concorde fallacy" or more recently the global cell-phone market.

In addition to the literature listed above, the non-simultaneous incurrence of sunk costs has been considered in escalation dynamics by Shubik (1971), O'Neill (1986), Leininger (1989), Demange (1992), Dekel et al. (2009), Hörner and Sahuguet (2011), Gul and Pesendorfer (2012). Two key differentiating factors is that they all assume only two rivals and introduce exogenous features to deter escalation, while we allow for more rivals and with no exogenous constraints allow escalation to go on indefinitely. A related but different literature is on races, initiated by Harris and Vickers (1987), with Clark and Nilssen (November 10, 2017) a recent work, considers a finite sequence of static contests. Different from dynamic war-of-attrition-type contests, this literature assumes an exogenous finish line so that competition cannot escalate for indefinitely long. They also assume to have only two players. Another more tangential literature, which considers the sunk cost feature but not the dynamics of contest escalation, is the all-pay auction models, e.g. Baye et al. (1993, 1996), Siegel (2009). In contrast to dynamic war-of-attrition these models settle the static "escalation" through one-shot simultaneous bids. Finally, in addition to the theory literature on the all-pay and war-of-attrition-type dynamics, there is an empirical and behavioral literature, including Teger et al. (1980), Haupert (1994), Murnighan (2002), Liu et al. (2014), Waniek et al. (2015), Morone et al. (2017), where escalation is attributed to psychological factors such as bounded rationality and spiteful bidding.

The remaining paper is organized as follows: Section 2 defines the game, the equilibrium concept, and constructs the surplus-dissipating bilateral and a surplus-enhancing m player rivalry equilibria. The stark contrast between the two equilibria, and the latter's reliance on a tightly packed positioning among the top three players, lead to the questions whether there exist multilateral equilibria robust to other kinds of positioning and whether such robust equilibria can Pareto dominate the bilateral equilibrium. Section 3 presents existence of such equilibria, which exhibit trilateral rivalry on path. Section 4 discusses two model extensions and finally Section 5 concludes. Supporting formal results are available in the online companion Appendix.

2. Multiplayer Rivalries

2.1. Dynamic Escalation With Ranked Players

Let there be $n \geq 3$ players, indexed by $i \in I := \{1, 2, \ldots, n\}$. The *state of the game* is an n-vector $x := (x_i)_{i \in I} \in \{0, 1, 2, 3, \ldots\}^n$ with the following properties:

a. there exists exactly one $i \in I$ for which $x_i = 0$; this i is called *frontrunner* at x;

b. there exists exactly one $j \in I$ for which $x_j = 1$; this j is called *follower* at x;

c. $k \neq k'$ implies $x_k \neq x_{k'}$.

At any state x of the game, x_i is interpreted as the *gap* between player i and the frontrunner. The rule of the game is: given any state x, every player other than the frontrunner chooses, simultaneously, whether to escalate or not; if no one escalates, then the game ends with the frontrunner getting a payoff equal to v; else one of those who chooses to escalate is randomly selected with equal probability; if player j is the selected one, then j pays a sunk cost equal to $(x_j + 1)\delta$ and has his gap changed to zero (i.e., $x_j := 0$) and the gap of everyone else is bumped up by one ($x_i := x_i + 1$ for all $i \in I \setminus \{j\}$), with those who choose to escalate but are not selected incurring no cost; then the game continues given the updated state. Note that the updated state satisfies Properties (a), (b) and (c). In order for an equilibrium to exists we assume the per increment cost δ satisfies $0 < \delta < v/2$.

In the proposed setup, the game assumes players start with different ranked positions. Alternatively, one can have the players start with equal footing and let gaps and ranked order emerge as some players choose to escalate, so that the above game corresponds to subgames. The analysis in such an alternative setup is identical to that in this paper, except for an additional multiplicity of equilibrium escalation probabilities at the initial rounds. We opt for the current model to abstract away from the start game coordination problem and focus on the escalation dynamics. However, illustrations are provided in the two model extensions discussed in Section 4; a formal discourse is provided in Ødegaard and Zheng (2018).

In our model, should a rival choose to escalate, he can only surpass the current frontrunner by an exogenous, small increment δ. This assumption, as explain in the Introduction, rules out the trivial outcome where escalation is preempted by a jump near to the value of the contested prize or slowed down to a stop by diminishing increments of escalation.

2.2. Markov Perfection, Symmetry, and Independence

By *equilibrium* we mean subgame perfect equilibrium subject to three conditions: Markov, symmetry and independence of nonparticipants. We add these conditions to exploit the stochastic recursive structure of the game.

The *Markov* condition means that the equilibrium strategy depends only on the state of the game, and reflects that any previously incurred escalation cost is sunk and does not affect the optimal strategy going forward. Furthermore, this highlights the time inconsistency issue players face. Unlike static or one-shot analysis of war-of-attrition games, where players determine the total amount they are willing to spend, in dynamic settings the cost to stay active is sunk and as such a trailing rival may have an incentive to spend incremental resources in hope of winning the payoff v. Consequently, an equilibrium is in the form of $(\sigma_i)_{i\in I}$ such that each player i's behavior strategy σ_i associates to any state x of the game where i is not the frontrunner a probability $\sigma_i(x)$ with which player i escalates in the current round of the game that starts with state x.

The *symmetry* condition means that a player's equilibrium strategy depends not on his name or identity but rather on his relative position with respect to other players. To formally state this condition, we need to define two notions. First is the ranking on the players: Given any state $x := (x_i)_{i\in I}$ of the game, by Properties (a)–(c) listed above, there exists a unique bijection $r_x : I \to \{1, 2, \ldots, n\}$ such that

$$0 = x_{r_x^{-1}(1)} < 1 = x_{r_x^{-1}(2)} < x_{r_x^{-1}(3)} < \cdots < x_{r_x^{-1}(n)}.$$

The bijection r_x is called *ranking* at x. For each $i \in I$, player i's *rank* at state x is $r_x(i)$. Note that the frontrunner and follower have ranks 1 and 2, respectively.

Second is the ordered vector of a state: Given any state $x := (x_i)_{i\in I}$, with r_x the ranking bijection at state x, for each $k = 1, 2, \ldots, n$, let $\hat{x}_{(k)}$ denote the k^{th}-smallest component of the vector x, i.e., $\hat{x}_{(k)} := x_{r_x^{-1}(k)}$; and the vector $\hat{x} := (\hat{x}_{(k)})_{k=1}^n$ is called *ordered state of the game*. Thus, at any ordered state $\hat{x}$, the gap between the k^{th}-ranked player and the frontrunner is equal to $\hat{x}_{(k)}$. Note that for the follower, the gap is always one ($\hat{x}_{(2)} = 1$), while for any other player with rank $k \geq 3$, the gap $\hat{x}_{(k)} \geq k - 1$. Furthermore, the distance between two consecutively ranked players need not be one: for any $k \in \{3, \ldots, n\}$, $\hat{x}_{(k-1)} - \hat{x}_{(k)} \geq 1$ and the inequality can be strict.

Now we state the symmetry condition for an equilibrium $(\sigma_i)_{i\in I}$: for any two states x and y of the game and any $k = 2, \ldots, n$,

$$\hat{x} = \hat{y} \Longrightarrow \sigma_{r_x^{-1}(k)}(x) = \sigma_{r_y^{-1}(k)}(y).$$

In other words, given the same configuration of gaps across players, if players i and j switch their positions then they simply switch their equilibrium probabilities of escalation.

With symmetry, we can index equilibrium behavior strategies by players' ranks rather than by the their identities: For any $k = 2, \ldots, n$, and any ordered state $\hat{x}$ of the game, pick any state y with $\hat{y} = \hat{x}$ and denote

$$\tilde{\sigma}_k(\hat{x}) := \sigma_{r_y^{-1}(k)}(y).$$

With $(\sigma_i)_{i\in I}$ symmetric, $\tilde{\sigma}_k(\hat{x})$ is identical across all states y such that $\hat{y} = \hat{x}$. Thus, an equilibrium can be equivalently expressed in the form of $(\tilde{\sigma}_k)_{k=2}^n$ such that $\tilde{\sigma}_k$ associates to each ordered state $\hat{x}$

of the game a probability with which the k^{th}-ranked player in $\hat{x}$ escalates in the current round of the game that starts with $\hat{x}$. The profile $(\tilde{\sigma}_k)_{k=2}^n$ excludes $k=1$ because the frontrunner, the first-ranked player, has no available move.

In a Markov and symmetric equilibrium a player's probability of escalation may depend not only on his rank but also on his gap, and other players' gaps, from the frontrunner. That includes the gaps of those who no longer escalate, which keep widening as the game continues. Since such players are no longer part of the competition, their gaps become the sunspot dimensions of the state. For simplicity and intuitive appeal, we introduce the next condition to rule out such sunspot effect.

The condition of *independence of nonparticipants* means that a player's equilibrium strategy does not vary with the gaps of those players who no longer escalate in the equilibrium. To formally state the condition, define a notation $\geq_k$: for any $k \in \{2,\ldots,n\}$ and any two ordered states $\hat{x}$ and $\hat{y}$, write $\hat{y} \geq_k \hat{x}$ to mean $\hat{y}_{(j)} \geq \hat{x}_{(j)}$ for all $j = k, k+1, \ldots, n$. Thus, the positions of ranks $k,\ldots,n$ are further behind the frontrunner in $\hat{y}$ than in $\hat{x}$. Given any equilibrium $(\tilde{\sigma}_k)_{k=2}^n$, call any $k \in \{2,\ldots,n\}$ *nonparticipant rank* starting from order state $\hat{x}$ if and only if $\tilde{\sigma}_k(\hat{y}) = 0$ for all ordered states $\hat{y}$ such that $\hat{y} \geq_k \hat{x}$. Note: if $k, k+1, \ldots, n$ are each nonparticipant ranks starting from ordered state $\hat{x}$, then the player currently on the k^{th} rank will no longer escalate in equilibrium, as any future ordered state $\hat{y}$ on the equilibrium path necessarily satisfies $\hat{y} \geq_k \hat{x}$. (By induction, those ranked behind the k^{th}-ranked player will not escalate. Hence the k^{th}-ranked player will remain in this rank and hence will not escalate according to $\tilde{\sigma}_k$.)

Now we state the "independence of nonparticipants" condition for equilibrium $(\tilde{\sigma}_k)_{k=2}^n$: for any $m \in \{2,\ldots,n\}$, if all $k \in \{m+1,\ldots,n\}$ are nonparticipant ranks starting from ordered state $\hat{x}$, then for any two ordered states $\hat{y}$ and $\hat{z}$ such that $\hat{y} \geq_{m+1} \hat{x}$ and $\hat{z} \geq_{m+1} \hat{x}$,

$$\forall k \in \{2,\ldots,m\} \left[\hat{y}_{(k)} = \hat{z}_{(k)}\right] \Longrightarrow \forall k \in \{2,\ldots,m\} \left[\tilde{\sigma}_k(\hat{y}) = \tilde{\sigma}_k(\hat{z})\right].$$

In other words, if the n^{th}-ranked player no longer escalates once his gap has reached $\hat{x}_{(n)}$ (and hence remaining being ranked n from now on), then his widening gap thereafter has no effect on any other player's equilibrium probability of escalation; if the $(n-1)^{\text{th}}$- and n^{th}-ranked players no longer escalate once their gaps have reached $\hat{x}_{(n-1)}$ and $\hat{x}_{(n)}$ respectively (and hence remaining in the ranks of $n-1$ and n thereafter), then any other player's equilibrium probability of escalation is independent of their gaps thereafter; etc.

2.3. Bilateral, Trilateral and m-Rivalry Equilibria

Roughly speaking, an m-rivalry equilibrium is an equilibrium on the path of which up to m different players escalate, though not necessarily simultaneously. More precisely, a player is said *active* in an equilibrium conditional on state x of the game if and only if, in the subgame starting from the state x, on the path of the equilibrium there exists a state of the game at which the player escalates with positive probability. For any $m \in \{2,\ldots,n\}$, an *m-rivalry equilibrium* conditional on state x is an equilibrium such that the set of active players in the equilibrium conditional on x consists exactly of the players with ranks $2,\ldots,m$. By the independence condition of nonparticipants, for any m-rivalry equilibrium $(\tilde{\sigma}_k)_{k=2}^n$, the domain of the equilibrium strategies $\tilde{\sigma}_k$ can be restricted to the set of truncated ordered states $(\hat{x}_{(k)})_{k=1}^m$, each consisting of the first m components of an ordered state $\hat{x}$.

By definition, a bilateral rivalry ($m=2$) equilibrium is in the form $(\tilde{\sigma}_k)_{k=2}^n$ such that $\tilde{\sigma}_k(\cdot) = 0$ for all $k \geq 3$. Thus, by independence of nonparticipants, the domain of $\tilde{\sigma}_k$ can be restricted to the set of truncated ordered states $(\hat{x}_{(1)}, \hat{x}_{(2)})$, which by definition of states is the singleton $\{(0,1)\}$. Thus, at any bilateral equilibrium $(\tilde{\sigma}_k)_{k=2}^n$, the equilibrium strategy $\tilde{\sigma}_2$ of the follower is a constant probability, and $\tilde{\sigma}_k(\cdot) = 0$ for all $k \geq 3$. Specifically, it can be verified that strategy $\tilde{\sigma}_2 = 1 - 2\delta/v$ constitutes a unique subgame perfect bilateral equilibrium, with continuation values for the frontrunner and the follower equal to 2δ and zero, respectively (Ødegaard and Zheng 2018).

This unique characterization of the *surplus-dissipating bilateral equilibrium* is in line with the usual intuition about two-player wars of attrition with complete information: the total surplus for the players is almost completely dissipated, leaving them merely 2δ. However, it begs the question whether the game admits other equilibria that Pareto dominate the bilateral one for the players?

By unicity of the bilateral equilibrium, the only possible way to generate larger total surplus for the players is to have more than two players escalate against one another. To see why a third rival may help, pick any $m \in \{3,\ldots,n\}$ such that $m \leq v/\delta$. Suppose for the moment that the state x of the game satisfies $\hat{x}_{(m)} = m-1$, i.e.,

$$\hat{x}_{(1)} = 0,\ \hat{x}_{(2)} = 1, \hat{x}_{(3)} = 2, \cdots, \hat{x}_{(m)} = m-1.$$

That is, the top-m players are consecutively positioned, one immediately following the other, such that the gap $\hat{x}_{(k)}$ of the k^{th}-ranked player is equal to $k-1$ for each $k \in \{1,2,\ldots,m\}$. Then the following can be verified to constitute an equilibrium for the game thereafter:

(i) If $\hat{x}_{(m)} = m-1$, then the m^{th}-ranked player escalates with probability $\tilde{\sigma}_m = 1 - m\delta/v$ and all other players stay put ($\tilde{\sigma}_k(\cdot) = 0,\ 2 \leq k < m$); if the m^{th}-ranked player does escalate, then $\hat{x}_{(m)} = m-1$ holds at the next state, and the subgame is played by repeating this step; else the game ends with the current frontrunner getting the prize v.

(ii) If $\hat{x}_{(m)} > m-1$, then find the largest $m' \in \{2,\ldots,m-1\}$ for which $\hat{x}_{(m')} = m'-1$ (which exists because $\hat{x}_{(2)} = 1$ by definition of states), and the subgame thereafter is played according to step (i) with m' replacing the m there.

This constructed m-rivalry equilibrium generates a total surplus $m\delta$ for the players, with the current frontrunner getting the entire $m\delta$ and everyone else getting zero. Note that the only condition the above construction requires about m is that $m \leq v/\delta$. That leads to a striking implication: despite complete information and the restriction to symmetric equilibria, the total surplus for the players can be almost as large as v:

Proposition 1 *If $n \geq \lfloor v/\delta \rfloor$, then at any state x of the game such that*

$$\hat{x}_{(\lfloor v/\delta \rfloor)} = \lfloor v/\delta \rfloor - 1,$$

the subgame thereafter admits a $\lfloor v/\delta \rfloor$-rivalry equilibrium generating total surplus $\lfloor v/\delta \rfloor \delta$; where the $\lfloor y \rfloor$ operator denotes the largest integer less than or equal to $y \in \mathbb{R}$.

The m-rivalry equilibrium achieves the larger surplus through having the m^{th}-ranked player be the only one to decide whether or not to escalate, with others avoiding the current escalation cost. In the bilateral equilibrium, the decision is made by the follower, who, almost neck and neck with the frontrunner, needs only pay 2δ to become the next frontrunner; thus the condition for an equilibrium, which has to be in mixed strategies, requires that any follower escalate with such a high probability that shrinks a frontrunner's continuation value to just 2δ. By contrast, when the escalation decision is made by the m^{th}-ranked player, it costs a larger amount $m\delta$ to become the next frontrunner; thus, the equilibrium condition requires that any m^{th}-ranked player escalate with such a low probability that enlarges a frontrunner's continuation value to $m\delta$.

