# Seeking Relationship Support: Strategic network formation and robust cooperation

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**Abstract**

We study cooperation on social networks with private monitoring and communication. For arbitrary networks, we construct a class of *multilateral restitution* equilibria that attain high cooperation on all *supported* links—i.e., all links that are in triangles. These equilibria are robust to social contagion, bilaterally renegotiation proof, and invariant to players' beliefs about the network outside their local neighborhoods. In these equilibria, guilty players are not ostracized, instead they remain to sustain the stability of the cooperation network by exerting high effort for their innocent partners, and they are willing to do so because they are compensated for their effort costs. Anticipating cooperation, players in a network formation game with random opportunities to form links will strategically form a network with realistic *small worlds* properties, including high support but relatively low clustering.

## 1 Introduction

Consider a social network in which each link is an ongoing productive relationship. Within such a relationship, the partners benefit from each others' efforts, but each has an individual

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temptation to shirk. Since these relationships are organized within a social network, there is scope for social enforcement: If a player shirks on one partner, and that partner informs other players in the network, then eventually the deviator may be punished by multiple partners. Because the multilateral punishment from the community is harsher than a bilateral punishment, it can help sustain a higher degree of cooperation.

On the other hand, as information transmission and community enforcement depend on the network, they may seem to rely on strong requirements on the players' knowledge or beliefs about the potentially large network outside their local neighborhoods. Indeed, most of the recent literature has assumed that the entire network is commonly known. In reality, the players may observe only a small subset of the entire network. For example, cooperations such as social insurance and risk sharing are particularly important in rural villages of developing countries, where people's knowledge and beliefs about the outer network could be very poor. Breza, Chandrasekhar, and Tahbaz-Salehi (2016) surveyed villages in Karnataka in southwest India, and found that 46% of respondents are not able to guess whether there is a link between a given pair of individuals, and conditional on making a guess, the accuracy is only 33%.[1] Our first goal is to understand how community enforcement can robustly sustain a high level of cooperation, but without requiring global knowledge or particular beliefs about the network.

Moreover, recent work on the theory of cooperation in social networks has largely taken a normative view on how to compare networks. Broadly speaking, the literature finds that denser networks yield more cooperation.[2] In contrast, the descriptive empirical side of the literature has found that social networks are not particularly dense. Compared to theoretically optimal networks, real social networks have similarly high *support*, but dramatically lower *clustering*. High support means that each pair of linked partners is likely to have at least one mutual neighbor; low clustering means that two players who share a mutual neighbor are unlikely to be linked to each other. As a result, for a given population and a given average degree, real social networks are much more "expansive" (Ambrus, Möbius, and Szeidl 2014) than is optimal under these theories. In this paper we take a positive theoretical approach, to study which networks players may form through

[1] This is consistent with other surveys on people' knowledge of their networks. For instance, Krackhardt (1990) finds that the accuracy of knowing other people's connections is 15%-48% in a startup of 36 people; Casciaro (1998) finds the accuracy is around 45% in a research center of 25 people.

[2] See, for example, Ali and Miller (2013); Ambrus, Möbius, and Szeidl (2014); Jackson, Rodriguez-Barraquer, and Tan (2012); Wolitzky (2013).

their strategic interactions. Anticipating cooperation, players are motivated to seek out support for their relationships.

This paper makes two significant contributions. Our first contribution is to introduce a class of *multilateral restitution* equilibria that implement multilateral enforcement with private monitoring on an arbitrary network, with several important robustness properties. First, each player's strategy depends only on the structure of his 1-neighborhood and the events that occur within his 2-neighborhood—containing his friends and friends of friends. Second, when a punishment arises, its ramifications do not cascade through the network like a contagion; instead the effect is contained within the deviator's 1-neighborhood. Third, multilateral restitution equilibria are not vulnerable to renegotiation between linked pairs of players.

Our second contribution is to introduce a simple network formation game with random match quality, in which players first get random opportunities to meet "strangers" and become "friends", and then get opportunities to meet friends of friends. If the players look forward to playing a multilateral restitution equilibrium after the network forms, then this network formation game makes transparent their motives to form a network with high support but relatively low clustering.

**Multilateral restitution** The difficulty in constructing robust equilibria with multilateral enforcement on arbitrary networks arises from interactions among overlapping neighborhoods. After player 1 deviates by shirking on player 2, for multilateral enforcement player 2 should inform player 3 of the deviation, and then both players 2 and 3 should punish player 1. However, these punishments may interfere with cooperation between players 2 and 3, since if their payoffs along links $1, 2$ and $1, 3$ are depressed, they have less to lose by shirking on link $2, 3$. If both of them are also connected to player 4, then shirking may start to cascade through the network as a contagion.

Multilateral restitution strategies solve this problem by preserving the payoffs of innocent players following a deviation. The deviator is not simply ostracized from the community; instead he remains active in the network to help sustain cooperation among his neighbors. Specifically, after player 1 deviates on player 2, as a *restitution* punishment player 1 must work hard when meeting player 2 in the future, while player 2 does just enough to motivate player 1 to do so. At the same time, players 2 and 3 communicate truthfully and continue cooperating at a high level. Player 3 eventually starts punishing player 1 as well—either because player 1 shirks on player 3, or because player 3 learns about player 1's guilt from player 2. The maximal level of cooperation that a triangle can sustain on the equilibrium path is that which makes each player indifferent between working and shirking when facing the threat of maximal punishment by both neighbors.[3] Observe that since innocent players 2 and 3 always expect to get their equilibrium payoffs in their relationships on the $1, 2, 3$ triangle (it is a surprise for them if player 1 shirks), restitution punishments do not initiate a contagion. Similarly, if players 2 and 3 are members of other triangles, cooperation on those triangles is not disrupted by player 1's deviation. In such an equilibrium, the players' effort level on each link only depends on whether the link belongs to a triangle, so they only need to know their 1-neighborhood. Moreover, the players who might be called upon to punish player 1 need to know behavior details within player 1's 1-neighborhood, but not beyond. Thus no player needs any information about the behaviors beyond her own 2-neighborhood. Finally, since innocent players expect to earn the same payoffs both on and off the equilibrium path, they are willing to rebuff any proposal to renegotiate (whether "internally" to another payoff vector in the same equilibrium, or "bilaterally" to any equilibrium that can be coordinated with a single partner).

To implement a restitution punishment, in which guilty player 1 works hard for innocent player 2, we allow the players to endogenously choose who acts first when they meet. On the equilibrium path, they randomize over who acts first. (This is strictly better than moving simultaneously, since whoever acts first can be punished immediately rather than waiting for their next meeting.) Once player 1 becomes guilty, he must always move first. He should work hard enough to give player 2 her equilibrium-path payoff; he is willing to do so because if he does she will immediately compensate him for his effort cost. Due to the endogenous sequencing, the punishment is stationary.[4]

**Network formation** We introduce a network formation game, prior to the repeated interaction game, in which players strategically form links over several stages, anticipating

[3] In contract law, restitution damages are calculated to erase the "unjust enrichment" obtained by the party who breached the contract (see Thompson 1984). Within a triangle, this is precisely the punishment that a deviator faces, since the prospect of punishment makes him indifferent between cooperating and deviating on the equilibrium path. On a denser network, however, the deviator may suffer restitution punishments with each of many partners, akin to the legal remedy of paying a multiple of the restitution damages. Notice that victims are never compensated for being shirked on—restitution punishments operate as deterrence, not for justice.

[4] If the players acted simultaneously, then attaining equilibrium-level continuation payoffs to the innocent partner and zero to the guilty partner and would require non-stationary continuation play.

the benefits of multilateral enforcement after the network forms. In each stage we focus on an equilibrium in which they coordinate to behave myopically with respect to the links they might form in later stages. In the first stage, random pairs of players are recognized to meet simultaneously. A pair of partners who meet will form a link if their idiosyncratic linking cost is sufficiently low compared to the benefit they expect from cooperating in the absence of relationship support. With a large population, the "backbone" network that forms is a uniform random network that is approximately a tree, with no clustering or support (Erdős and Rényi 1959).

