Destructive Bidding in All-Pay Auctions

Samuel Raisanen, Ph.D.*

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1 Introduction

All-pay auctions are used to model rent seeking behavior in contexts from elections, political lobbying, military conflict, and research and development. Missing from the existing literature, however, is the problem that participants in these contests often have the ability to take actions which reduce the value of the prize. During political campaigns negative advertising erodes the ability of the winner to pursue their legislative goals. In military conflicts, the winning side is faced with rebuilding damaged assets. In this work, two models are presented to examine the phenomenon of destructive bidding. In the first model, the bids themselves reduce the value of the prize symmetrically for all contestants, including the bidder. Then the highest bidder wins the remaining prize. In the second model, there is a pre-bidding round in which resources are committed to asymmetrically destroying value. Then in a bidding round, contestants bid on the prize. Rather than the highest bidder winning, the probability of winning is increasing with a contestant's bid and declining with opponent's bids.

**Address*: Central Michigan University, Department of Economics, SL 309, Mt Pleasant, MI 48859 USA

The all-pay auction has been of interest to economists since the 1980s. The standard common-valuation, all-pay auction is covered by Milgrom and Weber's 1982 revenue equivalence theorem. A full analysis of non-symmetric solutions appears in Baye et al. (1996). Closest in theme to this work is Kaplan et al. (2002) which examines an all-pay auction under which the rewards are dependent on one's own bid. This research considers the case where the value of the prize depends on other contestant's bids. Fibich et al. (2006) breaks revenue equivalence by considering risk-averse players. In theme, the second model presented here is similar to Franke et al. (2014) which compares a lottery to an all-pay auction. Finally, for a good summary of the literature regarding all-pay auctions, I recommend Siegel (2009).

2 Model 1: Common-Value All-Pay Auction with Destructive Bidding

In this first model of an all-pay auction with destructive bidding, their are N risk-neutral bidders with a common pre-bid valuation of v. Each bidder simultaneously submits a bid b_i. Each bidder then pays their bid and the highest bidder wins a prize worth $\tilde{v} = v - \gamma \sum_{i=1} N b_i$. As in the standard all-pay auction, there is no pure strategy Nash Equilibrium as the best response to any opponent's bid is to bid slightly higher as long as that is profitable and zero otherwise. To find the symmetric mixed strategy Nash Equilibrium we assume all other players bid with distribution $b_j \sim f(b)$ and a corresponding cumulative distribution function of $F(b)$. The expected surplus of player i bidding b_i is then

$$EU_i(b_i) = (v - \gamma b_i - (N-1)\gamma \int_0^{b_i} \frac{bf(b)}{F(b_i)} db - b_i) F^{N-1}(b_i) - b_i(1 - F^{N-1}(b_i)). \quad (1)$$

Because there is no pure strategy equilibrium, all bids b_i must give the same expected surplus.

Additionally, because a bid of zero yields a surplus of zero, all potential bids in the mixed strategy Nash equilibrium must yield a surplus of zero.

$$EU_i(b_i) = F^{N-1}(b_i)v - b_i - \gamma b_i F^{N-1}(b_i) - (N-1)\gamma F^{N-2}(b_i) \int_0^{b_i} bf(b)db = 0 \qquad (2)$$

Rearranging this equation, it can be seen that any bid equals the expected value of the prize:

$$b_i = F^{N-1}(b_i)(v - \gamma b_i - \frac{(N-1)\gamma}{F(b_i)} \int_0^{b_i} bf(b)db). \qquad (3)$$

Comparing this to the standard all-pay auction result, the bids are reduced by the expected value of the portion of the prize destroyed during the bidding. Differentiating equation 3 with respect to b_i and solving for $f(b_i)$ yields the characteristic differential equation for the problem:

$$f(b_i) = \frac{F(b_i) + \gamma F^N(b_i)}{(v - \gamma N b_i)F^{N-1}(b_i) + (N-2)b_i}. \qquad (4)$$

Differentiating again with respect to b_i shows that $F''(b_i) = f'(b_i) > 0$ and thus $f(b_i)$ is increasing and $F(b_i)$ is convex.[1]

Having the shape of the cumulative distribution function and the result that the bidding is the same as the standard all-pay auction adjusted for the expected loss in value due to the destructive bidding, we move on to our next model.

[1] $f'(b) = \frac{(f(b)+\gamma N F^{N-1}(b)f(b))D+((N-1)(v-\gamma N b)F^{N-2}(b)f(b)+(N-2)-\gamma N F^{N-1}(b))(F(b)+\gamma F^N(b))}{D^2}$ where $D = (v - \gamma N b)F^{N-1}(b)+(N-2)b$. $f'(b)$ can be shown to be positive since $\gamma N F^{N-1}(b)f(b) = \gamma N F^{N-1}(b)(F(b)+\gamma F^N(b))$ using equation 4.

3 Model 2: Pre-bidding Destructive Investment with a Stochastic Winner

In many situations to which all-pay auctions are applied, the winner of the contest is stochastically determined and bids increase the probability of winning the prize. As such, a second model of destructive bidding is presented here.

This game is played in two rounds by N risk-neutral contestants. In the pre-bidding round, contestants simultaneously choose an asymmetrically destructive investment d_i. The value of the prize for each player then becomes common knowledge: $v_i(d_1, ..., d_N)$ such that $\frac{\partial v_i}{\partial d_j} < \frac{\partial v_i}{\partial d_i} \leq 0$ for $j \neq i$. In the bidding round, each contestant then simultaneously submits a bid on the prize and wins with probability $\rho_i(b_1, ..., b_N)$ such that $\frac{\partial \rho_i}{\partial b_i} > 0$, $\frac{\partial \rho_i}{\partial b_j} < 0$, $\sum_{k=1}^{N} \rho_k = 1$. All players must pay the cost of their destructive investment and bid. Denote this cost as $c_i(b_i, d_i)$ such that $\frac{\partial c_i}{\partial b_i} = 1$.[2]

Using backwards induction to solve for Nash Equilibrium behavior, we first find the best response bidding function. Each bidder simultaneously maximizes their expected utility given the destructive investments and opponents' bids.