The contrast between multilateral and bilateral equilibria in Proposition 1 relies on the condition that there is some $m \geq 3$ for which the state of the game satisfies $\hat{x}_{(m)} = m - 1$. In other words, to enlarge the players' total surplus through such a multilateral equilibrium, we need at least the top three players to form a concentrated cohort within which the distance between any two consecutively ranked players is only one step. Whereas, if the game starts with a less concentrated positioning, say $\hat{x}_{(3)} > 2$ (hence the third-ranked player is at least two steps behind the follower, c.f. property (b) in the game setup), then provision (ii) above implies that the m-rivalry equilibrium coincides with the bilateral equilibrium. Thus we ask: Is there any m-rivalry equilibrium that is *robust* in the sense that it remains distinct from the bilateral equilibrium at some state of the game such that $\hat{x}_{(3)} > 2$? Do such equilibria outperform the bilateral one in generating players' total surplus?

Without further investigation, the answers to these questions are not intuitively obvious. Even if one manages to construct a robust m-rivalry equilibrium, it is not obvious that the equilibrium would outperform the bilateral one, because robustness may intensify the competition among players. To see that, recall the trilateral rivalry ($m = 3$) equilibrium above, where the follower stays put and the third-ranked player escalates with a low probability. Note how it keeps the follower from escalating: Should he escalate and become the next frontrunner, the follower would widen the distance between the next follower and the third-ranked player, so that $\hat{x}_{(3)} > 2$; consequently, the equilibrium not robust, the subgame equilibrium would collapse to the bilateral equilibrium, giving the deviator merely 2δ, no more than his escalation cost. Thus competition is suppressed by non-robustness. By contrast, in a robust trilateral equilibrium, if a follower escalates and overtakes the frontrunner, the widened gap for the third-ranked player need not collapse the subgame equilibrium into the detrimental bilateral one, hence the follower may escalate as well, competing away the total surplus. That, in turn, makes it harder to have a robust trilateral equilibrium, because robustness requires that the third-ranked player be willing to escalate despite wider gaps between him and the frontrunner and hence larger costs to escalate; it is not obvious that the larger costs of his escalation can be recovered by the continuation value of being the next frontrunner, given the competition pressure from the follower.

Despite such complications, the answers to both questions are Yes given a nonempty set of states of the game, and the robust m-rivalry equilibria we shall construct to demonstrate both answers are trilateral ones ($m = 3$): In the next section we characterize the set of such states in which robust trilateral equilibria exist and outperform the bilateral equilibrium.

3. Robust Trilateral Equilibria

The symmetry and Markov conditions together imply that there is no loss of generality to identify a player with the rank he belongs to currently. Therefore, at any ordered state $\hat{x}$, denote the player of rank one (frontrunner) by α, the player of rank two (follower) by β, and that of rank three by γ, whom we from now on call *underdog*. As explained in Section 2.3, any trilateral equilibrium is in the form $(\tilde{\sigma}_k)_{k=2}^n$ such that $\tilde{\sigma}_k = 0$ for all $k > 3$, and $(\tilde{\sigma}_2, \tilde{\sigma}_3)$ can vary only with $\hat{x}_{(3)}$. In other words, for any two ordered states $\hat{x}$ and $\hat{y}$, $\tilde{\sigma}_k(\hat{x}) = \tilde{\sigma}_k(\hat{y})$ if $\left(0, 1, \hat{x}_{(3)}\right) = \left(0, 1, \hat{y}_{(3)}\right)$. Hence we'll slightly abuse notation and write

$$\tilde{\sigma}_k(\hat{x}_{(3)}) := \tilde{\sigma}_k(\hat{x}).$$

Furthermore, for any ordered state $\hat{x}$, denote

$$s := \hat{x}_{(3)},$$

which we also call within this section the *state for trilateral equilibria*, or *state* for short. Note that $s \in \{2, 3, 4, \ldots\}$. For any such s, denote

$$\pi_{\beta,s} := \tilde{\sigma}_2(s) \quad \text{and} \quad \pi_{\gamma,s} := \tilde{\sigma}_3(s).$$

Thus, $\pi_{\beta,s}$ denotes the equilibrium probability for the follower β to escalate, and $\pi_{\gamma,s}$ that for the underdog γ to escalate, when the gap between the frontrunner α and γ is equal to s. Hence any trilateral equilibrium is equivalently in the form

$$(\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}.$$

Given any trilateral equilibrium $(\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}$, for every $s \geq 2$ and each $i \in \{\beta, \gamma\}$, let $q_{i,s}$ denote the probability with which the player currently in the role i becomes the α in the next round. By the equal-probability tie-breaking rule, we have at any $s \geq 2$

$$q_{i,s} = \pi_{i,s}\left(1 - \pi_{-i,s}/2\right), \tag{1}$$

with $-i$ being the element of $\{\beta, \gamma\} \setminus \{i\}$. Given this equilibrium and any state s, Let V_s denote the expected payoff for the current frontrunner α, M_s the expected payoff for the current β, and L_s that for the current γ. The law of motion is: for each $s \geq 2$,[4]

$$V_s \longrightarrow \begin{cases} v & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ M_{s+1} & \text{prob. } q_{\beta,s} \\ M_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{2}$$

$$M_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ V_{s+1} - 2\delta & \text{prob. } q_{\beta,s} \\ L_2 & \text{prob. } q_{\gamma,s}; \end{cases} \tag{3}$$

$$L_s \longrightarrow \begin{cases} 0 & \text{prob. } 1 - q_{\beta,s} - q_{\gamma,s} \\ L_{s+1} & \text{prob. } q_{\beta,s} \\ V_2 - (s+1)\delta & \text{prob. } q_{\gamma,s}. \end{cases} \tag{4}$$

3.1. The Dropout State

Since v is finite, at any trilateral equilibrium V_2 is finite and hence $V_2 < s\delta$ for all sufficiently large s. Thus, for any trilateral equilibrium

$$s_* := \max\{s \in \{2, 3, \ldots\} : V_2 \geq s\delta\}$$

exists and is unique. Call s_* the *dropout state* of the equilibrium. For example, the dropout state of the equilibrium constructed in Section 2.3, when $m = 3$, is equal to three. The next lemma, which follows from (4) coupled with the definition of s_*, justifies the appellation.

Lemma 1. *At any trilateral equilibrium with dropout state s_*, the γ-player (i) stays put for sure at state s if and only if $s \geq s_*$, and (ii) escalates for sure at state s if $2 \leq s < s_* - 1$.*

Proof Lemma 1 By definition of L_s, the equilibrium expected payoff for an underdog whose lag from the frontrunner is s, we know that $L_s = 0$ for all $s \geq v/\delta$. Starting from any such s and use backward induction towards smaller s, together with the law of motion (4) and the fact

$V_2-(s+1)\delta<0$ for all $s\geq s_*$ due to the definition of s_*, we observe that $L_s=0$ for all $s\geq s_*$. At any state $s\geq s_*$, by (4), an underdog gets zero expected payoff if he does not escalate; if he escalates then by Eq. (1) there is a positive probability with which he gets a negative payoff $V_2-(s+1)\delta$; hence his best response is uniquely to stay put. Hence

$$s\geq s_* \Longrightarrow L_s=0 \text{ and } \pi_{\gamma,s}=q_{\gamma,s}=0, \tag{5}$$

which proves Claim (i) of the lemma. Apply backward induction to (4) starting from $s=s_*$ and we obtain

$$2\leq s\leq s_*-1 \Longrightarrow V_2-(s+1)\delta\geq L_s\geq L_{s+1}\geq 0, \tag{6}$$

with the inequality $L_s\geq L_{s+1}$ being strict whenever $s<s_*-1$. Thus, for any $s<s_*-1$, $V_s-(s+1)\delta>L_{s+1}\geq 0$; hence Eqs. (1) and (4) together imply that an underdog's best response is uniquely to escalate for sure:

$$2\leq s<s_*-1 \Longrightarrow L_s>0 \text{ and } \pi_{\gamma,s}=1, \tag{7}$$

which proves Claim (ii) of the lemma. $\square$

By Lemma 1, once the game enters the dropout state or beyond, the player currently in the γ position will never catch up and only the frontrunner and follower may remain active. This has two immediate implications. First, the dropout state of a trilateral equilibrium is greater than or equal to three, as $s_*=2$ means that the equilibrium coincides with the bilateral one. Second, starting from $s\geq s_*$, the unicity observation in Section 2.3 implies that the surplus-dissipating bilateral equilibrium is the only on-path outcome in the subgame thereafter. Thus, the dropout state of a trilateral equilibrium can be viewed as the endogenous terminal node of the game, giving an expected payoff 2δ to the frontrunner, and zero expected payoff to everyone else (Lemma EC.1).

Reasoning backward from the dropout state s_*, we see that the game does not end if it is in any state $s\leq s_*-2$, because according to Lemma 1.ii the current underdog escalates for sure trying to catch up with the frontrunner. Thus the minimum state at which the game need not continue to the next round is the *critical state* s_*-1, at which the underdog need not escalate for sure. Furthermore, one can show that the follower at the critical state s_*-1 would rather be the underdog in the next round, should the game continue, than top the frontrunner right now, which would crowd out the underdog and get himself into the detrimental, bilateral equilibrium thereupon. Thus, the next lemma observes that, at the critical state s_*-1, the underdog solely determines whether the competition should continue or cease. Consequently, as the underdog has to mix between the two choices at equilibrium, his continuation value V_2 of being the next frontrunner is necessarily $s_*\delta$.

LEMMA 2. *In any robust trilateral equilibrium with dropout state s_*: (i) at the critical state $s_* - 1$ the β player stays put while the γ player escalates with a probability in $(0,1)$; and (ii) $V_2 = s_*\delta$.*

Proof of Lemma 2 We begin with some observations. By (4) and (6), L_2 is a convex combination between L_3 and $V_2 - 3\delta$, with $V_2 - 3\delta \geq L_3$ when $s_* \geq 3$. Thus,

$$s_* \geq 3 \Longrightarrow L_2 \leq V_2 - 3\delta. \tag{8}$$

Lemma EC.1, combined with (3) and (4), implies

$$M_{s_*-1} = q_{\gamma,s_*-1} L_2 \overset{(8)}{\leq} (V_2 - 3\delta)^+. \tag{9}$$

As explained immediately after Lemma 1, $s_* \geq 3$ for any trilateral equilibrium. If $s_* = 3$, then Lemma 1.i implies that an underdog stays put at all states $s \geq 3$, i.e., whenever his gap $\hat{x}_{(3)} > 2$, and hence the trilateral equilibrium is not robust. Thus, robustness implies $s_* \geq 4$. Now that $s_* \geq 4$, $L_2 > 0$ by Lemma EC.2. Thus, for the β player at $s = s_* - 1$, according to (3), the fact $V_{s_*} - 2\delta = 0$ (Lemma EC.1) and the fact that $L_2 > 0$ and $\pi_{\gamma,s_*-1} > 0$ (Lemma EC.3), it is the unique best response to not escalate at all, i.e., $\pi_{\beta,s_*-1} = 0$. Thus, the β player stays put for sure at state $s_* - 1$, as the lemma asserts.

Next we show that $0 < \pi_{\gamma,s_*-1} < 1$. The first inequality is implied by Lemma EC.3 since $s_* \geq 4$. To prove $\pi_{\gamma,s_*-1} < 1$, suppose to the contrary that $\pi_{\gamma,s_*-1} = 1$. Then by the fact $\pi_{\beta,s_*-1} = 0$ and (2) applied to the case $s = s_* - 1$, we have $V_{s_*-1} = M_2$. Consequently, by (3) applied to the case $s = s_* - 2$, $M_{s_*-2} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma EC.2. The supposition $\pi_{\gamma,s_*-1} = 1$ also implies $M_{s_*-1} = L_2$, which in turn implies, via (2) in the case $s = s_* - 2$, that $V_{s_*-2} \leq \max\{L_2, M_2\} \leq M_2$, with the last inequality again due to Lemma EC.2. Then (3) for the case $s = s_* - 3$ implies $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, and (2) implies $V_{s_*-3} \leq \max\{M_{s_*-2}, M_2\} \leq M_2$. Repeat the above reasoning on smaller s and we prove that $V_s \leq M_2$ for all $s \in \{2, 3, \ldots, s_* - 1\}$. Hence $V_3 \leq M_2$, which contradicts Lemma EC.2. Thus we have proved that $\pi_{\gamma,s_*-1} < 1$.

With $\pi_{\gamma,s_*-1} < 1$, escalating is not the unique best response for the γ player at state $s_* - 1$, hence $V_2 \leq s_*\delta$ (otherwise the bottom branch of (4) in the case $s = s_* - 1$ is strictly positive and, by (5), is strictly larger than the middle branch, so the γ player would strictly prefer to escalate). By definition of s_*, $V_2 \geq s_*\delta$. Thus $V_2 = s_*\delta$. □

3.2. Robust Dropout States Can Only Be Even

Lemma 2 implies that on the path of any trilateral equilibrium the game ends only when the state is $s_* - 1$, at which only the underdog γ may escalate: If he escalates (thereby becoming the next α) then the state returns to $s = 2$, else the game ends and the current α wins the good. Thus, in

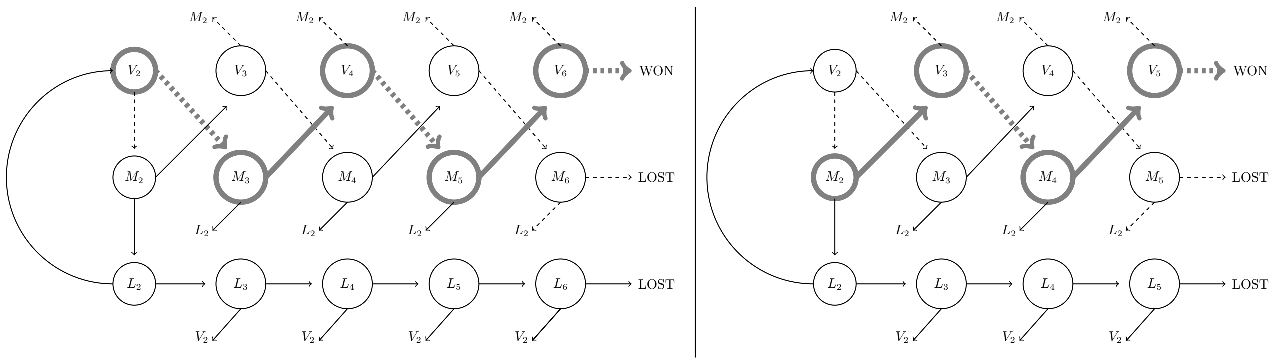

Figure 1 **The law of motions and equilibrium winning path if:** $s_* = 7$ **(left) and** $s_* = 6$ **(right).**

order to win, a player needs to be in the α position at the critical state $s_* - 1$. Consequently, if the dropout state s_* is an odd number, then on the path to winning a player must in the previous rounds have been the β player for all odd states $s < s_* - 1$, and the α player for all even states $s \le s_* - 1$. The left panel in Figure 1 illustrates the case of $s_* = 7$. Solid lines represent possible transitions if one escalates, and dashed lines if he does not escalate. The extra thick gray states and arrows indicate the winning path.

Thus, when s_* is odd, a player who happens to be in the β position at any even state $s < s_* - 1$ would in order to reach the winning path rather become the γ player in state $s = 2$ (through staying put) than become the superfluous α player in the odd state $s+1$ at the cost of 2δ (through escalating). In particular, in state $s = 2$, the β player would never escalate while the γ player would always escalate; hence the state $s = 2$ repeats itself, with the players switching roles according to $\gamma \to \alpha \to \beta \to \gamma$, thereby trapped in an infinite escalation loop. This contradiction, after being formalized, implies—

Theorem 1 *There does not exist any robust trilateral equilibrium whose dropout state s_* is an odd number.*

Proof of Theorem 1 Suppose, to the contrary, that there is a robust trilateral equilibrium whose dropout state s_* is an odd number. Since $s_* > 3$ by the robustness condition, $s_* \ge 5$. By Lemma EC.2, $L_2 \le M_2 \le V_3 - 2\delta$. Consider (2) in the case $s = s_* - 2$ together with the facts that $\pi_{\gamma, s_*-2} = 1$ (thereby ruling out $V_{s_*-2} \to v$) due to Lemma 1.ii and $s_* \ge 5$, that $M_{s_*-1} \le L_2$ due to (9), and that $M_2 \le V_3 - 2\delta$. Thus we have $V_{s_*-2} \le V_3 - 2\delta$. Then consider the decision of the β player at state $s = s_* - 3$, depicted by (3), to observe that M_{s_*-3} is between L_2 and $V_3 - 4\delta$. Thus, by (2) applied to the case $s = s_* - 4$, together with the facts $\pi_{\gamma, s_*-4} = 1$ and $M_2 \le V_3 - 2\delta$, we have $V_{s_*-4} \le V_3 - 2\delta$. Since s_* is an odd number and $s_* \ge 5$, this procedure of backward reasoning eventually reaches V_3, i.e., $3 = s_* - 2m$ for some positive integer m. Hence we obtain the contradiction $V_3 \le V_3 - 2\delta$. $\square$

When the dropout state s_* is an even number, by contrast, a β player is not in the predicament as in the previous case. First, in any even state $s < s_* - 1$ the β player wants to escalate in order to stay on the winning path and become the α in the odd state $s+1$. Second, in any odd state $s < s_* - 1$ the β player would rather escalate and become the α in the even state $s+1$ than stay put thereby becoming the γ player in state 2. With the former option, it takes a cost of 2δ (to become α in $s+1$) and two rounds for the player to have a chance to become the β player in state $s=2$ thereby landing on the winning path. With the latter option, it takes a cost of 3δ and three rounds for him to have such a chance of reaching the winning path. In the right panel of Figure 1, with $s_* = 6$, the situation of this odd-state β player is illustrated by the node M_3, from which the former option (becoming the next α) reaches the winning path state M_2 via $M_3 \to V_4 \to M_2$, while the latter option (being the next γ) reaches M_2 via the more roundabout route $M_3 \to L_2 \to V_2 \to M_2$.

