Mechanism Design with Ambiguous Transfers

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Abstract

This paper introduces ambiguous transfers to the problems of full surplus extraction and implementation in finite dimensional naive type spaces. The mechanism designer commits to one transfer rule but informs agents of a set of potential ones. Without knowing the adopted transfer rule, agents are assumed to make decisions based on the worst-case expected payoffs. A key condition in this paper is the Beliefs Determine Preferences (BDP) property, which requires an agent to hold distinct beliefs about others' information under different types. We show that full surplus extraction can be guaranteed via ambiguous transfers if and only if the BDP property is satisfied by all agents. With a common prior, all efficient allocations are implementable via individually rational and budget-balanced mechanisms with ambiguous transfers if and only if the BDP property holds for all agents. This necessary and sufficient condition is weaker than those for full surplus extraction and implementation via Bayesian mechanisms. Therefore, ambiguous transfers may achieve first-best outcomes that are impossible under the standard approach. In particular, with ambiguous transfers, efficient allocations become implementable generically in two-agent problems, a result that does not hold under a Bayesian framework.

Keywords: Full surplus extraction; Bayesian (partial) implementation; Ambiguous transfers; Correlated beliefs; Individual rationality; Budget balance

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1 Introduction

Many transaction mechanisms have uncertain rules. For instance, Priceline Express Deals offer travelers a known price for a hotel stay, but the exact name and location of the hotel remain unknown until the completion of payment. Alternatively, some stores run scratch-and-save promotions. Consumers receive scratch cards during check-out, which reveal discounts, and thus the costs of their purchases remain unknown at the time they decide to buy. As a third example, eBay allows sellers of auction-style listings to set hidden reserve prices.

In all the above mechanisms, the mechanism designer introduces uncertainty about the allocation and/or transfer rule without telling agents the underlying probability distribution. The subjective expected utility model can be adopted to describe agents' decision making without an objective probability. However, since **?**, many studies have challenged this model, arguing that decision makers tend to be ambiguity-averse.[1] Therefore, it is important to understand if and how a mechanism designer can benefit from agents' ambiguity aversion. More specifically, we would like to know whether engineering ambiguity on rules of mechanisms can help the designer achieve the first-best outcome.

This paper introduces ambiguous transfers to study two problems: full surplus extraction and interim individually rational and ex-post budget-balanced implementation of any ex-post efficient allocation rule. The analysis is based on finite dimensional naive type spaces where each agent's only private information is her payoff-relevant type. The problem of full surplus extraction aims to design a mechanism in which agents transfer the entire surplus to the designer. The efficient implementation problem constructs an incentive compatible, individually rational, and budget-balanced mechanism such that the socially optimal outcome emerges as an equilibrium. In our model, the mechanism designer informs agents of the exact allocation rule. She also commits to one transfer rule, but the communication is ambiguous so that agents only know a set of potential ones. Without knowing the adopted transfer rule, agents are assumed to be ambiguity-averse. More specifically, agents are maxmin expected utility maximizers who make decisions based on the worst-case scenario.

In this paper, the Beliefs Determine Preferences (BDP) property is the key condition for the existence of first-best mechanisms with ambiguous transfers. The property, introduced by Neeman (2004), requires that an agent should hold distinct beliefs about others' private information under different types. Correlated information is necessary in the BDP property.

[1]There is a huge literature studying ambiguity aversion from the perspective of different fields, including (but not limited to) decision theory (e.g., Gilboa and Schmeidler (1989), Klibanoff et al. (2005)), macroeconomics (e.g., **??**), finance (e.g. **?**, **?**), and experimental and behavioral economics (e.g., **?**, **?**).

We focus on a finite dimensional naive type space, in which case the BDP property holds for all agents generically.

We show that full surplus extraction can be guaranteed via ambiguous transfers if and only if the BDP property is satisfied by all agents. In addition, under a common prior, any efficient allocation rule is implementable via an interim individually rational and ex-post budget-balanced mechanism with ambiguous transfers if and only if the BDP property holds for all agents. By confining the analysis in private value common prior environments, we further show that efficient implementation can be guaranteed if and only if the BDP property fails for at most one agent. As an extension, we establish necessary and slightly stronger sufficient conditions for efficient implementation under environments potentially without a common prior. Lastly, we discuss the robustness of our sufficiency results under alternative models of ambiguity aversion.

The BDP property is weaker than Crémer and McLean (1988)'s Convex Independence condition, which is necessary and sufficient for full surplus extraction via a Bayesian mechanism. Convex Independence, together with the Identifiability condition established by Kosenok and Severinov (2008), is necessary and sufficient for implementing any efficient allocation rule via an interim individually rational and ex-post budget-balanced Bayesian mechanism. Under both problems, this paper requires a strictly weaker condition to obtain the first-best outcome compared to the Bayesian approach. As a result, when the conditions of Crémer and McLean (1988) or Kosenok and Severinov (2008) fail, engineering ambiguity deliberately may allow the designer to achieve first-best outcomes that are impossible under the Bayesian mechanisms.

Admittedly, some works that we will discuss in Section 1.1 have shown that Convex Independence is generic. But in applications, it may be of interest to study some non-generic cases where Convex Independence fails. In particular, the BDP property imposes weaker restrictions on the cardinality of the finite type space than Convex Independence and Identifiability. For example, when one agent has more types than the type profiles of all other agents, Convex Independence fails for this agent with positive probability. As another instance, Kosenok and Severinov (2008)'s necessary and sufficient conditions can never hold simultaneously for any common prior with only two agents, indicating an impossibility result on two-agent implementation problems. But the BDP property and ambiguous transfers can provide a solution to such problems generically.

In this paper, the mechanism designer announces an efficient allocation rule and introduces ambiguity in transfer rules only. As the ex-post efficient allocation rule is often unique in a finite-type framework, the mechanism designer may not have multiple allocation rules to choose from. In a related paper, Di Tillio et al. (2017) study how second-best revenue in an independent private value auction can be improved if the seller introduces ambiguity in both allocation and transfer rules. We discuss more on the relationship with this paper in Section 1.1.

The paper proceeds as follows. We review the literature in Section 1.1 and introduce the environment in Section 2. Our main result is presented in Section 3. Section 4 extends our main result along two directions. The Appendix collects all proofs and some examples.

1.1 Literature review

1.1.1 Efficient mechanism design

How to implement efficient allocations is a classical topic in mechanism design theory that has been widely studied in situations such as public good provision and bilateral trading. Individual rationality is a natural requirement as agents can opt out of the mechanism. Budget balance requires that agents should finance within themselves for the efficient outcome rather than rely on an outside budget-breaker. When either individual rationality or budget balance is required, the literature provides positive results for efficient mechanism design in private value environments. For instance, the VCG mechanism (Vickrey (1961), Clarke (1971), and Groves (1973)) is ex-post individually rational. The AGV mechanism (d'Aspremont and Gérard-Varet (1979)) is ex-post budget-balanced. However, the literature documents a tension between efficiency, individual rationality, and budget balance, when agents have independent information. For example, in a private value bilateral trading framework, Myerson and Satterthwaite (1983) prove that it is impossible to achieve efficiency with an individually rational and budget-balanced mechanism in general. With multi-dimensional and interdependent values, Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) prove that efficient allocations are generically non-implementable.

First-best mechanism design becomes possible in some correlated information environments. Crémer and McLean (1985, 1988) establish two conditions to fully extract agents' surplus in private value auctions, among which the Convex Independence condition is necessary and sufficient for full surplus extraction to be a Bayesian Nash equilibrium. In a fixed finite-dimensional type space, if no one has more types than all others' type profiles, the condition holds for all agents under almost every prior. Without restricting the dimension, different notions of genericity are adopted in the literature and various conclusions on genericity of Convex Independence (or the weaker BDP property) are made (e.g., Neeman (2004), Heifetz and Neeman (2006), Barelli (2009), Chen and Xiong (2011, 2013), **??**). With continuous types, McAfee and Reny (1992) show that approximate full surplus extraction can be achieved. In addition, the recent papers of Liu (2014) and Noda (2015) prove an intertemporal variant of Convex Independence is sufficient for first-best mechanism design in dynamic environments. By introducing ambiguous transfers, Section 3 of the current paper shows that a weaker condition, the BDP property, becomes necessary and sufficient for full surplus extraction under a finite, naive type space.

In an implementation problem, the allocation rule is exogenously given. Thus, the mechanism designer constructs incentive compatible transfers to achieve the desired outcome. Under the context of exchange economies, McLean and Postlewaite (2002, 2003a,b) propose the notion of informational size and prove the existence of incentive compatible and approximately efficient outcomes when agents have small informational size.[2] Under a mechanism design framework, McLean and Postlewaite (2004, 2015) implement efficient allocation rules via individually rational mechanisms under the BDP property. In their mechanisms, small outside money is needed even when agents are informationally small. Different from these papers, our mechanism for implementation in Section 3 is exactly efficient, individually rational, and budget-balanced without imposing any informational smallness assumption.

A few papers study budget-balanced mechanisms with or without independent information, including Matsushima (1991), Aoyagi (1998), Chung (1999), d'Aspremont et al. (2004), **?**, etc.[3] Among these works, d'Aspremont et al. (2004) propose necessary and sufficient conditions for budget-balanced mechanisms. None of these papers requires individual rationality. Also, they assume that there are at least three agents. In fact, d'Aspremont et al. (2004) indicate an impossibility result in implementing efficient allocations via budget-balanced mechanisms with two agents under correlated information. However, we do require individual rationality, and our mechanism with ambiguous transfers works for environments with at least two agents.

Matsushima (2007), Kosenok and Severinov (2008), and **?** among others design individually rational and budget-balanced mechanisms. Kosenok and Severinov (2008) propose the Identifiability condition, which along with the Convex Independence condition, is necessary and sufficient for implementing any ex-ante socially rational allocation rule via an interim individually rational and ex-post budget-balanced Bayesian mechanism. The Identifiability condition is generic with at least three agents and under some restrictions on the dimension of the type space, but Convex Independence and Identifiability never hold simultaneously in a two-agent setting. Thus Kosenok and Severinov (2008) imply an impossibility result in efficient, individually rational, and budget-balanced two-agent mechanism design. In our

[2]For related results, see also Sun and Yannelis (2007, 2008).

[3] Matsushima (1991), Chung (1999), d'Aspremont et al. (2004) only consider private value utility functions. In this case, incentive compatibility can be achieved via a VCG mechanism, rather than via information correlation. Thus, they allow for independent information.

paper, the BDP property is weaker than Convex Independence, and we do not need Identifiability. Moreover, the BDP property holds generically in a finite-dimensional type space with at least two agents, and thus we make the impossible possible for two-agent implementation problems.

1.1.2 Mechanism design under ambiguity

In the growing literature on mechanism design with ambiguity-averse agents, most of the works assume exogenously that agents hold ambiguous beliefs of others' types. For example, Bose et al. (2006), Bose and Daripa (2009), and Bodoh-Creed (2012) study optimal mechanism design with ambiguous averse agents. **?** and De Castro et al. (2017a,b) prove that all Pareto efficient allocations are incentive compatible and thus implementable when agents' ambiguous beliefs are unrestricted. Under the private value assumption, Wolitzky (2016) establishes a necessary condition for the existence of an efficient, individually rational, and weak budget-balanced mechanism. In an environment with multi-dimensional and interdependent values, Song (2016) quantifies the amount of ambiguity that is necessary and sometimes sufficient for efficient mechanism design. We do not assume exogenous ambiguity in agents' beliefs, which is the biggest difference between the above papers and our work.

Bose and Renou (2014) and Di Tillio et al. (2017) contrast the above works in that ambiguity is endogenously engineered by the mechanism designer. Before the allocation stage, Bose and Renou (2014) let the mechanism designer communicate with agents via an ambiguous device, which generates multiple beliefs. Their paper characterizes social choice functions that are implementable under this method. Our paper is different from Bose and Renou (2014), as we do not need multiple beliefs on other agents' private information.

Di Tillio et al. (2017) consider the problem of revenue maximization in a private value and independent belief auction. The seller commits to a simple mechanism, i.e., an allocation and transfer rule, but informs agents of a set of simple mechanisms. As all the simple mechanisms generate the same expected revenue (imposed by the Consistency condition), agents do not know the exact rule and thus make decisions based on the worst-case scenario. Compared to the Bayesian mechanism, their ambiguous approach yields a higher expected revenue.

In the current paper, ambiguity is engineered in a similar way to Di Tillio et al. (2017). However, instead of studying how ambiguous mechanisms improve second-best revenues under independent beliefs, our paper studies when the first-best outcome in surplus extraction or implementation can be achieved without restricting attention to independent beliefs. The essential factor that enables us to achieve the first-best outcome in a finite type space is the correlation in agents' beliefs and more particularly, the BDP property.

As mentioned before, we fix an efficient allocation rule and only allow for ambiguity in transfer rules, but in Di Tillio et al. (2017)'s mechanism both allocation and transfer rules are ambiguous. Our restriction on unambiguous allocation rule is compatible with Di Tillio et al. (2017)'s Consistency condition. In the full surplus extraction problem, each transfer rule leaves agents zero surplus and gives the designer the full surplus on path. In the efficient implementation problem, each transfer rule leads to the first-best efficiency on path. Therefore, all transfer rules are credible. The restriction on unambiguous allocation rule is closely related to two facts: (1) we aim to achieve the first-best outcome in full surplus extraction or implementation, and (2) our argument is confined to a finite type space. Allowing for ambiguity in allocation rules may fail full surplus extraction and implementation. To see this, consider a finite-type environment where the total surplus is maximized by a unique allocation rule. In this case, any other allocation rule is inefficient and has a lower surplus level. As the efficient allocation rule must be used in the mechanisms for full surplus extraction and implementation, and as agents know the designer's objective is to extract full surplus or maximize efficiency, any other rule with a lower surplus level is non-credible to the agents. Hence, multiple allocation rules are not used in our environment.

In Di Tillio et al. (2017)'s optimal mechanism under independent beliefs and finitely many types, ambiguity in allocation rules plays a role. Therefore, they cannot obtain the first-best revenue. In fact, in a screening or an independent private value auction framework, allowing for ambiguous transfers but not ambiguous allocations does not improve the seller's revenue compared to a standard unambiguous mechanism. However, according to Di Tillio et al. (2017)'s Appendix B, their approach works for full surplus extraction with continuous types. This is because there are infinitely many ex-ante efficient allocation rules, or infinitely many allocation rules that are ex-post efficient almost everywhere. Among them, every two rules are the same except in a null set of the type space. In a continuous type space, if an efficiency-maximizing social planner wants to implement an ex-post efficient allocation rule almost everywhere, she can follow the approach of Di Tillio et al. (2017)'s Appendix B as well. Hence, the current paper only focuses on environments with finitely many types.

2 Environment

We study an asymmetric information environment given by $\mathcal{E} = \{I, A, (\Theta_i, u_i, p_i)_{i=1}^N\}$.

- Let $I = \{1, ..., N\}$ be a finite set of **agents**. Assume $N \geq 2$.
- Denote the set of **feasible outcomes** by A.
- Let $\theta_i \in \Theta_i$ be agent i's **type** in her **type space**. Denote $\times_{i\in I}\Theta_i$ by Θ, $\times_{j\in I, j\neq i}\Theta_j$ by Θ_{-i}, and $\times_{k\in I, k\neq i,j}\Theta_k$ by Θ_{-i-j}. Let $|\Theta_i|$ be the cardinality of Θ_i. Assume $2 \leq |\Theta_i| < \infty$.

- Each agent i has a quasi-linear **utility function** $u_i(a,\theta)+b$, where $a \in A$ is a feasible outcome, $b \in \mathbb{R}$ is a monetary transfer, and $\theta \in \Theta$ is the realized type profile. In the special case that $u_i\big(a,(\theta_i,\theta_{-i})\big) = u_i\big(a,(\theta_i,\theta'_{-i})\big)$ for all $\theta_i \in \Theta_i$, $\theta_{-i},\theta'_{-i} \in \Theta_{-i}$, and $a \in A$, we say u_i has **private value** and denote $u_i\big(a,(\theta_i,\theta_{-i})\big)$ by $u_i(a,\theta_i)$.
- Let $p_i \in \Delta(\Theta)$ be agent i's **prior** on Θ. In the special case that $p_i(\theta) = p_j(\theta)$ for all $i,j \in I$ and $\theta \in \Theta$, we can drop the subscript and denote the **common prior** by p.

The structure of environment $\mathcal{E}$ is assumed to be common knowledge between the mechanism designer and agents, but every agent's realized type is her private information. As a type in this paper only concerns payoff-relevant information, such a type space is sometimes called a "naive" type space in the literature.

Let $p_i(\theta_i)$, $p_i(\theta_j)$, $p_i(\theta_i,\theta_j)$, and $p_i(\theta)$ represent the marginal distribution of p_i on θ_i, θ_j, (θ_i,θ_j), and θ respectively. Throughout this paper, we impose the following assumption.

Assumption 2.1: *For all $i,j \in I$ with $i \neq j$, and $(\theta_i,\theta_j) \in \Theta_i \times \Theta_j$, $p_i(\theta_i,\theta_j) > 0$.*

Type-θ_i agent i's **belief** is derived from Bayesian updating p_i: she believes that others have type profile $\theta_{-i} \in \Theta_{-i}$ with probability $p_i(\theta_{-i}|\theta_i) \equiv \frac{p_i(\theta)}{p_i(\theta_i)}$, which is well-defined since $p_i(\theta_i) \geq p_i(\theta_i,\theta_j) > 0$. For agent $i \neq j$ and types θ_i,θ_j, type-θ_i agent i's marginal belief on θ_j is $p_i(\theta_j|\theta_i) \equiv \frac{p_i(\theta_i,\theta_j)}{p_i(\theta_i)}$. When $N \geq 3$, further define type-θ_i agent i's marginal belief on θ_{-i-j} as $p(\theta_{-i-j}|\theta_i,\theta_j)$ given that θ_j is the type of agent j. When p is a common prior, we can use $p(\theta_{-i}|\theta_i)$, $p(\theta_j|\theta_i)$, and $p(\theta_{-i-j}|\theta_i,\theta_j)$ to denote these beliefs. For simplicity, let the vector $(p_i(\theta_{-i}|\theta_i))_{\theta_{-i}\in\Theta_{-i}}$ be denoted by $p_i(\cdot|\theta_i)$.