$$\max_{b_i \geq 0} \rho_i(b_i, b_{-i}) v_i(d_i, d_{-i}) - c_i(b_i, d_i) \tag{5}$$

This yields the first order condition, $\frac{\partial \rho_i}{\partial b_i} v_i(d_i, d_{-i}) = 1$. From the second order condition, this is a maximum as long as $\frac{\partial^2 \rho_i}{\partial b_i^2} < 0$. The best response function, $b_i^{br}(v_i, v_{-i}, b_{-i})$, is implicitly defined by the first order condition. Holding all else constant, this implies that higher valuations lead to higher bids, $\frac{\partial b_i^{br}}{\partial v_i}$ as long as the probability of winning is concave. The best response is non-monotonic in opponents bids. The change in the best response bid $\frac{\partial b_i^{br}}{\partial b_j}$ has the same sign as $\frac{\partial^2 \rho_i}{\partial b_i \partial b_j}$. To see that this can have either sign as b_i increases, consider the simple probability function $\rho_1(b_1, b_2) = \frac{b_1}{b_1 + b_2}$ which has a $\frac{\partial^2 \rho_1}{\partial b_1 \partial b_2} = \frac{b_1 - b_2}{(b_1 + b_2)^3}$ and thus a non-monotonic bidding best response function as shown in Figure 1. Finally, Nash Equilibrium bidding is characterized by the

[2]Costs are additively separable with $c_i(b_i, d_i) = b_i + c_i(d_i)$.

simultaneous solution to the first order condition for contestants $i = 1..N$. Denote this solution as $b_i^*(v_i, v_{-i})$.

The effect of destructive investment on bidding can be seen in Figure 2. As contestant 1 increases their destructive investment, lowering the value of contestant 2's prize, contestant 2's best response bid shifts downward. This lowers the bid of contestant 2 reducing their probability of winning. In response, contestant 1 is able to slightly reduce their bid as well but less than contestant 2: $\frac{\partial b_2^*}{\partial v_2} > \frac{\partial b_1^*}{\partial v_2}$.

Via backward induction, the equilibrium behavior in the first round can be solved by substituting the Nash equilibrium bids conditional on the destructive investments. Each contestant then simultaneously maximizes their expected surplus

$$\max_{d_i \geq 0} \rho_i(b_i^*(d_i, d_{-i}), b_{-i}^*(d_i, d_{-i}))v_i(d_i, d_{-i}) - c_i(b_i^*(d_i, d_{-i}), d_i) \tag{6}$$

Differentiating with respect the destructive investment and rearranging so each term is positive yields

$$-(\frac{\partial b_i^*}{\partial v_i}\frac{\partial v_i}{\partial d_i} + \sum_{j \neq i}\frac{\partial b_i^*}{\partial v_j}\frac{\partial v_j}{\partial d_i}) + \sum_{j \neq i}\frac{\partial \rho_i}{\partial b_j}(\sum_{k=1}^{N}\frac{\partial b_j^*}{\partial v_k}\frac{\partial v_k}{\partial d_i})v_i = \frac{\partial c_i}{\partial d_i} - \frac{\partial \rho_i}{\partial b_i}(\sum_{k=1}^{N}\frac{\partial b_i^*}{\partial v_k}\frac{\partial v_k}{\partial d_i})v_i - \rho_i\frac{\partial v_i}{\partial d_i}. \tag{7}$$

The left-hand side of equation 7 is the marginal benefit of the destructive investment. The first term denotes the amount the player will save by reducing their own bid because of the value they destroy as well as being able to bid less because other contestants lower their bids as their value is destroyed. The second term denotes the marginal benefit destructive investment resulting in an increased probability of winning due to lower bids by other contestants. The right-hand side of equation 7 is the marginal cost of the destructive investment. The first right-hand term is the marginal cost of the destructive investment itself. The second right-hand term is the marginal reduction in the probability of winning since the contestant bids less when their own valuation is reduced. The third right-hand term is the marginal reduction in value of the prize when the contestant wins due to the destructive investment. Nash equilibrium behavior is characterized by the simultaneous solution of equation 7 by all players $i = 1..N$.

Careful examination of equation 7 yields that destructive investment increases as the marginal effect of destructive investment on opponents valuations increases and as the marginal effect of destructive investment on one's own valuation declines. This observation implies that destructive investment will be most prevalent when actions which have a large negative effect on opponents and little effect on oneself are available to contestants.

3.1 Model 2: Sample Solution

Consider the following realization of Model 2 for two contestants. Let the probability of winning the contest be given by $\rho_i(b_1, b_2) = \frac{b_i}{b_1+b_2}$ for $i \in \{1, 2\}$. Each contestant values the good at $v_i = \bar{v} - \gamma_{own} d_i - \gamma_{opp} d_j$ for $i \in \{1, 2\}$ and $j \neq i$. Finally, let each contestant pay cost $c_i(b_i, d_i) = b_i + d_i^2$. In this formulation, the key parameters of interest are the effect of destructive investment on one's own valuation and it's effect on the opponent's valuation, γ_{own} and γ_{opp} respectively.

Solving the first order conditions from above[3], yields the following symmetric behavior:

$$
\begin{aligned}
d^* &= \max\{\frac{1}{8}(\gamma_{opp} - 2\gamma_{own}), 0\} &(8)\\
v^* &= \max\{\frac{1}{8}(8\bar{v} + 2\gamma_{own}^2 + \gamma_{own}\gamma_{opp} - \gamma_{opp}^2, 0\} &(9)\\
b^* &= \frac{v^*}{4} &(10)\\
E\pi^* &= \frac{1}{64}(16\bar{v} - 3\gamma_{opp}(\gamma_{opp} - 2\gamma_{own})) &(11)
\end{aligned}
$$

There are a number of interesting characteristics to this solution. First, equilibrium destructive

[3] With help from Mathematica for the pre-bidding, though verified by hand.

investment is non-zero only when the marginal effect of the investment on the opponent is sufficiently large relative to the effect on oneself ($\gamma_{opp} > 2\gamma_{own}$ in this case). Destructive investment is only optimal when the damage it does to other contestants is sufficiently large as compared to the damage it does to oneself.