After the backbone network is formed, in the second stage each player meets his or her "friends of friends"; i.e., the players at distance two in the backbone network. Each such pair forms a link if their idiosyncratic linking cost is sufficiently low compared to the benefit of cooperating in the presence of a supporting relationship, since every link formed in the second stage is supported. So in the second stage, the sparse backbone becomes more dense, and clustering increases.

In the third stage, players seek support for their unsupported relationships. Specifically, they can revisit their pairwise decisions over links they elected not to form in the second stage, in random order. This time, they take into account the externality that is provided—forming a new link brings support to one or two relationships that were unsupported after the second stage. We assume that these decisions are made to myopically maximize the joint surplus of the three players involved; i.e., as if forming the link gives them an opportunity to transfer utility among themselves.

While the equilibrium network inherits small average distances and a giant component from its backbone network, high support and moderate clustering arise from the fact that players seek support for their relationships, but don't benefit from having multiple supports for the same relationship.

### 1.1 Related literature

Our model of interaction builds on the private monitoring, variable effort models introduced by Ali and Miller (2013, 2016), which in turn built on the variable effort models of Ghosh and Ray (1996) and Kranton (1996). We allow partners to exchange messages about their past histories, following Lippert and Spagnolo (2011) and Ali and Miller (2016); like the latter, we assume that these messages are based on verifiable evidence rather than cheap talk. Like Ali and Miller (2018) we assume that partners act sequentially rather than simultaneously when they meet, enabling the first mover to be punished immediately. Our modeling innovation is to allow partners to endogenously select which of them moves first when they meet. By choosing who moves first as a function of the history, the players can attain higher equilibrium payoffs than if timing (either sequential or simultaneous) were imposed exogenously, and they can punish the guilty partner by making him or her move first.

The equilibrium properties on which we focus build on several strands of the prior literature. We seek equilibria that are "robust" to contagion, so that a deviation does not cause a breakdown of cooperation outside the deviator's neighborhood. This kind of robustness was first formalized by Jackson, Rodriguez-Barraquer, and Tan (2012). We seek equilibria in which players need only "local" knowledge, as pioneered by Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010), and employed by Nava and Piccione (2014) and Campbell (2014). For both robustness and local knowledge, we introduce stronger notions than used in the prior literature. We also seek equilibria that are "bilateral renegotiation proof" (Ali, Miller, and Yang 2016).

Our multilateral restitution equilibria, in which guilty players are not excluded, but rather must work hard to satisfy their punishers, build on the asymmetric, bilateral renegotiation proof punishments studied by Ali, Miller, and Yang (2016), which in turn built on the ideas of van Damme (1989).

Substantively, Jackson, Rodriguez-Barraquer, and Tan (2012) and Ali and Miller (2016) have studied questions that closely relate to ours. Jackson, Rodriguez-Barraquer, and Tan model each relationship as a fixed-effort repeated prisoners' dilemma, and assume that monitoring is public. They show that "social quilts"—networks in which cliques are arranged as nodes on trees—are optimal when community enforcement must be robust to contagion and invulnerable to renegotiation among innocent players. The Ali and Miller (2016) model, like ours, has private monitoring, variable efforts, and communication; they study "ostracism" equilibria in which guilty players are excluded while innocent partners continue to cooperate. While one of their main results is negative (no "permanent ostracism" equilibrium can support more cooperation than bilateral enforcement), their other main result shows how to construct a "temporary ostracism" equilibrium on a complete network that supports strictly more cooperation than bilateral enforcement. We complement these works by showing multilateral enforcement can work on an arbitrary network, and with agents holding only local knowledge of the network.

Much of the theoretical literature on social network formation has focused on Nash equilibrium or pairwise stable networks in symmetric, deterministic environments. Our network formation game is inspired instead by the stochastic but non-strategic model of Jackson and Rogers (2007), in which new players, arriving over time, are added to the network by first linking to random "strangers" (who become "friends"), and then linking to "friends of friends." At the stochastic limit, the resulting networks display "small worlds" properties. Closer to our network formation model are the stochastic *and* strategic models of Golub and Livne (2010) and Campbell (2014), where players first meet strangers and then meet friends of friends. Both models generate small worlds properties similar (Golub and Livne) or identical (Campbell) to the Jackson and Rogers model. However, Golub and Livne assume that each player makes an ex ante strategic decision regarding how intensively to socialize, and then the entire link formation process follows mechanically from these decisions. Campbell assumes that players cannot choose how many links to form or how to allocate them, other than to either form all of them with strangers or form a fixed fraction of them with friends of friends. Thus in both cases clustering and support are generated by the same mechanism. In contrast, we assume that players make case-by-case decisions over which links to form as the individual opportunities arrive; as a consequence our model contains not only a mechanism that generates both clustering and support, but also a mechanism that generates support without clustering. As for payoffs, players in the Golub and Livne model value only direct connections; clustering and support arise mechanically because making friends with strangers affords one more opportunities to make new friends. Players in the Campbell model value support in order to signal high patience; once revealed to be patient they no longer need support to sustain cooperation in their relationships.

Campbell (2014) features three other aspects that relate to our work. First, Campbell's players can choose the stakes of their relationships. However, they can choose only high or low, and use these stakes only as screening devices, not to take advantage of multilateral enforcement. Second, Campbell's players have only local knowledge of the network, which makes their inference problem of screening types non-degenerate. Third, Campbell also considers a variant of renegotiation proofness that is close to Jackson, Rodriguez-Barraquer, and Tan (2012).

The networks that form in our model can be interpreted as having both "strong ties" and "weak ties" (Granovetter 1973), where strong ties are links that are supported and exhibit high cooperation, while weak ties are links that are unsupported and exhibit low cooperation. However, in contrast to Granovetter's analysis it is not particularly likely that two players with strong ties to the same third player should seek to form a direct link in our model—while their link would be somewhat valuable because it would be supported, it does not bring new support to any existing unsupported links.

## 2 The repeated interaction model

A population of players $N = \{1, \dots, n\}$, arranged on a network, interact repeatedly along their network links. The network, $G$, is a collection of undirected bilateral links. If players $i$ and $j$ are linked in the network, we write $ij \in G$, or sometimes to avoid ambiguity, $i, j$. (Later, in Section 4, we examine how the prospect of multilateral enforcement influences how the network forms.)

We say a path of length $k$ between agent $i$ and $j$ is a sequence $\{i_0, i_1, \dots, i_k\}$ such that $i_0 = i$, $i_k = j$ and $i_l i_{l+1} \in G$ for any $l \in \{0, \dots, k-1\}$. Let $D(i,j)$ be the distance between agent $i$ and $j$ in the network, defined as the length of the shortest path between them. Let $D(i,i) = 0$ and $D(i,j) = \infty$ if $i$ and $j$ are not path-connected. Define agent $i$'s *$D$-neighborhood* as $(g_i^D, G_i^D)$: $g_i^D = \{j : D(i,j) \le D\}$ and $G_i^D = \{jk \in G : j, k \in g_i^D\}$. We assume agent $i$ only knows his or her 1-neighborhood at time $t = 0$.[5]

The repeated interaction game proceeds over time $t \in [0, \infty)$. Each pair of connected agents, $ij \in G$, is engaged in a partnership $ij$ that meets at random times generated by a Poisson process of rate $\lambda > 0$. Meetings are i.i.d. across partnerships and over time. Whenever partnership $ij$ meets, they play a stage game with four phases. Players outside the partnership cannot observe when the partners meet or how they behave when they meet. With this private monitoring, communication is crucial to sustain cooperation.