An **allocation rule** $q : \Theta \to A$ is a plan to assign a feasible outcome contingent on agents' realized type profile. An allocation rule q is said to be ex-post **efficient** if $\sum_{i\in I} u_i\big(q(\theta),\theta\big) \geq \sum_{i\in I} u_i\big(q'(\theta),\theta\big)$ for all $q' : \Theta \to A$ and $\theta \in \Theta$.

In this paper, a number $r \in \mathbb{R}$ is positive if $r > 0$. For any positive integer K, the vector $\mathbf{0} \in \mathbb{R}^K$ is a vector of K zeros. Let $\mathbb{R}^K_+$ denote $\{v \in \mathbb{R}^K | v_k \geq 0, \forall k = 1,...,K\}$.

Definition 2.1: *A **mechanism with ambiguous transfers** is a pair $\mathcal{M} = (q,\Phi)$, where $q : \Theta \to A$ is an allocation rule, and Φ is a set of transfer rules with a generic element $\phi : M \to \mathbb{R}^N$. We call the set Φ **ambiguous transfers**.*[4]

The mechanism works in the following way. The designer first commits to the allocation rule $q : M \to A$ and an arbitrary transfer rule $\phi \in \Phi$ secretly. Before reporting messages, agents are informed of the allocation rule q and ambiguous transfers Φ, but not ϕ. Af-

[4] We focus on direct mechanisms. One can follow Di Tillio et al. (2017) to establish a revelation principle, as thus the restriction on direct mechanisms is without loss of generality.

ter agents report their messages, the mechanism designer reveals ϕ. Then allocations and transfers are made according to the reported messages as well as q and ϕ.

As agents only know the set Φ, we follow the spirit of Gilboa and Schmeidler (1989)'s maxmin expected utility (MEU) and assume that agents make decisions based on the worst-case expected payoff. Hence, a type-θ_i agent i's interim payoff is

$$\inf_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta_i,\theta_{-i}),(\theta_i,\theta_{-i})\big)+\phi_i(\theta_i,\theta_{-i})]p_i(\theta_{-i}|\theta_i).^5$$

Throughout this paper, the outside option x_0 is normalized to give all agents zero payoffs at all type profiles. A mechanism with ambiguous transfers (q,Φ) is said to satisfy interim **individual rationality** (IR) if $\inf_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta_i,\theta_{-i}),(\theta_i,\theta_{-i})\big)+\phi_i(\theta_i,\theta_{-i})]p_i(\theta_{-i}|\theta_i)\geq 0$ for all $i\in I$ and $\theta_i\in\Theta_i$. It satisfies ex-post **budget balance** (BB) if $\sum_{i\in I}\phi_i(\theta)=0$ for all $\phi\in\Phi$ and $\theta\in\Theta$. The mechanism is said to satisfy interim **incentive compatibility** (IC) if $\inf_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta_i,\theta_{-i}),(\theta_i,\theta_{-i})\big)+\phi_i(\theta_i,\theta_{-i})]p_i(\theta_{-i}|\theta_i)\geq\inf_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta_i',\theta_{-i}),(\theta_i,\theta_{-i})\big)+\phi_i(\theta_i',\theta_{-i})]p_i(\theta_{-i}|\theta_i)$ for all $i\in I$ and $\theta_i,\theta_i'\in\Theta_i$.[6]

This paper studies two related but different objectives. One is full surplus extraction, and the other is implementation of an efficient allocation rule via an IR and BB mechanism.

In the sense of McAfee and Reny (1992), a mechanism with ambiguous transfers $\mathcal{M}=(q,\Phi)$ is said to **extract the full surplus** if it is IR and IC, q is ex-post efficient, and

$$\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta),\theta\big)+\phi_i(\theta)]p(\theta_{-i}|\theta_i)=0,\forall\phi\in\Phi.$$

The requirement that every $\phi\in\Phi$ extracts full surplus follows from Di Tillio et al. (2017)'s Consistency condition. In other words, since the designer's objective is to extract full surplus, any transfer rule that leaves agents a strictly positive surplus is not creditable.

Following Kosenok and Severinov (2008), we also want the mechanism to be BB in an implementation problem so that outside money is not needed to finance the efficient outcome. An allocation rule q is **implementable** by an IR and BB mechanism with ambiguous transfers if there exists an IC, IR, and BB mechanism with ambiguous transfers $\mathcal{M}=(q,\Phi)$.

[5] We follow Di Tillio et al. (2017) and adopt the infimum notation. In fact, the mechanisms we constructed in Theorem 3.1 and Proposition 4.1 have only finitely many transfer rules.

[6] Like many mechanism design works with ambiguity aversion, e.g., Wolitzky (2016), Di Tillio et al. (2017), Song (2016), we restrict attention to pure strategies. Depending on how the payoff of playing a mixed strategy is formalized, the restriction could be with or without loss of generality. See Wolitzky (2016) for more details.

3 Main result

Our key condition, the Beliefs Determine Preferences property, is introduced by Neeman (2004). It requires that an agent with different types should have distinct beliefs.

Definition 3.1: *Given a collection of priors $(p_i)_{i\in I}$, the **Beliefs Determine Preferences** (BDP) property holds for agent i if there does not exist $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ with $\bar{\theta}_i \neq \hat{\theta}_i$ such that $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$.*

For the BDP property to hold, it is necessary that agents' beliefs are correlated.

The following theorem is the main result of the paper.

Theorem 3.1:

1. *Given a collection of priors $(p_i)_{i\in I}$, full surplus extraction under any profile of utility functions can be achieved via a mechanism with ambiguous transfers if and only if the BDP property holds for all agents;*
2. *given a common prior p, any ex-post efficient allocation rule under any profile of utility functions is implementable via an IR and BB mechanism with ambiguous transfers if and only if the BDP property holds for all agents;*
3. *given a common prior p, any ex-post efficient allocation rule under any profile of **private value** utility functions is implementable via an IR and BB mechanism with ambiguous transfers if and only if the BDP property holds for at least $N-1$ agents.*

We remark that the number of agents, the dimension of types, whether the utility functions have private or interdependent value, and whether agents' beliefs are drawn from a common prior do not matter for Part 1 of the theorem.

Parts 2 and 3 of Theorem 3.1 focus on implementation via an IR and BB mechanism with ambiguous transfers, but full surplus extraction does not require the BB condition. To guarantee the BB condition and obtain unified necessary and sufficient conditions for implementation, we impose the common prior condition for Parts 2 and 3.[7] Part 2 does not restrict the environment to be a private value one while Part 3 does. When restricting the analysis on private value utility functions, Part 3 obtains a strictly weaker necessary and sufficient for implementation compared to Part 2. According to Part 3, even if ambiguous transfers are allowed and we confine our analysis to private value environments, we can always find non-implementable allocations when information is independent across agents.

To prove the necessity half of the three statements, when the BDP property fails for one

[7] The common prior condition is used explicitly in Lemmas A.4 and A.5 and thus the sufficiency direction of Parts 2 and 3 as well as the necessity direction of Part 3. Section 4.1 relaxes the common prior condition and presents necessary and (stronger) sufficient conditions on implementation.

agent, we construct a profile of utility functions under which full surplus extraction fails and an efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers.[8] When the BDP property fails for two agents, we construct private value utility functions under which efficient implementation fails.

We prove the sufficiency statements of Theorem 3.1 by constructing mechanisms consisting of two transfer rules. Although there are mechanisms with more transfers to extract the full surplus or implement efficient allocation rules, to be consistent with the spirit of minimal mechanisms of Di Tillio et al. (2017), we only present the ones with two rules.

To prove the sufficiency direction of Part 1, the Appendix begins with several lemmas. Lemma A.1 shows that for each $i \in I$ and $\bar{\theta}_i \neq \hat{\theta}_i$, there exists a transfer rule (a lottery) $\psi^{\bar{\theta}_i\hat{\theta}_i}$ such that (1) it gives every agent zero expected value when agents truthfully report, and (2) a type-$\bar{\theta}_i$ agent i achieves a negative expected value when she unilaterally misreports $\hat{\theta}_i$. This step is proven via Fredholm's theorem of the alternative. Lemmas A.2 and A.3 construct a linear combination of transfer rules $(\psi^{\bar{\theta}_i\hat{\theta}_i})_{i\in I,\bar{\theta}_i,\hat{\theta}_i\in\Theta_i,\bar{\theta}_i\neq\hat{\theta}_i}$, denoted by ψ, such that the lottery ψ gives all agents zero expected values on path, and gives any unilateral deviator a non-zero expected value.

Then pick an ex-post efficient allocation rule q and let $\eta_i(\theta) = -u_i(q(\theta), \theta)$ for all $i \in I$ and $\theta \in \Theta$. Let the set of ambiguous transfers for agent i be $\Phi_i = \{\eta_i + c\psi_i, \eta_i - c\psi_i\}$. As η_i transfers agent i's entire surplus to the mechanism designer and that ψ_i has zero expected value when every agent truthfully reports, each IR constraint binds. In addition, as ψ_i has non-zero expected value whenever i misreports unilaterally, the lower expected value from $\eta_i + c\psi_i$ and $\eta_i - c\psi_i$ is negative under a sufficiently large c. Thus, IC can be achieved. Intuitively, with multiple transfer rules, different IC constraints can be satisfied by distinct transfers. Namely, we do not need one transfer rule to satisfy all IC constraints, and thus the full surplus can be extracted under a weaker condition than under Bayesian mechanisms.

For the sufficiency direction of Part 2, Lemma A.5 constructs a BB transfer rule (a lottery) ψ that gives agents zero expected values on path, and gives any unilateral deviator a non-zero expected value. The common prior condition is adopted to guarantee BB of ψ. Then pick any BB and IR transfer rule η. The set of ambiguous transfers $\Phi = \{\eta + c\psi, \eta - c\psi\}$ can implement the efficient allocation rule for a sufficiently large c. The efficiency of the allocation rule does not play a role in the proof of Part 2.[9]

[8] The construction adopts interdependent value utility functions so that there is a unified necessity proof for Parts 1 and 2. However, one can also follow Crémer and McLean (1988) or Example 3.1 of the current paper to construct private value utility functions for the necessity proof of Part 1. By Part 3 of Theorem 3.1, when the BDP property fails for only one agent, efficient allocations under private value environments are implementable so the necessity direction of Part 2 has to rely on interdependent value utility functions.

[9] In fact, by combining our proof with that of Kosenok and Severinov (2008), Part 2 can be extended to

For the sufficiency direction of Part 3, since $N-1$ agents satisfy the BDP property, one can follow the previous argument and construct a lottery ψ such that IC of $N-1$ agents can be guaranteed with ambiguous transfers. Then by allocating the total surplus to the remaining agent and aligning her incentives with the mechanism designer, the agent will report truthfully in private value environments to maximize social surplus. Such a spirit has shown up in the VCG mechanisms. Unlike Part 2, efficiency of q plays a role in Part 3. Example A.1 in the Appendix illustrates that an inefficient allocation rule may not be implementable if just $N-1$ agents satisfy the BDP property.

3.1 Comparison

Part 1 of Theorem 3.1 is directly comparable to the result of Crémer and McLean (1988). Under Bayesian mechanisms, they show that full surplus extraction can be guaranteed if and only if the Convex Independence condition, defined below, holds for all agents.

Definition 3.2: *The **Convex Independence** (CI) condition holds for agent $i \in I$ if for any type $\bar{\theta}_i \in \Theta_i$ and non-negative coefficients $(c_{\hat{\theta}_i})_{\hat{\theta}_i \in \Theta_i}$, $p_i(\cdot|\bar{\theta}_i) \neq \sum_{\hat{\theta}_i \in \Theta_i \setminus \{\bar{\theta}_i\}} c_{\hat{\theta}_i} p_i(\cdot|\hat{\theta}_i)$.*

The CI condition has been proven to be generic by some papers. However, in a finite dimensional naive type space, the CI condition fails for i with positive probability when $|\Theta_i| > |\Theta_{-i}|$. For example, when $|\Theta_2| = 3 > |\Theta_1| = 2$, the CI condition fails for agent 2 under every prior. As another instance, if $N = 3$ and $(|\Theta_1|, |\Theta_2|, |\Theta_3|) = (5, 2, 2)$, it is easy to find a non-negligible set of priors under which agent 1's CI fails. The BDP property is weaker than CI in two aspects. Firstly, the BDP property holds for i generically even if $|\Theta_i| > |\Theta_{-i}|$. Secondly, the BDP property can address some linear cases of correlation that are ruled out by CI. When the BDP property holds for all agents but CI fails for someone, ambiguous transfers can perform better than Bayesian mechanisms in full surplus extraction.

Example 3.1: *This two agent example demonstrates how ambiguous transfers work. Suppose one agent has three types and the other has two. Their beliefs are drawn from a common prior $p \in \Delta(\Theta)$ below. The CI condition fails for agent 2.*

p	θ_2^1	θ_2^2	θ_2^3
θ_1^1	0.1	0.2	0.2
θ_1^2	0.2	0.1	0.2

implement any ex-ante **socially rational** allocation rule q, i.e., q satisfying $\sum_{\theta \in \Theta} \sum_{i \in I} u_i\big(q(\theta), \theta\big) p(\theta) \geq 0$.

In an auction, denote each type-θ_i agent i's private value by $v_i(\theta_i)$. Suppose $v_2(\theta_2^1) > v_2(\theta_2^2) > v_2(\theta_2^3) > v_1(\theta_1) > 0$ for all $\theta_1 \in \Theta_1$. Crémer and McLean (1988) has shown that full surplus extraction is impossible via a Bayesian mechanism.

Next, we see how ambiguous transfers can help. Let the set of ambiguous transfers be $\Phi = (\phi^1, \phi^2)$. Transfers $\phi^1 = (\phi_1^1, \phi_2^1)$ and $\phi^2 = (\phi_1^2, \phi_2^2)$ are defined as follows.

$$\phi_i^1(\theta_1, \theta_2) = \begin{cases} c\psi_1(\theta_1, \theta_2), & \text{if } i = 1, \\ -v_2(\theta_2) + c\psi_2(\theta_1, \theta_2), & \text{if } i = 2, \end{cases} \quad \phi_i^2(\theta_1, \theta_2) = \begin{cases} -c\psi_1(\theta_1, \theta_2), & \text{if } i = 1, \\ -v_2(\theta_2) - c\psi_2(\theta_1, \theta_2), & \text{if } i = 2, \end{cases}$$

where $c \geq 1.5(v_2(\theta_2^1) - v_2(\theta_2^3))$, $\psi_1 : \Theta \to \mathbb{R}$ is given below, and $\psi_2 = -\psi_1$.

ψ_1	θ_2^1	θ_2^2	θ_2^3
θ_1^1	-2	-1	2
θ_1^2	1	2	-2

For each type-$\bar{\theta}_i$ agent i, $\psi_i(\bar{\theta}_i, \cdot)$ has zero expected value under belief $p_i(\cdot|\bar{\theta}_i)$. When she unilaterally misreports $\hat{\theta}_i \neq \bar{\theta}_i$, $\psi_i(\hat{\theta}_i, \cdot)$ has non-zero expected value.

Full surplus extraction requires the good to be allocated to agent 2. Both ϕ^1 and ϕ^2 give agents zero expected payoffs on path. Hence, each IR constraint binds.

When type-$\bar{\theta}_2$ agent 2 misreports $\hat{\theta}_2 \neq \bar{\theta}_2$, her worst-case expected payoff is $v_2(\bar{\theta}_2) - v_2(\hat{\theta}_2) - c|\sum_{\theta_1 \in \Theta_1} \psi_2(\theta_1, \hat{\theta}_2) p_2(\theta_1|\bar{\theta}_2)| < v_2(\bar{\theta}_2) - v_2(\hat{\theta}_2)$. Therefore, any "upward" misreport of agent 2 results in a negative expected payoff. As $c \geq 1.5(v_2(\theta_2^1) - v_2(\theta_2^3))$ and $v_2(\theta_2^1) > v_2(\theta_2^2) > v_2(\theta_2^3)$, it is easy to verify the three "downward" IC constraints:

$$\begin{aligned} &IC(\theta_2^1\theta_2^2) && 0 \geq v_2(\theta_2^1) - v_2(\theta_2^2) - c|\tfrac{1}{3} \times (-1) + \tfrac{2}{3} \times 2| = v_2(\theta_2^1) - v_2(\theta_2^2) - c, \\ &IC(\theta_2^1\theta_2^3) && 0 \geq v_2(\theta_2^1) - v_2(\theta_2^3) - c|\tfrac{1}{3} \times 2 + \tfrac{2}{3} \times (-2)| = v_2(\theta_2^1) - v_2(\theta_2^3) - \tfrac{2}{3}c, \\ &IC(\theta_2^2\theta_2^3) && 0 \geq v_2(\theta_2^2) - v_2(\theta_2^3) - c|\tfrac{2}{3} \times 2 + \tfrac{1}{3} \times (-2)| = v_2(\theta_2^2) - v_2(\theta_2^3) - \tfrac{2}{3}c. \end{aligned}$$

Agent 1's IC constraints can also be verified. The ambiguous transfers extract full surplus.

Part 2 of Theorem 3.1 is comparable to the result of Kosenok and Severinov (2008), who prove that the conditions of CI and Identifiability (defined below) are necessary and sufficient for implementing any efficient allocation rules via an IR and BB Bayesian mechanism.

Definition 3.3: *The common prior $p(\cdot)$ satisfies the* ***Identifiability*** *condition if for any $\tilde{p}(\cdot) \neq p(\cdot)$, there exists an agent $i \in I$ and her type $\bar{\theta}_i \in \Theta_i$, with $\tilde{p}(\bar{\theta}_i) > 0$, such that for any non-negative coefficients $(c_{\hat{\theta}_i})_{\hat{\theta}_i \in \Theta_i}$, $\tilde{p}_i(\cdot|\bar{\theta}_i) \neq \sum_{\hat{\theta}_i \in \Theta_i} c_{\hat{\theta}_i} p_i(\cdot|\hat{\theta}_i)$.*

In a finite, naive type space with $N = 3$ and $|\Theta_i| \geq 3$ for some $i \in I$ or a space with $N > 3$, the Identifiability condition holds generically, but it fails if the cardinality restriction is not satisfied. In particular, Kosenok and Severinov (2008) have remarked that only independent beliefs satisfy this condition when $N = 2$, and thus CI and Identifiability can never hold simultaneously in two-agent settings. In a BB Bayesian mechanism without the Identifiability condition, some agent i may have the incentive to misreport in a way that makes the truthful report of some $j \neq i$ appear untruthful. This is because by BB, i can benefit from a low expected transfer to j, which is the punishment due to j's (seemingly) untruthful report. However, when the set of ambiguous transfers Φ is used, i does not have such an incentive, because it remains ambiguous whether misreport of j would result in a high or low expected transfer to j. Hence, with ambiguous transfers, we can relax the Identifiability condition.