Formalization of the above heuristic is based on the next observation, interesting in its own right, saying that when the top three players are consecutively positioned, the follower is a better position than the frontrunner (which also explains why a player going through the more roundabout route would like to carry out its last step, from V_2 to M_2):

Lemma 3. *At any equilibrium with any even number dropout state $s_* \geq 4$, $M_2 > V_2 + \delta/2$.*

Proof of Lemma 3 Let $m := \min\{k \in \{0,1,2\ldots\} : V_{2k+4} - 2\delta \leq L_2\}$. Note that m is well-defined because $s_*/2 - 2$ belongs to the set, as $V_{s_*} - 2\delta = 0 \leq L_2$ (Lemma EC.1). At any odd state $2k+1 \leq 2m+1$ (hence $k-1 < m$) we have $V_{2k+2} - 2\delta = V_{2(k-1)+4} - 2\delta > L_2$, with the last inequality due to the definition of m; hence by (2) in the state $s = 2k+1$ the β player escalates for sure, i.e. $\pi_{\beta,2k+1} = 1$. Thus, (EC.1) implies that $q_{\beta,s} = q_{\gamma,s} = 1/2$ at any such odd state. Coupled with (EC.2), that means the transition at every state s from 2 to $2m+2$ is that the current β and γ players each have probability 1/2 to become the next α player. Thus,[5]

$$V_2 = M_2 \sum_{k=0}^{m} 2^{-2k-1} + L_2 \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2} z_m \right) - 2\delta \sum_{k=1}^{m} 2^{-2k}, \tag{10}$$

where $z_m := 1$ if $2m+2 < s_* - 2$, and $z_m := 2\pi_{\gamma,s_*-1} - 1$ if $2m+2 = s_* - 2$; and the last series $\sum_{k=1}^{m}$ on the right-hand side uses the summation notation defined to be zero when $m = 0$.

To understand the term for M_2 on the right-hand side, note that M_2 enters the calculation of V_2 at the even states $s = 2,4,6,\ldots,2m-2$, and upon entry at state s and in every round transversing from states s to 2, the M_2 is discounted by the transition probability 1/2. The term for L_2 is similar, except that L_2 enters at the odd states $s = 3,5,7,\ldots,2m-1$, and that the transition probability for the L_2 at the last state $2m-1$ is equal to one if $2m-1 < s_* - 1$, and equal to π_{γ,s_*-1} if $2m-1 = s_* - 1$. That is why the last two terms within the bracket for L_2 are

$$2^{-2m-2} + 2^{-2m-2} z_m = \begin{cases} 2^{-2m-2} + 2^{-2m-2} = 2^{-2m-1} & \text{if } z_m = 1 \\ 2^{-2m-2} + 2^{-2m-2}\left(2\pi_{\gamma,s_*-1} - 1\right) = 2^{-2m-1}\pi_{\gamma,s_*-1} & \text{if } z_m = 2\pi_{\gamma,s_*-1} - 1. \end{cases}$$

The term for -2δ is analogous to that for M_2.

With $s_* \geq 4$, $V_2 - 4\delta \geq 0$. Thus, by the above-calculated transition probabilities,

$$L_2 = \frac{1}{2}(L_3 + V_2 - 3\delta) \leq \frac{1}{2}(V_2 - 4\delta + V_2 - 3\delta) = V_2 - \frac{7}{2}\delta.$$

This, combined with Eq. (10) and the fact $z_m \leq 1$ due to its definition, implies that

$$\begin{aligned} V_2 &\leq M_2 \sum_{k=0}^{m} 2^{-2k-1} + \left(V_2 - \frac{7}{2}\delta\right)\left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) - 2\delta \sum_{k=1}^{m} 2^{-2k} \\ &< M_2 \sum_{k=0}^{m} 2^{-2k-1} + V_2\left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) - \frac{7}{8}\delta. \end{aligned}$$

Thus, the lemma is proved if

$$1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) = \sum_{k=0}^{m} 2^{-2k-1}, \tag{11}$$

as the left-hand side of this equation is clearly strictly between zero and one. To prove (11), we use induction on m. When $m = 0$, (11) becomes $1 - 2^{-2} - 2^{-2} = 2^{-1}$, which is true. For any $m = 0, 1, 2, \ldots$, suppose that (11) is true. We shall prove that the equation is true when m is replaced by $m + 1$, i.e.,

$$1 - \left(\sum_{k=0}^{m+1} 2^{-2k-2} + 2^{-2(m+1)-2}\right) = \sum_{k=0}^{m+1} 2^{-2k-1}. \tag{12}$$

The left-hand side of (12) is equal to

$$\begin{aligned} &1 - \left(\sum_{k=0}^{m} 2^{-2k-2} + 2^{-2m-2}\right) + 2^{-2m-2} - 2^{-2(m+1)-2} - 2^{-2(m+1)-2} \\ &= \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-2} - 2^{-2(m+1)-1} \quad \text{(the induction hypothesis)} \\ &= \sum_{k=0}^{m} 2^{-2k-1} + 2^{-2m-3}, \end{aligned}$$

which is equal to the right-hand side of (12). Thus (11) is true in general, as desired. □

3.3. Existence of Robust Trilateral Equilibria

Lemmas 1–3 together have mostly pinned down the strategy profile for any robust trilateral equilibrium with dropout state s_*:

(A) The dropout state s_* is an even number greater than two; at each $s \in \{2, 3, \ldots, s_* - 2\}$ the β- and γ-players each escalate for sure; at state $s_* - 1$, β does not escalate and γ escalates with probability π_{γ, s_*-1}; should the game continue to any state $s \geq s_*$, β escalates with probability $1 - 2\delta/v$, and everyone else stays put for sure.

Next we pin down the strategy in the off-path event where a fourth player deviates to join the trilateral rivalry. Suppose that a player with rank $k \geq 4$ deviates to escalation and becomes the next frontrunner. By the Markov and symmetry conditions in our equilibrium concept, the subgame equilibrium thereafter is purely based on the starting state of the subgame or, more precisely, the positioning of players defined by that state, regardless of the history that the state has just resulted from a deviation or that the current frontrunner was a deviator. Thus, the deviation cannot alter the trilateral equilibrium strategy profile, hence the following provision is necessary for any trilateral equilibrium:

(B) No player with rank $k \geq 4$ escalates; if any such a player deviates and becomes the next frontrunner, then the top three players at the new state play the subgame according to the same trilateral equilibrium, with the same dropout state s_*.

Provisions *(A)* and *(B)* together define a set of strategy profiles of the game. In the following we locate a necessary and sufficient condition for any such a profile to constitute a robust trilateral equilibrium. Based on that condition, we obtain existence of such equilibria.

First, any trilateral equilibrium, by definition, needs to reach with positive probability an event where the underdog escalates. Thus, it is necessary that in the starting state of the game the underdog's gap be less than the dropout state:

$$\hat{x}_{(3)} \leq s_* - 1, \tag{13}$$

otherwise he never escalates (Lemma 1.i) so the equilibrium coincides with the bilateral one.

Second, we find a necessary and sufficient condition for players with ranks $k \geq 4$ to abide by Provision *(B)*. If such a player deviates and becomes the next frontrunner, then the new state is a trilateral rivalry consisting of the deviator, the previous frontrunner and the previous follower such that the third is only two steps behind the new frontrunner, i.e., $s = 2$. Hence the deviator, after paying the escalation cost $(\hat{x}_{(k)} + 1)\delta$, gets a continuation value V_2. Thus, a necessary condition for such players to not escalate is $V_2 \leq \left(\hat{x}_{(4)} + 1\right)\delta$, otherwise the fourth-ranked player would deviate to escalation. This inequality, coupled with another necessary condition $V_2 = s_*\delta$ (Lemma 2), implies

$$s_* \leq \hat{x}_{(4)} + 1. \tag{14}$$

Conversely, for any state x of the game, if (14) holds, then not only is it a best response for each player with rank $k \geq 4$ to not escalate currently, it is also his best response in any state y thereafter because $\hat{y}_{(k)} \geq \hat{y}_{(4)} > \hat{x}_{(4)}$ provided that he abides by *(B)* now.

Third, we find a necessary and sufficient condition for players β and γ to each abide by Provision *(A)*. By *(A)*, Eq. (1), and the equal-probability tie-breaking rule,

$$2 \leq s \leq s_* - 2 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{15}$$

Given any $\pi_{\gamma,s_*-1} \in [0,1]$, the value functions $(V_s, M_s, L_s)_s$ associated to any strategy profile satisfying Provision *(A)* can be calculated based on Eq. (15) and the law of motion, (2)–(4). The question is whether such a strategy profile constitutes an equilibrium. The crucial step in answering this question is to verify that, given *(A)* and *(B)*, escalation is a best response for the β player at every state below $s_* - 1$. Verification for all such states might sound cumbersome, but it turns out that we need only to check two inequalities: $V_3 - 2\delta \geq L_2$, and $V_{s_*-2} - 2\delta \geq L_2$ (Lemma EC.8).

The two sufficient conditions, one can show, are also necessary for any robust trilateral equilibrium. Thus the conditions, combined with the previous ones, imply a necessary and sufficient condition for any robust trilateral equilibrium: that the escalation probability π_{γ,s_*-1} at the critical state is determined by the equation $V_2 = s_*\delta$ (Lemma 2.ii), with V_2 as well as other value functions derived from the law of motion (2)–(4) and Eq. (15), such that $V_3 - 2\delta \geq L_2$, $V_{s_*-2} - 2\delta \geq L_2$, and Ineqs. (13) and (14) are each satisfied. From this condition we obtain a complete characterization of robust trilateral equilibria (Lemma EC.12), which in turn provides a set of states of the game in which robust trilateral equilibria exist:

Theorem 2 *Starting from any ordered state $\hat{x}$ of the game, there exists a robust trilateral equilibrium in the subgame thereupon if $\left\{\hat{x}_{(3)}+1, \hat{x}_{(3)}+2, \ldots, \hat{x}_{(4)}+1\right\}$ contains an even number s_* and at least one of the following conditions is satisfied:*

i. $s_* = 4$ and $v/\delta > 35/2$;

ii. $s_* = 6$ and $v/\delta > 6801/120$;

iii. $s_* \geq 8$ and $v/\delta \geq \left(\frac{1}{3}s_*^2 + \frac{5}{3}s_* - 8\right)2^{s_*-3}$.

Proof of Theorem 2 Let an even number s_* belong to $\left\{\hat{x}_{(3)}+1, \hat{x}_{(3)}+2, \ldots, \hat{x}_{(4)}+1\right\}$ and satisfy one of the three conditions, (i) or (ii) or (iii), in the theorem. That condition (i) or (ii) suffices existence is implied by Lemma EC.18. To prove sufficiency of condition (iii), pick any even number $s_* \in \{8, 10, 12, \ldots\}$. By Lemma EC.12.ii, s_* constitutes an equilibrium if Eq. (EC.9) admits a solution for $\pi \in [0,1]$ that satisfies Ineq. (EC.10). Denote $\phi(s_*, \pi)$ for the right-hand side of (EC.9), i.e., $\phi(s_*, \pi) := (2(1+\mu_*) - 3\mu_*\pi)(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*\pi))$. By Lemma EC.17, the left-hand side of (EC.9) is less than $\phi(s_*, \pi)$ when $\pi = 1$. Thus, it suffices to show that the left-hand side is greater than $\phi(s_*, \pi)$ when π is equal to some number greater than or equal to the right-hand side of Ineq. (EC.10). To that end, note from $s_* \geq 8$ that $\mu_* = 2^{-s_*+3} \leq 1/32$, hence $2 - \mu_* \geq 63/32$ and $1 + \mu_* < 33/32$; and recall that $\mu_* = 2^{-s_*+3}$. Thus, the left-hand side of (EC.9) is greater than $\frac{3\mu_* v}{\delta}(1-\pi)\frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6)$, and the right-hand side of Ineq. (EC.10), $1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)} < 1 - \frac{3\times 63/32}{2\times 1\times(s_*-3)}$. Thus, it suffices, for s_* to constitute an equilibrium, to have

$$\frac{3\mu_* v}{\delta}(1-\pi)\frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6) \geq \phi(s_*, \pi)$$

when $\pi = \pi_* := 1 - \frac{3\times 63}{64(s_*-3)}$. Note, from $0 < \pi < 1$, that $-1/32 < \mu_*(2-3\pi) < 1/16$. Hence

$$\begin{aligned}\frac{63}{32} = 2 - \frac{1}{32} \ &< 2(1+\mu_*) - 3\mu_*\pi < \ 2 + \frac{1}{16} = \frac{33}{16},\\ \frac{15}{16} = 1 - \frac{1}{16} \ &< \ 1 - 2\mu_* + 3\mu_*\pi \ < \ 1 + \frac{1}{32} = \frac{33}{32}.\end{aligned}$$

Thus, the first factor $2(1+\mu_*) - 3\mu_*\pi$ of $\phi(s_*,\pi)$ is positive for all $\pi \in (0,1)$. If the second factor of $\phi(s_*,\pi)$ is nonpositive when $\pi = \pi_*$ then $\phi(s_*,\pi_*) \le 0$ and we are done, as the left-hand side of (EC.9) is positive. Hence we may assume, without loss of generality, that $3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*\pi_*) > 0$. Consequently, $\phi(s_*,\pi_*)$ can only get bigger if we replace its first factor by the upper bound $33/16$, and the term $1 - 2\mu_* + 3\mu_*\pi$ in the second factor by its lower bound $15/16$ (note that, in the second factor, $s_* - 4 + \mu_* > 0$ because $s_* \ge 8$). I.e., $\phi(s_*,\pi_*)$ is less than

$$\begin{aligned}\frac{33}{16}\left(3s_* + 2(1-2\mu_*) - \frac{15}{16}(s_* - 4 + \mu_*)\right) &= \frac{33}{16}\left(\frac{33}{16}s_* + \frac{23}{4} - \frac{79}{16}\mu_*\right)\\ &< \frac{33}{16}\left(\frac{33}{16}s_* + \frac{23}{4}\right)\\ &< 5s_* + 12.\end{aligned}$$

Therefore, the above observations put together, we are done if $\frac{3\mu_* v}{\delta}(1-\pi_*)\frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6) \ge 5s_* + 12$. In other words, it suffices to have $\frac{3\mu_* v}{\delta} \cdot \frac{3\times 63}{64(s_*-3)} \cdot \frac{63}{32} + \left(\frac{63}{32}\right)^2(s_* - 6) \ge 5s_* + 12$, i.e.,

$$\frac{3^2\mu_* v}{\delta} \ge -2(s_*-6)(s_*-3) + \frac{32\times 64}{63^2}\left(5s_* + 12\right)(s_* - 3).$$

With $\frac{32\times 64}{63^2} \approx 0.516$, the above inequality holds if $\frac{3^2\mu_* v}{\delta} \ge -2(s_*-6)(s_*-3) + (5s_*+12)(s_*-3)$, i.e.,

$$\frac{9\mu_* v}{\delta} \ge 3s_*^2 + 15s_* - 72,$$

which is equivalent to the inequality $v/\delta \ge \left(\frac{1}{3}s_*^2 + \frac{5}{3}s_* - 8\right)2^{s_*-3}$ in Condition (iii). $\square$

Here, the condition that the set $\left\{\hat{x}_{(3)}+1, \hat{x}_{(3)}+2, \ldots, \hat{x}_{(4)}+1\right\}$ contains s_* is to guarantee Ineqs. (13) and (14), so that the third-ranked player is willing to escalate and anyone ranked behind him is not. Any of the three itemized conditions is to ensure $V_3 - 2\delta \ge L_2$ and $V_{s_*-2} - 2\delta \ge L_2$, so that the follower is willing to escalate at both even and odd states.

According to Theorem 2, when the third- and the fourth-ranked players are not far apart, say just one step between each other ($\hat{x}_{(4)} = \hat{x}_{(3)} + 1$), if the parameter v/δ is large enough for the even integer in $\{\hat{x}_{(3)}+1, \hat{x}_{(4)}+1\}$ to satisfy one of the three itemized conditions, then a robust trilateral equilibrium exists. Alternatively, if the third- and the fourth-ranked players are far apart enough for the set $\left\{\hat{x}_{(3)}+1, \ldots, \hat{x}_{(4)}+1\right\}$ to include an even number s_* meeting one of the three conditions, then again a robust trilateral equilibrium exists.