1. Pre-effort communication: each agent $i$ and $j$ simultaneously reveals verifiable evidence of any subset of his or her past interactions. We focus on evidentiary communication: agents can reveal or conceal the evidence, but they cannot present falsified evidence.

2. Sequencing: $i$ and $j$ simultaneously send messages indicating which partner they

[5] As we will explain, agent $i$ may learn about his or her indirect neighbors, after certain deviation occurs.

think should go first. If they agree, the agreed-upon partner moves first in the effort selection phase. If they disagree, then one is randomly selected to move first, with equal probabilities. The sequential moves with equal probabilities of being either mover yield strictly higher expected payoffs than the simultaneous moves.[6]

3. Effort selection: Let $i$ be the first mover. Then $i$ and $j$ sequentially choose effort levels $x_i, y_j$ in $[0, \infty)$, where $x_i$ indicates that $i$ is the first mover and $y_j$ indicates that $j$ is the second mover. Player $i$'s stage game payoff function when partnership $ij$ meets is $b(y_j) - c(x_i)$, where $b(y_j)$ is the benefit from her partner $j$'s effort and $c(x_i)$ is the cost she incurs from her own. Similarly, player $j$'s stage game payoff is $b(x_i) - c(y_j)$.

4. Post-effort communication: $i$ and $j$ have another opportunity to simultaneously reveal verifiable evidence about their past interactions, just as in the pre-effort communication phase.

Each pair has access to public randomization devices whose realizations are observed only by that pair. All players share a common discount rate $r > 0$.

We normalize the net value of effort $x$ to $b(x) - c(x) = x$. The following assumption articulates that higher effort levels increase the temptation to shirk.

**Assumption 1.** *The cost of effort $c$ is smooth, strictly increasing, and strictly convex, with $c(0) = c'(0) = 0$ and $\lim_{x\to\infty} c'(x) = \infty$. The "relative cost" $c(x)/x$ is strictly increasing.*

Strict convexity with the limit condition guarantees that in equilibrium effort is bounded (as long as continuation payoffs are bounded, which we assume below). Increasing relative cost means a player requires proportionally stronger incentives to exert higher effort.

**Solution concept** Our solution concept is *plain perfect Bayesian equilibrium*, or PPBE (Watson 2016). This refinement of "weak perfect Bayesian" equilibrium Mas-Colell, Whinston, and Green (1995) imposes Bayesian updating on off-path beliefs, but is less restrictive and simpler to verify than sequential equilibrium or perfect extended-Bayesian equilibrium (Fudenberg and Tirole 1991; Battigalli 1996). Since we will construct a particular class of equilibria without making any claims about optimality, we could adopt an even weaker

[6] Thus our results would remain unaffected if we allowed agents to also propose simultaneous moves. See remark 3 in the appendix for details.

solution concept. Our use of PPBE assures that our construction does not rely on the kinds of implausible off-path beliefs that are possible under weak perfect Bayesian equilibrium.

## 3 Robust community enforcement

In this section we show that community enforcement can sustain high cooperation in the repeated interaction era on whatever network arises from the network formation era, in a way that preserves cooperation among innocent players after a deviation, and without requiring a player to know anything about the network structure or behaviors outside of her local neighborhood.

To begin with, we seek to sustain high levels of cooperation in society even off the equilibrium path.

**Definition 1.** *A strategy profile is* ***robust*** *if partners who have not deviated always cooperate at the same level, on and off the path of play.*

This property is stronger than the robustness criterion used by Jackson, Rodriguez-Barraquer, and Tan (2012), which allowed for cooperation to break down among a bounded set of innocent players following a deviation by one of their neighbors.

Next, the strategy profile only requires local knowledge, including both the network topology as we defined in the model setup and the interactions.

**Definition 2.** *A strategy profile is* ***D-local*** *if each agent $i$'s strategy is invariant to her beliefs about interactions and network topology outside her $D$-neighborhood.*

We focus on strategy profiles that are 2-local. In fact, these strategy profiles use only 1-local information along the equilibrium path, but employ off-path punishments that can depend on 2-local information that players learn in the course of play. That even off-path behavior is invariant to beliefs about the wider network makes this property stronger than the invariance criterion used by Nava and Piccione (2014), which is invariant to non-local beliefs at the start of the game but allows players' behavior to depend on their beliefs about the global network in the course of play.[7]

[7]The equilibrium Nava and Piccione construct (for a game of local interaction rather than bilateral interaction) exploits that possibility by inducing incorrect beliefs about the network following a deviation. The equilibrium we construct satisfies both our 2-locality criterion and their invariance criterion.

### 3.1 Benchmark cooperation

Our benchmark for high cooperation is the maximum level of cooperation attainable by a stationary equilibrium on a triangle network. But first we introduce *bilateral cooperation*, the maximal cooperation attainable between two partners without the aid of community enforcement.

**Bilateral cooperation** Consider a strategy profile in which on the path of play first movers choose effort level $x$ and second movers choose effort level $y$; off the equilibrium path each exerts zero effort. The equilibrium path incentive constraints are:

$$0 \leq -c(x) + b(y) + \int_0^\infty e^{-rt} \lambda \frac{1}{2} \big(x + y\big) dt \tag{1}$$

$$0 \leq -c(y) + \int_0^\infty e^{-rt} \lambda \frac{1}{2} \big(x + y\big) dt \tag{2}$$

The bilateral cooperation levels $x^{\mathrm{B}}$ and $y^{\mathrm{B}}$ are the effort levels that bind these incentive constraints. Since the grim trigger punishment is a minmax punishment and each partner's effort relaxes the other partner's incentive constraint, these are the maximum efforts that can be supported by any stationary equilibrium that does not involve community enforcement. Note that this implies $-c(y^{\mathrm{B}}) = -c(x^{\mathrm{B}}) + b(y^{\mathrm{B}})$; i.e., the gain from shirking is the same regardless of whether a player moves first or second, and the first mover receives a negative payoff in the stage game.[8]

**Triangular cooperation** Consider a triangle network $\{i, j, k\}$, and a strategy profile in which on the path of play first movers choose effort level $x$ and second movers choose effort level $y$. Off the equilibrium path, if the first mover deviates, the second mover chooses zero effort in the current interaction, and both then choose zero effort in all future interactions. If the second mover deviates, both choose zero effort in all future interactions. In such a strategy profile, if $i$ deviates on $j$, both $i$ and $j$ will then shirk in their next meetings with $k$, so a "contagion" spreads until cooperation ceases over the whole network. Ali and Miller (2013) showed in a closely related model that such strategy profiles constitute equilibria if equilibrium-path incentive constraints bind. Moreover, since they implement minimax

[8] We could add another condition that no player receives negative payoff in the stage game, which would lower the level of efforts and the players' utilities, and the analysis of the model is analogous.

punishments, these equilibria maximize cooperation among all stationary equilibria. Here we focus only on the equilibrium-path constraints, to compute an upper bound on the cooperation that can be attained by any stationary equilibrium on a triangle network:

$$\int_0^\infty e^{-rt}e^{-2\lambda t}\lambda\frac{1}{2}b(x)\,dt \leq -c(x)+b(y)+2\int_0^\infty e^{-rt}\lambda\frac{1}{2}\big(x+y\big)dt \tag{3}$$

$$\int_0^\infty e^{-rt}e^{-2\lambda t}\lambda\frac{1}{2}b(x)\,dt \leq -c(y)+2\int_0^\infty e^{-rt}\lambda\frac{1}{2}\big(x+y\big)dt \tag{4}$$

These incentive constraints bind at effort levels $x^{\mathrm{T}}$ and $y^{\mathrm{T}}$. As with bilateral cooperation, the gain from shirking is the same for the first mover and the second mover.