In the current paper, the BDP property is weaker than the CI condition, and the Identifiability condition becomes irrelevant. When the BDP property holds for all agents, and the CI or Identifiability condition fails, ambiguous transfers can perform better than Bayesian mechanisms in implementing efficient allocations via IR and BB mechanisms. In particular, ambiguous transfers provide a solution to the impossibility of two-agent IR, BB, and efficient mechanism design in finite dimensional naive type spaces generically.

The following example illustrates how ambiguous transfers work.

Example 3.2: *Consider the same prior p as in Example 3.1. Recall the CI condition fails for agent 2. The Identifiability condition also fails. Following Kosenok and Severinov (2008), one can construct utility functions under which an efficient allocation rule is not Bayesian implementable. However, the rule is implementable via ambiguous transfers.*

Let the feasible set of alternatives A be $\{x_0, x_1, x_2\}$. The outcome x_0 gives both agents zero payoffs at all type profiles. The payoffs given by x_1 and x_2 are presented below, where the first component denotes agent 1's payoff and the second denotes 2's. Assume $0 < a < B$.

x_1	θ_2^1	θ_2^2	θ_2^3
θ_1^1	$a, 0$	a, a	a, a
θ_1^2	$a, 0$	a, a	a, a

x_2	θ_2^1	θ_2^2	θ_2^3
θ_1^1	a, a	$a - 2B, a + B$	$a, 0$
θ_1^2	a, a	$a - 2B, a + B$	$a, 0$

The efficient allocation rule is $q(\theta_1, \theta_2^1) = x_2$ and $q(\theta_1, \theta_2^2) = q(\theta_1, \theta_2^3) = x_1$ for all $\theta_1 \in \Theta_1$. Suppose by contradiction that a BB transfer rule $\phi = (-\phi_2, \phi_2)$ implements q. Then

$$IC(\theta_1^1\theta_1^2) \qquad a - 0.2\phi_2(\theta_1^1, \theta_2^1) - 0.4\phi_2(\theta_1^1, \theta_2^2) - 0.4\phi_2(\theta_1^1, \theta_2^3)$$
$$\geq a - 0.2\phi_2(\theta_1^2, \theta_2^1) - 0.4\phi_2(\theta_1^2, \theta_2^2) - 0.4\phi_2(\theta_1^2, \theta_2^3),$$

$$
\begin{aligned}
&IC(\theta_1^2\theta_1^1) && a-0.4\phi_2(\theta_1^2,\theta_2^1)-0.2\phi_2(\theta_1^2,\theta_2^2)-0.4\phi_2(\theta_1^2,\theta_2^3)\\
& && \geq a-0.4\phi_2(\theta_1^1,\theta_2^1)-0.2\phi_2(\theta_1^1,\theta_2^2)-0.4\phi_2(\theta_1^1,\theta_2^3),\\
&IC(\theta_2^1\theta_2^2) && a+\tfrac{1}{3}\phi_2(\theta_1^1,\theta_2^1)+\tfrac{2}{3}\phi_2(\theta_1^2,\theta_2^1)\geq 0+\tfrac{1}{3}\phi_2(\theta_1^1,\theta_2^2)+\tfrac{2}{3}\phi_2(\theta_1^2,\theta_2^2),\\
&IC(\theta_2^2\theta_2^1) && a+\tfrac{2}{3}\phi_2(\theta_1^1,\theta_2^2)+\tfrac{1}{3}\phi_2(\theta_1^2,\theta_2^2)\geq a+B+\tfrac{2}{3}\phi_2(\theta_1^1,\theta_2^1)+\tfrac{1}{3}\phi_2(\theta_1^2,\theta_2^1).
\end{aligned}
$$

Multiply the inequalities by $0.5, 0.5, 0.3,$ *and* 0.3 *respectively and sum up. We have* $1.6a \geq 1.3a + 0.3B$*, a contradiction.*

For each $i \in I$ *and* $\theta \in \Theta$*, define* $\phi_i^1(\theta) = c\psi_i(\theta)$ *and* $\phi_i^2(\theta) = -c\psi_i(\theta)$*, where* $\psi = (\psi_1, \psi_2)$ *is defined in Example 3.1 and* $c \geq B$*. Let* $\Phi = \{\phi^1, \phi^2\}$ *be ambiguous transfers.*

Both ϕ^1 *and* ϕ^2 *satisfy the BB condition. Each type-*$\bar{\theta}_i$ *agent* i *obtains an interim payoff of* $a > 0$ *on path, and thus the IR condition holds. Suppose type-*θ_2^2 *agent* 2 *misreports* θ_2^1*, her worst-case expected payoff is* $a + B - c|\frac{2}{3} \times (-2) + \frac{1}{3} \times (1)| = a + B - c \leq a$*. Thus, we have established* $IC(\theta_2^2\theta_2^1)$*. The other IC constraints can be verified similarly. Therefore, the ambiguous transfers implement* q*.*

This example can also demonstrate the necessity of the BDP property. Suppose the beliefs satisfy $\tilde{p}_2(\cdot|\theta_2^1) = \tilde{p}_2(\cdot|\theta_2^2)$ *and an IR and BB mechanism with ambiguous transfers* $(q, \tilde{\Phi})$ *implements* q*. By adding the following expressions*

$$
\begin{aligned}
&IC(\theta_2^1\theta_2^2) && \inf_{\tilde{\phi}\in\tilde{\Phi}}\{a+\sum_{\theta_1\in\Theta_1}\tilde{\phi}_2(\theta_1,\theta_2^1)\tilde{p}_2(\theta_1|\theta_2^1)\}\geq\inf_{\tilde{\phi}\in\tilde{\Phi}}\sum_{\theta_1\in\Theta_1}\tilde{\phi}_2(\theta_1,\theta_2^2)\tilde{p}_2(\theta_1|\theta_2^1),\\
&IC(\theta_2^2\theta_2^1) && \inf_{\tilde{\phi}\in\tilde{\Phi}}\{a+\sum_{\theta_1\in\Theta_1}\tilde{\phi}_2(\theta_1,\theta_2^2)\tilde{p}_2(\theta_1|\theta_2^2)\}\geq\inf_{\tilde{\phi}\in\tilde{\Phi}}\{a+B+\sum_{\theta_1\in\Theta_1}\tilde{\phi}_2(\theta_1,\theta_2^1)\tilde{p}_2(\theta_1|\theta_2^2)\},
\end{aligned}
$$

and taking into account $\tilde{p}_2(\cdot|\theta_2^1) = \tilde{p}_2(\cdot|\theta_2^2)$*, we have* $2a \geq a + B$*, a contradiction. Hence, implementation via ambiguous transfers cannot be guaranteed without the BDP property.*

To compare ambiguous transfers with Bayesian mechanisms in the context of Part 3 of Theorem 3.1, we present the following necessary condition for Bayesian implementation under private value environments.

Proposition 3.1: *Given a common prior* p*, if any ex-post efficient allocation rule* q *under any profile of private value utility functions is implementable via an IR and BB Bayesian mechanism, then the CI condition holds for at least* $N - 1$ *agents.*

The necessary condition of Proposition 3.1 is stronger than the necessary and sufficient one in Part 3 of Theorem 3.1. Hence, there are cases when ambiguous transfers perform strictly better than Bayesian mechanisms in implementation under private value environments. The example below shows how ambiguous transfers work.

Example 3.3: *Consider a three-by-three bilateral trading example with a common prior p. Agent 1's BDP property fails and agent 2's holds. We demonstrate that the efficient allocation rule is implementable via ambiguous transfers in a private value environment.*

Suppose agent 1 is the seller of one unit of good and 2 is the buyer. A type-θ_i agent i's private evaluation of trading is $v_i(\theta_i)$. Assume that $v_2(\theta_2^1) > -v_1(\theta_1^1) > v_2(\theta_2^2) > -v_1(\theta_1^2) > v_2(\theta_2^3) > -v_1(\theta_1^3) \geq 0$. Outcomes in $A = \{x_0, x_1\}$ are feasible, where x_0 represents no trade and x_1 represents trading. The efficient allocation rule q, to trade whenever the total surplus is positive, is summarized below.

p	θ_2^1	θ_2^2	θ_2^3
θ_1^1	0.1	0.1	0.1
θ_1^2	0.1	0.1	0.1
θ_1^3	0.2	0.15	0.05

q	θ_2^1	θ_2^2	θ_2^3
θ_1^1	x_1	x_0	x_0
θ_1^2	x_1	x_1	x_0
θ_1^3	x_1	x_1	x_1

Since both agents' CI conditions fail, from the proof of Proposition 3.1, there exist functions v_1 and v_2 such that q is not implementable via an IR and BB Bayesian mechanism.

To see how ambiguous transfers work, let $\Phi = \{\phi^1, \phi^2\}$. Define

$$\phi_i^1(\theta) = \begin{cases} \eta(\theta) + c\psi_1(\theta), & \text{if } i = 1, \\ -\eta(\theta) + c\psi_2(\theta), & \text{if } i = 2, \end{cases} \qquad \phi_i^2(\theta) = \begin{cases} \eta(\theta) - c\psi_1(\theta), & \text{if } i = 1, \\ -\eta(\theta) - c\psi_2(\theta), & \text{if } i = 2, \end{cases}$$

where c is a sufficiently large number, and functions η and $\psi_1 = -\psi_2$ are given below.

η	θ_2^1	θ_2^2	θ_2^3
θ_1^1	$v_2(\theta_2^1)$	0	0
θ_1^2	$v_2(\theta_2^1)$	$v_2(\theta_2^2)$	0
θ_1^3	$v_2(\theta_2^1)$	$v_2(\theta_2^2)$	$v_2(\theta_2^3)$

ψ_1	θ_2^1	θ_2^2	θ_2^3
θ_1^1	9	-18	9
θ_1^2	-3	6	-3
θ_1^3	-3	8	-12

It is easy to verify the conditions of IR and BB for Φ. For simplicity, we only establish $IC(\theta_1^1\theta_1^2)$ and $IC(\theta_1^2\theta_1^1)$ below and omit the details of verifying other IC constraints.

$$\begin{aligned} IC(\theta_1^1\theta_1^2) \quad & \tfrac{1}{3}v_1(\theta_1^1)+\tfrac{1}{3}v_2(\theta_2^1)-c|\tfrac{1}{3}(9)+\tfrac{1}{3}(-18)+\tfrac{1}{3}(9)| = \tfrac{1}{3}v_1(\theta_1^1)+\tfrac{1}{3}v_2(\theta_2^1) > \\ & \tfrac{2}{3}v_1(\theta_1^1)+\tfrac{1}{3}v_2(\theta_2^1)+\tfrac{1}{3}v_2(\theta_2^2)-c|\tfrac{1}{3}(-3)+\tfrac{1}{3}(6)+\tfrac{1}{3}(-3)| = \tfrac{2}{3}v_1(\theta_1^1)+\tfrac{1}{3}v_2(\theta_2^1)+\tfrac{1}{3}v_2(\theta_2^2), \\ IC(\theta_1^2\theta_1^1) \quad & \tfrac{2}{3}v_1(\theta_1^2)+\tfrac{1}{3}v_2(\theta_2^1)+\tfrac{1}{3}v_2(\theta_2^2)-c|\tfrac{1}{3}(-3)+\tfrac{1}{3}(6)+\tfrac{1}{3}(-3)| = \tfrac{2}{3}v_1(\theta_1^2)+\tfrac{1}{3}v_2(\theta_2^1)+\tfrac{1}{3}v_2(\theta_2^2) > \\ & \tfrac{1}{3}v_1(\theta_1^2)+\tfrac{1}{3}v_2(\theta_2^1)-c|\tfrac{1}{3}(9)+\tfrac{1}{3}(-18)+\tfrac{1}{3}(9)| = \tfrac{1}{3}v_1(\theta_1^2)+\tfrac{1}{3}v_2(\theta_2^1). \end{aligned}$$

The strict inequalities above follow from the fact that $v_1(\theta_1^1)+v_2(\theta_2^2) < 0$ and $v_1(\theta_1^2)+v_2(\theta_2^2) > 0$. Therefore, the IR and BB mechanism with ambiguous transfers implements q.

4 Extension

4.1 No common prior

In this section, we study implementation via ambiguous transfers when agents do not necessarily have a common prior. We also demonstrate with examples that ambiguous transfers may implement Bayesian non-implementable allocation rules when there is no common prior.

The common prior condition is used in Parts 2 and 3 of Theorem 3.1. In fact, without a common prior, the following example shows that the BDP property is no longer sufficient for implementation via an IR and BB mechanism with ambiguous transfers.

Example 4.1: *Consider an adaptation of Example 3.2 where each agent has two types. In* $A = \{x_0, x_1, x_2\}$*, the payoffs of* x_1 *and* x_2 *are presented below. Assume* $0 < 16a < B$*.*

x_1	θ_2^1	θ_2^2
θ_1^1	$a, 0$	a, a
θ_1^2	$a, 0$	a, a

x_2	θ_2^1	θ_2^2
θ_1^1	a, a	$a - 2B, a + B$
θ_1^2	a, a	$a - 2B, a + B$

The efficient allocation rule is $q(\theta_1^1, \theta_2^1) = q(\theta_1^2, \theta_2^1) = x_2$ *and* $q(\theta_1^1, \theta_2^2) = q(\theta_1^2, \theta_2^2) = x_1$*. Let the beliefs satisfy* $p_1(\theta_2^1|\theta_1^1) = 0.75$*,* $p_1(\theta_2^1|\theta_1^2) = 0.25$*,* $p_2(\theta_1^1|\theta_2^1) = 0.7$*, and* $p_2(\theta_1^1|\theta_2^2) = 0.3$*, which cannot be generated from a common prior. The BDP property holds for both agents.*

Suppose by contradiction that an IR and BB mechanism with ambiguous transfers (q, Φ) *implements* q*. By* $IC(\theta_2^2\theta_2^1)$ *and the IR condition, for all* $\epsilon > 0$*, there exists a BB transfer rule* $\phi = (-\phi_2, \phi_2) \in \Phi$ *such that:*

$$
\begin{array}{ll}
IC(\theta_2^2\theta_2^1) & a + 0.3\phi_2(\theta_1^1, \theta_2^2) + 0.7\phi_2(\theta_1^2, \theta_2^2) + \epsilon \geq a + B + 0.3\phi_2(\theta_1^1, \theta_2^1) + 0.7\phi_2(\theta_1^2, \theta_2^1), \\
IR(\theta_1^1) & a - 0.75\phi_2(\theta_1^1, \theta_2^1) - 0.25\phi_2(\theta_1^1, \theta_2^2) \geq 0, \\
IR(\theta_1^2) & a - 0.25\phi_2(\theta_1^2, \theta_2^1) - 0.75\phi_2(\theta_1^2, \theta_2^2) \geq 0, \\
IR(\theta_2^1) & a + 0.7\phi_2(\theta_1^1, \theta_2^1) + 0.3\phi_2(\theta_1^2, \theta_2^1) \geq 0, \\
IR(\theta_2^2) & a + 0.3\phi_2(\theta_1^1, \theta_2^2) + 0.7\phi_2(\theta_1^2, \theta_2^2) \geq 0.
\end{array}
$$

Multiply the expressions by 4*,* 18*,* 14*,* 21*, and* 11 *respectively, add them up, and let* ϵ *go to zero. It follows that* $0 \geq 4B - 64a > 0$*, a contradiction.*

Hence, q *is not implementable via an IR and BB mechanism with ambiguous transfers.*

In the Bayesian mechanism design literature, Bergemann et al. (2012), Smith (2010), and Börgers et al. (2015) have documented results related to ex-post efficiency maximization without a common prior. Without requiring IR and BB, Bergemann et al. (2012) show that the BDP property is sufficient for Bayesian implementation of efficient allocations, but the current paper requires the conditions of IR and BB. Smith (2010) compares the welfare of two mechanisms on public good provision, and Börgers et al. (2015) provide a sufficient condition on when agents' interim payoffs can be arbitrarily increased in an IC mechanism. Different from Smith (2010) and Börgers et al. (2015), the current section provides a general condition on when the first-best efficiency is implementable via an IR and BB mechanism.

Below we introduce two properties to generalize Parts 2 and 3 of Theorem 3.1 to potentially non-common prior environments. For all $i \neq j$, θ_i, and θ_j, by slightly abusing notations, we let $p_j(\theta_i, \cdot|\theta_j)$ be the $|\Theta_{-i-j}|$-dimensional vector $\big(p_j(\theta_i, \theta_{-i-j}|\theta_j)\big)_{\theta_{-i-j}\in\Theta_{-i-j}}$ when $N \geq 3$, and be the number $p_j(\theta_i|\theta_j)$ when $N = 2$.

Definition 4.1: *Given a collection of priors $(p_i)_{i\in I}$, an agent $i \in I$, two types $\bar{\theta}_i \neq \hat{\theta}_i$, a prior $\mu \in \Delta(\Theta)$, and two constants $\bar{C}, \hat{C}$, define two conditions below:*

***Condition** $N1$: $\mu(\theta_j) > 0$ and $\mu(\theta_{-j}|\theta_j) = p_j(\theta_{-j}|\theta_j)$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$ and θ_{-j};*

***Condition** $N2$: $\hat{C}p_i(\theta_j, \cdot|\hat{\theta}_i) = p_i(\theta_j, \cdot|\bar{\theta}_i) + \bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\hat{\theta}_i|\theta_j)}p_j(\hat{\theta}_i, \cdot|\theta_j)$ for all $j \neq i$ and θ_j.*

*The priors $(p_i)_{i\in I}$ satisfy the **Weak No Common Prior*** (WNCP*) property for agent i if there do not exist types $\bar{\theta}_i \neq \hat{\theta}_i$, a prior $\mu \in \Delta(\Theta)$, and constants $\bar{C} \geq 1$ and $\hat{C} > 1$ such that Conditions $N1$ and $N2$ hold.*

*The priors $(p_i)_{i\in I}$ satisfy the **No Common Prior*** (NCP*) property for agent i if there do not exist types $\bar{\theta}_i \neq \hat{\theta}_i$, a prior $\mu \in \Delta(\Theta)$, and constants $\bar{C} > 0$ and $\hat{C} > 1$ such that Conditions $N1$ and $N2$ hold.*

Note that both sides of the equation in Condition N2 are $|\Theta_{-i-j}|$-dimensional vectors or numbers. In the definitions of the WNCP* and NCP* properties, the only difference is the size of $\bar{C}$. The NCP* property is stronger than the WNCP* property.