Corollary 1 *There are at most finitely many robust trilateral equilibria.*

Proof Corollary 1 Since $v/\delta \geq s_*$ for any dropout state of any equilibrium, there are only finitely many equilibrium-feasible dropout states. Given any dropout state of any equilibrium, each player's action at every state is uniquely determined, according to Lemmas 1–EC.4, except the γ player's escalation probability π_{γ,s_*-1} at the critical state. Thus, it suffices to prove that for each dropout state s_* there are only finitely many compatible π_{γ,s_*-1} at the equilibrium. To that end, since π_{γ,s_*-1} is determined by Eq. (EC.9) given s_*, we need only to show that for each s_* Eq. (EC.9) admits at most two solutions for π, the shorthand for π_{γ,s_*-1}. To show that, note that the left-hand side of Eq. (EC.9) is a linear function of π, whereas the right-hand side is strictly convex in π: The derivative of the right-hand side with respect to π is equal to

$$-3\mu_*\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*\pi)\right) + \left(2(1+\mu_*) - 3\mu_*\pi\right)\left(-(s_* - 4 + \mu_*)3\mu_*\right),$$

whose derivative with respect to π is equal to

$$3\mu_*(s_* - 4 + \mu_*)3\mu_* + 3\mu_*(s_* - 4 + \mu_*)3\mu_* = 18\mu_*(s_* - 4 + \mu_*) > 0,$$

with the inequality due to the fact that $s_* - 4 + \mu_* = s_* - 4 + 2^{-s_*+4} > 0$ as $s_* \geq 4$ (Theorem 1). Thus, Eq. (EC.9) admits at most two solutions for π, as desired. $\square$

The reason for this corollary is that the parameter v/δ implies an upper bound for equilibrium-feasible dropout states s_*, which can only be integers (Lemma 1.ii). Given each dropout state the Bellman equation corresponding to $V_2 = s_*\delta$ admits at most two solutions for π_{γ,s_*-1}, which in turn determines the equilibrium strategy profile uniquely.

Numerical Illustration - Figure 2 shows the trilateral equilibria, with dropout states $s_* \in \{4, 6, 8, 10\}$, computed within the case where $\delta = \$1$ and v ranges from \$0 to \$1,000. Each curve in the figure corresponds to one such equilibrium, graphing the γ-player's (underdog's) equilibrium probability of escalation in the critical state $s_* - 1$ as a function of the underlying value v (or v/δ, as $\delta = 1$ here). The vertical lines indicate the points at which additional equilibria are admitted. For instance, starting at $v = \$57$ ($\approx 6801/102$) the equilibrium with dropout state $s_* = 6$ becomes permissible. Note that, within each equilibrium, the bidding probability is increasing in the underlying value v. On the other hand, when an equilibrium with a higher dropout state becomes permissible, had the players switched to the new one, the corresponding equilibrium bidding probability would drastically reduce. Furthermore, each additional equilibrium requires an order of magnitude increase in v, consistent with the inequalities in the three itemized conditions in Theorem 2.

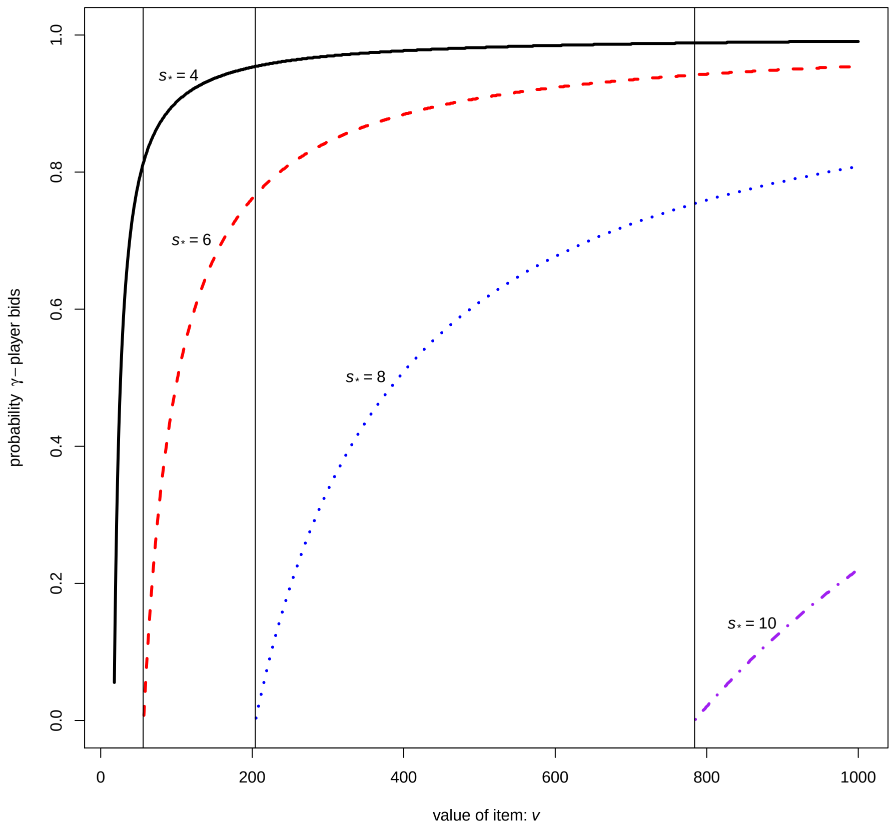

Figure 2 **Equilibrium bidding probability for the underdog in the critical state** $s_* - 1$; $\delta = 1$.

3.4. Pareto Superiority of Robust Trilateral Equilibria

An equilibrium $\mathscr{E}$ is said *Pareto superior* to another equilibrium $\mathscr{E}'$ if and only if two conditions are met: (i) at any state of the game where both equilibria are valid but different on their paths, some players active in $\mathscr{E}$ are better-off in $\mathscr{E}$ than they are in $\mathscr{E}'$, and no player is worse-off in the former than in the latter; and (ii) there exists a state of the game conditional on which every player active in $\mathscr{E}$ is better-off in $\mathscr{E}$ than he is in $\mathscr{E}'$, and no player is worse-off in the former than in the latter. We observe here that any robust trilateral equilibrium, despite its intensified competition between the follower and the underdog before the critical state is reached, is Pareto superior to the bilateral equilibrium.

Theorem 3 *Any robust trilateral equilibrium is Pareto superior to the bilateral equilibrium.*

Proof Theorem 3 Note that the set of active players in any trilateral equilibrium consists of the top three players: the frontrunner, follower and underdog. Recall that the bilateral equilibrium gives a surplus 2δ to the frontrunner and zero surplus to everyone else. Now consider any robust trilateral equilibrium with dropout state s_*, described by Provisions *(A)* and *(B)*, Section 3.3. If the game continues to any state such that $s \geq s_*$, the two equilibria do not differ in path. Thus it suffices to consider only those states where $s \leq s_* - 1$. At any such a state, in the trilateral equilibrium the frontrunner's surplus, by (2), is equal to either (i) $(M_2 + M_{s+1})/2$ (if $s < s_* - 1$) or (ii) $(1 - \pi_{\gamma,s_*-1})v + \pi_{\gamma,s_*-1}M_2$ (if $s = s_* - 1$), the follower's surplus is equal to a convex combination between $V_{s+1} - 2\delta$ and L_2, and the underdog's equal to L_s. If $s \leq s_* - 1$, then in Case (i), the frontrunner's surplus is $(M_2 + M_{s+1})/2 > M_2/2 > (V_2 + \delta/2)/2 \geq (4\delta + \delta/2)/2 > 2\delta$, with the second inequality due to Lemma 3, and the third inequality due to $V_2 = s_*\delta$ and $s_* \geq 4$, while in Case (ii), his surplus is $(1 - \pi_{\gamma,s_*-1})v + \pi_{\gamma,s_*-1}M_2 > (1 - \pi_{\gamma,s_*-1})2\delta + \pi_{\gamma,s_*-1}4\delta > 2\delta$, with the first inequality due to $v > 2\delta$ by assumption, the fact $M_2 > V_2$ by Lemma 3, and the fact $V_2 \geq 4\delta$ as in Case (i). Hence the frontrunner's surplus in the trilateral equilibrium is greater than the bilateral equilibrium surplus 2δ.

Only remain to show that $M_s, L_s > 0$ for any $s \leq s_* - 1$. When $s = s_* - 1$, $M_s = \pi_{\gamma,s_*-1}L_2 > 0$, and $L_s = \pi_{\gamma,s_*-1}V_2 > 0$; where the inequalities holds due to Lemmas 2 and EC.2. If $s \leq s_* - 2$, then by Eq. (4) and Provision (A) in Section 3.3, $L_s \geq \frac{1}{2}(V_2 - (s+1)\delta) = \frac{1}{2}(s_*\delta - (s+1)\delta) > 0$, with the last inequality due to $s \leq s_* - 2$. By (3), the follower can secure an expected payoff no less than L_2, through staying put at s (while the underdog escalates for sure by Provision (A)), hence $M_2 \geq L_2 > 0$. By contrast, both players get zero surplus in the bilateral equilibrium in whatever state. □

Theorem 3 implies that, at any state x of the game, the players can avoid the detrimental bilateral rivalry if a robust trilateral equilibrium exists and has not reached the end of its path (i.e., $\hat{x}_{(3)} \leq s_* - 1$). In other words, a third rival can help the players avoid the detrimental outcome of bilateral escalation if the condition of Theorem 2 is satisifed.

4. Model Extensions

The above analysis is based on an initial ranked positioning of the players, complete information, and equilibria where only trilateral rivalry emerges on path. While a comprehensive relaxation of these and other restrictions is beyond the scope of this paper, this section presents partial extensions of the listed restrictions individually. Subsection 4.1 considers a case of four players and constructs a quadrilateral-rivalry equilibrium based on trilateral-rivalry ones. Subsection 4.2 considers a two-player model with asymmetric information and constructs a perfect Bayesian equilibrium that resembles a purification of the bilateral-rivalry equilibrium. In both extensions we also relax the assumption of the initial ranked positioning, and instead, as in the dollar auction, let all players start with equal footing, each having committed zero payment.[6]

4.1. A Quadrilateral Rivalry Equilibrium

The analysis in Section 3 has highlighted two effects of adding a third rival to an otherwise bilateral rivalry: first, once a play has reached the bilateral-rivalry subgame, it is Pareto dominated by any trilateral-rivalry equilibrium (Theorem 3); second, participation of the third rival eliminates the bilateral-rivalry in the sense that in any trilateral equilibrium, the bilateral rivalry can only be an off-path event (Lemmas 1 and 2). But then would a trilateral-rivalry equilibrium become fragile given the participation of a *fourth* rival, as the bilateral equilibrium being fragile given the participation of a third rival? Here we construct an example to show that the answer is No.

Let there be four players, denoted as frontrunner (α), follower (β), underdog (γ_1), and "bottomdog" (γ_2) for the fourth-place bidder. Let $t \in \{0, 1, 2, 3, \ldots\}$ denote the gap between the bottomdog and the frontrunner. In the quadrilateral-rivalry equilibrium constructed below, the dropout state for the bottomdog is when $t = 4$. In the subgame once this dropout state is reached, the other three players play the trilateral-rivalry equilibrium whose dropout state is $s_* = 4$. Specifically, the quadrilateral strategy profile is:

a. In the initial round ($t = 0$) when no one is the frontrunner, everyone bids for sure.

b. In the second round ($t = 1$), with one frontrunner, and third round ($t = 2$), with one frontrunner and one follower, every non-frontrunner bids for sure.

c. In any round where $t = 3$ (and a ranked positioning has emerged):

i. if the current configuration is

$$\begin{bmatrix} \alpha & p \\ \beta & p-\delta \\ \gamma_1 & p-2\delta \\ \gamma_2 & p-3\delta \end{bmatrix} \tag{16}$$

for some $p \geq 3\delta$, then β, γ_1 and γ_2 each bid for sure;

ii. else then it is the fourth round and the configuration is in the form

$$\begin{bmatrix} \alpha & 3\delta \\ \beta & 2\delta \\ \varnothing & \delta \\ \{\gamma_1, \gamma_2\} & 0 \end{bmatrix}, \tag{17}$$

then the play mimics the above-specified trilateral-rivalry equilibrium at the critical state $s_* - 1 = 3$: β stays put, and γ_1 and γ_2 each bid with probability $1 - (1 - \pi_{\gamma,3})^{1/2}$; where $\pi_{\gamma,3}$, as defined by Provision *(A)* in subsection 3.3, is the trilateral equilibrium probability that escalation continues.

d. If $t \geq 4$, then γ_2 quits from now on, and the other players play the above-specified trilateral-rivalry equilibrium; in the off-path event where γ_2 leapfrogs to the top, then he and the previous frontrunner and follower constitute a consecutive 3-player configuration, and the three play the trilateral-rivalry equilibrium from now on, with γ_1 the new bottomdog quitting from now on.

Appendix EC.6 proves that the above strategy profile constitutes a subgame perfect equilibrium. Note that trilateral rivalry occurs on path with positive probability in this equilibrium. Thus, adding a fourth rival does not eliminate trilateral rivalry as adding a third rival does to bilateral rivalry. To check if this quadrilateral equilibrium dominates any trilateral one, define, given the consecutive configuration (16), let $W_\alpha, W_\beta, W_{\gamma_1}, W_{\gamma_2}$ denote the continuation value for the α, β, γ_1, and γ_2 player, respectively. By Appendix EC.6,

$$4\delta < W_\alpha < 5\delta, \quad 5\delta < W_\beta < 6\delta, \quad 3\delta/8 < W_{\gamma_1} < 2\delta/5, \quad \delta/9 < W_{\gamma_2} < 5\delta/27.$$

In the configuration (16), if the players stick to the trilateral-rivalry equilibrium with dropout state $s_* = 4$, then α gets $V_2 = 2\delta$, β gets M_2, which by Eqs. (EC.15) and (EC.16) in Appendix EC.6 is equal to $\left(8 - \frac{1}{2}\pi_{\gamma,3}\right)\delta$, γ_1 gets $L_2 = \delta/2$, and γ_2, not supposed to participate, gets zero. By contrast, if they switch to the quadrilateral-rivalry equilibrium from now on, α gets $W_\alpha > 4\delta = V_2$, β gets W_β, γ_1 gets W_{γ_1}, and γ_2 gets $W_{\gamma_2} > 0$. While the change of equilibrium would make α and γ_2 better-off, it would make β and γ_1 worse-off: By Eq. (EC.27) in Appendix EC.6, $M_2 - W_\beta > 2\delta$, and by Eq. (EC.26) in Appendix EC.6, $L_2 - W_{\gamma_1} > \delta/10$. Thus, given configuration (16), the quadrilateral equilibrium does not Pareto dominate the trilateral one. In any other configuration where a γ_2's leapfrogging is commonly seen as a deviation from the trilateral equilibrium, the position of γ_2 can only be lower (i.e., $t \geq 4$) and hence the leapfrog would cost him at least 5δ; since $W_\alpha < 5\delta$, it is an unprofitable deviation for γ_2 even if the deviation could switch the equilibrium to the quadrilateral one. Thus, this quadrilateral equilibrium does not dynamically Pareto dominate the trilateral one.

The above example suggests that the effect of adding a third rival to bilateral rivalry might be more critical than adding an $(n+1)$th rival to an n-bidder play: while the bilateral-rivalry subgame equilibrium is surplus-dissipating, n-rivalry subgame equilibriums need not be so detrimental to the bidders.

4.2. Asymmetric Information in Bilateral Rivalry

Here we relax both the complete-information assumption and the assumption that the game starts with a ranked order among the players. For simplicity, assume that there are only two bidders. For each $i \in \{1, 2\}$, bidder i's type is drawn from a commonly known distribution F_i, absolutely continuous and strictly increasing on its support $[a_i, z_i]$, with $z_i > a_i \geq 0$. The realized type t_i of bidder i is i's private information at the outset; if b_i is the highest level among bidder i's committed bid, then i's payoff from the game is equal to $v - b_i/t_i$ if he wins the prize, and equal to $-b_i/t_i$ if he does not win it. Recall that δ denotes the exogenous increment price ascension. The tie-breaking rule is: if no one bids in the initial round, then the game ends with the good not sold; if exactly one player bids in the initial round, then the game ends with the good sold to the only bidder at the price equal to δ; else one of the two bidders is selected randomly with probability 1/2 to be the frontrunner in the second round, after which no tie will occur.

A perfect Bayesian equilibrium: The idea is that at each round where a player is supposed to make a move, he bids if and only if his type is above a cutoff in the support of the posterior belief about his type, and the cutoff is so chosen that his opponent, now the frontrunner, would have been indifferent about bidding in the previous round if the opponent's type is equal to the opponent's cutoff in the previous round.