**Lemma 1.** *There exist triangular effort levels $x^{\mathrm{T}} > x^{\mathrm{B}}$ and $y^{\mathrm{T}} > y^{\mathrm{B}}$ that bind constraints (3) and (4).*

All omitted proofs are in Appendix A. The triangle can sustain higher levels of effort because each player faces more punishment if he or she deviates.

### 3.2 Robust cooperation with local knowledge

Our main result shows that high levels of cooperation can be sustained in a robust manner, with players needing only local information about the network and other players' behavior.

A link $ij$ is *supported* if there exists $k$ such that $ik \in G$ and $jk \in G$; i.e., if $i$ and $j$ have at least one common friend.

**Theorem 1.** *For any network, there exists a robust and 2-local equilibrium for the repeated interaction game that supports triangular effort levels on every supported link along the path of play.*

Moreover, we identify a class of *multilateral restitution* equilibria that attain these properties. Intuitively, robustness requires innocent partners to continue to cooperate with each other, in which case they do not have to worry about deviations that may occur outside their local knowledge. In order to achieve this robustness, innocent partners need the help of their guilty mutual friends to sustain cooperation at a high level off the equilibrium path. Accordingly, they should not use ostracism as illustrated in the top panel in Figure 1—instead they should punish the deviator in a less socially wasteful way. We describe the class of strategy profiles here, and provide the full proof in Section 3.4.

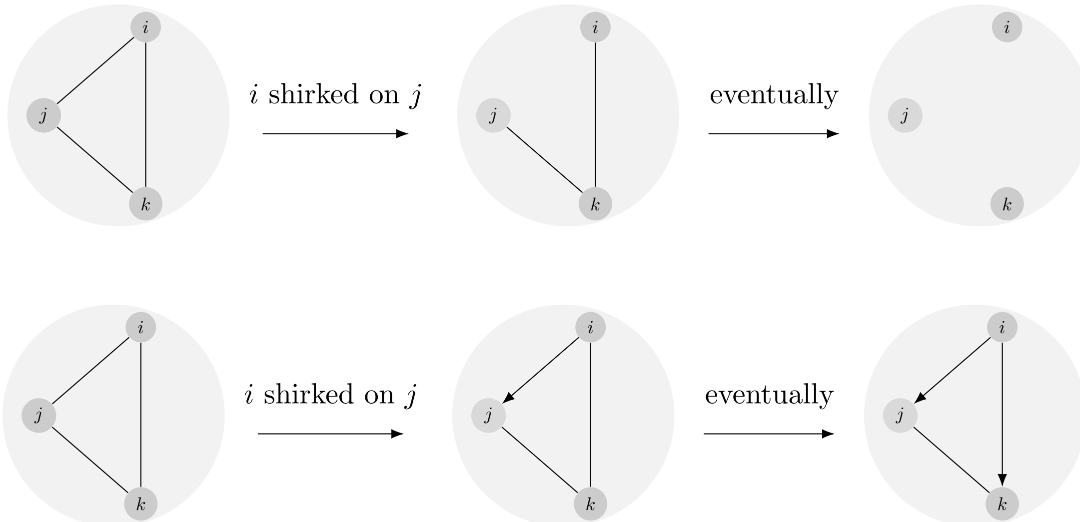


Figure 1: (Top) Network enforcement by ostracism; (Bottom) Network enforcement by multilateral restitution.

Multilateral restitution strategies operate as follows. Along the equilibrium path partners communicate "truthfully"; i.e., when partners $i$ and $j$ meet, partner $i$ reveals all the information about deviations she has that involves the intersection of her 2-neighborhood and partner $j$'s 2-neighborhood. In the sequencing phase, they never nominate themselves to move first, with the result that the first mover is always randomly selected.[9]

Partners on any unsupported link cooperate at the bilateral levels $x^{\mathrm{B}}$ and $y^{\mathrm{B}}$. Their cooperation is supported by the threat of letting the guilty player receive zero utility. Ali, Miller, and Yang (2016) have shown that their bilateral interaction can be made both weakly renegotiation proof and bilateral renegotiation proof. Since play along their link is measurable with respect to the interactions along their link, we do not need to consider them for the remainder of this discussion.

Now consider partners on any supported link. On the equilibrium path they cooperate at the triangular levels $x^{\mathrm{T}}$ and $y^{\mathrm{T}}$. Moreover, if they have never deviated they continue cooperating at the triangular level even off the equilibrium path. We say that players who have never deviated are "innocent."

[9] As Section 3.1 noted, the first mover receives a lower payoff than the second mover, so no punishment is needed to deter deviations on the equilibrium path in the sequencing phase.

Now consider a supported link $ij$ and suppose that player $i$ has deviated on player $j$. Then we say that $i$ is "guilty." If $i$'s deviation occurs in the effort phase while moving first, $j$ immediately chooses zero effort when moving second in the same interaction. In either case, thereafter they punish $i$ on the $ij$ link by requiring $i$ to always be the first mover (that is, both nominate $i$ in the sequencing phase), and choosing effort levels $x^{\mathrm{P}}$ and $y^{\mathrm{P}}$ that are calibrated to deliver the equilibrium path payoff to $j$ while delivering a zero payoff to $i$:

$$-c(x^{\mathrm{P}}) + b(y^{\mathrm{P}}) = 0, \text{ and } b(x^{\mathrm{P}}) - c(y^{\mathrm{P}}) = \frac{1}{2}\big(x^{\mathrm{T}} + y^{\mathrm{T}}\big). \tag{5}$$

There exists a solution satisfying $0 < x^{\mathrm{P}} < x^{\mathrm{T}}$ and $0 < y^{\mathrm{P}} < y^{\mathrm{T}}$, as shown in Lemma 2.[10]

In addition to player $i$'s punishment on the $ij$ link, to support triangular levels of cooperation $i$ must also be punished by at least one other neighbor, such as player $k$ whose $jk$ link supports $ij$. However, once player $i$ is being punished by both players $j$ and $k$, $i$ may no longer have an incentive to cooperate at triangular levels with other neighbors $l$, $m$, and so on. To accommodate this potential collapse of incentives, the equilibrium specifies that guilty $i$ should also eventually be punished by every neighbor reachable via a path from $j$ that is contained in $i$'s 1-neighborhood $N_i$ but does not pass through $i$. The set of such players—including $j$—is denoted $\Sigma_{ij}$. Formally,

$$\Sigma_{ij} = \{k : \exists \text{ a path } (j_0, j_1, \ldots, j_l) \subset N_i \text{ s.t. } j_0 = j, j_l = k\}. \tag{6}$$

The punishments within individual relationships may of course be delayed, since information about $i$'s deviation must be passed through the network. When $i$ meets such a neighbor $k \in \Sigma_{ij}$, in the pre-effort communication phase $k$ will reveal if she knows $i$ is guilty, in which case $i$'s punishment on the $ik$ link starts immediately. If instead $k$ still thinks $i$ is innocent, $i$ may either shirk on $k$ or pretend to be innocent by working (and concealing any incriminating information). Regardless, eventually all players in $\Sigma_{ij}$ will learn that $i$ is guilty. Since $i$ is punished along at least one other link after deviating on $j$, the punishment is sufficiently severe to support triangular cooperation.

If another player $l$ in $i$'s neighborhood deviates when $i$ is already guilty, $i$ then becomes

[10] One could alternatively modify the multilateral restitution strategies we have constructed to provide strict incentives for the guilty player to continue playing during his or her punishment, by carefully adjusting assigned effort levels at every history. However, doing so would reduce the players' equilibrium-path payoffs, and would conflict with bilateral renegotiation proofness that we will discuss below.

innocent with respect to $\Sigma_{il}$ and $l$ becomes guilty with respect to $\Sigma_{li}$. When $i$ learns of $l$'s guilt, $i$ can then present evidence of his newly established innocence to each neighbor in $\Sigma_{il}$. Similarly, if $j$ (who already knows $i$ is guilty) learns of $l$'s guilt, he is willing to present that evidence to $i$ because $j$ earns equilibrium-path payoffs on the $ij$ link regardless.