When the properties fail for agent i, Condition N1 requires the existence of types $\bar{\theta}_i \neq \hat{\theta}_i$ such that beliefs of all agents except the one of $\hat{\theta}_i$ can be generated by a common prior. To see the requirement of Condition N2 more clearly, when agents have full-support priors, we can divide both sides of the equation in Condition N2 by $p_i(\theta_j \cdot |\bar{\theta}_i)$ and take into account Condition N1. Then the equation in Condition N2 becomes $\hat{C}\frac{p_i(\theta_{-i}|\hat{\theta}_i)}{p_i(\theta_{-i}|\bar{\theta}_i)} = 1 + \bar{C}\frac{\mu(\hat{\theta}_i, \theta_{-i})}{\mu(\bar{\theta}_i, \theta_{-i})}, \forall \theta_{-i} \in \Theta_{-i}$, i.e., the ratio of i's beliefs under types $\hat{\theta}_i$ and $\bar{\theta}_i$ has a linear relationship with the ratio of the prior μ at types $\hat{\theta}_i$ and $\bar{\theta}_i$.

Let $(i, \bar{\theta}_i, \hat{\theta}_i) = (2, \theta_2^2, \theta_2^1)$, $\mu(\theta_1^1, \theta_2^1) = \frac{27}{64}$, $\mu(\theta_1^1, \theta_2^2) = \frac{9}{64}$, $\mu(\theta_1^2, \theta_2^1) = \frac{7}{64}$, $\mu(\theta_1^2, \theta_2^2) = \frac{21}{64}$, $\bar{C} = \frac{15}{4}$, and $\hat{C} = \frac{21}{4}$, one can see that both WNCP* and NCP* properties fail for agent 2 in Example 4.1.

In the special case that agents have a common prior, we have the following lemma.

Lemma 4.1: *Given a common prior p, the following three statements are equivalent:*

1. *the BDP property holds for agent i;*
2. *the NCP* property holds for agent i;*
3. *the WNCP* property holds for agent i.*

When $N = 2$, for Condition N1 to hold for agent i, we must have $\hat{C}\frac{p_i(\theta_j|\hat{\theta}_i)}{p_i(\theta_j|\bar{\theta}_i)} = 1 + \bar{C}\frac{p_j(\hat{\theta}_i|\theta_j)}{p_j(\bar{\theta}_i|\theta_j)}$ for all $\theta_j \in \Theta_j$. There are $|\Theta_j|$ linear equations. When $|\Theta_j| > 2$, it is generically impossible to find $\bar{C}$ and $\hat{C}$ satisfying all $|\Theta_j|$ equations, and thus, the NCP* and WNCP* properties hold for agent i under almost all priors $(p_i \in \Delta(\Theta))_{i\in I}$ over a finite, naive type space.

When $N \geq 3$, we say the **NCP**** property holds if there are agents $i \neq j$ and types $\bar{\theta}_i \neq \hat{\theta}_i$, and $\bar{\theta}_j \neq \hat{\theta}_j$, such that the marginal beliefs over Θ_{-i-j} satisfies $p_i(\cdot|\bar{\theta}_i, \bar{\theta}_j) \neq p_j(\cdot|\bar{\theta}_i, \bar{\theta}_j)$ and $p_i(\cdot|\hat{\theta}_i, \hat{\theta}_j) \neq p_j(\cdot|\hat{\theta}_i, \hat{\theta}_j)$. Namely, there should be two agents whose marginal beliefs towards the rest of the agents are different at two type profiles. Notice this property is stated across agents instead of for a particular agent. Without imposing the common prior condition, the NCP** property holds for almost all priors $(p_i \in \Delta(\Theta))_{i\in I}$ over a finite naive type space. When the NCP** property holds, for any agent, there does not exist μ such that Condition N1 holds, and thus the NCP* and WNCP* properties are satisfied by all agents.

In view of Lemma 4.1, the following proposition generalizes Parts 2 and 3 of Theorem 3.1 to the case when there may not exist a common prior. Since Theorem 3.1 is more elegant and only involves the BDP property in the characterization, we leave it as the main result and Proposition 4.1 as an extension.

Proposition 4.1: *Given a collection of priors $(p_i)_{i\in I}$,*

1. *if the BDP or WNCP* property fails for some agent, then there exists a profile of utility functions under which an efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers; if the BDP and NCP* properties hold for all agents, then any ex-post efficient allocation rule under any profile of utility functions is implementable via an IR and BB mechanism with ambiguous transfers;*
2. *if there exist agents $i \neq j$ such that the WNCP* property fails for i and the BDP property fails for j, then there exists a profile of* ***private value*** *utility functions under which an ex-post efficient allocation rule is not implementable via an IR and BB mechanism with ambiguous transfers; if there do not exist agents $i \neq j$ such that the WNCP* property fails for i and the BDP property fails for j, then any ex-post efficient allocation rule under any profile of* ***private value*** *utility functions is implementable via an IR and BB mechanism with ambiguous transfers.*

When the BDP property holds for all agents, the necessary and (stronger) sufficient conditions in Part 2 of the proposition hold. Hence, when the BDP property holds for all agents, with or without a common prior, efficient implementation via ambiguous transfers can be guaranteed under private value environments.

Similar to Theorem 3.1, the efficiency of an allocation rule q does not play a role in the proof of Part 1 of Proposition 4.1, but plays a role in Part 2. The non-common prior condition is used explicitly when we prove the sufficiency direction of Part 2.

In Example 4.2, the first group of sufficiency conditions of Theorem 4.1 holds. As a result, any efficient allocation rule is implementable via ambiguous transfers. Since we find an efficient allocation rule that is not implementable via Bayesian mechanisms, we demonstrate that ambiguous transfers may perform better than Bayesian mechanisms.

Example 4.2: *Under the following beliefs without a common prior, the efficient allocation rule q is not Bayesian implementable, but it is implementable via ambiguous transfers.*

$p_1(\tilde{\theta}_2\vert\tilde{\theta}_1)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	$\frac{7}{28}$	$\frac{12}{28}$	$\frac{9}{28}$
θ_1^2	$\frac{13}{28}$	$\frac{12}{28}$	$\frac{3}{28}$

$p_2(\tilde{\theta}_1\vert\tilde{\theta}_2)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
θ_1^2	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$

The feasible set of outcomes, payoffs, and the efficient allocation rule are identical to those in Example 3.2, except that $0 < 8.5a < B$ is imposed. Suppose by contradiction that there exists a BB Bayesian transfer rule $\phi = (-\phi_2, \phi_2) : \Theta \to \mathbb{R}^2$ implementing q. As in Example 3.2, by multiplying $IR(\theta_1^1)$, $IR(\theta_1^2)$, $IR(\theta_2^1)$, $IR(\theta_2^2)$, $IR(\theta_2^3)$, $IC(\theta_1^1\theta_1^2)$, $IC(\theta_1^2\theta_1^1)$, $IC(\theta_2^1\theta_2^2)$, $IC(\theta_2^1\theta_2^3)$, $IC(\theta_2^2\theta_2^1)$, and $IC(\theta_2^3\theta_2^2)$ by 7, 7, 3, 8, 3, 3.5, 3.5, 3, 3, 4, and 3, and summing up, we obtain $0 \geq 4B - 34a$, a contradiction. Hence, q is not Bayesian implementable.

It is easy to see that both agents satisfy the BDP property. We demonstrate below that the NCP property holds for agent 1 and omit the verification for agent 2. When $N = 2$, Condition N2 becomes $\hat{C}\frac{p_i(\theta_j|\hat{\theta}_i)}{p_i(\theta_j|\bar{\theta}_i)} = 1 + \bar{C}\frac{p_j(\hat{\theta}_i|\theta_j)}{p_j(\bar{\theta}_i|\theta_j)}$ for all θ_j. Consider $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^1, \theta_1^2)$, the NCP* property holds because there does not exist $\bar{C} > 0$ and $\hat{C} > 1$ such that $\hat{C}(\frac{13}{7}, 1, \frac{1}{3}) = (1, 1, 1) + \bar{C}(2, 1, 0.5)$. A symmetric argument applies to $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^2, \theta_1^1)$.*

By Part 1 of Proposition 4.1, q is implementable via ambiguous transfers.

In the following private value bilateral trading example, there exists an efficient allocation rule q that is not Bayesian implementable. However, the second group of sufficient conditions of Proposition 4.1 holds, and thus q is implementable via ambiguous transfers. Hence, when we confine our analysis to private value environments without a common prior, ambiguous transfers may perform better than Bayesian mechanisms.

Example 4.3: *Agent* 1 *is the seller of a unit of indivisible good and* 2 *is the buyer. Outcomes in* $A = \{x_0, x_1\}$ *are feasible. The outcome* x_0 *represents no trade. The payoffs of* x_1*, trading, are given below. The efficient allocation rule is* $q(\theta_1^1, \theta_2^2) = x_0$ *and* $q(\theta) = x_1$ *for all other* θ*.*

x_1	θ_2^1	θ_2^2
θ_1^1	-3.5, 4	-3.5, 1
θ_1^2	-0.5, 4	-0.5, 1

Let the beliefs satisfy $p_1(\theta_2^1|\theta_1^1) = 0.3$*,* $p_1(\theta_2^1|\theta_1^2) = 0.25$*,* $p_2(\theta_1^1|\theta_2^1) = 0.3$*, and* $p_2(\theta_1^1|\theta_2^2) = 0.2$*, which cannot be generated by a common prior.*

Suppose by way of contradiction that there exists an IR and BB Bayesian transfer rule $\phi = (-\phi_2, \phi_2) : \Theta \to \mathbb{R}^2$ *that implements* q*. As in Example 3.2, multiply* $IR(\theta_1^1)$*,* $IC(\theta_1^2\theta_1^1)$*,* $IC(\theta_2^1\theta_2^2)$*,* $IR(\theta_2^2)$*, and* $IC(\theta_2^2\theta_2^1)$ *by* $10, 8, 4, 10$*, and* 1 *respectively and add them up. We obtain* $0 \geq 0.9$*, a contradiction. Therefore,* q *is not Bayesian implementable.*

However, as the BDP property holds for both agents, by Part 2 of Proposition 4.1, q *is implementable via an IR and BB mechanism with ambiguous transfers.*

4.2 Other ambiguity aversion preferences

To check the robustness of our result, we look at alternative preferences of ambiguity aversion in this subsection. One is the α-maxmin expected utility (α-MEU) as in Ghirardato and Marinacci (2002), and the other is the smooth ambiguity aversion preferences of Klibanoff et al. (2005). Even though these preferences differ from Gilboa and Schmeidler (1989), the mechanism designer can still benefit from agents' ambiguity aversion.

Ghirardato and Marinacci (2002) introduce the α-MEU, which is a generalization of the MEU. Under an environment described in Section 2, a type-θ_i agent i with **α-maxmin expected utility** has the following interim utility level from participating and reporting truthfully when Φ is the set of ambiguous transfers:

$$\alpha \inf_{\phi \in \Phi} \{ \sum_{\theta_{-i} \in \Theta_{-i}} u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) p_i(\theta_{-i}|\theta_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) \}$$
$$+ (1-\alpha) \sup_{\phi \in \Phi} \{ \sum_{\theta_{-i} \in \Theta_{-i}} u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) p_i(\theta_{-i}|\theta_i) + \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) \},$$

where $\alpha \in [0,1]$. An agent is said to be ambiguity-averse if $\alpha > 0.5$. All previous sections adopt the MEU preferences, which correspond to the case $\alpha = 1$.

Alternatively, an agent i with **smooth ambiguity aversion** has a utility function of

$$\int_{\pi \in \Delta(\Phi)} v\Big(\int_{\phi \in \Phi} \Big(\sum_{\theta_{-i} \in \Theta_{-i}} [u_i\big(q(\theta_i, \theta_{-i}), (\theta_i, \theta_{-i})\big) + \phi_i(\theta_i, \theta_{-i})] p_i(\theta_{-i}|\theta_i) \Big) d\pi \Big) d\mu,$$

where

- for each distribution $\pi \in \Delta(\Phi)$, $\pi(\phi)$ measures the subjective density that ϕ is the true transfer rule chosen by the mechanism designer;
- for each distribution $\mu \in \Delta(\Delta(\Phi))$, $\mu(\pi)$ measures the subjective density that $\pi \in \Delta(\Phi)$ is the right density function the mechanism designer uses to choose the transfer rule;
- $v : R \to R$ is a strictly increasing function that characterizes ambiguity attitude, where a strictly concave v implies ambiguity aversion.

Under the α-MEU preferences with $\alpha > 0.5$ or the smooth ambiguity aversion preferences with a strictly concave v, the sufficiency part of Theorem 3.1 still holds. We can construct ambiguous transfers in the same way as those under MEU except for choosing a potentially different multiplier c.

As an illustration, we demonstrate with the Example 3.2. Let v be a strictly increasing and strictly concave function. Consider the same transfers as ϕ^1 and ϕ^2 except for a potentially different multiplier c. Then it is easy to verify individual rationality and budget balance. A generic element of $\Delta(\Phi)$ is a Bernoulli distribution between ϕ^1 and ϕ^2. Let μ be the uniform distribution over $\Delta(\Phi)$ for example. As an illustration, we check $IC(\theta_2^2\theta_2^1)$. Truth-telling always gives agent 2 an expected utility of

$$\int_0^1 v(\mu a + (1-\mu)a)d\mu = v(a).$$

By misreporting from θ_2^2 to θ_2^1, agent 2 gets an interim utility of

$$\int_0^1 v\big(\mu(a+B+c) + (1-\mu)(a+B-c)\big)d\mu.$$

For v sufficiently concave or c sufficiently large, the above expression has a value no more than $v(a)$, implying that $IC(\theta_2^2\theta_2^1)$ holds. One can verify other IC constraints as well.

A Appendix

Lemma A.1: *Given a collection of priors $(p_i)_{i\in I}$, if the BDP property holds for agent i, then for all $\bar\theta_i, \hat\theta_i \in \Theta_i$ with $\bar\theta_i \neq \hat\theta_i$, there exists a transfer rule $\psi^{\bar\theta_i\hat\theta_i} : \Theta \to \mathbb{R}^N$ such that*

1. $\sum_{\theta_{-j}\in\Theta_{-j}} \psi_j^{\bar\theta_i\hat\theta_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j}|\theta_j) = 0$ *for all $j \in I$ and $\theta_j \in \Theta_j$;*

2. $\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{\bar\theta_i\hat\theta_i}(\hat\theta_i, \theta_{-i}) p_i(\theta_{-i}|\bar\theta_i) < 0.$

Proof. We first define vectors $p_{\theta_j\theta'_j}$ for all $j \in I$ and $\theta_j, \theta'_j \in \Theta_j$. Each $p_{\theta_j\theta'_j}$ has $N \times |\Theta|$ dimensions and every dimension corresponds to an agent and a type profile. For any $j \in I$

and $\theta_j, \theta_j' \in \Theta_j$, whenever there exists $\theta_{-j} \in \Theta_{-j}$ such that a dimension of $p_{\theta_j \theta_j'}$ corresponds to agent j and type profile (θ_j', θ_{-j}), let this dimension be $p_j(\theta_{-j}|\theta_j)$. Thus we have defined $|\Theta_{-j}|$ dimensions of the vector $p_{\theta_j \theta_j'}$. Let all other dimensions of $p_{\theta_j \theta_j'}$ be 0.[10]

Suppose by way of contradiction that the BDP property holds for agent i, but there exist different types $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$, such that no $\psi^{\bar{\theta}_i \hat{\theta}_i}$ satisfies the two conditions. By Fredholm's theorem of the alternative, there exist coefficients $(a_{\theta_j})_{j \in I, \theta_j \in \Theta_j}$ such that

$$p_{\bar{\theta}_i \hat{\theta}_i} = \sum_{j \in I} \sum_{\theta_j \in \Theta_j} a_{\theta_j} p_{\theta_j \theta_j}.$$

By focusing on each dimension of $p_{\bar{\theta}_i \hat{\theta}_i}$ that corresponds to agent i and type profile $(\hat{\theta}_i, \theta_{-i})$, we know that $p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i)$ for all $\theta_{-i} \in \Theta_{-i}$. Adding this expression over $\theta_{-i} \in \Theta_{-i}$ yields $a_{\hat{\theta}_i} = 1$. Hence, $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, contradicting the BDP property. □

Lemma A.2: *For any $K \times K$ matrix $X = (x_{k\tilde{k}})$ whose diagonal elements are all negative, there exists a vector $\lambda \in \mathbb{R}_+^K \backslash \{\mathbf{0}\}$ such that $\sum_{\tilde{k}=1}^{K} x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$ for all $k = 1, ..., K$.*

Proof. We prove the result by induction.

When $K = 1$. Pick an arbitrary $\lambda_1 > 0$. As $x_{11} < 0$, the statement holds for 1.

Suppose the statement holds for $K - 1$, where $K \geq 2$. Consider any $K \times K$ matrix X with negative diagonal elements. By the supposition for the northwest $K-1$ by $K-1$ block, there exists a non-zero vector $(\lambda_1, ..., \lambda_{K-1}) \in \mathbb{R}_+^{K-1} \backslash \{\mathbf{0}\}$ such that $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$ for all $k = 1, ..., K - 1$.

Case 1. Suppose $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda_{\tilde{k}} \neq 0$. Let $\lambda_K = 0$, and thus the statement holds for K.