The cutoffs for the initial round: Let $(s_1^0, s_2^0) \in (a_1, z_1) \times (a_2, z_i)$ satisfy

$$\forall i \in \{1,2\} : \frac{\delta}{v s_{-i}^0} = F_i(s_i^0). \tag{18}$$

For example, if F_i is the uniform distribution on $[0,1]$ for each i, then $s_1^0 = s_2^0 = \sqrt{\delta/v}$ constitutes such a pair.

The cutoffs for the second round: For each $i \in \{1,2\}$, define $s_i^1 \in (s_i^0, z_i)$ by

$$\frac{F_i(s_i^1) - F_i(s_i^0)}{1 - F_i(s_i^0)} = \frac{\delta}{v s_{-i}^0}. \tag{19}$$

The cutoffs for any round after the second one: For any $(s_1, s_2) \in (a_1, z_1) \times (a_2, z_i)$, if $[s_i, z_i]$ is the support of the posterior distribution of i's type at the start of this round for each $i \in \{1,2\}$, then define the cutoff $s_i' \in (s_i, z_i)$ for each player i in this round by

$$\frac{F_i(s_i') - F_i(s_i)}{1 - F_i(s_i)} = \frac{2\delta}{v s_{-i}}. \tag{20}$$

The equilibrium: Initialize $s_i := a_i$ for each $i \in \{1,2\}$.

a. In the initial round, for each player i of type t_i, i bids if and only if $t_i \geq s_i^0$. If player i bids then the posterior about i becomes $F_i(\cdot)/(1 - F_i(s_i^0))$, hence his infimum type is updated to $s_i := s_i^0$; else the game ends, with the good either sold at price δ to the other bidder if the latter has bid, or not sold if neither has bid, and hence there is no need for updating.

b. In the second round, with the frontrunner α selected among those who bid in the initial round, the follower β of type t_β bids if and only if $t_\beta \geq s_\beta^1$. If β does bid, then the posterior about bidder β becomes $F_\beta(\cdot)/(1 - F_\beta(s_\beta^1))$, hence his infimum type is updated to $s_\beta := s_\beta^1$; else the game ends and there is no need for updating. If β does not bid, the game ends and there is no need for updating.

c. If the game continues to any round after the second round, with s_i denoting the updated infimum type of player i at the start of the current round (hence the posterior distribution about i is $F_i(\cdot)/(1 - F_i(s_i))$), the current follower β of type t_β bids if and only if $t_\beta \geq s_\beta'$, where s_β' is derived from (s_1, s_2) by Eq. (20). If β does bid then the posterior about bidder β becomes $F_\beta(\cdot)/(1 - F_\beta(s_\beta'))$, hence his infimum type is updated to $s_\beta := s_\beta'$; else the game ends and there is no need for updating.

This equilibrium exhibits two interesting features. First, the allocation is not ex post efficient, as the winner need not be the bidder with the stronger realized type. Second, in the case where $z_1 < a_2$, it is commonly known ex ante that bidder 1 is weaker than bidder 2, yet at the equilibrium their bidding competition may escalate for many rounds, especially when their realized types are near the corresponding supremums.

Verification of the equilibrium: At the start of any round after the initial one, let (s_1, s_2) denote the pair of current updated type infimums of the two players and, for any $i \in \{1,2\}$, let $M_i(t_i|s_i, s_{-i})$ denote the expected payoff for player i of type t_i if i is the current follower, given the continuation equilibrium described above. Then

$$M_i(t_i|s_i, s_{-i}) = \max\left\{0, -\frac{2\delta}{t_i} + V_i(t_i|s_i', s_{-i})\right\}, \tag{21}$$

note that $V_i(t_i|s_i', s_{-i})$ denotes i's expected payoff from being the frontrunner in the next round, with his type infimum updated to s_i', derived from (s_i, s_{-i}) by Eq. (20). In general, $V_i(t_i|s_i, s_{-i})$ denotes the expected payoff for player i of type t_i if i is the current frontrunner in any round after the second one such that the updated type infimums at the start of the current round are s_i and s_{-i} respectively. Then

$$\begin{aligned} V_i(t_i|s_i, s_{-i}) &= \frac{F_{-i}(s'_{-i}) - F_{-i}(s_{-i})}{1 - F_{-i}(s_{-i})} v + \frac{1 - F_{-i}(s'_{-i})}{1 - F_{-i}(s_{-i})} M_i(t_i|s_i, s'_{-i}) \\ &= \frac{2\delta}{s_i} + \left(1 - \frac{2\delta}{s_i}\right) M_i(t_i|s_i, s'_{-i}), \end{aligned} \tag{22}$$

where s'_{-i} is derived from (s_i', s_{-i}) by Eq. (20), and the second line follows from Eq. (20), with the roles of i and $-i$ switched. By contrast, if, after both players bid in the initial round, player i is selected the frontrunner in the second round, then i's expected payoff, viewed at the start of this round, is equal to

$$\begin{aligned} V_i^0(t_i|s_i^0, s_{-i}^0) &= \frac{F_{-i}(s_{-i}^1) - F_{-i}(s_{-i}^0)}{1 - F_{-i}(s_{-i}^0)} v + \frac{1 - F_{-i}(s_{-i}^1)}{1 - F_{-i}(s_{-i}^0)} M_i(t_i|s_i^0, s_{-i}^1) \\ &= \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1), \end{aligned} \tag{23}$$

where the second line follows from Eq. (19), with the roles of i and $-i$ switched. Based on the Bellman equations (21), (22) and (23), one can prove that the above-described bidding strategy and updating rule constitute a perfect Bayesian equilibrium (Section EC.7).

5. Conclusion

War-of-attrition-like contests and conflicts, such as online crowdsourcing challenges, lobbying, R&D races, arms races, online penny auctions, bargaining for technology standards, and contests among and within political parties, often involve a multitude of contenders. Such multiplicity of rivals, despite the cutthroat competition it intensifies, also provides an opportunity for a player to preserve its resources for a while, free-riding other contenders' escalation efforts without conceding. To gain insight into these opposite effects, this paper proposes and analyzes a new, stochastic and recursive model for escalation among n ranked players.

In contrast to the extant literature, mostly based on two-rival assumptions, which coupled with complete information would have predicted that war of attrition is detrimental to the rivals, our results demonstrate that conflict escalations with more than two rivals result in surplus-enhancement despite complete information. We demonstrate this first by constructing an m-rivalry equilibrium that, when a large cohort of ranked players are closely positioned, generates a total surplus near the full value of the contested prize. While this equilibrium is not robust to other kinds of positioning, we further demonstrate the thesis by proving existence of other multilateral equilibria that are robust, and Pareto superior to the bilateral equilibrium. These are the trilateral equilibria, where the top three players escalate against one another despite the third-ranked player often having to overcome a wider distance from the front. Such normative advantage of multiplayer rivalry suggests, for instance, that adding a viable third political party to the United States de facto two-party system may help mitigate the more and more acute partisan conflict.

The trilateral equilibria each exhibit three interesting, dynamic features. First, as the trilateral escalation continues, the gap between the frontrunner and the third-ranked rival may collapse or expand, depending on whether the third-ranked rival manages to overtake the frontrunner. Second, the escalation can possibly end only when this gap reaches its maximum that the equilibrium can sustain, at which point it is the third-ranked rival that decides, randomizing between leapfrogging and staying put, whether the escalation shall continue. Third, at the onset of a trilateral rivalry before any of the three rivals lag behind, the ideal position for a player is not to be the frontrunner but rather be the follower, wedged in between the frontrunner and the third-ranked rival.

Our dynamic model of war-of-attrition-type escalation, with its stochastic yet tractable recursive structure and generality in the number of rivals, is conducive to further investigations of multiplayer attritional dynamics. First, it would be interesting to explore robust equilibria involving more than three rivals. One could start by constructing m-rivalry equilibria based on a trilateral equilibrium: Instead of restricting active rivals to the top three players, let any k^{th}-ranked player escalate if his gap from the frontrunner is sufficiently narrow; as the escalation continues, the gaps of those who fail to leapfrog eventually become too wide for them to participate any more, hence the set of active players eventually reduces to the top three, who then play the trilateral equilibrium thereafter. We conjecture that such equilibria, like the trilateral ones, Pareto dominate the bilateral equilibrium. An open question is how they compare to the trilateral ones in terms of the players' welfare.

Endnotes

1. There was a minimum requirement for the winning submission to outperform Netflix own Cinematch algorithm by at least 10%; see: http://www.netflixprize.com. In May 2017, real estate valuation firm Zillow initiated a similar challenge, focusing on home sales price predictions; https://www.kaggle.com/c/zillow-prize-1. Accessed: 2018-10-27.

2. In online penny auctions bidders pay a fee to nominally increment the price (e.g. 70 cents to increase the price with a penny), and the bidder who places the last bid wins the auction and pays, in addition to the sunk bidding cost, the final auction price. Different from a dynamic war-of-attrition-type escalation, in online penny auctions regardless of previous bidding activity all contenders only have to pay the minimal incremental cost to assume leadership. Furthermore, the winner has to pay, in addition to the sunk bidding cost, the final ending price of the good.

3. Leapfrogging phenomena and attritional wars is also displayed within US polical parties. For instance, during the US presidential primary elections, where prospective candidates tend not to spend their resources simultaneously to stay on the game but rather strategize on the timing of spending—to figure out, along the sequence of state-wise primary elections, whether to spend big on the state that comes next in the sequence. Such political candidates, ranked by the polls, are often seen leapfrogging.

4. Note, similar to the analysis in Section 2.3, discounting is not required to ensure convergence of the dynamic program. Therefore, as we are implicitly considering relative short time-intervals between escalations, and without loss of generality, we omit discounting.

5. In the following, we extend the summation notation by defining, for any sequence $(a_k)_{k=1}^{\infty}$, $i > j \Longrightarrow \sum_{k=i}^{j} a_k := 0$. In particular, $\sum_{k=1}^{0} a_k = 0$ according to this notation.

6. Another interesting extension to consider is to relax the tie-breaking rule. For instance, rather than ties broken randomly with equal probability, one may consider scenarios where in each round, all bidders who simultaneously bid incur the sunk cost and all become co-frontrunners; with the game ending once no more bids are submitted, but the prize awarded only if the frontrunner is unique. A discussion for the bilateral and trilateral rivalry is provided in Ødegaard and Zheng (2018).

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Proofs of Formal Results

EC.1. Lemmas EC.1, EC.2, and EC.3

LEMMA EC.1. *At any trilateral equilibrium with dropout state s_*, if $s \geq s_*$ then $V_s = 2\delta$ and $M_s = L_s = 0$.*

Proof Lemma EC.1 Take any trilateral equilibrium, with dropout state s_* and value functions V_s, M_s and L_s. By Lemma 1.i, at any state $s \geq s_*$ the player who is the current underdog stays put for all future rounds. This, combined with the property of trilateral equilibrium that none of the players ranked behind the underdog would ever participate in escalation and the "independence of nonparticipants" condition, implies that the subgame equilibrium thereafter is the bilateral equilibrium characterized in Section 2.3. Thus, $V_s = V_* = 2\delta$ and $M_s = M_* = 0$. Since $s \geq s_*$ implies $L_s = 0$ (Lemma 1), the lemma is proved. $\square$

LEMMA EC.2. *At any trilateral equilibrium with dropout state $s_* \geq 4$, $V_3 - 2\delta \geq M_2 \geq L_2 > 0$.*

Proof Lemma EC.2 Suppose that $V_3 - 2\delta < L_2$. Then, by the fact $\pi_{\gamma,2} = 1$ (Lemma 1.ii and $s_* \geq 4$) and Eq. (1), the β player at state $s = 2$ would rather stay put, to get L_2, than escalate to get $V_3 - 2\delta$. Thus $\pi_{\beta,2} = 0$. This, combined with (2) in the case $s = 2$ and the fact $\pi_{\gamma,2} = 1$, implies that $V_2 = M_2$. Since $V_3 - 2\delta < L_2$ coupled with (3) implies $M_2 \leq L_2$, we have a contradiction $V_2 \leq L_2 < V_2$, with the last inequality due to (4). Thus we have proved $V_3 - 2\delta \geq L_2$. Therefore, with M_2 a convex combination between $V_3 - 2\delta$ and L_2 (since $\pi_{\gamma,2} = 1$), $V_3 - 2\delta \geq M_2 \geq L_2$. Finally, to show $L_2 > 0$, note from the hypothesis $s_* \geq 4$ and definition of s_* that $V_2 - 3\delta > 0$. This positive payoff the underdog at state $s = 2$ can secure with a positive probability through escalating. Hence $L_2 > 0$ follows from (4). $\square$

LEMMA EC.3. *At any trilateral equilibrium with dropout state $s_* \geq 4$, $\pi_{\gamma,s_*-1} > 0$.*

Proof Lemma EC.3 Suppose, to the contrary, that $\pi_{\gamma,s_*-1} = 0$ at equilibrium. Then $M_{s_*-1} = 0$ according to (3), with $s = s_* - 1$, and the fact $V_{s_*} - 2\delta = 0$ by Lemma EC.1. Consequently, (2) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.i), implies that $V_{s_*-2} \leq M_2$, which in turn implies, by (3) in the case $s = s_* - 3$ and the fact $\pi_{\gamma,s_*-3} = 1$, that $M_{s_*-3} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$, with the last inequality due to Lemma EC.2. That in turn implies $V_{s_*-4} \leq M_2$ by (2) and the fact $\pi_{\gamma,s_*-4} = 1$. Thus $V_{s_*-2} \leq M_2$, $V_{s_*-4} \leq M_2$ and $M_{s_*-3} \leq M_2$.

The supposition $\pi_{\gamma,s_*-1} = 0$, coupled with the fact that $\pi_{\gamma,s} = 0$ at all $s > s_* - 1$ (Lemma 1.ii), also implies that γ drops out of the race starting from the state $s_* - 1$. Thus, by the "independence of nonparticipants" condition, $V_{s_*-1} = V_{s_*}$, hence Lemma EC.1 implies $V_{s_*-1} = 2\delta$. Then (3) applied to the case $s = s_* - 2$, coupled with the fact $\pi_{\gamma,s_*-2} = 1$, implies $M_{s_*-2} \leq L_2$. Thus, by (2) and the fact $\pi_{\gamma,s_*-3} = 1$, we have $V_{s_*-3} \leq \max\{L_2, M_2\} \leq M_2$, the last inequality again due to Lemma EC.2. With $V_{s_*-3} \leq M_2$, (3) implies $M_{s_*-4} \leq \max\{M_2 - 2\delta, L_2\} \leq M_2$. Thus $V_{s_*-1} = 2\delta$, $V_{s_*-3} \leq M_2$, $M_{s_*-2} \leq L_2 \leq M_2$ and $M_{s_*-4} \leq M_2$.