Second, while this solution makes it seem that destructive investment sufficient to reduce valuations to zero are feasible, the second order conditions prevent this. Due to the complexity of the closed form solution, the second order condition is omitted here. For larger than this critical threshold, $\gamma_{opp} > \gamma^*_{opp}$, symmetric, pure strategy equilibria fail to exist.[4].

3.2 Risk-Aversion

Consider the following realization of model 2 which adds risk-averse contestants. This game is played in two rounds by two contestants with weakly risk-averse preferences. In the first round, players simultaneously choose an asymmetrically destructive investment d_i. To simplify the analysis, destructive investment only affects other contestants. The value of the prize for each player is then common knowledge and depends only on the opponent's destructive investment: $v_i(d_j) = \bar{v} - \gamma d_j$ for $j \neq i$. In the second round, each player then simultaneously submits a bid and wins with probability $\rho_i(b_i, b_j) = \frac{b_i}{b_i + b_j}$. All players must pay the cost of their destructive investment and bid: $c_i(b_i, d_i) = b_i + d_i^2$. To study the effect of risk-aversion on contestant behavior, utility over wealth is assumed to have a constant coefficient of relative risk-aversion: $u_i(w_i) = w_i^{1-\alpha_i}$. Contestants have constant relative risk-aversion of $\alpha_i \in (0, 1]$.[5] Finally, bidders receive Von-Neuman Morganstern utility $u(\bar{w} + v_i - c_i)$ if they win the prize and $u(\bar{w} - c_i)$ if they lose the prize.

Due to the complexity of finding a closed form solution to this model when $\alpha \neq 0$, numerical solutions are presented in Figures 3-8.[6] All figures have a risk-neutral contestant 1 and display

[4] Work to characterize the asymmetric and/or mixed strategy equilibria in this case is incomplete at this time.

[5] Coefficient of Relative Risk Aversion: $CRRA = \frac{-wu''(w)}{u'(w)} = \alpha$

[6] Numerical solutions found by repeatedly and sequentially finding roots to the first order conditions to each round

ten graphs of Nash equilibrium behavior for varying parameter values: Equilibrium bids b_1 and b_2, the sum of bids $b_1 + b_2$, equilibrium destructive investments d_1 and d_2, the sum of destructive investments $d_1 + d_2$, equilibrium round 2 valuations v_1 and v_2, the sum of valuations $v_1 + v_2$, and contestant 1's probability of winning. Figures 3-5 show the effect making contestant 2 increasingly risk-averse for differing strengths of destructive investment. Figures 6-8 show the effect increasing the strength of the destructive investment for differing levels of risk-aversion by contestant 2.

Examining graphs in Figures 3-5 demonstrates a number of key results. First, as contestant 2 becomes more risk-averse, they reduce their destructive investment and eventual bid. Contestant-1 however has a non-monotonic response to an increase in contestant 2's risk-aversion. Initially, increasing opponent risk-aversion results in a larger investment in reducing the value of contestant-2's prize. Additionally, contestant 1 increases their bid as contestant 2 becomes more risk-averse because contestant 2 destroys less of contestant 1's value. For high levels of relative risk-aversion by contestant 2, contestant 2 will reduce their bid enough that contestant 1's probability of winning increases sufficiently that they destroy less and bid less. As destructive investment becomes more effective however (increasing γ), closer to monotonic contestant-1's destructive investment and bidding become with respect to contestant-2's risk-aversion. Effectively, when destructive investment is highly effective, the increase in destructive investment reduces the valuation of the second contestant sufficiently to make changes in risk-aversion less important.

Figure 6 shows how increasing the effectiveness of destructive investment impacts destructive investment and bids for the symmetric, risk-neutral case. As expected, when destructive investment becomes more effective, that investment increases and bids decline. Increased risk-aversion by contestant 2 results in more destructive investment by contestant 1 (see scales on (γ, d_1) graphs) and non-monotonicity in contestant 2's destructive investment response to investment effectiveness. For risk-averse contestant, as destructive investment becomes highly effective, they will

until convergence is reached. A tolerance of 10^{-7} between iterations was used to determine convergence between iterations. Mathematica code is available upon request.

reduce investment. Because the less risk-averse contestant highly invests in destroying the more risk-averse contestant's value, the more risk-averse contestant reduces their bid and thus their probability of winning. When a contestant loses, any destruct investment is merely a cost. this cost is reduced by investing less in the destruction of opponent's value.

4 Conclusion

There are a few results explicitly omitted from prior discussion. First, in any case where the destructive investment is costly and chosen simultaneous to the bid, the Nash equilibrium is to choose no destructive investment. This is because the chosen destructive investment is unobserved by the opponent and thus cannot affect their bid to increase the probability of the player winning the contest. Second, the Nash equilibrium surplus in model 2 is lower than if both contestants choose no destructive investment. This is a classic prisoner's dilemma problem and thus cooperative solutions which ban or minimize destructive investment will increase the expected surplus for each contestant. In military conflicts, this can take the form of bans on particular weapons or tactics. In corporate competition, many countries have at times outlawed comparative advertising. In politics, there have been calls to ban attack ads.

The explicit results from models 1 and 2 admit a number of conclusions which have implications for all-pay auctions in the real world. If the bidding itself reduces the common value of a prize, as in model 1, then bidding behavior is consistent with existing literature but with the bids decreased by the expected value of destruction to the prize. As such, in a military conflict, the more powerful the weaponry is, the less that will be committed to the battle. If a separate investment which reduces the value of the prize for contestants asymmetrically is required, as in model 2, then the deciding factor in how large such a destructive investment will be the relative effect of the investment on opponents' valuation compared to one's own valuation. Negative advertising in political campaigns reduce the ability of the winning candidate to effectively govern. Research has suggested that voters do view the attacker, as well as the target, more negatively. Model 2 predicts that such a tactic is only optimal when voters view the recipient of the negative ads significantly worse than the attacker in the process. If the use of negative advertising had a nearly symmetric effect on both parties then such tactics would be eschewed. Additionally, risk-aversion is a competitive disadvantage in this game. Risk-averse contestants will bid less aggressively and be faced with opponents who are more aggressive in destroying value.