Case 2. Suppose $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda_{\tilde{k}} = 0$ and there exists $k_0 \in \{1, ..., K-1\}$ such that $x_{Kk_0} \lambda_{k_0} \neq 0$. Let $(\lambda_1', ..., \lambda_{K-1}') = (\lambda_1, ..., \lambda_{k_0 - 1}, \lambda_{k_0} + \epsilon, \lambda_{k_0+1}, ..., \lambda_{K-1})$ for $\epsilon > 0$. Then $\sum_{\tilde{k}=1}^{K-1} x_{K\tilde{k}} \lambda_{\tilde{k}}' \neq 0$. When ϵ is sufficiently close to zero, as $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}} \neq 0$ for all $k = 1, ..., K - 1$, we also have $\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}}' \neq 0$ for all $k = 1, ..., K - 1$. Thus, we can replace $(\lambda_1, ..., \lambda_{K-1})$ with $(\lambda_1', ..., \lambda_{K-1}')$ and go back to Case 1.

Case 3. Suppose $x_{K\tilde{k}} \lambda_{\tilde{k}} = 0$ for all $\tilde{k} = 1, ..., K - 1$. Pick any $\lambda_K > 0$ such that $\lambda_K \neq -\frac{\sum_{\tilde{k}=1}^{K-1} x_{k\tilde{k}} \lambda_{\tilde{k}}}{x_{kK}}$ for all $k = 1, ..., K - 1$ with $x_{kK} \neq 0$. The statement thus holds for K. □

Lemma A.3: *Given a collection of priors $(p_i)_{i \in I}$, if the BDP property holds for all agents, then there exists a transfer rule $\psi : \Theta \to \mathbb{R}^N$ such that*

1. $\displaystyle\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) = 0$ *for all $i \in I$ and $\theta_i \in \Theta_i$;*

[10] As an illustration, consider $I = \{1, 2\}$ and $\Theta = \{(\theta_1^1, \theta_2^1), (\theta_1^1, \theta_2^2), (\theta_1^2, \theta_2^1), (\theta_1^2, \theta_2^2)\}$. For each $p_{\theta_j \theta_j'}$, its first (last) four dimensions correspond to agent 1 (2) and the type profile $(\theta_1^1, \theta_2^1), (\theta_1^1, \theta_2^2), (\theta_1^2, \theta_2^1)$, and (θ_1^2, θ_2^2) respectively. Then for example, $p_{\theta_2^2 \theta_2^1} = (0, 0, 0, 0, p_2(\theta_1^1|\theta_2^2), 0, p_2(\theta_1^2|\theta_2^2), 0)$.

2. $\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) \neq 0$ *for all* $i \in I$ *and* $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ *with* $\bar{\theta}_i \neq \hat{\theta}_i$.

Proof. Let K be the cardinality of the set $\mathcal{K} = \{(\bar{\theta}_i, \hat{\theta}_i) | i \in I, \bar{\theta}_i, \hat{\theta}_i \in \Theta_i, \bar{\theta}_i \neq \hat{\theta}_i\}$. Let $f : \mathcal{K} \to \{1, ..., K\}$ be a one-to-one mapping, which allows us to index the elements of $\mathcal{K}$.

For all $k, \tilde{k} \in \{1, ..., K\}$ ($k, \tilde{k}$ may be equal), where $f^{-1}(k) = (\bar{\theta}_i, \hat{\theta}_i)$ and $f^{-1}(\tilde{k}) = (\tilde{\bar{\theta}}_j, \tilde{\hat{\theta}}_j)$, define a number $x_{k\tilde{k}} = \sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{\tilde{\bar{\theta}}_j\tilde{\hat{\theta}}_j}(\hat{\theta}_i, \theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)$, where each transfer rule $\psi^{\tilde{\bar{\theta}}_j\tilde{\hat{\theta}}_j}$ is defined and proved to exist in Lemma A.1. Recall the second condition of $\psi^{\tilde{\bar{\theta}}_j\tilde{\hat{\theta}}_j}$ implies that $x_{\tilde{k}\tilde{k}} < 0$ for all $\tilde{k} = 1, ..., K$. Then $X \equiv (x_{k\tilde{k}})$ is a $K \times K$ matrix. By Lemma A.2, there exists $\lambda \in \mathbb{R}_+^K \backslash \{\mathbf{0}\}$ such that $\sum_{\tilde{k}=1}^K x_{k\tilde{k}}\lambda_{\tilde{k}} \neq 0$ for all $k = 1, ..., K$. Hence, for all $(\bar{\theta}_i, \hat{\theta}_i) \in \mathcal{K}$,

$$\sum_{\tilde{k}=1}^{K}[\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{f^{-1}(\tilde{k})}(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)]\lambda_{\tilde{k}} = \sum_{\theta_{-i}\in\Theta_{-i}} [\sum_{\tilde{k}=1}^{K} \lambda_{\tilde{k}}\psi_i^{f^{-1}(\tilde{k})}(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i) \neq 0. \quad (1)$$

Define a new transfer rule ψ by making a linear combination of the rules $(\psi^{f^{-1}(\tilde{k})})_{\tilde{k}=1,...,K}$ such that $\psi \equiv \sum_{\tilde{k}=1}^K \lambda_{\tilde{k}}\psi^{f^{-1}(\tilde{k})}$. Thus by expression (1), the transfer rule ψ satisfies the second condition of this lemma. The first condition also holds for ψ because each $\psi^{f^{-1}(\tilde{k})}$ satisfies this condition. □

Lemma A.4: *Given a common prior p, if the BDP property holds for agent i, then for all* $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ *with* $\bar{\theta}_i \neq \hat{\theta}_i$, *there exists a transfer rule* $\psi^{\bar{\theta}_i\hat{\theta}_i} : \Theta \to \mathbb{R}^N$ *such that,*
1. $\sum_{j\in I} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta) = 0$ *for all* $\theta \in \Theta$;
2. $\sum_{\theta_{-j}\in\Theta_{-j}} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta_j,\theta_{-j})p_j(\theta_{-j}|\theta_j) = 0$ *for all* $j \in I$ *and* $\theta_j \in \Theta_j$;
3. $\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) < 0$.

Proof. For each $\theta \in \Theta$, define a $N \times |\Theta|$-dimensional vector e_θ below. Every dimension corresponds to an agent and a type profile. Let all dimensions of e_θ that correspond to an agent and type profile θ be 1 and other dimensions be 0.[11] Vectors $(p_{\theta_j\theta_j'})_{j\in I,\theta_j,\theta_j'\in\Theta_j}$ has been defined in Lemma A.1.

Suppose by way of contradiction that the BDP property holds for agent i, but there exist types $\bar{\theta}_i \neq \hat{\theta}_i$, such that no transfer rule $\psi^{\bar{\theta}_i\hat{\theta}_i}$ satisfies the three conditions. By Fredholm's theorem of the alternative, there exist coefficients $(a_{\theta_j})_{j\in I,\theta_j\in\Theta_j}$ and $(b_\theta)_{\theta\in\Theta}$ such that

$$p_{\bar{\theta}_i\hat{\theta}_i} = \sum_{j\in I}\sum_{\theta_j\in\Theta_j} a_{\theta_j}p_{\theta_j\theta_j} + \sum_{\theta\in\Theta} b_\theta e_\theta. \quad (2)$$

[11]As an illustration, recall the same example as in footnote [10]. One has $e_{(\theta_1^2,\theta_2^1)} = (0,0,1,0,0,0,1,0)$.

Fix any agent $j \neq i$. All elements of $p_{\bar{\theta}_i\hat{\theta}_i}$ corresponding to agent j are zero. All those corresponding to agent i and $\bar{\theta}_i$ are zero. Those corresponding to agent i and $\hat{\theta}_i$ may not be zero. The three observations, along with expression (2), imply that

$$0 = a_{\theta_j} p_j(\theta_{-j}|\theta_j) + b_\theta, \forall \theta \in \Theta, \tag{3}$$

$$0 = a_{\bar{\theta}_i} p_i(\theta_{-i}|\bar{\theta}_i) + b_{\bar{\theta}_i,\theta_{-i}}, \forall \theta_{-i} \in \Theta_{-i}, \tag{4}$$

$$p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) + b_{\hat{\theta}_i,\theta_{-i}}, \forall \theta_{-i} \in \Theta_{-i}. \tag{5}$$

Choosing $\theta_i = \bar{\theta}_i$ in expression (3) and cancelling $b_{\bar{\theta}_i,\theta_{-i}}$ in expressions (3) and (4) yield $a_{\theta_j} p_j(\bar{\theta}_i, \theta_{-i-j}|\theta_j) = a_{\bar{\theta}_i} p_i(\theta_{-i}|\bar{\theta}_i)$ for all θ_{-i}. We remark that when $N = 2$, this expression abuses notations slightly as the left-hand side should be $a_{\theta_j} p_j(\bar{\theta}_i|\theta_j)$. Summing across all $\theta_{-i-j} \in \Theta_{-i-j}$ when $N \geq 3$ or ignoring any θ_{-i-j} when $N = 2$ yields $a_{\theta_j} p_j(\bar{\theta}_i|\theta_j) = a_{\bar{\theta}_i} p_i(\theta_j|\bar{\theta}_i)$. As p is a common prior, we further know $a_{\theta_j} = a_{\bar{\theta}_i} \frac{p(\theta_j)}{p(\bar{\theta}_i)}$ for all $\theta_j \in \Theta_j$.

By choosing $\theta_i = \hat{\theta}_i$ in expression (3) and plugging in a_{θ_j} derived in the previous paragraph, we know $b_{\hat{\theta}_i,\theta_{-i}} = - a_{\bar{\theta}_i} \frac{p(\theta_j)}{p(\bar{\theta}_i)} p_j(\hat{\theta}_i, \theta_{-i-j}|\theta_j) = -a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)} p_i(\theta_{-i}|\hat{\theta}_i)$ for all θ_{-i}.

Plugging $b_{\hat{\theta}_i,\theta_{-i}}$ above into expression (5) yields $p_i(\theta_{-i}|\bar{\theta}_i) = (a_{\hat{\theta}_i} - a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)}) p_i(\theta_{-i}|\hat{\theta}_i)$ for all θ_{-i}. Hence, $a_{\hat{\theta}_i} - a_{\bar{\theta}_i} \frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)} = 1$ and $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, a contradiction. □

Lemma A.5: *When there is a common prior p, if the BDP property holds for all agents, then there exists a transfer rule $\psi : \Theta \to \mathbb{R}^N$ such that*

1. $\sum_{i \in I} \psi_i(\theta) = 0$ *for all* $\theta \in \Theta$;
2. $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\theta_i, \theta_{-i}) p_i(\theta_{-i}|\theta_i) = 0$ *for all* $i \in I$ *and* $\theta_i \in \Theta_i$;
3. $\sum_{\theta_{-i} \in \Theta_{-i}} \psi_i(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i}|\bar{\theta}_i) \neq 0$ *for all* $i \in I$ *and* $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ *with* $\bar{\theta}_i \neq \hat{\theta}_i$.

Proof. One can construct a linear combination of transfer rules developed in Lemma A.4 such that the combination satisfies the three conditions here. The detailed argument is omitted as it is analogous to Lemma A.3. □

Proof of Theorem 3.1. **Necessity of Parts 1 and 2**. Suppose there exists $i \in I$ and $\bar{\theta}_i \neq \hat{\theta}_i$ such that $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$. Consider an adaptation of the utility functions constructed by Kosenok and Severinov (2008). Let the set of feasible outcomes be $A = \{x_0, x_1, x_2\}$, where agents' payoffs of consuming x_0 are zero. The payoffs for agent i and all $j \neq i$ to consume x_1 and x_2 are given below with $0 < a < B$.

	$u_i\big(x_1,(\theta_i,\theta_j)\big)$	$u_j\big(x_1,(\theta_i,\theta_j)\big)$	$u_i\big(x_2,(\theta_i,\theta_j)\big)$	$u_j\big(x_2,(\theta_i,\theta_j)\big)$
$\theta_i=\bar{\theta}_i$	a	a	a+B	a-2B
$\theta_i=\hat{\theta}_i$	0	a	a	a
$\theta_i\neq\bar{\theta}_i,\hat{\theta}_i$	a	a	0	a

The efficient allocation rule is $q(\theta)=x_2$ if $\theta_i=\hat{\theta}_i$ and $q(\theta)=x_1$ elsewhere.

Suppose by way of contradiction that full surplus extraction can be achieved by a mechanism with ambiguous transfers (q,Φ). By $IC(\bar{\theta}_i\hat{\theta}_i)$ and $IC(\hat{\theta}_i\bar{\theta}_i)$,

$$\inf_{\phi\in\Phi}\{a+\sum_{\theta_{-i}\in\Theta_{-i}}\phi_i(\bar{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)\}\geq\inf_{\phi\in\Phi}\{a+B+\sum_{\theta_{-i}\in\Theta_{-i}}\phi_i(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)\},$$
$$\inf_{\phi\in\Phi}\{a+\sum_{\theta_{-i}\in\Theta_{-i}}\phi_i(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\hat{\theta}_i)\}\geq\inf_{\phi\in\Phi}\{0+\sum_{\theta_{-i}\in\Theta_{-i}}\phi_i(\bar{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\hat{\theta}_i)\}.$$

Recall that $p_i(\cdot|\bar{\theta}_i)=p_i(\cdot|\hat{\theta}_i)$. Adding these two inequalities gives $2a\geq a+B$, a contradiction. Therefore, the condition that the BDP property holds for all agents is necessary to guarantee full surplus extraction via a mechanism with ambiguous transfers.

To prove that the same condition is necessary for IR and BB implementation via a mechanism with ambiguous transfers, we can adopt the same argument.

Necessity of Part 3. In view of Lemma 4.1, this result is a corollary of the necessity result of Part 2 of Proposition 4.1.

Sufficiency of Part 1. Pick an arbitrary ex-post efficient allocation rule q. Define two transfer rules ϕ and ϕ' by $\phi_i=-\eta_i+c\psi_i$ and $\phi'_i=-\eta_i-c\psi_i$ for all $i\in I$, where ψ is defined and proved to exist in Lemma A.3, $\eta_i(\theta)=u_i(q(\theta),\theta)$ for all $\theta\in\Theta$, and c is no less than

$$\max_{\substack{i\in I,\bar{\theta}_i,\hat{\theta}_i\in\Theta_i,\\ \bar{\theta}_i\neq\hat{\theta}_i}}\frac{\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})\big)-u_i\big(q(\hat{\theta}_i,\theta_{-i}),(\hat{\theta}_i,\theta_{-i})\big)]p_i(\theta_{-i}|\bar{\theta}_i)}{|\sum_{\theta_{-i}\in\Theta_{-i}}\psi_i(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)|}.$$

Define $\Phi=\{\phi,\phi'\}$. All IR constraints bind because $-\eta_i$ extracts agent i's full surplus on path, and $c\psi_i$ has zero interim expected value under agent i's belief. The choice of c gives any unilateral deviator a non-positive worst-case expected payoff, and thus the IC condition also holds. Hence, (q,Φ) extracts the full surplus.

Sufficiency of Part 2. In view of Lemma 4.1, sufficiency of Parts 2 and 3 can be viewed as a corollary of the sufficiency results of Proposition 4.1. However, with a common prior, we are able to construct mechanisms with two transfer rules to fulfill our goal, which are particularly simple and make the robustness check in Section 4.2 easier. As a result, we present the proofs of sufficiency direction of Parts 2 and 3 of Theorem 3.1 here.

Pick any BB transfer rule $\eta : \Theta \to \mathbb{R}^N$ such that $\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\theta_i,\theta_{-i}),(\theta_i,\theta_{-i})\big) + \eta_i(\theta_i,\theta_{-i})]p_i(\theta_{-i}|\theta_i) \geq 0$ for all $i \in I$ and $\theta_i \in \Theta_i$. For example, we can choose $\eta_i(\theta) = \frac{1}{N}\sum_{j\in I} u_j(q(\theta),\theta) - u_i(q(\theta),\theta)$ for all $i \in I$ and $\theta \in \Theta$ so that all agents have equal surplus. By Lemma A.5, there exists a BB transfer rule ψ which gives all agents zero expected values on path and gives any unilateral deviator a non-zero expected value.

Pick any c that is no less than

$$\frac{\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})\big) + \eta_i(\hat{\theta}_i,\theta_{-i}) - u_i\big(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})\big) - \eta_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)}{|\sum_{\theta_{-i}\in\Theta_{-i}}\psi_i(\hat{\theta}_i,\theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i)|}$$

for all i and $\bar{\theta}_i \neq \hat{\theta}_i$, where the denominator is positive. Let $\mathcal{M}$ be $(q, \{\eta + c\psi, \eta - c\psi\})$.

The IR condition follows from the choice of η and the fact that ψ gives agents zero expected values on path. For all i and $\bar{\theta}_i \neq \hat{\theta}_i$, the choice of c indicates that

$$\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})\big) + \eta_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i) \geq$$
$$\min\{\sum_{\theta_{-i}\in\Theta_{-i}}[u_i\big(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i})\big) + \eta_i(\hat{\theta}_i,\theta_{-i}) \pm c\psi_i(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\},$$

and thus $\mathcal{M}$ satisfies the IC condition. The BB condition of $\mathcal{M}$ follows from BB of η and ψ. Therefore, $\mathcal{M}$ is an IR and BB mechanism with ambiguous transfers that implements q.

Sufficiency of Part 3. Given a collection of common prior p, when the BDP property holds for all agents, the sufficiency part has been proven in Part 2. When there is exactly one agent, i, for whom the BDP property fails, following a similar argument as Lemmas A.4 and A.5, one can prove that there exists a transfer rule $\psi : \Theta \to \mathbb{R}^N$ such that

1. $\sum_{j\in I}\psi_j(\theta) = 0$ for all $\theta \in \Theta$;
2. $\sum_{\theta_{-j}\in\Theta_{-j}}\psi_j(\theta_j,\theta_{-j})p_j(\theta_{-j}|\theta_j) = 0$ for all $j \in I$ and $\theta_j \in \Theta_j$;
3. $\sum_{\theta_{-j}\in\Theta_{-j}}\psi_j(\hat{\theta}_j,\theta_{-j})p_j(\theta_{-j}|\bar{\theta}_j) \neq 0$ for all $j \neq i$ and $\bar{\theta}_j, \hat{\theta}_j \in \Theta_j$ satisfying $\bar{\theta}_j \neq \hat{\theta}_j$.