Repeat the above reasoning on (2) and (3) for smaller and smaller s and we obtain the fact that $V_{s_*-1} = 2\delta$, $V_s \leq M_2$ and $M_s \leq M_2$ for all $s \in \{2, \dots, s_* - 2\}$. Thus, either $V_3 = 2\delta$ (when $s_* = 4$) or $V_3 \leq M_2$ (when $s_* \geq 5$). Either case contradicts Lemma EC.2. □

EC.2. Lemmas EC.4, EC.5, EC.6, and EC.7

Lemma EC.4. *At any robust trilateral equilibrium with dropout state s_* being an even number and $s_* \geq 4$, at any state $s \in \{2, \dots, s_* - 2\}$ the β player escalates for sure.*

Lemma EC.4 follows from Lemmas EC.6 and EC.7, established in this section. The former shows that escalation is a follower's unique best response at even-number states; the latter shows that for odd-number states. Lemma 3 is proved in Subsection EC.2.2 here. We start with—

Lemma EC.5. *At any equilibrium with dropout state an even number $s_* \geq 4$, $L_2 < V_2 \leq V_3 - 2\delta$.*

Proof Lemma EC.5 Since $\pi_{\beta,s_*-1} = 0$ (Lemma 2), $M_{s_*-1} \leq L_2$. Thus, since $\pi_{\gamma,s_*-2} = 1$ (Lemma 1.ii), V_{s_*-2} is a convex combination between M_{s_*-1}, which is less than L_2, and M_2, which is a convex combination between $V_3 - 2\delta$ and L_2, as $\pi_{\gamma,2} = 1$. Thus V_{s_*-2} is between L_2 and $V_3 - 2\delta$. Consequently, M_{s_*-3}, a convex combination between L_2 and $V_{s_*-2} - 2\delta$ (since $\pi_{\gamma,s_*-3} = 1$), is between L_2 and $V_3 - 2\delta$. Repeating this reasoning, with s_* being an even number, we eventually reach $2 = s_* - 2m$ for some integer $m \geq 1$, and obtain the fact that V_2 is a number between L_2 and $V_3 - 2\delta$. Thus, $L_2 < V_3 - 2\delta$, otherwise the fact $L_2 < V_2$ by (4) would be contradicted. Hence $L_2 < V_2 \leq V_3 - 2\delta$. □

EC.2.1. Escalation at Even States

Lemma EC.6. *At any equilibrium with any even number dropout state $s_* \geq 4$, $\pi_{\beta,s} = 1$ if $2 \leq s \leq s_* - 2$ such that s is an even number.*

Proof Lemma EC.6 First, by Lemma EC.5, $L_2 < V_3 - 2\delta$. Thus at state $s = 2$ the β player strictly prefers to escalate, i.e., $\pi_{\beta,2} = 1$. Second, pick any even number s such that $4 \leq s \leq s_* - 2$ and suppose, to the contrary of the lemma, that $\pi_{\beta,s} < 1$, which means that the β player at state s does not strictly prefer to escalate. Thus $M_s \leq L_2$ (as the transition $M_s \to 0$ is ruled out by the fact $\pi_{\gamma,s} = 1$). Consequently, V_{s-1}, a convex combination between M_s and M_2, is weakly less than M_2, as $L_2 \leq M_2$ by Lemma EC.2. Furthermore, M_{s-2}, a convex combination between $V_{s-1} - 2\delta$ and L_2, is less than M_2, and that in turns implies $V_{s-3} \leq M_2$. Repeating this reasoning, with s an even number, we eventually obtain the conclusion that $V_3 \leq M_2$, which contradicts Lemma EC.2. Thus, $\pi_{\beta,s} = 1$. $\square$

At any equilibrium with any even number dropout state $s_* \geq 4$, since $\pi_{\gamma,s} = 1$ for all $s \leq s_* - 2$ (Lemma 1.ii), Eq. (1) and the equal-probability tie-breaking rule together imply

$$\forall s \in \{2, 3, 4, \ldots, s_* - 2\} : [\pi_{\beta,s} = 1 \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2] . \tag{EC.1}$$

By Lemma EC.6,

$$2 \leq s \leq s_* - 2 \text{ and } s \text{ is even} \Longrightarrow q_{\beta,s} = q_{\gamma,s} = 1/2. \tag{EC.2}$$

EC.2.2. Escalation at Odd States

LEMMA EC.7. *At any equilibrium with any even number dropout state $s_* \geq 4$ and at any state $1 \leq s \leq s_* - 2$ such that s is an odd number, $\pi_{\beta,s} = 1$.*

Proof Lemma EC.7 Pick any odd number s such that $s \leq s_* - 2$. It suffices to prove that $V_{s+1} - 2\delta > L_2$. Since $s+1$ is even, it follows from (EC.2) that

$$V_{s+1} = \frac{1}{2}\left(M_2 + M_{s+2}\right) \geq \frac{1}{2}\left(M_2 + L_2\right),$$

with the inequality due to the fact $M_{s+2} \geq L_2$, which in turn is due to the fact that the β player at state $s+2$ can always secure the payoff L_2 through not escalating at all. Thus,

$$\begin{aligned}
V_{s+1} - 2\delta - L_2 &\geq \frac{1}{2}\left(M_2 + L_2\right) - 2\delta - L_2 \\
&= \frac{1}{2}M_2 - \frac{1}{2}L_2 - 2\delta \\
&= \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}L_3 + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
&\geq \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{1}{2}(V_2 - 4\delta) + \frac{1}{2}(V_2 - 3\delta)\right) - 2\delta \\
&= \frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta,
\end{aligned}$$

with the second inequality due to the definition of L_s and the fact $V_2 - 4\delta \geq 0$ ($s_* \geq 4$). Since $\frac{1}{2}M_2 - \frac{1}{2}V_2 - \frac{1}{4}\delta > 0$ by Lemma 3, $V_{s+1} - 2\delta - L_2 > 0$, as desired. $\square$

EC.3. Lemmas EC.8, EC.9, EC.10, and EC.11

LEMMA EC.8. *For any even number $s_* \geq 4$ and any strategy profile satisfying (A) and (B), escalation is a best response for the β player at state $s \in \{2, 3, \ldots, s_* - 2\}$ if either (i) s is even and $V_3 - 2\delta \geq L_2$, or (ii) s is odd and $V_{s_*-2} - 2\delta \geq L_2$.*

All lemmas in this subsection assume the hypotheses in Lemma EC.8, that $s_* \geq 4$ is an even number and a strategy profile $(\pi_{\beta,s}, \pi_{\gamma,s})_{s=2}^{\infty}$ satisfying Provision (A) is given, with the associated value functions $(V_s, M_s, L_s)_s$ derived from (2)–(4) and Eq. (15).

LEMMA EC.9. *For any positive integer m such that $2m+1 \leq s_* - 1$, if $V_{2m+1} - 2\delta \leq L_2$ then $V_3 - 2\delta < L_2$.*

Proof Lemma EC.9 Pick any m specified by the hypothesis such that $V_{2m+1} - 2\delta \leq L_2$. Suppose, to the contrary of the lemma, that $V_3 - 2\delta \geq L_2$. Thus, the law of motion (2) in the case $s = 2$, with $\pi_{\gamma,2} = 1$, implies that M_2 is between L_2 and $V_3 - 2\delta$, hence $V_3 - 2\delta \geq M_2 \geq L_2$. By the law of motion (3) in the case $s = 2m$, M_{2m} is a convex combination among zero, $V_{2m+1} - 2\delta$ and L_2. Thus the hypothesis implies that $M_{2m} \leq L_2$. Consequently, the law of motion (2) in the case $s = 2m-1$, together with $\pi_{\gamma,2m-1} = 1$ and $M_2 \geq L_2$, implies that $V_{2m-1} \leq M_2$ and hence $V_{2m-1} - 2\delta \leq M_2 - 2\delta$. Then (3) in the case $s = 2m-2$ implies $M_{2m-2} \leq L_2$. Repeating this reasoning backward, with 3 being odd, we eventually reach state $s = 3$ and obtain $V_3 \leq M_2$. But since $V_3 - 2\delta \geq M_2$, we have a contradiction $V_3 - 2\delta \geq M_2 \geq V_3$. □

LEMMA EC.10. *Denote $\pi := \pi_{\gamma,s_*-1}$. For any integer m such that $1 \leq m \leq s_*/2 - 1$,*

$$M_{s_*-(2m-1)} = -\delta \sum_{k=1}^{m-1} 2^{-2k+2} + M_2 \sum_{k=1}^{m-1} 2^{-2k} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}\pi \right), \tag{EC.3}$$

$$V_{s_*-2m} = -\delta \sum_{k=1}^{m-1} 2^{-2k+1} + M_2 \sum_{k=1}^{m} 2^{-2k+1} + L_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1}\pi \right), \tag{EC.4}$$

$$\begin{aligned} V_{s_*-(2m-1)} = &-\delta \sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}(1-\pi)v + L_2 \sum_{k=1}^{m-1} 2^{-2k} \\ &+ M_2 \left(\sum_{k=1}^{m-1} 2^{-2k+1} + 2^{-2(m-1)}\pi \right), \end{aligned} \tag{EC.5}$$

$$\begin{aligned} M_{s_*-2m} = &-\delta \sum_{k=0}^{m-1} 2^{-2k} + 2^{-2m+1}(1-\pi)v + L_2 \sum_{k=1}^{m} 2^{-2k+1} \\ &+ M_2 \left(\sum_{k=1}^{m-1} 2^{-2k} + 2^{-2m+1}\pi \right), \end{aligned} \tag{EC.6}$$

$$L_2 = \delta \left(s_* - 4 + 2^{-s_*+3} \right). \tag{EC.7}$$

Proof Lemma EC.10 First, we prove Eqs. (EC.3) and (EC.4). When $m = 1$, Eq. (EC.3), coupled with the summation notation defined in (**??**), becomes $M_{s_*-1} = \pi L_2 = \pi_{\gamma,s_*-1} L_2$, which follows from (3) and the fact that $V_s = 2\delta$ and $M_s = 0$ for all $s \geq s_*$, due to Provision (A). This coupled with Eq. (15) implies that

$$V_{s_*-2} = (M_{s_*-1} + M_2)/2 = M_2/2 + \pi L_2/2,$$

which is Eq. (EC.4) when $m=1$ (using again the summation notation in (**??**)). Suppose, for any integer m' with $1 \le m' \le s_*/2-2$, that Eqs. (EC.3) and (EC.4) are true with $m=m'$. By the induction hypothesis of (EC.4) and Eq. (15),

$$\begin{aligned} M_{s_*-(2m'+1)} &= \frac{1}{2}\left(V_{s_*-2m'}-2\delta+L_2\right) \\ &= -\delta\left(1+\frac{1}{2}\sum_{k=1}^{m'-1}2^{-2k+1}\right)+\frac{M_2}{2}\sum_{k=1}^{m'}2^{-2k+1}+\frac{L_2}{2}\left(1+\sum_{k=1}^{m'-1}2^{-2k}+2^{-2m'+1}\pi\right), \end{aligned}$$

which is Eq. (EC.3) when $m=m'+1$. By the above calculation of $M_{s_*-(2m'+1)}$ and Eq. (15),

$$\begin{aligned} V_{s_*-(2m'+2)} &= \frac{1}{2}\left(M_{s_*-(2m'+1)}+M_2\right) \\ &= -\frac{\delta}{2}\left(1+\frac{1}{2}\sum_{k=1}^{m'-1}2^{-2k+1}\right)+\frac{M_2}{2}\left(1+\sum_{k=1}^{m'}2^{-2k+1}\right) \\ &\quad +\frac{L_2}{4}\left(1+\sum_{k=1}^{m'-1}2^{-2k}+2^{-2m'+1}\pi\right), \end{aligned}$$

which is Eq. (EC.4) in the case $m=m'+1$. Thus Eqs. (EC.3) and (EC.4) are proved.

Next we prove Eqs. (EC.5) and (EC.6). When $m=1$, Eq. (EC.5), coupled with the notation $\sum_{k=1}^{0}a_k=0$, becomes $V_{s_*-1}=(1-\pi)v+\pi M_2$, which is true by definition of π and the fact $\pi_{\beta,s_*-1}=0$ (Provision (A)). Then by Eq. (15)

$$M_{s_*-2}=\left(V_{s_*-1}-2\delta+L_2\right)/2=\left((1-\pi)v+\pi M_2-2\delta+L_2\right)/2,$$

which is Eq. (EC.6) when $m=1$ (again using the notation $\sum_{k=1}^{0}a_k=0$). Suppose, for any integer m' with $1\le m'\le s_*/2-2$, that Eqs. (EC.5) and (EC.6) are true with $m=m'$. By the induction hypothesis and Eq. (15),

$$\begin{aligned} V_{s_*-(2m'+1)} &= \frac{1}{2}\left(M_{s_*-2m'}+M_2\right) \\ &= -\frac{\delta}{2}\sum_{k=0}^{m'-1}2^{-2k}+2^{-1}2^{-2m'+1}(1-\pi)v+\frac{L_2}{2}\sum_{k=1}^{m'}2^{-2k+1} \\ &\quad +M_2\left(2^{-1}+2^{-1}\sum_{k=1}^{m'-1}2^{-2k}+2^{-1}2^{-2m'+1}\pi\right), \end{aligned}$$

which is Eq. (EC.5) in the case $m=m'+1$. By the above calculation and Eq. (15),

$$\begin{aligned} M_{s_*-(2m'+2)} &= \frac{1}{2}\left(V_{s_*-(2m'+1)}-2\delta+L_2\right) \\ &= -\delta\left(1+\frac{1}{2}\sum_{k=1}^{m'}2^{-2k+1}\right)+2^{-1}2^{-2m'}(1-\pi)v \\ &\quad +L_2\left(\frac{1}{2}+2^{-1}\sum_{k=1}^{m'}2^{-2k}\right)+\frac{M_2}{2}\left(\sum_{k=1}^{m'}2^{-2k+1}+2^{-2m'}\pi\right), \end{aligned}$$

which is Eq. (EC.6) in the case $m = m' + 1$. Hence Eqs. (EC.5) and (EC.6) are proved.

Finally we prove Eq. (EC.7). Applying Eq. (15) to (4) recursively we obtain, for any integer $s_* \geq 4$, that

$$\begin{aligned} L_2 &= \frac{1}{2}\left(V_2 - 3\delta + \frac{1}{2}\left(V_2 - 4\delta + \frac{1}{2}\left(\cdots + \frac{1}{2}\left(V_2 - (s_* - 1)\delta\right)\right)\right)\right) \\ &= \frac{\delta}{2}\left(s_* - 3 + \frac{1}{2}\left(s_* - 4 + \frac{1}{2}\left(\cdots + \frac{1}{2}\cdot 1\right)\right)\right) \\ &= \delta\left(\frac{1}{2}(s_* - 3) + \frac{1}{2^2}(s_* - 4) + \frac{1}{2^3}(s_* - 5) + \cdots + \frac{1}{2^{s_*-3}}\right), \end{aligned}$$

which is equal to the right-hand side of (EC.7). In the above multiline calculation, the first and second lines are due to $V_2 = s_*\delta$ (Lemma 2.ii). □

LEMMA EC.11. $V_{s_*-2} - 2\delta \geq L_2 \Longrightarrow \forall m \in \{1, \ldots, s_*/2 - 1\} : V_{s_*-2m} - 2\delta \geq L_2$.

Proof Lemma EC.11 By the law of motion and Eq. (15), Eqs. (EC.3), (EC.4), (EC.5), (EC.6) and (EC.7) hold. Denote

$$\begin{aligned} \mu(m) &:= 2^{-2m+1}, \\ \mu_* &:= 2^{-s_*+3}. \end{aligned}$$

With the fact $\sum_{k=1}^{m-1} 2^{-2k} = (1 - 2^{-2m+2})/3$, Eq. (EC.4) becomes

$$V_{s_*-2m} = -\delta \cdot \frac{2}{3}(1 - 2\mu(m)) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) + L_2\left(\frac{1}{3}(1 - 2\mu(m)) + \mu(m)\pi\right).$$

Hence

$$\begin{aligned} V_{s_*-2m} - 2\delta - L_2 &= -\delta\left(\frac{2}{3}(1 - 2\mu(m)) + 2\right) + M_2\left(\frac{2}{3}(1 - 2\mu(m)) + \mu(m)\right) \\ &\quad - L_2\left(1 - \frac{1}{3}(1 - 2\mu(m)) - \mu(m)\pi\right) \\ &= -\frac{4}{3}(2 - \mu(m))\delta + \frac{1}{3}(2 - \mu(m))M_2 \\ &\quad - (s_* - 4 + \mu_*)\,\delta\left(\frac{2}{3}(1 + \mu(m)) - \mu(m)\pi\right), \end{aligned}$$

with the second equality due to (EC.7). Thus, $V_{s_*-2m} - 2\delta \geq L_2$ is equivalent to

$$\frac{1}{3}(2 - \mu(m))M_2 \geq \delta\left(\frac{4}{3}(2 - \mu(m)) + (s_* - 4 + \mu_*)\left(\frac{2}{3}(1 + \mu(m)) - \mu(m)\pi\right)\right),$$

i.e.,

$$\frac{M_2}{\delta} \geq 4 + \frac{2(1 + \mu(m)) - 3\mu(m)\pi}{2 - \mu(m)}(s_* - 4 + \mu_*). \tag{EC.8}$$

Since $s_* - 4 \geq 0$ by hypothesis, and

$$\begin{aligned}\frac{d}{d\mu(m)}\left(\frac{2(1+\mu(m)) - 3\mu(m)\pi}{2-\mu(m)}\right) &= \frac{(2-\mu(m))(2-3\pi) + 2(1+\mu(m)) - 3\mu(m)\pi}{(2-\mu(m))^2} \\ &= \frac{6(1-\pi)}{(2-\mu(m))^2} \geq 0,\end{aligned}$$

the right-hand side of (EC.8) is weakly increasing in $\mu(m)$, which in turn is strictly decreasing in m. Thus the right-hand side of (EC.8) is weakly decreasing in m. Consequently, $V_{s_*-2m} - 2\delta - L_2 \geq 0$ is satisfied for all m if the inequality holds at the minimum $m = 1$, i.e., if $V_{s_*-2} - 2\delta - L_2 \geq 0$, as claimed. □

Proof of Lemma EC.8 Let $s \in \{1, 2, \ldots, s_* - 2\}$. If s is even and $V_3 - 2\delta \geq L_2$, then Lemma EC.9 implies $V_{s+1} - 2\delta > L_2$; thus, by (3) and by the fact that $\pi_{\gamma,s} = 1$ due to Provision (A), the β player at s gets L_2 if he does not escalate, and $\frac{1}{2}(V_{s+1} - 2\delta) + \frac{1}{2}L_2$ if he does. Hence escalating is the unique best response for β at s. If s is odd and $V_{s_*-2} - 2\delta \geq L_2$, then Lemma EC.11 implies that $V_{s+1} - 2\delta \geq L_2$; thus, by the same token as in the previous case, the β player at s weakly prefers to escalate. □

EC.4. Lemmas EC.12, EC.13, EC.14, EC.15, and EC.16

Lemma EC.12. *Starting from any state x, the game thereupon admits a robust trilateral equilibrium with dropout state s_* if and only if all the following conditions hold: (a) s_* is an even number greater than two; (b) $\hat{x}_{(3)} \leq s_* - 1 \leq \hat{x}_{(4)}$; and (c) one of the following holds:*

i. either $s_* \leq 6$ and the equation

$$\begin{aligned}&\frac{3\mu_* v}{\delta}(1-\pi)(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) \\ &= (2(1+\mu_*) - 3\mu_*\pi)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1 - 2\mu_* + 3\mu_*\pi)\right),\end{aligned} \tag{EC.9}$$

where $\mu_* := 2^{-s_*+3}$, admits a solution for $\pi \in [0,1]$;

ii. or $s_* \geq 8$ and Eq. (EC.9) admits a solution for $\pi \in [0,1]$ such that

$$\pi \geq 1 - \frac{3(2-\mu_*)}{2(1-2\mu_*)(s_* - 4 + \mu_*)}. \tag{EC.10}$$

The proof consists of the following lemmas. The solution π to equation (EC.9) corresponds to the γ player's escalation probability π_{γ,s_*-1} in the critical state $s_* - 1$. The bifurcation in Lemma EC.12.c is due to an implication of the following lemmas that at the solution for $V_2 = s_*\delta$, neither $V_3 - 2\delta \geq L_2$ nor $V_{s_*-2} - 2\delta \geq L_2$ are binding when $s_* \leq 6$, and only one of the inequalities is binding when $s_* \geq 8$.