Multilateral restitution equilibria are by construction robust and 2-local. They also satisfy a somewhat stronger, but more subtle, locality property: Players need to know only their 1-neighborhoods on the equilibrium path. A player needs to know about her 2-neighborhood only off the equilibrium path after a deviation, when she becomes responsible for punishing one of her neighbors for shirking on a victim who is in her 2-neighborhood but not her 1-neighborhood. Since communication is evidentiary, players can start the game knowing only their 1-neighborhoods, and learn additional information about the network only as needed.[11]

Finally, to establish that there exists a multilateral restitution equilibrium, we need to tackle several difficulties stemming from the fact that the set of players $\Sigma_{ij}$ that punishes player $i$ for deviations on the $ij$ link will generally intersect, but not coincide with, the similarly defined sets for other supported partnerships. Before proving the theorem in Section 3.4, we address renegotiation proofness. Weak renegotiation proofness (Farrell and Maskin 1989) is straightforward.

**Proposition 1.** *For any network, a multilateral restitution equilibrium for the repeated interaction game is weakly renegotiation-proof.*

*Proof.* Off path, a guilty player cannot get out of her punishment by renegotiating with her neighbors—they are still receiving their maximum payoffs within the equilibrium and are therefore willing to reject any renegotiation proposal to obtain the payoff vector associated with a different history in the equilibrium. □

Jackson, Rodriguez-Barraquer, and Tan (2012) construct renegotiation-proof equilibria for certain networks, under the assumptions that monitoring is public and any deviation along a link automatically severs that link; in their context only some types of networks

[11]Since the players exchange verifiable messages, there is some additional subtlety with regard to 2-locality. In each interaction, multilateral restitution specifies that partners should reveal all their verifiable information about the intersection of their 2-neighborhoods, which information contains messages passed within the 2-neighborhoods that themselves contain evidence of events outside the 2-neighborhoods. Our interpretation is that when a player receives evidence from a partner, she ignores or discards any part of it that does not pertain to her own 2-neighborhood.

are renegotiation-proof. Our result is stronger, because in our model monitoring is private and links with deviators are not severed. The stronger result is enabled by the fact that when one player on a link is being punished, her partner is rewarded with a high payoff rather than harmed by severing the link.

Due to the private monitoring in our environment, "strong renegotiation proofness" (Farrell and Maskin 1989) and related concepts do not logically apply: players do not have common knowledge of the history, and therefore would have to renegotiate while holding payoff-relevant private information. Instead we use the notion of *bilateral renegotiation proofness* (Ali, Miller, and Yang 2016), which allows players to renegotiate only bilaterally with their neighbors, and agree to jointly deviate from the equilibrium only to bilateral continuation play and only if they have common knowledge that they both will gain from deviating.

**Proposition 2.** *For any network, there exists a robust, 2-local, and bilateral renegotiation proof equilibrium for the repeated interaction game that supports triangular effort levels on every supported link along the path of play.*

First, observe that a multilateral restitution strategy profile may not be bilateral renegotiation-proof, because once guilty player $i$ is being punished by innocent player $j$, they may be able to bilaterally renegotiate to play bilaterally along their link in a way that gives $i$ a strictly positive payoff while giving $j$ a payoff strictly better than triangular cooperation. To solve this problem, we modify the strategies so that guilty players are punished using bilaterally renegotiation-proof punishments, as calibrated in the proof below.

*Proof.* Consider a pair of partners, at a history such that one is guilty and the other is innocent. The guilty agent always goes first and chooses effort $x$ and then the innocent one chooses $y$. Let $\tilde{x}^B$ and $\tilde{y}^B$ be the effort levels such that the innocent partner gets the maximal bilaterally renegotiation proof utility while the guilty parter gets zero, and the innocent partner does not want to deviate. That is, $\tilde{x}^B$ and $\tilde{y}^B$ solve

$$\begin{aligned} &\max_{x,y} b(x) - c(y) \\ \text{s.t.} &- c(x) + b(y) = 0 \\ 0 \leq &- c(y) + \int_0^\infty e^{-rt}\lambda\big(b(x) - c(y)\big)dt. \end{aligned} \tag{7}$$

Then let $\tilde{x}^P = \max(\tilde{x}^B, x^P)$ and $\tilde{y}^P = \max(\tilde{y}^B, y^P)$. In equilibrium they choose effort according to $(\tilde{x}^P, \tilde{y}^P)$ instead of $(x^P, y^P)$. By construction they cannot both gain from bilateral renegotiation. Between two guilty agents, the one whose deviation is more recent moves first, and they choose according to $(\tilde{x}^B, \tilde{y}^B)$. This construction ensures that no pair of agents can renegotiate any bilateral improvement, but punishes deviations at least as harshly as a multilateral restitution equilibrium. □

### 3.3 Communication and the diffusion of information

With private monitoring, it is important to establish that information of a player's deviation diffuses through the network fast enough to provide sufficient punishment. Truthful communication cannot be taken for granted: Ali and Miller (2016) show that players would not communicate truthfully under permanent ostracism if they were cooperating above the bilateral effort level. In this part, we investigate players' incentives for truthful communication.

First, a guilty player not only does not have incentives to communicate truthfully, but also may have the incentive to try to slow down the diffusion of information by working with some partners rather than shirking.

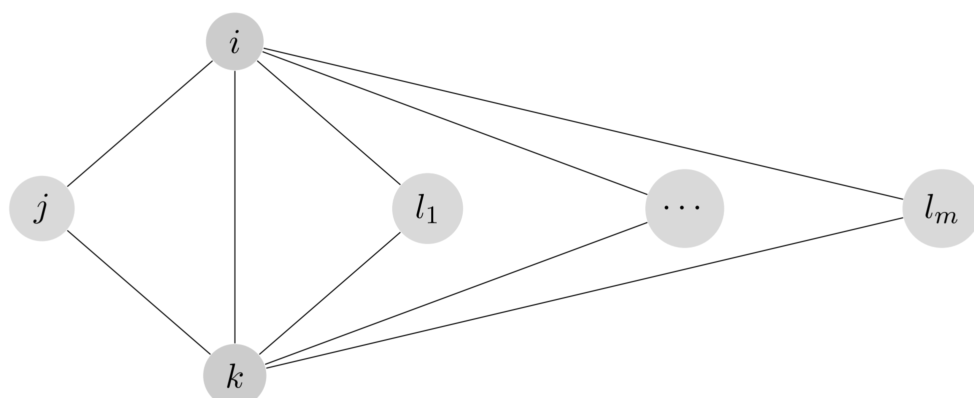


Figure 2: A generalized diamond

**Example 1.** *Consider a generalized diamond with diagonal players $i$ and $k$ and common neighbors $j, l_1, \ldots, l_m$, as shown in Figure 2. In a multilateral restitution equilibrium, if $m$ is sufficiently large then $i$, after shirking on $j$ along the equilibrium path, will not immediately start shirking on $k$.*

*Proof.* Consider the scenario that player $i$ meets player $k$ immediately after she shirks on player $j$, so $i$ has not met any of $l_1, \ldots, l_m$ since deviating. In the pre-effort communication phase, $k$ indicates she does not know that $i$ has shirked on $j$; then suppose nature chooses $i$ to move first. If $i$ shirks, her expected payoff is

$$0 + m \int_0^\infty e^{rt} e^{-\lambda t} \lambda \frac{1}{2} b(x) dt. \tag{8}$$

Now, we consider another strategy for $i$, which we will show guarantees a strictly higher payoff than (8) for $m$ sufficiently large. In this strategy, $i$ cooperates with $k$ in the current meeting, and shirks on all her neighbors who have not learned about her shirking in the future. First, by cooperating with $k$ she earns $b(y^T) - c(x^T) < 0$ in the current meeting, and at least zero thereafter on link $ik$. But on each link $il_z$, for $z = 1, \ldots, m$, $i$ expects a strictly higher payoff than she would if $k$ knew about her shirking. It suffices to show that for $m$ sufficiently large, the improvement is strictly greater than $c(x^T) - b(y^T)$.