In the future, this research will examine the point at which the symmetric pure strategy Nash equilibrium in model 2 breaks down and potential variations on the set-up of model 1.

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5 Figures

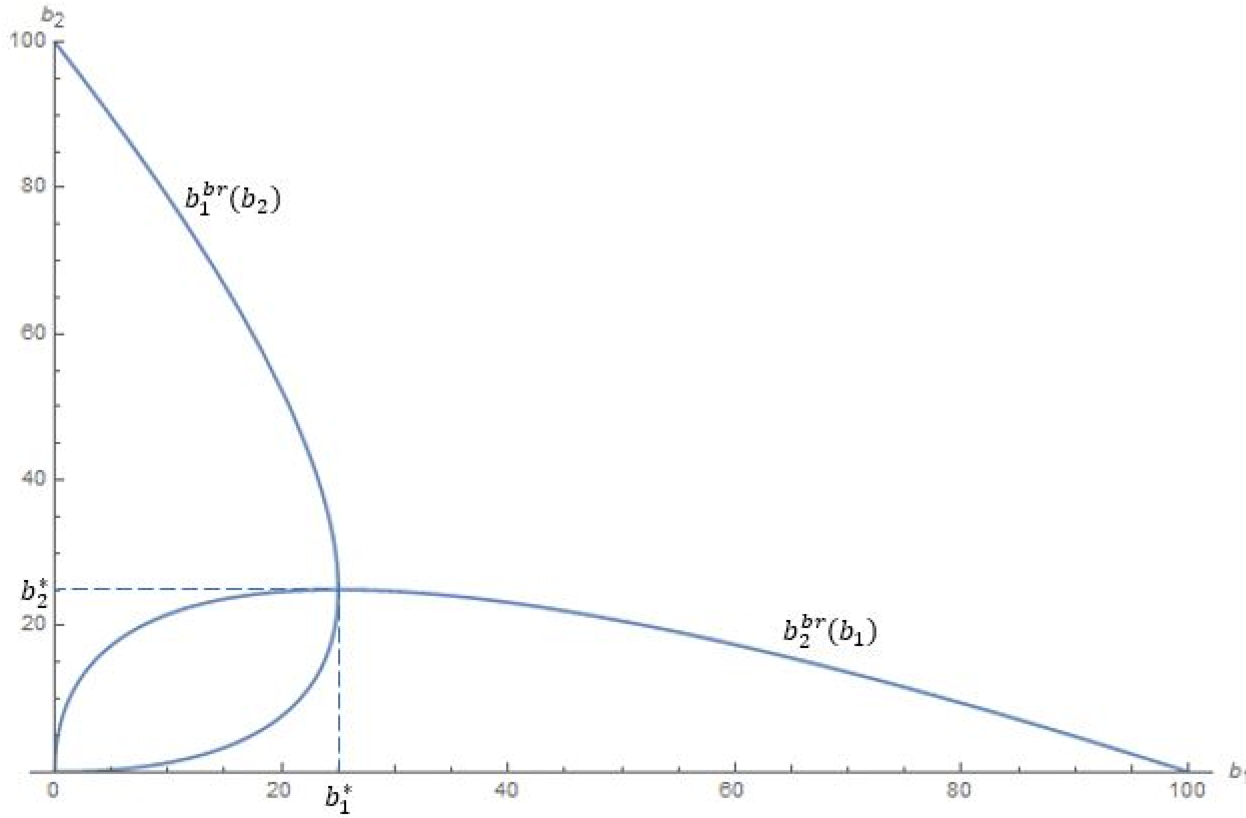

Figure 1: Best Responses and Nash Equilibrium Bidding for $\rho_1(b_1, b_2) = \frac{b_1}{b_1+b_2}$ and $v_1 = v_2 = 100$, yielding $b_i^{br} = \sqrt{v_i b_j} - b_j$