Notice that the third statement is different from the one in Lemma A.5, as the BDP property fails for agent i here.

We construct a mechanism where agent i obtains all the surplus on path. For all $\theta \in \Theta$ and $j \in I$ with $j \neq i$, let $\eta_j(\theta) = -u_j(q(\theta),\theta_j)$, and $\eta_i(\theta) = -\sum_{j\neq i}\eta_j(\theta)$.

Pick any c that is no less than

$$\max_{\substack{j\neq i, \bar{\theta}_j, \hat{\theta}_j\in\Theta_j, \\ \bar{\theta}_j\neq\hat{\theta}_j}} \frac{\sum_{\theta_{-j}\in\Theta_{-j}}[u_j\big(q(\hat{\theta}_j,\theta_{-j}),\bar{\theta}_j\big) - u_j\big(q(\hat{\theta}_j,\theta_{-j}),\hat{\theta}_j\big)]p_j(\theta_{-j}|\bar{\theta}_j)}{|\sum_{\theta_{-j}\in\Theta_{-j}}\psi_j(\hat{\theta}_j,\theta_{-j})p_j(\theta_{-j}|\bar{\theta}_j)|}.$$

Let the set of ambiguous transfers be $\Phi = \{\eta + c\psi, \eta - c\psi\}$, which is IR and BB. The choice of η, ψ, and c implies that agent $j \neq i$ obtains zero worst-case expected payoffs in the interim stage and a unilateral misreporting gives her non-positive ones. Therefore, j's IC constraints are satisfied.

For any $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ with $\bar{\theta}_i \neq \hat{\theta}_i$, the argument below verifies $IC(\bar{\theta}_i\hat{\theta}_i)$:

$$\begin{aligned}
&\min\{\sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\bar{\theta}_i,\theta_{-i}),\bar{\theta}_i\big) + \eta_i(\bar{\theta}_i,\theta_{-i}) \pm c\psi_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\} \\
=&\min\{\sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\bar{\theta}_i,\theta_{-i}),\bar{\theta}_i\big) + \sum_{j\neq i} u_j\big(q(\bar{\theta}_i,\theta_{-i}),\theta_j\big) \pm c\psi_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\} \\
=&\sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\bar{\theta}_i,\theta_{-i}),\bar{\theta}_i\big) + \sum_{j\neq i} u_j\big(q(\bar{\theta}_i,\theta_{-i}),\theta_j\big)]p_i(\theta_{-i}|\bar{\theta}_i) \\
\geq&\sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i\big) + \sum_{j\neq i} u_j\big(q(\hat{\theta}_i,\theta_{-i}),\theta_j\big)]p_i(\theta_{-i}|\bar{\theta}_i) \\
\geq&\min\{\sum_{\theta_{-i}\in\Theta_{-i}} [u_i\big(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i\big) + \sum_{j\neq i} u_j\big(q(\hat{\theta}_i,\theta_{-i}),\theta_j\big) \pm c\psi_i(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\},
\end{aligned}$$

where the first equality comes from the definition of η, the second equality follows from the second property of ψ, the first inequality comes from ex-post efficiency of q at each type profile $(\bar{\theta}_i, \theta_{-i})$, and the second inequality comes from the minimization operation.

Therefore, the IR and BB mechanism with ambiguous transfers implements q. □

Example A.1: *In this private value example, the BDP property holds for $N-1$ agents. But an inefficient allocation rule q is not implementable via an IR and BB mechanism with ambiguous transfers.*

Define a common prior p by $p(\theta_1^3, \theta_2^2) = 2/7$, and $p(\theta) = 1/7$ for all other θ. Only agent 2 satisfies the BDP property. Let feasible allocations be $A = \{x_0, x_1, x_2\}$. The payoffs of x_1 and x_2 are presented below.

x_1	θ_2^1	θ_2^2
θ_1^1	0,0	0,0
θ_1^2	2,0	2,0
θ_1^3	0,0	0,0

x_2	θ_2^1	θ_2^2
θ_1^1	2,0	2,0
θ_1^2	0,0	0,0
θ_1^3	0,0	0,0

Consider an allocation rule $q(\theta) = x_2$ if $\theta_1 = \theta_1^2$, and $q(\theta) = x_1$ elsewhere. Suppose by way of contradiction that q is implemented by $\mathcal{M} = (q, \Phi)$. Let $U_{\theta_1^1}$ and $U_{\theta_1^2}$ denote type-θ_1^1 and type-θ_1^2 agent 1's worst-case expected payoff from participation.

As $IR(\theta_1^1)$ and $IC(\theta_1^2\theta_1^1)$ hold, for any $\epsilon > 0$, there exists $\phi^1 \in \Phi$ such that

$$IR(\theta_1^1) \qquad\qquad 0.5\phi_1^1(\theta_1^1, \theta_2^1) + 0.5\phi_1^1(\theta_1^1, \theta_2^2) \geq U_{\theta_1^1},$$

$$IC(\theta_1^2\theta_1^1) \qquad U_{\theta_1^2}+\epsilon \geq 2+0.5\phi_1^1(\theta_1^1,\theta_2^1)+0.5\phi_1^1(\theta_1^1,\theta_2^2).$$

Similarly, by $IR(\theta_1^2)$ *and* $IC(\theta_1^1\theta_1^2)$, *for any* $\epsilon > 0$, *there exists* $\phi^2 \in \Phi$ *such that*

$$IR(\theta_1^2) \qquad 0.5\phi_1^2(\theta_1^2,\theta_2^1)+0.5\phi_1^2(\theta_1^2,\theta_2^2) \geq U_{\theta_1^2},$$
$$IC(\theta_1^1\theta_1^2) \qquad U_{\theta_1^1}+\epsilon \geq 2+0.5\phi_1^2(\theta_1^2,\theta_2^1)+0.5\phi_1^2(\theta_1^2,\theta_2^2).$$

Add the above inequalities pairwise and let ϵ *go to zero. Thus we have* $U_{\theta_1^2} \geq 2+U_{\theta_1^1}$ *and* $U_{\theta_1^1} \geq 2+U_{\theta_1^2}$. *These two expressions imply* $0 \geq 4$, *which is a contradiction.*

Proof of Proposition 3.1. For each $i \in I$, let θ_i be a generic element of Θ_i. By relabeling the indices, we assume without loss of generality there are non-negative coefficients $(\beta_{\theta_1})_{\theta_1\neq\theta_1^1}, (\beta_{\theta_2})_{\theta_2\neq\theta_2^2}$ such that $p_1(\cdot|\theta_1^1)=\sum_{\theta_1\neq\theta_1^1}\beta_{\theta_1}p_1(\cdot|\theta_1)$ and $p_2(\cdot|\theta_2^2)=\sum_{\theta_2\neq\theta_2^2}\beta_{\theta_2}p_2(\cdot|\theta_2)$, and

$$\frac{\beta_{\theta_2^1}}{p(\theta_2^1)} \geq \frac{\beta_{\theta_2}}{p(\theta_2)}, \forall\theta_2 \neq \theta_2^1,\theta_2^2. \tag{6}$$

Now we construct a profile of private value utility functions $(v_i(\cdot))_{i\in I}$ and an efficient allocation rule q such that q is not implementable via a Bayesian mechanism. In this way we can prove the necessity of the condition in the current proposition.

Let agent 1 own a unit of private good and all others be potential buyers. For each agent $i \in I$, assume that type-θ_i agent i's private value of trading is $v_i(\theta_i)$, where the parameters satisfy that $v_2(\theta_2^1) > -v_1(\theta_1^1) > v_2(\theta_2^2) > -v_1(\theta_1^2) > ... > v_2(\theta_2^{\min\{|\Theta_1|,|\Theta_2|\}}) > -v_1(\theta_1^{\min\{|\Theta_1|,|\Theta_2|\}})$. When $|\Theta_2| \geq |\Theta_1|$, let $-v_1(\theta_1^{|\Theta_1|}) > v_i(\theta_i) > 0$ for any other agent-type pair $(i \neq 1,\theta_i)$ not in the ranking. When $|\Theta_2| < |\Theta_1|$, let $-v_1(\theta_1^{|\Theta_2|}) > -v_1(\theta_1) > v_i(\theta_i)$ for any other agent-type pairs $(1,\theta_1)$ and $(i\neq 2,\theta_i)$ not in the ranking. No trade gives all agents zero payoffs. The efficient allocation rule q is that at a type profile θ, agent 1 should sell the good to agent 2 if and only if $v_1(\theta_1)+v_2(\theta_2) > 0$ (note that $v_1(\theta_1)+v_2(\theta_2)\neq 0$ by construction).

Suppose by way of contradiction there exists an IR and BB Bayesian transfer ϕ that implements q. Then by IR and IC, for all $i \in I$, $\bar\theta_i \neq \hat\theta_i$,

$$IR(\bar\theta_i) \qquad \sum_{\theta_{-i}}\phi_i(\bar\theta_i,\theta_{-i})p_i(\theta_{-i}|\bar\theta_i) \geq -\sum_{\theta_{-i}}u_i(q(\bar\theta_i,\theta_{-i}),\bar\theta_i)p_i(\theta_{-i}|\bar\theta_i),$$
$$IC(\bar\theta_i\hat\theta_i) \qquad \sum_{\theta_{-i}}\phi_i(\bar\theta_i,\theta_{-i})p_i(\theta_{-i}|\bar\theta_i) - \sum_{\theta_{-i}}\phi_i(\hat\theta_i,\theta_{-i})p_i(\theta_{-i}|\bar\theta_i)$$
$$\geq -\sum_{\theta_{-i}}u_i(q(\bar\theta_i,\theta_{-i}),\bar\theta_i)p_i(\theta_{-i}|\bar\theta_i)+\sum_{\theta_{-i}}u_i(q(\hat\theta_i,\theta_{-i}),\bar\theta_i)p_i(\theta_{-i}|\bar\theta_i).$$

Notice there is a common prior p. We choose a constant $\delta > 0$ sufficiently large such that

$$\frac{\delta\beta_{\theta_2^1}p(\theta_2^2)}{p(\theta_2^1)} \geq \frac{\beta_{\theta_1}p(\theta_1^1)}{p(\theta_1)}, \forall\theta_1 \neq \theta_1^1, \tag{7}$$

and then denote the left-hand-side term by γ. Now we compute the weighted sum of the above individual rationality and incentive compatibility constraints where (1) the weight of $IR(\theta_1^1)$ is $p(\theta_1^1)(\gamma+1)$, (2) for each $\theta_1 \neq \theta_1^1$ the weight of $IR(\theta_1)$ is $p(\theta_1)\gamma - \beta_{\theta_1} p(\theta_1^1)$, (3) the weight of $IR(\theta_2^2)$ is $p(\theta_2^2)(\gamma+\delta)$, (4) for each $\theta_2 \neq \theta_2^2$ the weight of $IR(\theta_2)$ is $p(\theta_2)\gamma - \delta\beta_{\theta_2} p(\theta_2^2)$, (5) for each $i \neq 1, 2$ and $\theta_i \in \Theta_i$ the weight of $IR(\theta_i)$ is $p(\theta_i)\gamma$, (6) for each $\theta_1 \neq \theta_1^1$ the weight of $IC(\theta_1\theta_1^1)$ is $p(\theta_1^1)\beta_{\theta_1}$, (7) for each $\theta_2 \neq \theta_2^2$ the weight of $IC(\theta_2\theta_2^2)$ is $\delta\beta_{\theta_2} p(\theta_2^2)$, and (8) every other inequality has weight zero. From expressions (6) and (7), we know all the weights are non-negative.

By the BB condition, the left-hand side of the weighted sum is zero. On the right-hand side, the coefficients of $v_1(\theta_1^1)$ and $v_2(\theta_2^1)$ are $-(\gamma+1)p(\theta_1^1, \theta_2^1)$ and $-\gamma p(\theta_1^1, \theta_2^1)$ respectively. Let $-v_1(\theta_1^1)$ and $v_2(\theta_2^1)$ approach each other and let all other $v_i(\theta_i)$ approach zero. Then the weighted sum implies that $0 \geq -v_1(\theta_1^1)p(\theta_1^1, \theta_2^1) > 0$ (due to Assumption 2.1), a contradiction. Hence, q cannot be implemented by an IR and BB Bayesian mechanism. $\square$

Lemma A.6: *Given a collection of priors $(p_i)_{i\in I}$, there exist a collection of coefficients $(b_\theta)_{\theta\in\Theta}$ and non-negative coefficients $(a_{\theta_j})_{j\in I, \theta_j\in\Theta_j}$ such that equation*

$$p_{\bar{\theta}_i\hat{\theta}_i} = \sum_{j\in I}\sum_{\theta_j\in\Theta_j} a_{\theta_j} p_{\theta_j\theta_j} - \sum_{\theta\in\Theta} b_\theta e_\theta \tag{8}$$

holds if and only if $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$ or there exists $\mu \in \Delta(\Theta)$, $\bar{C} > 0$, and $\hat{C} > 1$ such that
1. $\mu(\theta_j) > 0$ and $\mu(\theta_{-j}|\theta_j) = p_j(\theta_{-j}|\theta_j)$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$;
2. $\hat{C} p_i(\theta_j, \cdot|\hat{\theta}_i) = p_i(\theta_j, \cdot|\bar{\theta}_i) + \bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\hat{\theta}_i|\theta_j)} p_j(\hat{\theta}_i, \cdot|\theta_j)$ for all $j \neq i$ and θ_j.[12]

Proof. **Necessity**. Expression (8) implies

$$0 = a_{\theta_i} p_i(\theta_{-i}|\theta_i) - b_\theta, \forall \theta_i \neq \hat{\theta}_i, \theta_{-i}, \tag{9}$$

$$p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) - b_{\hat{\theta}_i, \theta_{-i}}, \forall \theta_{-i}, \tag{10}$$

$$0 = a_{\theta_j} p_j(\theta_{-j}|\theta_j) - b_\theta, \forall j \neq i, \theta. \tag{11}$$

If $N = 2$, we ignore any term θ_{-i-j} to avoid introducing additional notations.

Case 1. Suppose $a_{\tilde{\theta}_i} = 0$ for some $\tilde{\theta}_i \neq \hat{\theta}_i$. The argument below shows that $a_{\hat{\theta}_i} = 1$, $a_{\theta_j} = 0$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$, $b_\theta = 0$ for all θ, and $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$.

Canceling $b_{\tilde{\theta}_i, \theta_{-i}}$ in (9) and (11) yields $0 = a_{\tilde{\theta}_i} p_i(\theta_{-i}|\tilde{\theta}_i) = a_{\theta_j} p_j(\tilde{\theta}_i, \theta_{-i-j}|\theta_j)$ for all $j \neq i$ and $\theta_{-i} \in \Theta_{-i}$. For all $j \neq i$ and θ_j, it follows that $a_{\theta_j} = 0$ by Assumption 2.1.

By expression (11), the previous paragraph implies $b_\theta = 0$ for all θ. From expression (9), we further know $a_{\theta_i} = 0$ for all $\theta_i \neq \hat{\theta}_i$.

[12] Vectors $(p_{\theta_j\theta_j'})_{j\in I, \theta_j, \theta_j'\in\Theta_j}$ and $(e_\theta)_{\theta\in\Theta}$ are defined in Lemmas A.1 and A.4.

By canceling $b_{\hat{\theta}_i,\theta_{-i}}$ in (10) and (11), we have $a_{\hat{\theta}_i} p_i(\theta_{-i}|\hat{\theta}_i) - p_i(\theta_{-i}|\bar{\theta}_i) = a_{\theta_j} p_j(\hat{\theta}_i, \theta_{-i-j}|\theta_j) = 0$ for all θ_{-i}. Summing the equation across all θ_{-i}, we get $a_{\hat{\theta}_i} = 1$ and thus $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$.

Case 2. Suppose $a_{\theta_i} > 0$ for all $\theta_i \neq \hat{\theta}_i$. Similar to the argument of the previous case, we know $a_{\hat{\theta}_i} > 1$ and $a_{\theta_j} > 0$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$. Subsequently, we will establish that there exists $\mu \in \Delta(\Theta)$, $\bar{C} > 0$, and $\hat{C} > 1$ such that both conditions in Definition 4.1 hold so that the NCP* property fails for i.

Define $\mu \in \Delta(\Theta)$ by $\mu(\theta) = \frac{b_\theta}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}}$ for all $\theta \in \Theta$. Then from expressions (9) and (11), we know $\mu(\cdot|\theta_j) = p_j(\cdot|\theta_j)$ and $\mu(\theta_j) = \frac{a_{\theta_j}}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}} > 0$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$. Hence, Condition N1 in Definition 4.1 holds. By canceling $b_{\hat{\theta}_i \theta_{-i}}$ in expressions (10) and (11), we have $a_{\hat{\theta}_i} p_i(\theta_j, \cdot|\hat{\theta}_i) = p_i(\theta_j, \cdot|\bar{\theta}_i) + a_{\theta_j} p_j(\hat{\theta}_i, \cdot|\theta_j)$ for all $j \neq i$ and θ_j, where $a_{\theta_j} = \mu(\theta_j) \sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}} = \mu(\bar{\theta}_i) \frac{\mu(\theta_j|\bar{\theta}_i)}{\mu(\bar{\theta}_i|\theta_j)} \sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}} = a_{\bar{\theta}_i} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)}$. Recall $a_{\bar{\theta}_i} > 0$ and $a_{\hat{\theta}_i} > 1$. Thus by defining $\bar{C} = a_{\bar{\theta}_i}$ and $\hat{C} = a_{\hat{\theta}_i}$, Condition N2 in Definition 4.1 holds. Hence, the NCP* property fails for i.