We claim that when $m$ is sufficiently large, with a probability above $e^{-1}$, agent $i$ meets $\sqrt{m}+1$ players from the set $\{l_1, \ldots, l_m\}$ before $k$ knows about $i$'s shirking. The probability that $i$ meets $\sqrt{m}+1$ of them before $k$ meets $i$ (again), $j$, or, any other player $i$ has shirked on, is

$$\frac{m}{m+2} \cdot \frac{m-1}{m+2} \cdots \frac{m-\sqrt{m}}{m+2} \geq \left( \frac{m-\sqrt{m}}{m+2} \right)^{\sqrt{m}},$$

which converges to $e^{-1}$ as $m \to \infty$. Conditional on $i$ meeting these $\sqrt{m}+1$ players before $k$ learns $i$ is guilty, she expects her first meeting with each of them to occur earlier on average than the unconditional expected first meeting time. Therefore $i$'s expected payoff, from the perspective of her initial meeting with $k$ after shirking on $j$, is at least $\sqrt{m}\, e^{-1} \int_0^\infty e^{-rt} e^{-\lambda t} \frac{1}{2} b(x^{\mathrm{T}})\, dt$. Since this lower bound is strictly increasing and linear in $\sqrt{m}$, the claim is proven. ◇

Based on this example, we cannot assume that information always diffuses through every feasible channel. However, in a multilateral restitution equilibrium, innocent players are always willing share information truthfully. The restitution is specifically calibrated to deliver them their equilibrium path payoffs, even off the equilibrium path. Therefore they cannot benefit by slowing the diffusion of information.[12]

The truthful communication of innocent players puts a lower bound on the speed of information diffusion. In the generalized diamond network shown in Figure 2, while $i$ cannot be relied upon to shirk on $k$ after shirking on $j$, knowledge of $i$'s deviation will be passed from $j$ to $k$ to each $l_z$. We use this bound to show that if players cooperate at triangular levels on all supported links along the equilibrium path, then $i$'s continuation payoff on each link $il_z$ is strictly greater on the equilibrium path than off the equilibrium path after $i$ shirks on $j$. (See Lemma 3.) This implies that even if information diffuses only due to truthful communication by innocent players, it still suffices to deter deviations on generalized diamond networks. In Lemma 4 we then use an induction argument to extend this conclusion to arbitrary networks.

### 3.4 Proof of Theorem 1

We prove that a multilateral restitution equilibrium exists and satisfies the desired properties, first on a triangle network, then on a "diamond" network, and then on a "generalized diamond". Ultimately, we prove by induction that it works on any arbitrary network.

1. Triangle network
2. Diamond network
3. Generalized diamond
4. Arbitrary network

**Triangle network.** Consider a triangle $ijk$. The incentives on the equilibrium path have been verified when deriving the effort levels $x^{\mathrm{T}}$ and $y^{\mathrm{T}}$. We need to examine incentives off the equilibrium path. Start with verifying that player $i$, after initially deviating on player $j$, wants to shirk on player $k$ when $k$ does not know he is guilty, even if he moves first:

$$-c(x^{\mathrm{T}}) + b(y^{\mathrm{T}}) + \int_0^{\infty} e^{-rt} e^{-2\lambda t} \lambda \frac{1}{2} b(x^{\mathrm{T}})\, dt \leq 0 \tag{9}$$

[12] Something about the bilateral renegotiation proof equilibrium.

This is implied by Equations (3) and (4). It follows that $i$ also wants to shirk on $k$ when moving second.

Once player $i$ has deviated on player $j$, $j$'s immediate gain from a deviating on the $ij$ link decreases, as shown in the lemma below. (Here we invoke the restrictions on off-path beliefs imposed by PPBE: all players' beliefs must accord with the behavior specified by the strategy profile after $i$'s deviation. In this case, $j$ must believe that if she deviated on $i$ and became guilty, $k$ would learn of her guilt according to the stochastic process generated by the strategy profile. Henceforth we invoke these restrictions without further note.)

**Lemma 2.** $x^{\mathrm{P}} < x^{\mathrm{T}}$ *and* $y^{\mathrm{P}} < y^{\mathrm{T}}$.

This lemma implies that innocent players prefer not to shirk even off the equilibrium path, when some of their partners are guilty. Moreover, innocent players cannot gain by being untruthful, and guilty players by construction cannot gain by deviating.

**Diamond network.** Consider a diamond network of 4 players, $\{i, j, k, l\}$, depicted in Figure 3; i.e., all pairs are connected except for $jl$. In this network, we call link $ik$ the "diagonal."

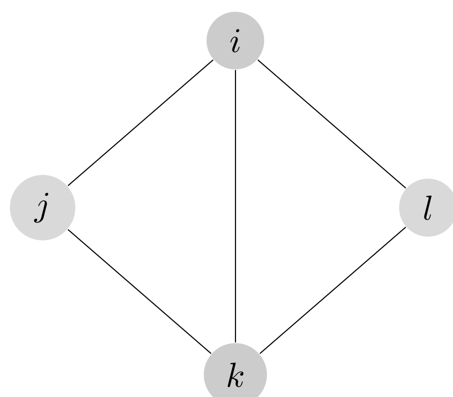


Figure 3: A diamond

**Lemma 3.** *On a diamond network, there exists a 2-local multilateral restitution equilibrium that supports triangular efforts on every link along the path of play.*

*Proof.* Note that the analysis after a deviation by $j$ or $l$ is identical to the analysis on a triangle network. Moreover, innocent players are always willing to implement the prescribed punishments (since they still get their equilibrium path payoffs, but would be punished themselves for any deviation), and to communicate truthfully. So it suffices to consider only equilibrium path deviations by $i$.

First, consider whether player $i$ could gain by shirking on player $j$, on the equilibrium path. Once player $k$ knows of $i$'s deviation, incentives on the $ik$ link are straightforwardly similar to the triangle case; this is also true for the $il$ link. So we focus on what happens when $i$ meets a partner (either $k$ or $l$) who does not know of $i$'s deviation. (Note that a partner who does know of $i$'s deviation should demonstrate that knowledge in the pre-play communication phase.) Our class of strategy profiles does not specify whether $i$ should work or shirk in such meetings. What is important is that whatever happens should not lead to payoffs for $i$ that are high enough to justify shirking on $j$ in the first place.

Suppose, after shirking on player $j$, player $i$ meets player $l$; $l$ does not know of $i$'s deviation, and $i$ does not know whether $k$ knows of $i$'s deviation. Observe that on the equilibrium path, if $j$ were not present then $i$ would have been just indifferent between working (at effort $x^{\mathrm{T}}$ or $y^{\mathrm{T}}$, depending on whether he moved first) and shirking on $l$. Now off the equilibrium path, $i$ expects zero future payoffs on the $ij$ link, but $j$'s presence means $k$ and ultimately $l$ will learn of $i$'s original deviation sooner in expectation. This loss of future social collateral strictly reduces $i$'s incentive to work with $l$, compared to the equilibrium path on a triangle network. Hence $i$ strictly prefers to shirk on $l$, regardless of $i$'s belief about the probability that $k$ knows of $i$'s deviation. (Moreover, it follows from analysis of the triangle that after shirking on $l$, $i$ subsequently strictly prefers to shirk on $k$.)