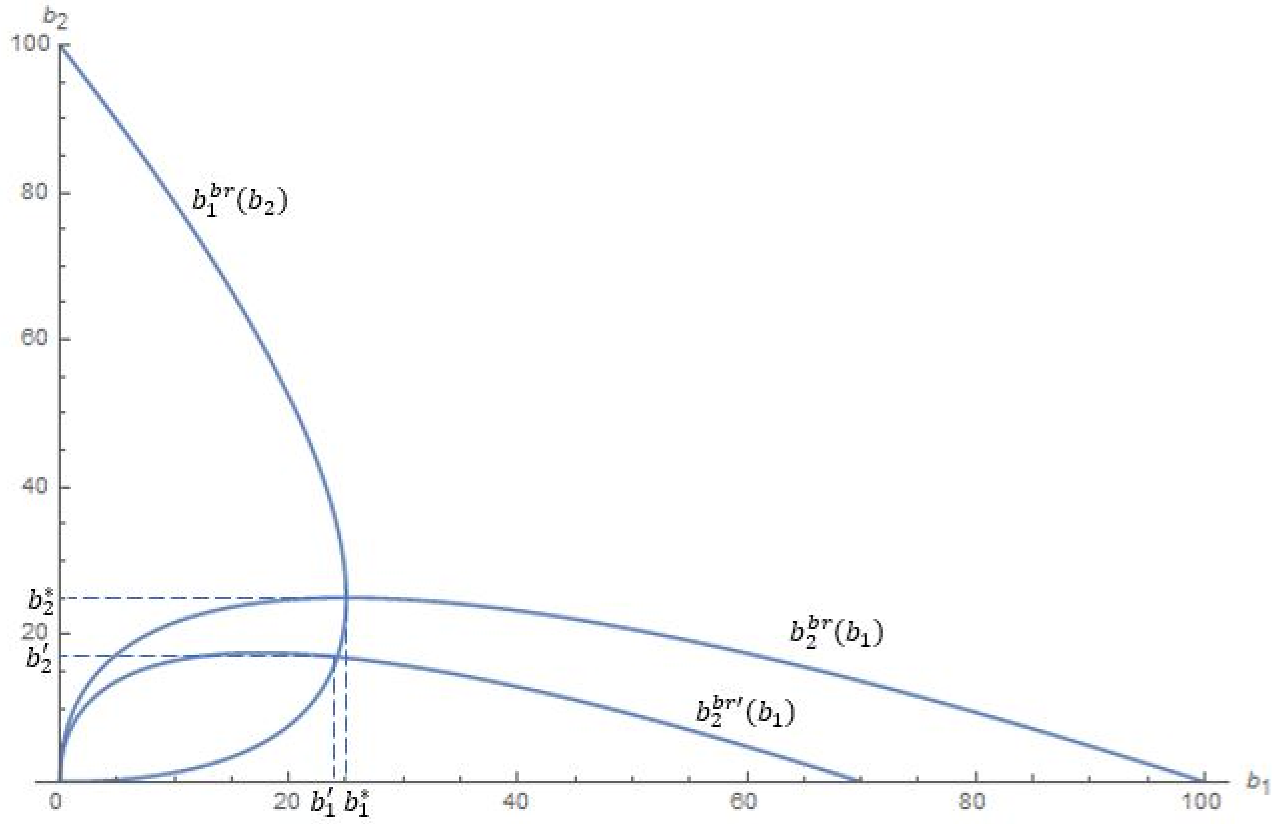

Figure 2: Affect of Destructive Investment by Contestant 1 on Equilibrium Bids

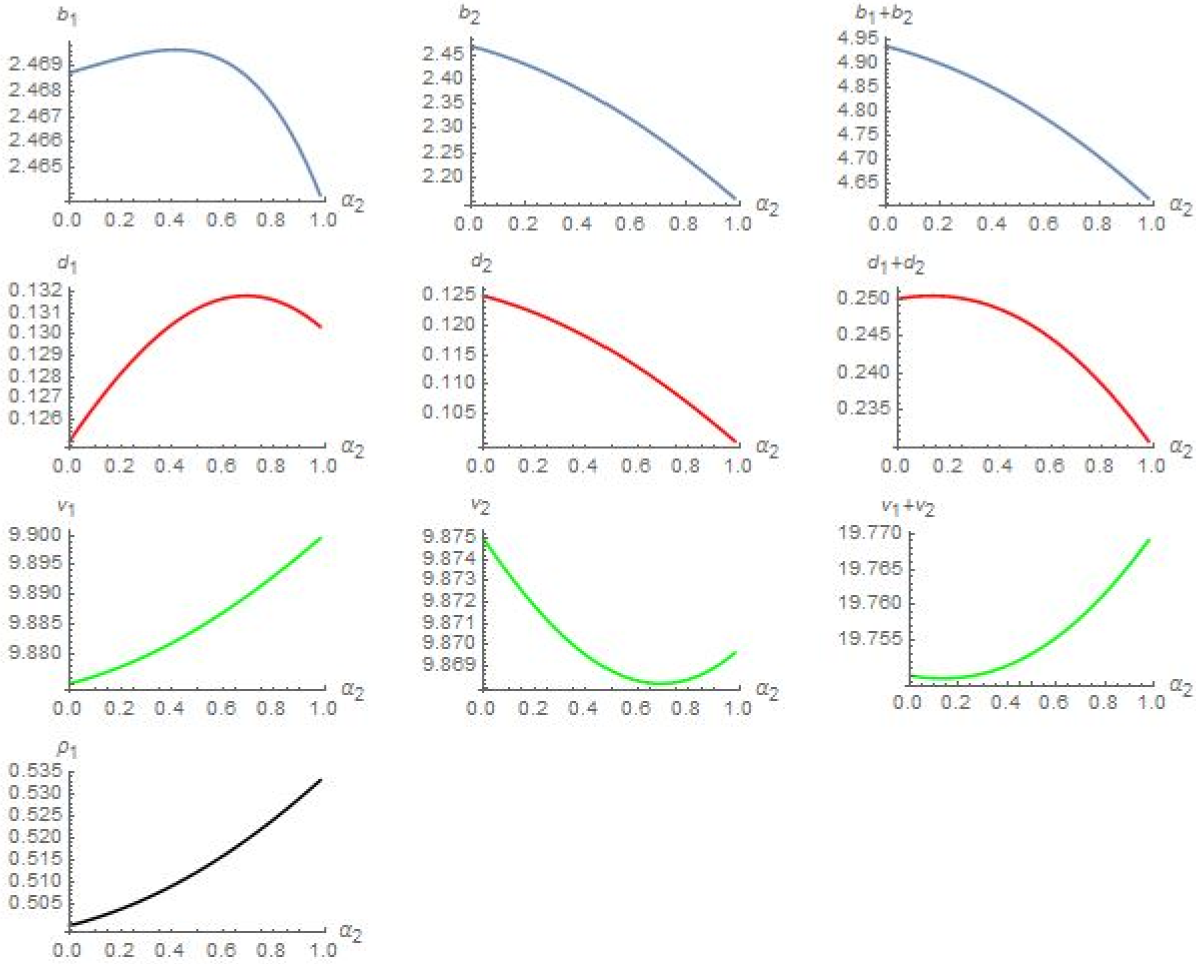