LEMMA EC.13. *Any integer $s_* \geq 3$ constitutes a robust trilateral equilibrium if and only if s_* is an even number and there exists $(M_2, \pi, L_2) \in \mathbb{R}^3_+$ satisfying the following conditions:*

a. $(M_2, \pi, L_2) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}_+$ and it solves simultaneously Eq. (EC.4) such that $m = s_*/2 - 1$, Eq. (EC.6) such that $m = s_*/2 - 1$ and $V_2 = s_*\delta$, and Eq. (EC.7);

b. $M_2 \geq s_*\delta$;

c. Ineq. (EC.8) is satisfied in the case $m = 1$;

d. $\hat{x}_{(3)} \leq s_* - 1 \leq \hat{x}_{(4)}$.

Proof of Lemma EC.13 The necessity that s_* is even for an equilibrium follows from Theorem 1. In Condition (a), the necessity of $V_2 = s_*\delta$ follows from Lemma 2, and the rest from Lemma EC.10, which in turn follows from Provision (A), necessary due to Lemmas 1–EC.4. With $V_2 = s_*\delta$, the condition $M_2 \geq s_*\delta$ is equivalent to $M_2 \geq V_2$; hence the necessity of Condition (b) follows from Lemma 3. The necessity of Condition (c) follows from Lemma EC.4, which implies the necessity of $V_{s_*-2m} - 2\delta \geq L_2$, which as shown in the proof of Lemma EC.11 requires Ineq. (EC.8). The necessity of Condition (d) has been explained immediately before Ineqs. (13) and (14).

To prove that these conditions together suffice an equilibrium, pick any even number $s_* \geq 4$ and assume Conditions (a)–(d). Consider the strategy profile according to provisions (A) and (B), Section 3.3. This strategy profile implies Eq. (15), which allows calculation of the value functions $(V_s, M_s, L_s)_{s=2}^{s_*}$ via the law of motions. The incentive for each player to abide by the strategy profile at any state $s \geq s_*$ is the same as in the bilateral equilibrium. At the state $s_* - 1$, escalating with probability π is a best response for the γ player because he is indifferent about escalation, since $V_2 - s_*\delta = 0 = L_{s_*}$, and staying put is the best response for the β player because $V_{s_*} - 2\delta = 0 < L_2$. At any state s with $2 \leq s \leq s_* - 2$, escalating is the best response for the γ player because $V_2 - (s+1)\delta > L_{s+1}$ (by Eq. (4)); Condition (c) by Lemma EC.11 suffices the incentive for the β player at every odd state to escalate. To incentivize the β player at every even state $s \leq s_* - 2$ to escalate, Lemma EC.9 says that it suffices to have $V_3 - 2\delta \geq L_2$, which is equivalent to $M_2 \geq L_2$ since, by the law of motion and Eq. (15), M_2 is the midpoint between $V_3 - 2\delta$ and L_2. Since $L_2 < s_*\delta$ by Eq. (EC.7), the condition $M_2 \geq L_2$ is guaranteed by Condition (b), $M_2 \geq s_*\delta$. Finally, by the explanation immediately after Ineq, (14), Condition (d) ensures that any player ranked behind! γ has no incentive to deviate to escalate. $\square$

LEMMA EC.14. *For any $s_* \geq 4$, Condition (c) in Lemma EC.13 implies Condition (b) in Lemma EC.13.*

Proof of Lemma EC.14 Condition (c) in Lemma EC.13 is Ineq. (EC.8) in the case $m = 1$, i.e., when $\mu(m) = 2^{-2m+1} = 1/2$. Hence the condition is equivalent to

$$\frac{M_2}{\delta} \geq 4 + (2 - \pi)(s_* - 4 + \mu_*). \tag{EC.11}$$

To prove that this inequality implies Condition (b), i.e., $M_2/\delta \geq s_*$, it suffices to show

$$4+(2-\pi)(s_*-4+\mu_*)>s_*,$$

i.e.,

$$(1-\pi)(s_*-4)+\mu_*(2-\pi)>0,$$

which is true because $s_* \geq 4$, $\mu_* = 2^{-s_*+3} > 0$ and $\pi \leq 1$. □

Lemma EC.15. *Condition (a) in Lemma EC.13 is equivalent to the existence of an $\pi \in [0,1]$ that solves Eq. (EC.9).*

Proof of Lemma EC.15 Condition (a) requires existence of $(M_2,\pi,L_2)\in\mathbb{R}_+\times[0,1]\times\mathbb{R}_+$ that satisfies Eqs. (EC.4), (EC.6) and (EC.7) in the case of $m=s_*/2-1$ and $V_2=s_*\delta$. Combine (EC.4) with (EC.7) and use the notation $\mu_*:=2^{-s_*+3}$ and the fact $\sum_{k=1}^{m-1}2^{-2k}=(1-2^{-2m+2})/3$ to obtain

$$s_*\delta=V_2=-\delta\cdot\frac{2}{3}(1-2\mu_*)+M_2\left(\frac{2}{3}(1-2\mu_*)+\mu_*\right)+\underbrace{\delta(s_*-4+\mu_*)}_{L_2}\left(\frac{1}{3}(1-2\mu_*)+\mu_*\pi\right),$$

i.e.,

$$\frac{M_2}{\delta}=\frac{1}{2-\mu_*}\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_*\pi)\right). \tag{EC.12}$$

By the same token, (EC.6) coupled with (EC.7) is equivalent to

$$M_2\left(1-\frac{1}{3}(1-2\mu_*)-\mu_*\pi\right)=-\delta\left(1+\frac{1}{3}(1-2\mu_*)\right)+(1-\pi)\mu_*v+\delta(s_*-4+\mu_*)\left(\frac{2}{3}(1-2\mu_*)+\mu_*\right),$$

i.e.,

$$\frac{M_2}{\delta}\left(2(1+\mu_*)-3\mu_*\pi\right)=\frac{3\mu_*v}{\delta}(1-\pi)+(2-\mu_*)(s_*-6+\mu_*). \tag{EC.13}$$

Plug (EC.12) into (EC.13) and we obtain Eq. (EC.9). □

Lemma EC.16. *For any even number $s_*\geq 4$, suppose that Eq. (EC.12) holds. Then Condition (c) in Lemma EC.13 is equivalent to Ineq. (EC.10), which is implied by $\pi\geq 0$ if and only if $s_*\leq 6$.*

Proof of Lemma EC.16 Condition (c) in Lemma EC.13 has been shown to be equivalent to Ineq. (EC.11). Provided that Eq. (EC.12) is satisfied, Ineq. (EC.11) is equivalent to

$$4+(2-\pi)(s_*-4+\mu_*)\leq\frac{1}{2-\mu_*}\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_*\pi)\right).$$

This inequality, given the fact $1-2\mu_*\geq 0$, is equivalent to

$$\pi\geq\frac{1}{2(1-2\mu_*)}\left(5-4\mu_*-\frac{3(s_*-2)}{s_*-4+\mu_*}\right),$$

i.e., Ineq (EC.10). Given Condition (a) in Lemma EC.13, which implies $\pi \geq 0$, Ineq. (EC.10) is redundant if and only if the right-hand side of (EC.10) is nonpositive, i.e.,

$$\frac{3(2-\mu_*)}{2(1-2\mu_*)(s_*-4+\mu_*)} \geq 1,$$

i.e.,

$$s_* \leq 4 - 2^{-s_*+3} + \frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)}.$$

This inequality is satisfied when $s_* \in \{4,6\}$, as its right-hand side is equal to ∞ when $s_*=4$, and $61/8$ when $s_*=6$. The inequality does not hold, by contrast, when $s_* \geq 8$, as

$$\begin{aligned} s_* \geq 8 \Rightarrow 2^{-s_*+2} \leq 2^{-6} \;\Rightarrow\; &\frac{1-2^{-s_*+2}}{1/4-2^{-s_*+2}} \leq \frac{1-2^{-6}}{1/4-2^{-6}} = \frac{63}{15} \\ \Rightarrow 4 - 2^{-s_*+3} + &\frac{3\left(2-2^{-s_*+3}\right)}{2\left(1-2^{-s_*+4}\right)} < 4 + \frac{3}{2}\cdot\frac{2}{4}\cdot\frac{63}{15} < 8 \leq s_*. \end{aligned}$$

Thus, for all even numbers $s_* \geq 4$, Ineq. (EC.11) follows if and only if $s_* \leq 6$. □

Proof of Lemma EC.12 The theorem follows from Lemma EC.13, where Condition (a) has been characterized by Lemma EC.15, Condition (b) by Lemmas EC.14 can be dispensed with, and Condition (c), by Lemma EC.16, can be dispensed with when $s_* \leq 6$ (hence Claim (i) of the theorem) and is equivalent to Ineq (EC.10) when $s_* > 6$ (hence Claim (ii) of the theorem). Finally, Condition (d) in Lemma EC.13 is the same as Condition (b) in Lemma EC.12. □

EC.5. Lemmas EC.17 and EC.18

LEMMA EC.17. *If $\pi = 1$, the left-hand side of (EC.9) is less than the right-hand side of (EC.9).*

Proof of Lemma EC.17 When $\pi = 1$, the left-hand side of (EC.9) is equal to $(2-\mu_*)^2(s_*-6+\mu_*)$, and the right-hand side equal to

$$\begin{aligned} &(2(1+\mu_*)-3\mu_*)\left(3s_*+2(1-2\mu_*)-(s_*-4+\mu_*)(1-2\mu_*+3\mu_*)\right) \\ =\;&(2-\mu_*)\left(2s_*+6-\mu_*-\mu_*s_*-\mu_*^2\right). \end{aligned}$$

Thus, the lemma follows if

$$(2-\mu_*)(s_*-6+\mu_*) < 2s_*+6-\mu_*-\mu_*s_*-\mu_*^2,$$

i.e., $9\mu_* < 18$, which is true because $\mu_* = 2^{-s_*+3}$. □

LEMMA EC.18. *Starting at state x such that $\hat{x}_{(3)} \leq s_* \leq \hat{x}_{(4)}$, $s_*=4$ constitutes a robust trilateral equilibrium if and only if $v/\delta > 35/2$, and $s_*=6$ constitutes such an equilibrium if and only if $v/\delta > 6801/120$ $(=56.675)$.*

Proof of Lemma EC.18 By Lemma EC.12, given $s_* \le 6$ and $\hat{x}_{(3)} \le s_* \le \hat{x}_{(4)}$, the necessary and sufficient condition for equilibrium is that Eq. (EC.9) admits a solution for $\pi \in [0,1]$. By Lemma EC.17, the left-hand side of that equation is less than its right-hand side when $\pi = 1$. Thus, it suffices to show that the left-hand side is greater than the right-hand side when $\pi = 0$, i.e.,

$$\frac{3\mu_* v}{\delta}(2-\mu_*) + (2-\mu_*)^2(s_* - 6 + \mu_*) > 2(1+\mu_*)\left(3s_* + 2(1-2\mu_*) - (s_* - 4 + \mu_*)(1-2\mu_*)\right),$$

which is equivalent to

$$\frac{v}{\delta}(2-\mu_*) > s_*(4+\mu_*) + (6-\mu_*)(2/\mu_* - 2 - \mu_*).$$

Since μ_* is equal to $1/2$ when $s_* = 4$, and equal to $1/8$ when $s_* = 6$, the above inequality is equivalent to $v/\delta > 35/2$ when $s_* = 4$, and $v/\delta > 6801/120$ when $s_* = 6$. $\square$

EC.6. Verification of the Quadrilateral Equilibrium in Section 4.1

First, from our characterization of trilateral-rivalry equilibria, one can obtain the associated value function for the trilateral-rivalry equilibrium with dropout state $s_* = 4$:

$$V_2 = 4\delta, \tag{EC.14}$$

$$V_3 = (16 + 3/2 - \pi_{\gamma,3})\,\delta, \tag{EC.15}$$

$$M_2 = V_3/2 - 3\delta/4, \tag{EC.16}$$

$$L_2 = \delta/2, \tag{EC.17}$$

$$M_3 = \pi_{\gamma,3} L_2 = \pi_{\gamma,3}\delta/2, \tag{EC.18}$$

$$L_3 = 0. \tag{EC.19}$$

Second, recall the notation W_α, W_β, W_{γ_1} and W_{γ_2}: In the consecutive configuration (16) such that $m = 2$, let W_α denote the continuation value for α, W_β the continuation value for β, W_{γ_1} for γ_1, and W_{γ_2} for γ_2. By Provision (c.i) of the proposed strategy profile, players β, γ_1 and γ_2 each bid for sure, with others staying put and hence omitted. Thus the configuration in the next round is one of the following three, each with probability $1/3$:

$$[\beta, \alpha, \square, \gamma_1, \gamma_2],\ t = 4; \tag{EC.20}$$

$$[\gamma_1, \alpha, \beta, \square, \gamma_2],\ t = 4; \tag{EC.21}$$

$$[\gamma_2, \alpha, \beta, \gamma_1],\ t = 3. \tag{EC.22}$$

If it is (EC.20) or (EC.21), player γ_2 quits and the other three play the trilateral-rivalry equilibrium with $s_* = 4$; if it is (EC.22) then each non-frontrunner bids for sure, as in (16).

Given any consecutive configuration in the of (16), then

$$W_{\gamma_2} = \frac{1}{3}(W_\alpha - 4\delta) \tag{EC.23}$$

because γ_2 quits, thereby getting zero payoff, unless (EC.22) happens. Since (EC.20), (EC.21) and (EC.22) each happen with probability 1/3,

$$\begin{aligned} W_\alpha &= \frac{1}{3}(M_3 + M_2 + W_\beta), && \text{(EC.24)}\\ W_\beta &= \frac{1}{3}\left((V_3 - 2\delta) + L_2 + W_{\gamma_1}\right) \overset{(EC.17)}{=} \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + W_{\gamma_1}\right), && \text{(EC.25)} \end{aligned}$$

and

$$\begin{aligned} W_{\gamma_1} &= \frac{1}{3}(L_3 + (V_2 - 3\delta) + W_{\gamma_2})\\ &= \frac{1}{3}\left(\delta + \frac{1}{3}(W_\alpha - 4\delta)\right) \quad \text{by (EC.14), (EC.19) and (EC.23)}\\ &= \frac{1}{9}\left(-\delta + \frac{1}{3}(M_3 + M_2 + W_\beta)\right) \quad \text{by (EC.24)}\\ &= \frac{1}{27}\left(-3\delta + \frac{\pi_{\gamma,3}}{2}\delta + \left(\frac{1}{2}V_3 - \frac{3}{4}\delta\right) + \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + W_{\gamma_1}\right)\right) \quad \text{by (EC.18), (EC.16) and (EC.25).} \end{aligned}$$

Thus,

$$\begin{aligned} W_{\gamma_1} &= \frac{1}{80}\left(\frac{5}{2}V_3 + \left(-9 + \frac{1}{4}(6\pi_{\gamma,3} - 15)\right)\delta\right)\\ &\overset{(EC.15)}{=} \frac{1}{80}\left(\frac{5}{2}\left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta + \left(-9 + \frac{1}{4}(6\pi_{\gamma,3} - 15)\right)\delta\right)\\ &= \frac{1}{80}(31 - \pi_{\gamma,3})\,\delta. && \text{(EC.26)} \end{aligned}$$

This plugged into Eq. (EC.25) gives

$$\begin{aligned} W_\beta &= \frac{1}{3}\left(V_3 - \frac{3}{2}\delta + \frac{1}{80}(31 - \pi_{\gamma,3})\,\delta\right)\\ &\overset{(EC.15)}{=} \frac{1}{3}\left(\left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta - \frac{3}{2}\delta + \frac{1}{80}(31 - \pi_{\gamma,3})\,\delta\right)\\ &= \frac{1}{3}\left(16 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta. && \text{(EC.27)} \end{aligned}$$