Sufficiency. When $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, define (1) $a_{\hat{\theta}_i} = 1$, (2) $a_{\theta_j} = 0$ for all $(j, \theta_j) \neq (i, \hat{\theta}_i)$, and (3) $b_\theta = 0$ for all $\theta \in \Theta$. When Conditions N1 and N2 hold for $i, \bar{\theta}_i, \hat{\theta}_i, \mu, \bar{C} > 0$, and $\hat{C} > 1$, define (1) $a_{\bar{\theta}_i} = \bar{C}$, $a_{\hat{\theta}_i} = \hat{C}$, (2) $b_\theta = \bar{C}\frac{\mu(\theta)}{\mu(\bar{\theta}_i)}$, $\forall\theta$, and (3) $a_{\theta_k} = \bar{C}\frac{\mu(\theta_k)}{\mu(\bar{\theta}_i)}$, $\forall (k, \theta_k) \neq (i, \bar{\theta}_i), (i, \hat{\theta}_i)$. For both cases, it is easy to verify expression (8). □

Lemma A.7: *Given a collection of priors $(p_i)_{i\in I}$, if the BDP and NCP* properties hold for agent i , then for all $\bar{\theta}_i, \hat{\theta}_i \in \Theta_i$ with $\bar{\theta}_i \neq \hat{\theta}_i$, there exists a transfer rule $\psi^{\bar{\theta}_i\hat{\theta}_i} : \Theta \to \mathbb{R}^N$ such that*

1. $\displaystyle\sum_{j\in I} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta) = 0$ *for all* $\theta \in \Theta$;
2. $\displaystyle\sum_{\theta_{-j}\in\Theta_{-j}} \psi_j^{\bar{\theta}_i\hat{\theta}_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j}|\theta_j) \geq 0$ *for all* $j \in I$, $\theta_j \in \Theta_j$;
3. $\displaystyle\sum_{\theta_{-i}\in\Theta_{-i}} \psi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i}|\bar{\theta}_i) < 0.$

Proof. We prove by contraposition. Suppose there exists $\bar{\theta}_i \neq \hat{\theta}_i$ such that no $\psi^{\bar{\theta}_i\hat{\theta}_i}$ satisfies these conditions. By Motzkin's theorem of the alternative, there exist coefficients $(b_\theta)_{\theta\in\Theta}$ and non-negative coefficients $(a_{\theta_j})_{j\in I, \theta_j\in\Theta_j}$ such that expression (8) holds. By Lemma A.6, the BDP property or the NCP* property fails. □

Lemma A.8: *Given a collection of priors $(p_i)_{i\in I}$, if the BDP property holds for all agents, then the NCP* property holds for at least $N - 1$ agents.*

Proof. Let all agents satisfy the BDP property. Suppose by way of contradiction that there are agents $i \neq j$, types $\bar{\theta}_i \neq \hat{\theta}_i$, and types $\bar{\theta}_j \neq \hat{\theta}_j$ such that for both $(i, \bar{\theta}_i, \hat{\theta}_i)$ and $(j, \bar{\theta}_j, \hat{\theta}_j)$ the NCP* property fails. By the two-case argument in the proof of Lemma

A.6, there exist coefficients $(a_{\theta_k} > 0)_{k\in I,\theta_k\in\Theta_k}$ where $a_{\hat{\theta}_i} > 1$, $(b_\theta)_{\theta\in\Theta}$, $(c_{\theta_k} > 0)_{k\in I,\theta_k\in\Theta_k}$ where $c_{\hat{\theta}_j} > 1$, and $(d_\theta)_{\theta\in\Theta}$ such that $p_{\bar{\theta}_i\hat{\theta}_i} = \sum_{k\in I}\sum_{\theta_k\in\Theta_k} a_{\theta_k}p_{\theta_k\theta_k} - \sum_{\theta\in\Theta} b_\theta e_\theta$ and $p_{\bar{\theta}_j\hat{\theta}_j} = \sum_{k\in I}\sum_{\theta_k\in\Theta_k} c_{\theta_k}p_{\theta_k\theta_k} - \sum_{\theta\in\Theta} d_\theta e_\theta$. Thus, the following equations hold.

$$0 = a_{\theta_i}p_i(\theta_{-i}|\theta_i) - b_\theta, \forall\theta_i \neq \hat{\theta}_i \text{ and } \theta_{-i}, \qquad 0 = c_{\theta_j}p_j(\theta_{-j}|\theta_j) - d_\theta, \forall\theta_j \neq \hat{\theta}_j \text{ and } \theta_{-j},$$
$$p_i(\theta_{-i}|\bar{\theta}_i) = a_{\hat{\theta}_i}p_i(\theta_{-i}|\hat{\theta}_i) - b_{\hat{\theta}_i,\theta_{-i}}, \forall\theta_{-i}, \qquad p_j(\theta_{-j}|\bar{\theta}_j) = c_{\hat{\theta}_j}p_j(\theta_{-j}|\hat{\theta}_j) - d_{\hat{\theta}_j,\theta_{-j}}, \forall\theta_{-j},$$
$$0 = a_{\theta_j}p_j(\theta_{-j}|\theta_j) - b_\theta, \forall\theta, \qquad 0 = c_{\theta_i}p_i(\theta_{-i}|\theta_i) - d_\theta, \forall\theta.$$

In the argument below, we ignore θ_{-i-j} if $N = 2$ to avoid introducing additional notations. Canceling all b_θ, d_θ, and $p_j(\theta_{-j}|\theta_j)$ in the above equations yields:

$$[\frac{a_{\theta_i}}{a_{\theta_j}} - \frac{c_{\theta_i}}{c_{\theta_j}}]p_i(\theta_{-i}|\theta_i) = 0, \forall\theta_i \neq \hat{\theta}_i, \theta_j \neq \hat{\theta}_j, \text{ and } \theta_{-i-j}, \tag{12}$$

$$[\frac{a_{\hat{\theta}_i}}{a_{\theta_j}} - \frac{c_{\hat{\theta}_i}}{c_{\theta_j}}]p_i(\theta_{-i}|\hat{\theta}_i) = \frac{p_i(\theta_{-i}|\bar{\theta}_i)}{a_{\theta_j}}, \forall\theta_j \neq \hat{\theta}_j \text{ and } \theta_{-i-j}, \tag{13}$$

$$[\frac{a_{\theta_i}}{a_{\hat{\theta}_j}} - \frac{c_{\theta_i}}{c_{\hat{\theta}_j}}]p_i(\hat{\theta}_j, \theta_{-i-j}|\theta_i) = \frac{c_{\theta_i}p_i(\bar{\theta}_j, \theta_{-i-j}|\theta_i)}{c_{\hat{\theta}_j}c_{\bar{\theta}_j}}, \forall\theta_i \neq \hat{\theta}_i \text{ and } \theta_{-i-j}, \tag{14}$$

$$[\frac{a_{\hat{\theta}_i}}{a_{\hat{\theta}_j}} - \frac{c_{\hat{\theta}_i}}{c_{\hat{\theta}_j}}]p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{a_{\hat{\theta}_j}} + \frac{c_{\hat{\theta}_i}p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}{c_{\hat{\theta}_j}c_{\bar{\theta}_j}}, \forall\theta_{-i-j}. \tag{15}$$

Step 1. We want to prove for all $\theta_{-i-j} \in \Theta_{-i-j}$, either all the four numbers $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, $p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$, $p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, and $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$ are positive, or they are all equal to zero.

By Assumption 2.1, there exists $\tilde{\theta}_{-i-j}$ such that $p_i(\bar{\theta}_j, \tilde{\theta}_{-i-j}|\bar{\theta}_i) > 0$. Hence, expressions (13) and (14) imply $\frac{a_{\hat{\theta}_i}}{a_{\bar{\theta}_j}} - \frac{c_{\hat{\theta}_i}}{c_{\bar{\theta}_j}}, \frac{a_{\bar{\theta}_i}}{a_{\hat{\theta}_j}} - \frac{c_{\bar{\theta}_i}}{c_{\hat{\theta}_j}} > 0$. Thus for each θ_{-i-j}, either $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)$, $p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)$, $p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$, or $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) = p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) = 0$.

In the previous case, expression (15) implies that we also have $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) > 0$.

In the latter case, we must have $p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) = 0$. Because otherwise expression (15) would imply $\frac{a_{\hat{\theta}_i}}{a_{\hat{\theta}_j}} = \frac{c_{\hat{\theta}_i}}{c_{\hat{\theta}_j}}$, which further implies that $p_i(\hat{\theta}_j, \cdot|\bar{\theta}_i) = p_i(\bar{\theta}_j, \cdot|\hat{\theta}_i) = \mathbf{0}$, a contradiction to Assumption 2.1.

Step 2. We want to prove that for all $\theta_{-i-j} \in \Theta_{-i-j}$ such that $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$,

$$\frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)} = \frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}.$$

When $p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) > 0$, canceling $a_{\hat{\theta}_j}$, $c_{\hat{\theta}_j}$, and $c_{\hat{\theta}_i}$ in expressions (12) through (15) yields

$$\frac{c_{\bar{\theta}_j}p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i) + p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{c_{\bar{\theta}_j}p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i) + p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)} = \frac{a_{\hat{\theta}_i} - \frac{p_i(\bar{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}}{a_{\hat{\theta}_i} - \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}} \times \frac{p_i(\hat{\theta}_j, \theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j, \theta_{-i-j}|\hat{\theta}_i)}.$$

By Condition N2, we also know that $a_{\hat{\theta}_i} - \frac{p_i(\bar{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)} \geq 0$ and $a_{\hat{\theta}_i} - \frac{p_i(\hat{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)} \geq 0$. Suppose $\frac{p_i(\bar{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)} > (<) \frac{p_i(\hat{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)}$. The left-hand side of the above equation is greater (less) than $\frac{p_i(\hat{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)}$ and the right-hand side is less (greater) than $\frac{p_i(\hat{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)}$, a contradiction. Hence, $\frac{p_i(\bar{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)} = \frac{p_i(\hat{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)}{p_i(\hat{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)}$. Rearranging terms yields the desired result.

Step 3. We want to prove that $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, which contradicts the BDP property.

Expression (12) implies that $\frac{a_{\bar{\theta}_i}}{a_{\theta_j}} = \frac{c_{\bar{\theta}_i}}{c_{\theta_j}}$ for all $\theta_j \neq \hat{\theta}_j$. Plugging it into expression (13) yields $(\frac{a_{\hat{\theta}_i}}{a_{\bar{\theta}_i}} - \frac{c_{\hat{\theta}_i}}{c_{\bar{\theta}_i}})p_i(\theta_{-i}|\hat{\theta}_i) = \frac{1}{a_{\bar{\theta}_i}} p_i(\theta_{-i}|\bar{\theta}_i)$ for all $\theta_j \neq \hat{\theta}_j$ and θ_{-i-j}. Hence,

$$\frac{p_i(\theta_j,\tilde{\theta}_{-i-j}|\bar{\theta}_i)}{p_i(\bar{\theta}_j,\theta_{-i-j}|\bar{\theta}_i)} = \frac{p_i(\theta_j,\tilde{\theta}_{-i-j}|\hat{\theta}_i)}{p_i(\bar{\theta}_j,\theta_{-i-j}|\hat{\theta}_i)}, \forall \theta_j \neq \hat{\theta}_j, \theta_{-i-j} \text{ s.t. } p_i(\bar{\theta}_j,\theta_{-i-j}|\bar{\theta}_i) > 0, \text{ and } \tilde{\theta}_{-i-j}.$$

Combining this expression with Step 1 and Step 2, we have established the desired result. □

Lemma A.9: *Let q be an efficient allocation rule under a private value environment. For any $i \in I$, $\tilde{\Theta}_i \subseteq \Theta_i$ with $|\tilde{\Theta}_i| \geq 2$, and distribution $\pi \in \Delta(\Theta_{-i})$, there exist values $(U_{\theta_i} \geq 0)_{\theta_i \in \tilde{\Theta}_i}$ such that $U_{\theta_i} - U_{\theta_i'} \geq \sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\theta_i',\theta_{-i}),\theta_i) - u_i(q(\theta_i',\theta_{-i}),\theta_i')]\pi(\theta_{-i})$ for all $\theta_i, \theta_i' \in \tilde{\Theta}_i$.*

Proof. Let a loop be a sequence $(\theta_i^1, \theta_i^2, ..., \theta_i^K)$ in $\tilde{\Theta}_i$ with length $K \geq 2$ and $\theta_i^1 = \theta_i^K$. As q is ex-post efficient, $u_i(q(\theta_i^{k+1},\theta_{-i}),\theta_i^{k+1}) + \sum_{j\neq i} u_j(q(\theta_i^{k+1},\theta_{-i}),\theta_j) \geq u_i(q(\theta_i^k,\theta_{-i}),\theta_i^{k+1}) + \sum_{j\neq i} u_j(q(\theta_i^k,\theta_{-i}),\theta_j)$ for all $k = 1, ..., K-1$ and $\theta_{-j} \in \Theta_{-j}$. Summing the inequalities across $k = 1, ..., K-1$ and taking into account $\theta_i^1 = \theta_i^K$, we obtain that $\sum_{k=1}^{K-1}[u_i(q(\theta_i^k,\theta_{-i}),\theta_i^{k+1}) - u_i(q(\theta_i^k,\theta_{-i}),\theta_i^k)] \leq 0$. This is the "cyclical monotonicity" condition is the literature.

Fix an arbitrary $\tilde{\theta}_i \in \tilde{\Theta}_i$. For each $(\theta_i,\theta_{-i}) \in \tilde{\Theta}_i \times \Theta_{-i}$, define the function $V_i(\cdot) : \tilde{\Theta}_i \times \Theta_{-i} \to \mathbb{R}$ by:

$$V_i(\theta_i,\theta_{-i}) \equiv \sup_{\substack{(\theta_i^1,\ldots,\theta_i^k) \text{ is any finite sequence} \\ \text{starting with } \tilde{\theta}_i \text{ and ending with } \theta_i}} \sum_{k=1}^{K-1}[u_i(q(\theta_i^k,\theta_{-i}),\theta_i^{k+1}) - u_i(q(\theta_i^k,\theta_{-i}),\theta_i^k)].$$

Then by Theorem 1 of **?** or Proposition 5.2 of Börgers et al. (2015), $V_i(\cdot)$ is a well-defined function satisfying

$$V_i(\theta_i,\theta_{-i}) - V_i(\theta_i',\theta_{-i}) \geq \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i',\theta_{-i}),\theta_i) - u_i(q(\theta_i',\theta_{-i}),\theta_i'), \forall \theta_i, \theta_i' \in \tilde{\Theta}_i.$$

When $C > 0$ is sufficiently large, we have $U_{\theta_i} \equiv \sum_{\theta_{-i}\in\Theta_{-i}} V_i(\theta_i,\theta_{-i})\pi_i(\theta_{-i}) + C$ is non-negative for all $\theta_i \in \tilde{\Theta}_i$. The values $(U_{\theta_i} \geq 0)_{\theta_i \in \tilde{\Theta}_i}$ satisfy the desired condition. □

Proof of Lemma 4.1. Suppose there is a common prior p. Then the equation in Condition N2 can be rewritten as $\big(\hat{C} - \bar{C}\frac{p(\hat{\theta}_i)}{p(\bar{\theta}_i)}\big)p(\cdot|\hat{\theta}_i) = p(\cdot|\bar{\theta}_i)$.

1 $\implies$ *2*. We prove by contrapositive. Suppose the NCP* property fails for i. Then there exists $i \in I$, $\bar{\theta}_i \neq \hat{\theta}_i$, $\mu \in \Delta(\Theta)$, $\bar{C} > 0$, and $\hat{C} > 1$ such that Conditions N1 and N2 hold. By the common prior condition, it is easy to see that the only μ satisfying Condition N1 is $\mu = p$. The equation in Condition N2 implies that the BDP property fails for agent i.

2 $\implies$ *3*. This step is trivial.

3 $\implies$ *1*. We prove by contrapositive. Suppose the BDP property fails for agent i as $p(\cdot|\bar{\theta}_i) = p(\cdot|\hat{\theta}_i)$. Define a prior $\mu = p$, which makes Condition N1 hold. Pick any $\bar{C} \geq 1$ and define $\hat{C} = 1 + \bar{C}\frac{p(\hat{\theta}_i)}{p(\theta_i)}$. We know $\hat{C} > 1$ from Assumption 2.1. Thus Conditions N2 also holds. Hence, the WNCP* property fails for agent i. □

Proof of Proposition 4.1. **Necessity of Part 1**. Suppose there exists an agent $i \in I$ for whom the BDP property fails. The same construction as in the necessity proof of Parts 1 and 2 of Theorem 3.1 can establish the necessity of the BDP property.

To prove the necessity of the WNCP* property, suppose there exists $i \in I$, $\bar{\theta}_i \neq \hat{\theta}_i$, $\mu \in \Delta(\Theta)$, $\bar{C} \geq 1$, and $\hat{C} > 1$ such that the two conditions in Definition 4.1 hold so that the WNCP* property fails for i. Fix any agent $j \neq i$. Consider the same construction except that $0 < [\hat{C} + \bar{C}\sum_{(k,\theta_k)\neq(i,\hat{\theta}_i)} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)}\frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)} - 1]a < B$. Suppose by way of contradiction an IR and BB mechanism with ambiguous transfers (q, Φ) implements q. Then for all $\epsilon > 0$ there exists $\phi \in \Phi$ such that:

$$\begin{aligned}
&IR(\theta_k) && \sum_{\theta_{-k\in\Theta_{-k}}} \phi_k(\theta_k, \theta_{-k})p_k(\theta_{-k}|\theta_k) \geq -a, \forall k \in I \text{ and } \theta_k \in \Theta_k \\
&BB(\theta) && -\sum_{k\in I} \phi_k(\theta) = 0, \forall\theta \in \Theta \\
&IC(\bar{\theta}_i\hat{\theta}_i) && \sum_{\theta_{-i\in\Theta_{-i}}} \phi_i(\bar{\theta}_i, \theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) - \sum_{\theta_{-i\in\Theta_{-i}}} \phi_i(\hat{\theta}_i, \theta_{-i})p_i(\theta_{-i}|\bar{\theta}_i) + \epsilon \geq B.
\end{aligned}$$

Multiply $IR(\bar{\theta}_i)$ by $\bar{C} - 1$, $IR(\hat{\theta}_i)$ by $\hat{C}$, each $IR(\theta_k)$ where $(k, \theta_k) \neq (i, \bar{\theta}_i), (i, \hat{\theta}_i)$ by $\bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)}\frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)}$, each $BB(\theta)$ by $\bar{C}\frac{p_i(\theta_j|\bar{\theta}_i)p_j(\theta_{-j}|\theta_j)}{p_j(\bar{\theta}_i|\theta_j)}$, and $IC(\bar{\theta}_i\hat{\theta}_i)$ by 1. Add up and let ϵ go to zero. We have $0 \geq B - [\hat{C} + \bar{C}\sum_{(k,\theta_k)\neq(i,\hat{\theta}_i)} \frac{p_i(\theta_j|\bar{\theta}_i)}{p_j(\bar{\theta}_i|\theta_j)}\frac{p_j(\theta_{-j}|\theta_j)}{p_k(\theta_{-k}|\theta_k)} - 1]a > 0$, a contradiction.