Figure 3: $\alpha_1 = 0, \gamma_1 = 1$

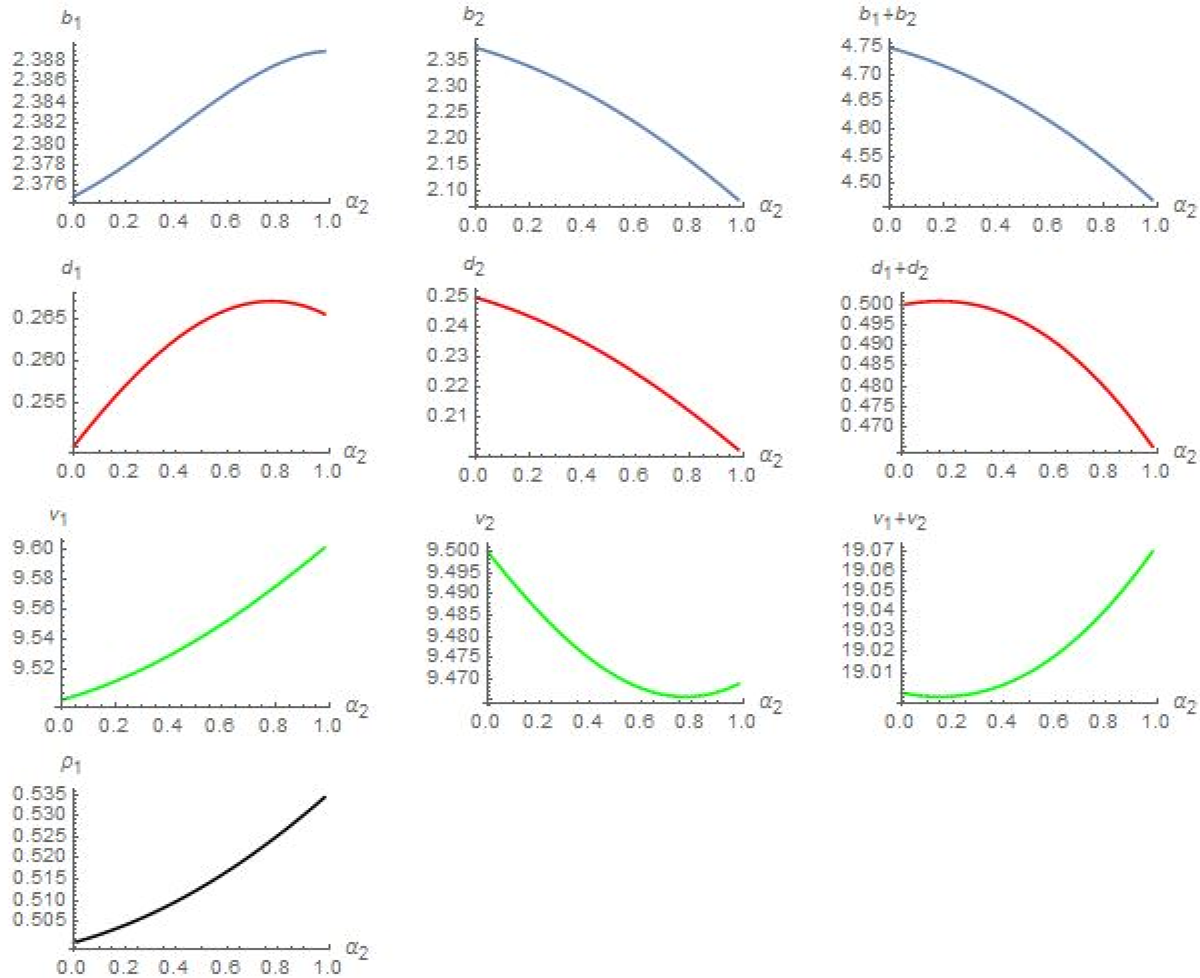

Figure 4: $\alpha_1 = 0, \gamma_1 = 2$

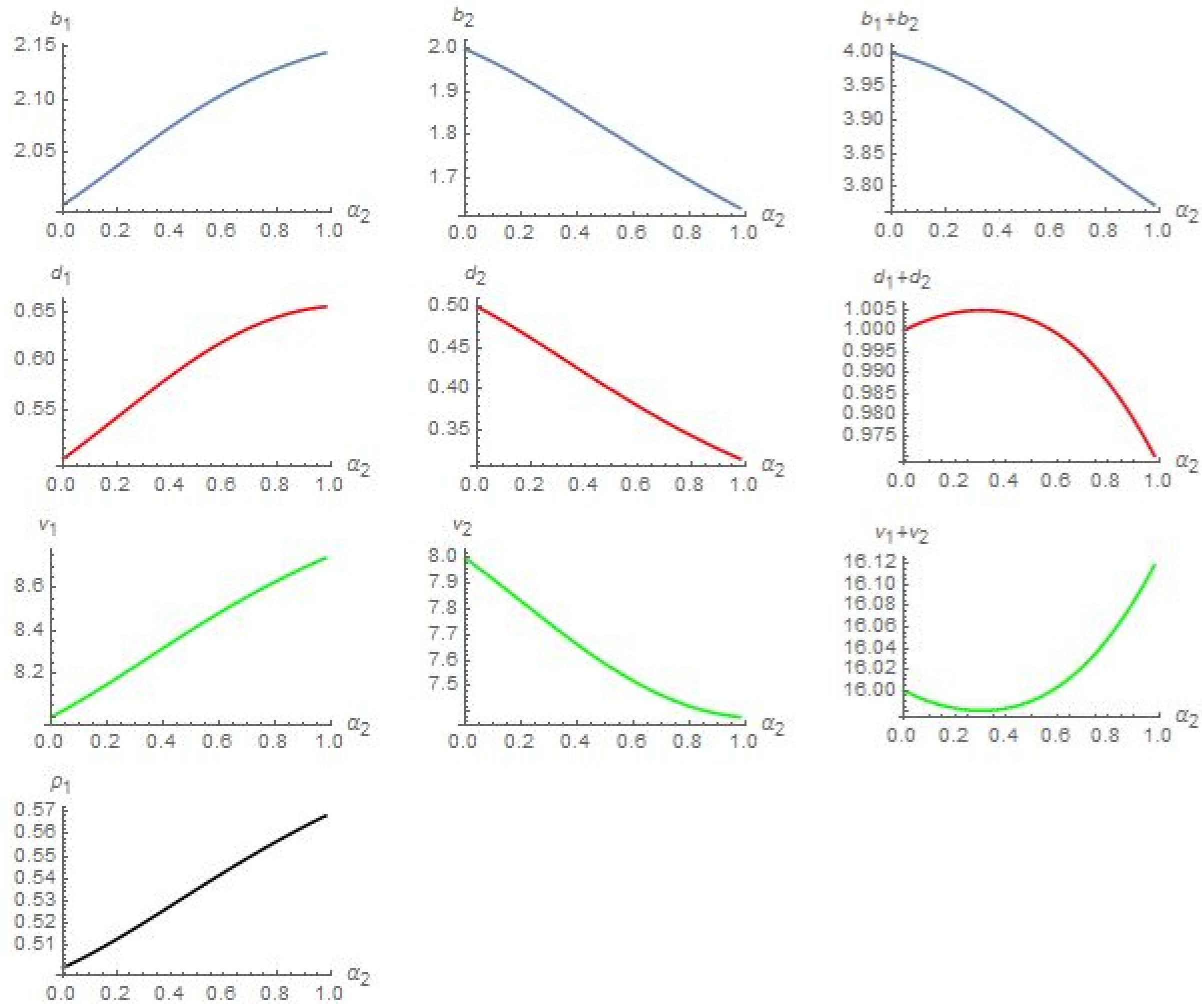