Next consider player $i$ (after shirking on player $j$) meeting player $k$, when $k$ does not know of $i$'s deviation, and when $i$ has not yet shirked on $l$. Now matters are a bit more complicated—by working rather than shirking on $k$, $i$ can slow down the rate at which $l$ learns that $i$ has deviated. The most $i$ can slow down the rate at which $l$ learns of the deviation is to work with $k$ until either $k$ learns of $i$'s deviation from $j$, or $i$ shirks on $l$. We already know this makes $i$ strictly worse off on the $ik$ link than shirking immediately (since $i$ would be indifferent on the equilibrium path of a triangle, but here $j$ will spread the news to $k$).

The key is to show that $i$ is also weakly worse off on the $il$ link. That is, the payoff $i$ gets on the $il$ link on the equilibrium path is weakly higher than when $i$ has deviated on $j$;

this is satisfied if

$$\int_0^\infty e^{-rt}\lambda\frac{1}{2}\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)dt \geq \int_0^\infty e^{-rt}e^{-2\lambda t}\Big(\lambda\frac{1}{2}b(x^{\mathrm{T}})+\lambda\int_0^\infty e^{-r\tau}e^{-2\lambda\tau}\lambda\frac{1}{2}b(x^{\mathrm{T}})\,d\tau\Big)dt, \tag{10}$$

which we verify below. Note that we consider the slowest information transmission, such that the news about $i$'s deviation goes only from $j$ to $k$ then to $l$, but not from $i$ to $k$ then to $l$. So the value on the RHS of (10) is a loose upper bound on $i$'s expected payoff on the link $il$ after his deviation.

We now verify that (10) holds. From summing the binding incentive constraints in (3) and (4) with $b(x^{\mathrm{T}})$, we have

$$b(x^{\mathrm{T}})+\int_0^\infty e^{-rt}e^{-2\lambda t}\lambda b(x^{\mathrm{T}})\,dt = x^{\mathrm{T}}+y^{\mathrm{T}}+2\int_0^\infty e^{-rt}\lambda\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)dt. \tag{11}$$

Then, we simplify the RHS of equation (10):

$$\begin{aligned}
&\int_0^\infty e^{-rt}e^{-2\lambda t}\Big(\lambda\frac{1}{2}b(x^{\mathrm{T}})+\lambda\int_0^\infty e^{-r\tau}e^{-2\lambda\tau}\lambda\frac{1}{2}b(x^{\mathrm{T}})\,d\tau\Big)dt\\
&=\frac{\lambda}{r+2\lambda}\cdot\frac{1}{2}\left(b(x^{\mathrm{T}})+\int_0^\infty e^{-rt}e^{-2\lambda t}\lambda b(x^{\mathrm{T}})\,dt\right)\\
&=\frac{\lambda}{r+2\lambda}\cdot\frac{1}{2}\left(x^{\mathrm{T}}+y^{\mathrm{T}}+2\int_0^\infty e^{-rt}\lambda\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)dt\right)\\
&=\frac{\lambda}{r+2\lambda}\cdot\frac{1}{2}\cdot\frac{r+2\lambda}{r}\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)\\
&=\frac{\lambda}{r}\cdot\frac{1}{2}\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)\\
&=\int_0^\infty e^{-rt}\lambda\frac{1}{2}\big(x^{\mathrm{T}}+y^{\mathrm{T}}\big)dt
\end{aligned}$$

where the second equality is from (11). Thus equation (10) holds with equality.

Then, consider whether player $i$ could gain by shirking on player $k$ along the equilibrium path. Since $k$ then spreads the news to both players $j$ and $l$, this is strictly worse for $i$ than shirking on $j$ on the equilibrium path.

Finally, consider players other than the original deviator $i$: as in our analysis of the triangle network above, they expect to receive equilibrium-path payoffs on all their links (being shirked on is always a surprise) and therefore have no incentive to deviate on or off the equilibrium path. □

**Generalized diamond.** A generalized diamond has a diagonal link $ik$ and two or more common neighbors of players $i$ and $k$. That is, letting $l_1 = l$, we can add more players $l_2, \ldots, l_m$ to the network in Figure 3 such that $il_z \in G$ and $kl_z \in G$ for each $z \in \{1, \ldots, m\}$. It is straightforward to see that $i$ does not benefit from deviating on $j$ along the equilibrium path in any generalized diamond: first, she is punished by $k$, and then, by equation (10), she gets a weakly lower payoff on each link $il_z$ after shirking on $j$. Similarly, if she deviates on $k$ then she will be punished by all $l_1, \ldots, l_m$.

**Arbitrary network.** Now we complete the proof of Theorem 1 by the following lemma.

**Lemma 4.** *Consider any arbitrary neighborhood $\Sigma_{ij}$. Triangular efforts on supported link $ij$ are sustained by a multilateral restitution equilibrium.*

*Proof.* Because $ij$ is supported, $|\Sigma_{ij}| \geq 2$. We prove the lemma by induction on the number of agents in $\Sigma_{ij}$. When $|\Sigma_{ij}| = 2$, the local neighborhood is a triangle, and $i$'s incentive to cooperate with $j$ in a multilateral restitution strategy profile has been verified. Suppose $i$ has incentives to cooperate with $j$ under a multilateral restitution strategy profile when facing any punishing set such that $|\Sigma_{ij}| \leq m \geq 2$. Then we consider when the punishing set has size $|\Sigma_{ij}| = m + 1 \geq 3$.

We consider two separate cases. First, if $i$ has is at least one partner $k$ whose link to $i$ is uniquely supported by $j$ (i.e., if the $ij$ link were removed, the $ik$ link would be unsupported), then we partition $\Sigma_{ij}$ as follows. $\Sigma^1_{ij}$ includes $j$ and all agents whose links to $i$ are uniquely supported by $j$, and $\Sigma^2_{ij} = \Sigma_{ij} \setminus \Sigma^1_{ij}$. Note that players in $\Sigma^1_{ij} \setminus \{j\}$ do not have any links to $\Sigma^2_{ij}$. Therefore once $i$ deviates on $j$, there is no incentive for $i$ to delay shirking on neighbors in $\Sigma^1_{ij}$ in order to slow the rate at which neighbors in $\Sigma^2_{ij}$ learn he is guilty, nor vice versa. Hence it suffices to show that $i$ does not gain from deviating on $j$ separately on each subnetwork $\{i\} \cup \Sigma^1_{ij}$ and $\{i, j\} \cup \Sigma^2_{ij}$. As for $\{i\} \cup \Sigma^1_{ij}$, it is a generalized diamond with $ij$ being its diagonal, so by our previous analysis $\Sigma^1_{ij}$ itself suffices to deter $i$ from deviating on $j$. As for $\{i, j\} \cup \Sigma^2_{ij}$, it is a punishing set of size $|\Sigma^2_{ij}| \leq m$, which by the induction hypothesis itself suffices to deter $i$ from deviating on $j$.

The second case is when there is no player whose link to player $i$ is uniquely supported by $ij$. Then we arbitrarily choose player $k$ (a common neighbor of $i$ and $j$, at least one of which exists since $ij$ is supported), and partition $\Sigma_{ij}$ as follows. $\Sigma^1_{ij}$ includes $j$, $k$, and any neighbor of $i$ whose link to $i$ is unsupported in $\Sigma_{ij} \setminus \{i, j\}$, while $\Sigma^2_{ij} = \Sigma_{ij} \setminus \Sigma^1_{ij}$. Then $\{i\} \cup \Sigma^1_{ij}$ is a generalized diamond combined with zero or more links among $\Sigma^1_{ij}$. Since the added links (none of which directly involve $i$) aid in distributing information about $i$'s deviation on $j$ among the players in $\Sigma^1_{ij}$, $\Sigma^1_{ij}$ itself suffices to deter $i$ from deviating on $j$. As in the first case, $\Sigma^2_{ij}$ is a punishing set of size $|\Sigma^2_{ij}| \leq m$, which by the induction hypothesis itself suffices to deter $i$ from deviating on $j$. So the overall punishment from $\Sigma_{ij}$ deters $i$ from deviating on $j$.