Plugging this into Eq. (EC.24), we have

$$\begin{aligned} W_\alpha &= \frac{1}{3}\left(M_3 + M_2 + \frac{1}{3}\left(16 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right)\\ &\overset{(EC.18),(EC.16)}{=} \frac{1}{3}\left(\frac{\pi_{\gamma,3}}{2}\delta + \left(\frac{1}{2}V_3 - \frac{3}{4}\delta\right) + \frac{1}{3}\left(16 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right)\\ &\overset{(EC.15)}{=} \frac{1}{3}\left(\frac{\pi_{\gamma,3}}{2}\delta + \frac{1}{2}\left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta - \frac{3}{4}\delta + \frac{1}{3}\left(16 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta\right)\\ &= \frac{1}{9}\left(40 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta && \text{(EC.28)} \end{aligned}$$

Then Eq. (EC.23) implies

$$W_{\gamma_2} = \frac{1}{27}\left(4 + \frac{31}{80} - \left(1 + \frac{1}{80}\right)\pi_{\gamma,3}\right)\delta. \tag{EC.29}$$

Recall that $0 < \pi_{\gamma,3} < 1$, which plugged into Eqs. (EC.26), (EC.27), (EC.28) and (EC.29) implies

$$\tfrac{3}{8}\delta < W_{\gamma_1} < \tfrac{32}{80}\delta = \tfrac{2}{5}\delta, \tag{EC.30}$$

$$5\delta = \tfrac{1}{3}\cdot 15\delta < W_\beta < \tfrac{1}{3}\cdot 17\delta < 6\delta, \tag{EC.31}$$

$$4\delta < \tfrac{1}{9}\cdot 39\delta < W_\alpha < \tfrac{1}{9}\cdot 41\delta < 5\delta, \tag{EC.32}$$

$$\tfrac{1}{9}\delta = \tfrac{3}{27}\delta < W_{\gamma_2} < \tfrac{5}{27}\delta. \tag{EC.33}$$

We verify the equilibrium conditions through backward induction. Let us start with any subgame with state $t \geq 4$. Expecting the trilateral-rivalry equilibrium to be played in the subgame (Provision (e)), the current frontrunner, follower and third-place bidder each finds it a best response to abide by it, as verified in our paper. The lowest-place bidder γ_2 cannot profit from the deviation of leapfrogging to the top: being on the top gives him a continuation value equal to $V_2 = 4\delta$ because he and the previous frontrunner and follow form a consecutive trilateral configuration, while the leapfrog costs him a payment at least as large as 5δ. Thus, the proposed strategy profile is an equilibrium in this subgame.

Next consider any subgame with $t = 3$, i.e., the consecutive configuration (16). For player γ_2: if he does not bid then the state becomes $t = 4$ next round, in which he will quit and get zero; whereas if he bids now and becomes the next frontrunner, he gets a payoff $W_\alpha - 4\delta$, which is positive by (EC.32); thus bidding is his best response. For player γ_1: if he becomes the next frontrunner (through bidding), then the next configuration becomes (EC.21), giving him a payoff equal to $V_2 - 3\delta = \delta$; whereas, if γ_1 does not bid, the next round is either (EC.20), giving him a payoff $L_3 = 0$, or (EC.22), giving him a payoff $W_{\gamma_2} < 5\delta/27$ by (EC.33). Thus, bidding is the best response for player γ_1. For player β: if he bids and becomes the next frontrunner, the next configuration is (EC.20), giving him a payoff equal to $V_3 - 2\delta$, which is larger than 14δ by Eq. (EC.15); whereas, if β does not bid, then the next configuration is either (EC.21), giving him a payoff $L_2 = \delta/2$, or (EC.22), giving him a payoff $W_{\gamma_1} < 2\delta/5$ by (EC.30). Thus bidding is the best response for player β. Therefore, the proposed strategy profile constitutes an equilibrium in any subgame that starts with the consecutive configuration (16).

We next consider any subgame with $t = 2$, which means the third round, with configuration in the form

$$\begin{bmatrix} \alpha \\ \beta \\ \{\gamma_1, \gamma_2\} \end{bmatrix}. \tag{EC.34}$$

For player β: if he becomes the next frontrunner, then according to Provision (c.ii) he gets

$$V_3 - 2\delta \overset{(EC.15)}{=} \left(16 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta - 2\delta = \left(14 + \frac{3}{2} - \pi_{\gamma,3}\right)\delta;$$

by contrast, if he does not bid then, since both of the γ players are bidding according to Provision (b), β in the next round will become the third-place bidder in the consecutive configuration and hence gets payoff $W_{\gamma_1} < 2\delta/5$ by (EC.30). Thus bidding is the best response for β. For each of the γ players: if he does not bid then he becomes the fourth-place player in the next round thereby getting only $W_{\gamma_2} < 5\delta/27$ by (EC.33); whereas bidding and becoming the next frontrunner gives him $W_\alpha - 3\delta > \delta$ by (EC.32); thus bidding is the best response for each of γ_1 and γ_2. Hence the proposed strategy profile constitutes an equilibrium in any subgame with $t = 2$.

Next consider any subgame with $t = 1$, which means the second round, at which the configuration is in the form

$$\begin{bmatrix} \alpha \\ \{\gamma_1, \gamma_2, \gamma_3\} \end{bmatrix}. \tag{EC.35}$$

If a γ player does not bid, he in the third round will become one of the two γ players in the configuration (EC.34) and hence his payoff will be equal to

$$\frac{1}{3}\cdot 0 + \frac{1}{3}(W_\alpha - 3\delta) + \frac{1}{3}W_{\gamma_2} \overset{(EC.32),(EC.33)}{<} \frac{1}{3}(5\delta - 3\delta) + \frac{5}{81}\delta = \frac{59}{81}\delta < \delta,$$

where the zero term on the left-hand side is his payoff in the event where the β player in the third round gets to become the next frontrunner thereby starting the subgame equilibrium according to Provision (c.ii), rendering zero expected payoff for both bottom-row players, this γ one of them. By contrast, if this γ player bids and becomes the next frontrunner, then in the third round he will be the α in configuration (EC.34), which in the fourth round will become one of the following, each with probability 1/3:

$$\begin{bmatrix} \beta \\ \alpha \\ \varnothing \\ \{\gamma_1, \gamma_2\} \end{bmatrix}, \quad \begin{bmatrix} \gamma_1 \\ \alpha \\ \beta \\ \gamma_2 \end{bmatrix}, \quad \begin{bmatrix} \gamma_2 \\ \alpha \\ \beta \\ \gamma_1 \end{bmatrix}; \tag{EC.36}$$

thus, when $t = 1$, his expected payoff from bidding and becoming the next frontrunner is

$$\frac{1}{3}M_3 + \frac{2}{3}W_\beta - 2\delta \overset{(EC.31)}{>} \frac{2}{3}\cdot 5\delta - 2\delta = \frac{4}{3}\delta > \delta.$$

Hence bidding is the best response for the player. Thus, the proposed strategy constitutes an equilibrium in any subgame with $t = 1$.

Finally, consider $t = 0$, i.e., the initial round. Consider any bidder i. If i bids and becomes the frontrunner (in the second round), then he will become the β in the configuration (EC.34) in the third round, and then in the fourth round, his position one of the three configurations in (EC.36) occupied by the β there. Thus his payoff from bidding in the initial round and becoming the first frontrunner is equal to

$$-\delta+\frac{1}{3}(V_3-2\delta)+\frac{2}{3}W_{\gamma_1}\overset{(EC.15)}{=}-\delta+\frac{1}{3}\left(\left(16+\frac{3}{2}-\pi_{\gamma,3}\right)\delta-2\delta\right)+\frac{2}{3}W_{\gamma_1}>\frac{23}{6}\delta+\frac{2}{3}\cdot\frac{3}{8}\delta=\frac{49}{12}\delta.$$

If i does not bid in the initial round, he becomes in the second round one of the γ players in the configuration (EC.35); then he either (i) pays 2δ to become the frontrunner in the third round (and become the second-place player in the fourth round to get either M_3 or W_β), or (ii) becomes one of the two γ players in the configuration (EC.34) in the third round. In Case (ii), as shown in the previous step on $t=2$, the best outcome for the bidder is to get $W_\alpha-3\delta$. Since $M_3<\delta/2$ by (EC.18) and $W_\beta-2\delta<4\delta$ by (EC.31), Case (i) renders less than 4δ for him; since $W_\alpha-3\delta<2\delta$ by (EC.32), Case (ii) gives him less than 2δ. Thus, either case in the alternative of not bidding produces less than 4δ, while bidding and becoming the initial frontrunner yields more than 4δ. Thus, bidding is the best response for each player i in the initial round. Therefore, the quadrilateral-rivalry strategy profile is a subgame perfect equilibrium.

EC.7. Verification of the Perfect Bayesian Equilibrium in Section 4.2

First, consider any round after the second one, and let (s_1,s_2) be the current updated pair of infimum types. Let i be the current follower. If i does not bid now, the game ends and he gets zero, with the cost of the payment he has committed in the past already sunk. If i bids then he adds 2δ to his committed payment and becomes the next frontrunner, with his infimum type updated to s_i'; thus, given type t_i, his expected payoff from bidding is equal to

$$\begin{aligned}-\frac{2\delta}{t_i}+V_i(t_i|s_i',s_{-i})&\overset{(22)}{=}-\frac{2\delta}{t_i}+\frac{2\delta}{s_i'}+\left(1-\frac{2\delta}{s_i'}\right)M_i(t_i|s_i',s_{-i}')\\&\overset{(21)}{=}-\frac{2\delta}{t_i}+\frac{2\delta}{s_i'}+\left(1-\frac{2\delta}{s_i'}\right)\max\left\{0,-\frac{2\delta}{t_i}+V_i(t_i|s_i'',s_{-i}')\right\},\end{aligned}\tag{EC.37}$$

where s_{-i}' is derived from (s_i',s_{-i}) by Eq. (20), and s_i'' from (s_i',s_{-i}') analogously; and Eq. (22) is applicable to the first line because $V_i(t_i|s_i',s_{-i})$ is player i's expected payoff from being the frontrunner in the next round, which is after at least the second round.

Claim: For any $t_i\in[a_i,z_i]$ there exists an integer $N_i(t_i)$ such that at the start of the $N_i(t_i)$th round $M_i(t_i|s_i^{N_i(t_i)},s_{-i}^{N_i(t_i)})=0$, with (s_i^n,s_{-i}^n) denoting the updated infimums at the start of the nth round. Otherwise, Eq. (EC.37), applied iteratively, implies that for any $n=1,2,3,\ldots$

$$0<-\frac{2\delta}{t_i}+V_i(t_i|s_i',s_{-i})=-\frac{2\delta}{t_i}n+\left(1-\frac{2\delta}{s_i^n}\right)M_i(t_i|s_i^n,s_{-i}^n)\leq-\frac{2\delta}{t_i}n+v,$$

which is impossible given v a finite constant.

The claim established above, coupled with the first line of (EC.37), implies that, at the start of the $N_i(t_i)$th round, if the updated infimums thereof are denoted by (s'_i, s'_{-i}), then

$$-\frac{2\delta}{t_i} + V_i(t_i|s'_i, s_{-i}) \begin{cases} > 0 \text{ if } t_i > s'_i \\ \leq 0 \text{ if } t_i \leq s'_i. \end{cases} \tag{EC.38}$$

Then, at the start of the $(N_i(t_i) - 1)$th round, Eq. (21) implies

$$M_i(t_i|s_i, s_{-i}) \begin{cases} > 0 \text{ if } t_i > s'_i \\ = 0 \text{ if } t_i \leq s'_i. \end{cases} \tag{EC.39}$$

In other words, in the $(N_i(t_i) - 1)$th round, at the start of which the updated infimum types are (s_i, s_{-i}), and the current follower i would bid if and only if his type is above s'_i (i.e., if and only if the continuation value of currently being the follower is positive).

At the start of the $(N_i(t_i) - 2)$th round, the updated infimum of player i, the frontrunner now and soon to become the follower next, is still s_i, while that of player $-i$ is some s^{-1}_{-i} such that her updated infimum s_{-i} at the $(N_i(t_i) - 1)$th round is derived from (s_i, s^{-1}_{-i}) by Eq. (20); by Eq. (22),

$$V_i(t_i|s_i, s^{-1}_{-i}) = \frac{2\delta}{s_i} + \left(1 - \frac{2\delta}{s_i}\right) M_i(t_i|s_i, s^{-1}_{-i}) \geq \frac{2\delta}{s_i},$$

with the inequality due to Eq. (21). Thus,

$$t_i > s_i \Longrightarrow -\frac{2\delta}{t_i} + V_i(t_i|s_i, s^{-1}_{-i}) \geq -\frac{2\delta}{t_i} + \frac{2\delta}{s_i} > 0; \tag{EC.40}$$

if $t_i \leq s_i$ then $t_i < s'_i$, as $s_i < s'_i$ by Eq. (20), then Eq. (EC.39) implies $M_i(t_i|s_i, s^{-1}_{-i}) = 0$ and hence

$$t_i \leq s_i \Longrightarrow -\frac{2\delta}{t_i} + V_i(t_i|s_i, s^{-1}_{-i}) = -\frac{2\delta}{t_i} + \frac{2\delta}{s_i} \leq 0. \tag{EC.41}$$

Thus, (EC.38) is extended from the $N_i(t_i)$th round to the $(N_i(t_i) - 2)$th round. Thus, in the $(N_i(t_i) - 3)$th round, where the current follower i contemplates whether or not to bid, (EC.40) and (EC.41) together imply that player i would bid if and only if his type is above s_i, as prescribed by the proposed equilibrium.

We can repeat the above reasoning, thereby extending (EC.38) backward round by round, as long as the value function V_i obeys Eq. (22). Hence by backward induction we extend (EC.38) down to the third round, with (s'_i, s_{-i}) denoting the updated infimum types at the start of the third round. That means, in the second round, the follower i finds it a best response to bid if and only if his type is above s'_i, as prescribed by the proposed equilibrium.

Thus we need only to justify the equilibrium strategy for the initial round. Consider the decision of any player $i \in \{1, 2\}$ in the initial round. If player i bids and becomes the frontrunner in the second round, then he commits the first increment δ; if the other player $-i$ does not bid, then player i wins and gets the payoff

$$-\frac{\delta}{t_i} + v;$$

if player $-i$ also bids in the initial round (and fails to be selected the frontrunner), player i's expected payoff is equal to

$$-\frac{\delta}{t_i} + V_i^0(t_i|s_i^0, s_{-i}^0) \overset{(23)}{=} -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1). \tag{EC.42}$$

If player i bids but is not selected the frontrunner, then his expected payoff is equal to

$$M_i(t_i|s_i^0, s_{-i}^0) = \max\left\{0, -\frac{2\delta}{t_i} + V_i(t_i|s_i^1, s_{-i}^0)\right\}.$$

If player i stays put, then the game ends, either with no sale if player $-i$ also stays put, or with $-i$ bidding and winning the good at price δ; in either case player i gets zero. Ineq. (EC.38), applied to the case where $(s_i', s_{-i}) = (s_i^1, s_{-i}^0)$ in the third round, means

$$-\frac{2\delta}{t_i} + V_i(t_i|s_i^1, s_{-i}^0) \begin{cases} > 0 \text{ if } t_i > s_i^1 \\ \leq 0 \text{ if } t_i \leq s_i^1. \end{cases}$$

Thus, in the second round, Eq. (21) implies

$$M_i(t_i|s_i^0, s_{-i}^0) \begin{cases} > 0 \text{ if } t_i > s_i^1 \\ = 0 \text{ if } t_i \leq s_i^1. \end{cases}$$

If $t_i \leq s_i^0$, $t_i < s_i^1$ and hence $M_i(t_i|s_i^0, s_{-i}^0) = 0$, so Eq. (EC.42) implies

$$-\frac{\delta}{t_i} + V_i^0(t_i|s_i^0, s_{-i}^0) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} \leq 0;$$

by contrast, if $t_i > s_i^0$,

$$-\frac{\delta}{t_i} + V_i(t_i|s_i^0, s_{-i}^0) = -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} + \left(1 - \frac{\delta}{s_i^0}\right) M_i(t_i|s_i^0, s_{-i}^1) \geq -\frac{\delta}{t_i} + \frac{\delta}{s_i^0} \geq 0.$$

Thus, as long as $-\delta/s_i^0 + v \geq 0$, it is a best response for player i to bid in the initial round if and only $t_i > s_i^0$. Note that $-\delta/s_i^0 + v \geq 0$ is equivalent to $\delta/s_i^0 \leq v$, which is guaranteed by Eq. (18), because Eq. (18), with the roles of i and $-i$ switched, implies

$$\frac{\delta}{s_i^0} = vF_{-i}(s_{-i}^0) \leq v.$$

Hence $-\delta/s_i^0 + v \geq 0$ is true, so the strategy in the initial round prescribed by the proposed equilibrium is a best response for i.