Figure 5: $\alpha_1 = 0, \gamma_1 = 4$

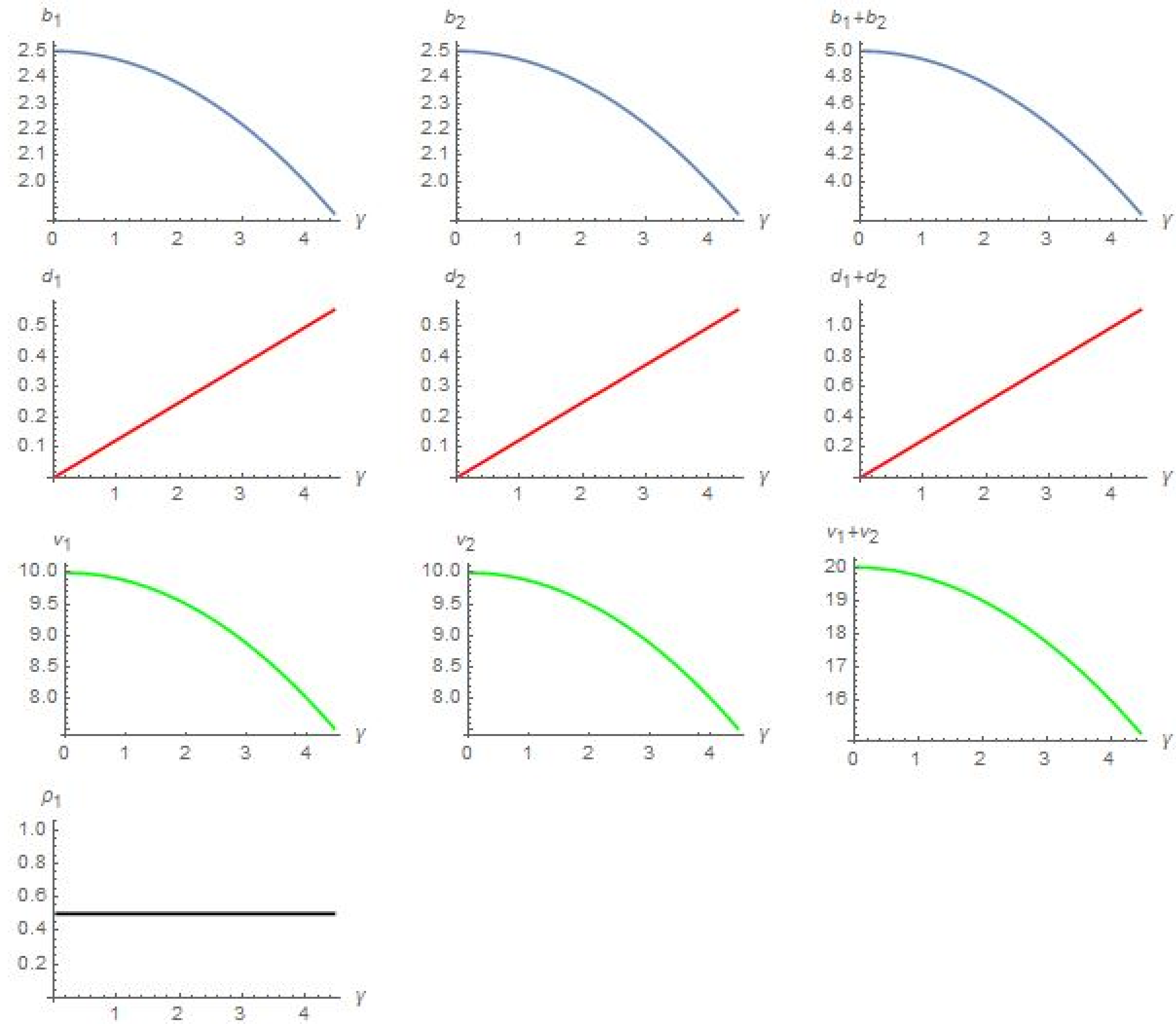

Figure 6: $\alpha_1 = 0, \alpha_2 = 0$