Necessity of Part 2. Suppose the WNCP* property fails for one agent and the BDP property fails for another. By relabelling the indices, assume without loss of generality that $p_2(\cdot|\theta_2^1) = p_2(\cdot|\theta_2^2)$ and there exists $\mu \in \Delta(\Theta)$, $\bar{C} \geq 1$, and $\hat{C} > 1$ such that the two conditions in the WNCP* property hold for $(i, \bar{\theta}_i, \hat{\theta}_i) = (1, \theta_1^2, \theta_1^1)$. Consider the same private value functions $(v_i(\cdot))_{i\in I}$ and efficient allocation rule q as in the necessity direction of Proposition 3.1. We will prove that q is not implementable via an IR and BB mechanism with ambiguous transfers, which establishes the necessity of the condition in Part 2 of Proposition 4.1.

Suppose by contradiction that q is implementable by an IR and BB mechanism with ambiguous transfers (q, Φ). Hence, for each $i \in I$ and $\theta_i \in \Theta_i$, there exists an equilibrium interim payoff of participation $U_{\theta_i} \geq 0$, such that for all $\epsilon > 0$, there exists a transfer rule $\phi \in \Phi$ such that

$$
\begin{aligned}
&IR(\theta_i) \quad && \sum_{\theta_{-i}\in\Theta_{-i}} \phi_i(\theta_i,\theta_{-i})p_i(\theta_{-i}|\theta_i) \geq U_{\theta_i} - \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\theta_i), \forall i \in I, \theta_i \in \Theta_i,\\
&BB(\theta) && -\sum_{i\in I}\phi_i(\theta) = 0, \forall \theta \in \Theta,\\
&IC(\theta_1^2\theta_1^1) && U_{\theta_1^2} + \epsilon \geq \sum_{\theta_{-1}\in\Theta_{-1}} \phi_1(\theta_1^1,\theta_{-1})p_1(\theta_{-1}|\theta_1^2) + \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2).
\end{aligned}
$$

Multiply $IR(\theta_1^1)$ by $\hat{C}$, each $IR(\theta_i)$ where $(i,\theta_i) \neq (1,\theta_1^1)$ by $\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)}$, each $BB(\theta)$ by $\bar{C}\frac{p_1(\theta_2|\theta_1^2)p_2(\theta_{-2}|\theta_2)}{p_2(\theta_1^2|\theta_2)}$, and $IC(\theta_1^2\theta_1^1)$ by 1. Add them up and let ϵ go to zero. Since the WNCP* property fails, we have

$$
\begin{aligned}
U_{\theta_1^2} \geq & \sum_{(i,\theta_i)\neq(1,\theta_1^1)} [U_{\theta_i} - \sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\theta_i)]\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)}\\
& [U_{\theta_1^1} - \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^1)p_1(\theta_{-1}|\theta_1^1)]\hat{C} + \sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2). \qquad (16)
\end{aligned}
$$

From $IC(\theta_2^1\theta_2^2)$, we know for all $\epsilon > 0$, there exists a transfer rule $\tilde{\phi} \in \Phi$ such that

$$
\begin{aligned}
&IR(\theta_2^2) \quad && \sum_{\theta_{-2}\in\Theta_{-2}} \tilde{\phi}_2(\theta_2^2,\theta_{-2})p_2(\theta_{-2}|\theta_2^2) \geq U_{\theta_2^2} - \sum_{\theta_{-2}\in\Theta_{-2}} u_2(q(\theta_2^2,\theta_{-2}),\theta_2^2)p_2(\theta_{-2}|\theta_2^2)\\
&IC(\theta_2^1\theta_2^2) && U_{\theta_2^1} + \epsilon \geq \sum_{\theta_{-2}\in\Theta_{-2}} \tilde{\phi}_2(\theta_2^2,\theta_{-2})p_2(\theta_{-2}|\theta_2^1) + \sum_{\theta_{-2}\in\Theta_{-2}} u_2(q(\theta_2^2,\theta_{-2}),\theta_2^1)p_2(\theta_{-2}|\theta_2^1).
\end{aligned}
$$

In view of $p_2(\cdot|\theta_2^1) = p_2(\cdot|\theta_2^2)$, add up the two inequalities and let ϵ approach zero. We obtain the following inequality.

$$
U_{\theta_2^1} \geq U_{\theta_2^2} + \sum_{\theta_{-2}\in\Theta_{-2}} [u_2(q(\theta_2^2,\theta_{-2}),\theta_2^1)) - u_2(q(\theta_2^2,\theta_{-2}),\theta_2^2)]p_2(\theta_{-2}|\theta_2^1).
$$

Plug the above expression into expression (16). Since the coefficient of $IR(\theta_1^2)$ is $\bar{C} \geq 1$ and each $U_{\theta_i} \geq 0$, we have

$$
\begin{aligned}
0 \geq & -\sum_{(i,\theta_i)\neq(1,\theta_1^1),(2,\theta_2^1)} [\sum_{\theta_{-i}\in\Theta_{-i}} u_i(q(\theta_i,\theta_{-i}),\theta_i)p_i(\theta_{-i}|\theta_i)]\bar{C}\frac{p_1(\theta_2|\theta_1^2)}{p_2(\theta_1^2|\theta_2)}\frac{p_2(\theta_{-2}|\theta_2)}{p_i(\theta_{-i}|\theta_i)}\\
& + \sum_{\theta_{-2}\in\Theta_{-2}} [u_2(q(\theta_2^2,\theta_{-2}),\theta_2^1) - u_2(q(\theta_2^2,\theta_{-2}),\theta_2^2) - u_2(q(\theta_2^1,\theta_{-2}),\theta_2^1)]p_2(\theta_{-2}|\theta_2^1)\bar{C}\frac{p_1(\theta_2^1|\theta_1^2)}{p_2(\theta_1^2|\theta_2^1)}
\end{aligned}
$$

$$-\sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^1)p_1(\theta_{-1}|\theta_1^1)\hat{C}+\sum_{\theta_{-1}\in\Theta_{-1}} u_1(q(\theta_1^1,\theta_{-1}),\theta_1^2)p_1(\theta_{-1}|\theta_1^2).$$

Plug the explicit form of q (agent 1 sells to 2 if and only if $v_1(\theta_1)+v_2(\theta_2)>0$) and Condition N2 into the above expression. By letting $-v_1(\theta_1^1)$ and $v_2(\theta_2^1)$ approach each other and letting all other $v_i(\theta_i)$ approach zero, we have $0\geq -v_1(\theta_1^1)p(\theta_2^1|\theta_1^2)$, which is impossible in view of Assumption 2.1.

Therefore, q is not implementable via an IR and BB mechanism with ambiguous transfers.

Sufficiency of Part 1. Suppose the BDP and NCP* properties hold for all agents. For all i and $\bar{\theta}_i\neq\hat{\theta}_i$, there exists $\psi^{\bar{\theta}_i\hat{\theta}_i}:\Theta\to\mathbb{R}^N$ satisfying the conditions of Lemma A.7.

Let η be any IR and BB transfer rule. Define $\Phi=\{\eta,\eta+c\psi^{\bar{\theta}_j\hat{\theta}_j}: j\in I,\bar{\theta}_j,\hat{\theta}_j\in\Theta_j,\bar{\theta}_j\neq\hat{\theta}_j\}$, where c is sufficiently large such that for all $j\in I$ and $\bar{\theta}_j\neq\hat{\theta}_j$, the expression $\sum_{\theta_{-j}\in\Theta_{-j}}[u_j(q(\hat{\theta}_j,\theta_{-j}),(\bar{\theta}_j,\theta_{-j}))-u_j(q(\bar{\theta}_j,\theta_{-j}),(\bar{\theta}_j,\theta_{-j}))+\eta_j(\hat{\theta}_j,\theta_{-j})-\eta_j(\bar{\theta}_j,\theta_{-j})+c\psi_j^{\bar{\theta}_j\hat{\theta}_j}(\hat{\theta}_j,\theta_{-j})]p_j(\theta_{-j}|\bar{\theta}_j)$ is non-positive.

For any type-$\bar{\theta}_i$ agent i, the inequality below shows that misreporting $\hat{\theta}_i$ is not profitable:

$$\begin{aligned}
&\min_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i}))+\phi_i(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
=&\sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\bar{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i}))+\eta(\bar{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
\geq&\sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i}))+\eta(\hat{\theta}_i,\theta_{-i})+c\psi_i^{\bar{\theta}_i\hat{\theta}_i}(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i)\\
\geq&\min_{\phi\in\Phi}\sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\hat{\theta}_i,\theta_{-i}),(\bar{\theta}_i,\theta_{-i}))+\phi_i(\hat{\theta}_i,\theta_{-i})]p_i(\theta_{-i}|\bar{\theta}_i),
\end{aligned}$$

where the equality follows from the second requirement of Lemma A.7 and the composition of ambiguous transfers, the first inequality comes from the choice of c, and the second inequality comes from the composition of ambiguous transfers again. The conditions of IR and BB follow from corresponding properties of η and each $\phi\in\Phi$.

Sufficiency of Part 2. We focus on the non-common prior case. In view of Lemma 4.1, the common prior case has been established by Part 3 of Theorem 3.1.

Suppose there do not exist agents $i\neq j$ such that the NCP* property fails for i and the BDP property fails for j. Then either of the following is true. Case 1: there are at least $N-1$ agents satisfying both the BDP and NCP* properties. Note by Lemma A.8, a special situation in this case is that all agents satisfy the BDP property. Case 2: all agents satisfy the NCP* property.

Case 1. Suppose there are at least $N-1$ agents satisfying both the BDP and NCP* properties. By Lemma A.7, there exists $I'\subseteq I$ with $|I'|\geq N-1$ such that for all $i\in I'$ and $\bar{\theta}_i\neq\hat{\theta}_i$, there exists $\psi^{\bar{\theta}_i\hat{\theta}_i}:\Theta\to\mathbb{R}^N$ satisfying the three conditions in the lemma.

Pick an agent $i \in I$, where $\{i\} = I \backslash I'$ if $I \backslash I'$ is a singleton and $i \in I$ is arbitrary if $I \backslash I' = \emptyset$. As in the proof of Part 3 of Theorem 3.1, let η be an IR and BB transfer rule such that agent i obtains all the surplus. Define $\Phi = \{\eta\} \cup \{\eta + c\psi^{\bar{\theta}_j \hat{\theta}_j} : j \in I, j \neq i, \bar{\theta}_j, \hat{\theta}_j \in \Theta_j, \bar{\theta}_j \neq \hat{\theta}_j\}$, where c is sufficiently large such that for all $j \neq i$ and $\bar{\theta}_j \neq \hat{\theta}_j$,

$$0 \geq \sum_{\theta_{-j} \in \Theta_{-j}} [u_j(q(\hat{\theta}_j, \theta_{-j}), \bar{\theta}_j) - u_j(q(\hat{\theta}_j, \theta_{-j}), \hat{\theta}_j) + c\psi_j^{\bar{\theta}_j \hat{\theta}_j}(\hat{\theta}_j, \theta_{-j})] p_j(\theta_{-j} | \bar{\theta}_j).$$

For agent $j \neq i$ with type θ_j, truthfully reporting gives her a worst-case expected utility level of zero because the worst transfer rule, η, extracts all her surplus. Thus, j's IR condition binds. The choice of c makes misreporting unprofitable. Therefore, her IC condition holds.

When all agents truthfully report, a type-$\bar{\theta}_i$ agent i obtains a worst-case expected payoff of $\min_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \phi_i(\bar{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i) = \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\bar{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\bar{\theta}_i, \theta_{-i}), \theta_j)] p_i(\theta_{-i} | \bar{\theta}_i) \geq 0$. Hence, agent i's IR condition holds. By efficiency of q, this term is weakly higher than $\sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \sum_{j \neq i} u_j(q(\hat{\theta}_i, \theta_{-i}), \theta_j)] p_i(\theta_{-i} | \bar{\theta}_i)$ for all $\hat{\theta}_i \neq \bar{\theta}_i$. Note the latter expression is weakly higher than the worst-case expected payoff of misreporting $\hat{\theta}_i$, $\min_{\phi \in \Phi} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) + \phi_i(\hat{\theta}_i, \theta_{-i})] p_i(\theta_{-i} | \bar{\theta}_i)$. Hence, we have also verified agent i's IC constraints.

The BB condition is easy to see. Therefore, the IR and BB mechanism with ambiguous transfers implements q.

Case 2. Suppose all agents satisfy the NCP* property. For any $j \in I$, let $\mathcal{P}_j$ be the partition of Θ_j such that $p_j(\cdot | \theta_j) = p_j(\cdot | \theta'_j)$ if and only if θ_j and θ'_j are in the same $\tilde{\Theta}_j \in \mathcal{P}_j$. For each $\tilde{\Theta}_j$ with $|\tilde{\Theta}_j| \geq 2$ and $\theta_j \in \tilde{\Theta}_j$, define U_{θ_j} according to Lemma A.9. For a singleton $\tilde{\Theta}_j \in \mathcal{P}_j$ and $\{\theta_j\} = \tilde{\Theta}_j$, define $U_{\theta_j} = 0$.

We will demonstrate that for each i and $\bar{\theta}_i \neq \hat{\theta}_i$ the following system has a solution $\phi^{\bar{\theta}_i \hat{\theta}_i}$.

$$\begin{aligned}
&IR(\bar{\theta}_i) \quad \sum_{\theta_{-j} \in \Theta_{-j}} \phi_i^{\bar{\theta}_i \hat{\theta}_i}(\bar{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) = U_{\bar{\theta}_i} - \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\bar{\theta}_i, \theta_{-i}), \theta_i) p_i(\theta_{-i} | \bar{\theta}_i), \\
&IR(\theta_j) \quad \sum_{\theta_{-j} \in \Theta_{-j}} \phi_j^{\bar{\theta}_i \hat{\theta}_i}(\theta_j, \theta_{-j}) p_j(\theta_{-j} | \theta_j) \geq U_{\theta_j} - \sum_{\theta_{-j} \in \Theta_{-j}} u_j(q(\theta_j, \theta_{-j}), \theta_j) p_j(\theta_{-j} | \theta_j), \forall (j, \theta_j) \neq (i, \bar{\theta}_i), \\
&BB(\theta) \quad\quad -\sum_{j \in I} \phi_j^{\bar{\theta}_i \hat{\theta}_i}(\theta) = 0, \forall \theta \in \Theta, \\
&IC(\bar{\theta}_i \hat{\theta}_i) \quad -\sum_{\theta_{-j} \in \Theta_{-j}} \phi_i^{\bar{\theta}_i \hat{\theta}_i}(\hat{\theta}_i, \theta_{-i}) p_i(\theta_{-i} | \bar{\theta}_i) \geq -U_{\bar{\theta}_i} + \sum_{\theta_{-i} \in \Theta_{-i}} u_i(q(\hat{\theta}_i, \theta_{-i}), \bar{\theta}_i) p_i(\theta_{-i} | \bar{\theta}_i).
\end{aligned}$$

Suppose by way of contradiction that the system does not have a solution. By Gale's theorem of the alternative, there exist coefficients $a_{\bar{\theta}_i}$ of $IR(\bar{\theta}_i)$, $a_{\theta_j} \geq 0$ of $IR(\theta_j)$ for each $(j, \theta_j) \neq (i, \bar{\theta}_i)$, b_θ of $BB(\theta)$ for each $\theta \in \Theta$, and $\gamma_{\bar{\theta}_i \hat{\theta}_i} \geq 0$ of $IC(\bar{\theta}_i \hat{\theta}_i)$, that are not all zero, such that the weighted sum of the left-hand sides of the expressions is cancelled and the weighted sum of the right-hand sides is positive.[13]

Suppose $\gamma_{\bar{\theta}_i\hat{\theta}_i} = 0$. Following the argument of Lemma A.7, we know (1) $(a_{\theta_j} > 0)_{j\in I,\theta_j\in\Theta_j}$ and (2) $(b_\theta \geq 0)_{\theta\in\Theta}$ is a non-zero vector. Define $\mu(\theta) = \frac{b_\theta}{\sum_{\tilde{\theta}\in\Theta} b_{\tilde{\theta}}}$ for all θ, which is a common prior, contradicting the non-common prior assumption stated at the beginning of this part of the proof.

Suppose $\gamma_{\bar{\theta}_i\hat{\theta}_i} > 0$. From Lemma A.7 and that the NCP* property holds for all agents, we know: (1) $p_i(\cdot|\bar{\theta}_i) = p_i(\cdot|\hat{\theta}_i)$, and (2) among all the coefficients, $a_{\hat{\theta}_i} = \gamma_{\bar{\theta}_i\hat{\theta}_i} > 0$ and everything else is zero. According to Lemma A.9, the choice of $U_{\bar{\theta}_i}$ and $U_{\hat{\theta}_i}$ satisfies $U_{\hat{\theta}_i} - U_{\bar{\theta}_i} + \sum_{\theta_{-i}\in\Theta_{-i}}[u_i(q(\hat{\theta}_i,\theta_{-i}),\bar{\theta}_i) - u_i(q(\hat{\theta}_i,\theta_{-i}),\hat{\theta}_i)]p_i(\theta_{-i}|\bar{\theta}_i) \leq 0$. Hence, the weighted sum of the right-hand sides is non-positive, a contradiction.

Therefore, for each i and $\bar{\theta}_i \neq \hat{\theta}_i$, the system has a solution $\phi^{\bar{\theta}_i\hat{\theta}_i}$. Let the set of ambiguous transfers be $\Phi = \{\phi^{\bar{\theta}_i\hat{\theta}_i}, \forall i, \bar{\theta}_i, \hat{\theta}_i \in \Theta_i, \bar{\theta}_i \neq \hat{\theta}_i\}$. It is easy to see that (q,Φ) is an IR and BB mechanism with ambiguous transfers that implements q. □

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[13]Each equation in the system can be written as two weak inequalities: $LHS = RHS$ is equivalent to $LHS \geq RHS$ and $-LHS \geq -RHS$. Then one can apply Gales' Theorem of inequality to weak inequalities.

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