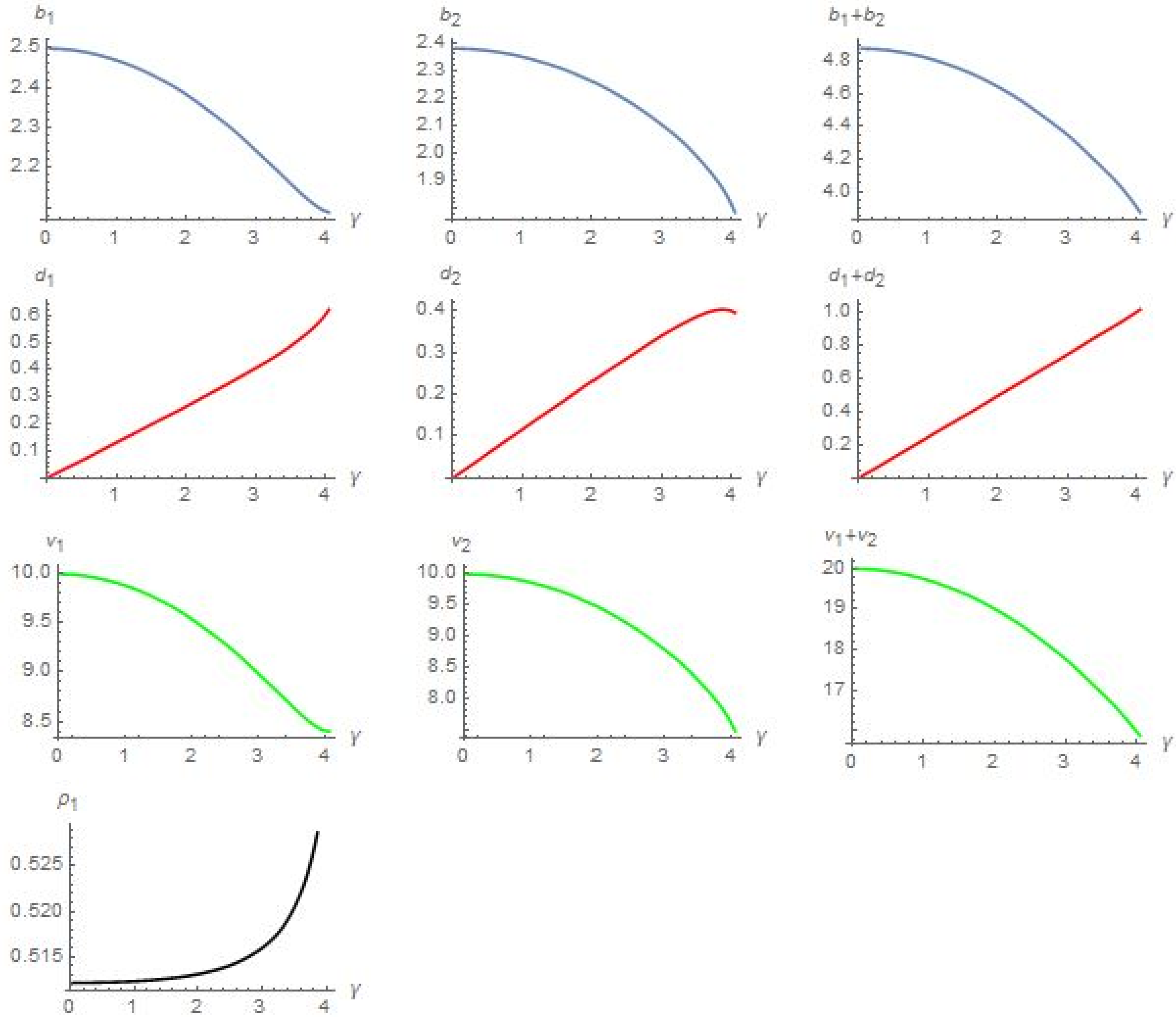

Figure 7: $\alpha_1 = 0, \alpha_2 = 0.5$

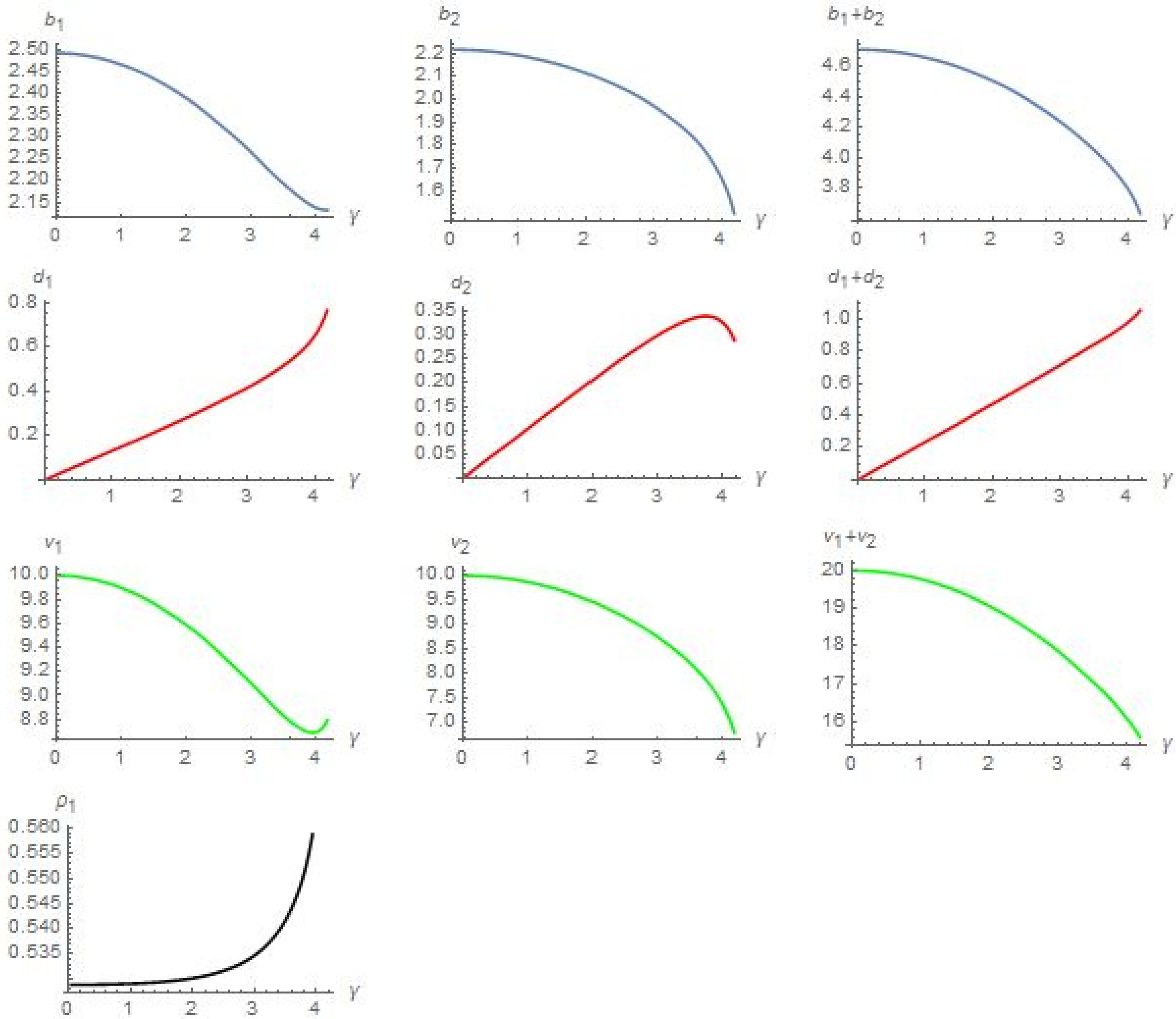

Figure 8: $\alpha_1 = 0, \alpha_2 = 0.9$