Off the equilibrium path, by Lemma 2 the same analysis applies to any innocent player, even if he or she has guilty neighbors.

Evidently the strategies form an equilibrium, which by construction is robust and 2-local. □

# 4 Strategic network formation

Since partnerships supported by multilateral enforcement are particularly valuable, players may strategically seek them out. In this section we introduce a dynamic network formation game with random linking opportunities and random link formation costs. We identify a simple equilibrium in this game in which agents form a network with realistic *small worlds* properties, because they anticipate our multilateral enforcement equilibrium once the network has formed. We focus in particular on measures of "clustering" and "support": *clustering* is the fraction of paths of length two that are contained in triangles; *support* is the fraction of links that are contained in triangles.

## 4.1 Dynamic network formation model

We follow the setup of a growing network in Jackson and Rogers (2007). It starts at period $n$ with a complete network of $n$ players. Then at each period $i > n$, player $i$ is born and the network formation game proceeds as follows:

- Stage 0: An i.i.d. random cost $\gamma_{ij}$ is drawn for each pair of players $\{i, j\}$ with $j < i$, from a distribution with CDF $F$. These costs are not revealed to the players.

- Stage 1: $m_r$ players are uniformly randomly selected from a pool of players born before $i$. When $i$ meets each of them, say $j$, they learn their cost $\gamma_{ij}$. Then, $i$ decides whether to form or not form the link. If $i$ forms the link, he or she pays $2\gamma_{ij}$.

- Stage 2: $m_n$ players are uniformly randomly selected from the union of $i$'s stage-1 neighbors' neighbors. When $i$ meets each of them, say $k$, they learn their cost $\gamma_{ik}$. Then, $i$ decides whether to form or not form the link. If $i$ forms the link, he or she pays $2\gamma_{ik}$.[13]

- Stage 3: In uniform random sequence, each triple $\{i, j, k\}$ (such that $ij$ is formed in stage 1 and $jk$ is connected) that $k$ was recognized in Stage 2 but the link $ik$ did not form is recognized again. When their triple is recognized, player $i$ learns if $ij$ and $jk$ belong to any triangle. Then, $i$ decides whether to form or not form the link. If $i$ forms the link, he or she pays $2\gamma_{ik}$.

- Stage 4: All players who $i$ forms a link to pay $i$ their share of the cost.

The network that arises at the end of Stage 1 is called the *backbone network* $G_1$, and its links are called "backbone links." The network that arises at the end of stage 2 is $G_2$, and the network that arises at the end of the network formation game is simply "the network", denoted $G$.

### 4.2 Network formation equilibrium

While a multilateral restitution equilibrium exists in the repeated interaction game for any network, in this section, we show that players who anticipate playing a multilateral restitution equilibrium will form a network with certain realistic characteristics in the network formation game. For tractability, in our analytic results we focus on an equilibrium that is somewhat naive from the players' collective perspective, due to some coordination failures. We denote $u^{\mathrm{B}} = \frac{1}{2}(x^{\mathrm{B}} + y^{\mathrm{B}})$ be the utility of a link with bilateral cooperation and $u^{\mathrm{T}} = \frac{1}{2}(x^{\mathrm{T}} + y^{\mathrm{T}})$ be the utility of a link with triangular cooperation.

[13] It worths noting that our network-based search differs slightly from Jackson and Rogers (2007) because in their model these $m_n$ agents are chosen from the union of all $m_r$ stage-1 players' neighbors regardless of whether $i$ forms a link to them or not in Stage 1. In our setup, we assume these $m_n$ agents are chosen from neighbors of those who $i$ forms links to in Stage 1. This is because anticipating cooperation, it is more beneficial to search through linked neighbors.

- Stage 1: When a pair $\{i, j\}$ is recognized to meet, $i$ forms the link if and only if $\gamma_{ij} \leq u^{\mathrm{B}}$.
- Stage 2: When a pair $\{i, k\}$ is recognized to meet, $i$ forms the link if and only if $\gamma_{ik} \leq u^{\mathrm{T}}$.
- Stage 3: When a triple $\{i, j, k\}$ is recognized to meet, $i$ forms the link $ik$ if and only if $\gamma_{ik} \leq u^{\mathrm{T}} + z_{ijk}(u^{\mathrm{T}} - u^{\mathrm{B}})$, where $z_{ijk} \in \{0, 1, 2\}$ is the number of unsupported links in the triple at the time they meet.
- Stage 4: If the link $ik$ should not be formed according to $i$'s strategy profile in Stage 1-3, player $k$ (also agent $j$ if it is formed in Stage 3) does not need to pay the cost. Otherwise, if the link $ik$ is formed in stage 1 or 2, $k$ pays $\gamma_{ik}$ to player $i$. In addition, if the link $ik$ is formed for the triple $\{i, j, k\}$, $j$ and $k$ pay $i$ the amounts that allocate to each of them one third of their net gains, calculated as if the final network $G$ were to be the one that forms immediately from their decision.

Any player who has deviated from the strategy profile above, becomes guilty with respect to that link. More precisely, if $i$ forms a link $ij$ that she should not form, that she immediately becomes guilty to $j$. If $j$ does not pay his or her share of the cost to $i$, she immediately becomes guilty to $i$ (and also to $k$ if it is the triple $\{i, j, k\}$ in Stage 3).

It is clear that each link that forms is strictly beneficial to the (innocent) players who form it. Since coordination is needed to form each link, there is no incentive for any player to deviate from this strategy profile. However, the players suffer from some coordination failure in Stages 1 and 3 that causes them to forego profitable linking opportunities. In Stage 1, they fail to anticipate links that may form in Stages 2 and 3; optimally anticipating later links would lead them to form some links in Stage 1 for which $\gamma_{ij} > u^{\mathrm{B}}$. In Stage 3, they fail to anticipate links that may form later in Stage 3: since supporting links are formed by a greedy algorithm rather than an optimal algorithm, a less beneficial link may be chosen early over a more beneficial one that is recognized later. Focusing on an equilibrium with these coordination failures aids in finding closed form expressions for network statistics like support and clustering.

The next result shows that equilibrium cooperation behaviors in Theorem 1 are not affected by the linked being formed at the same time.

**Theorem 2.** *During the dynamic network formation process, there exists a robust and 2-local equilibrium for the repeated interaction game that supports triangular effort levels on every supported link along the path of play.*

*Proof.* We verify that no player has incentives to shirk and also all innocent players share information truthfully.

First, consider players $i > j > k$. Suppose $j$ has shirked on $k$ before player $i$ is born. We claim that player $j$ gets less value from $i$'s entry compared to that when $j$ is innocent. The same set of links are formed by $i$ regardless of $j$'s guilt, because $j$'s deviation is a surprise to $i$. Let $G_i$ be the network after $i$ has formed his or her links. If $j$ pays her share of the cost for link $ij$, then the link formation is identical to that when $j$ is innocent. However, $j$ faces weakly more punishment because $\Sigma_{jk}(G_{i-1}) \subset \Sigma_{jk}(G_i)$. If $j$ does not pay her share of the cost for link $ij$, $j$ immediately becomes guilty to $i$ and thus gets zero utility from the link $ij$. Moreover, the set of players to punish $j$ becomes strictly larger: $\Sigma_{jk}(G_{i-1}) \subsetneq \Sigma_{jk}(G_i) \cup \Sigma_{ji}(G_i)$.

Second, each innocent player $l \in \Sigma_{jk}(G_{i-1})$ shares the information about $j$'s shirking truthfully. This is straightforward because agent $i$ forms the same set of links, and each innocent player pays the same amount of his or her share of the cost for the link to $i$. As $l$'s utility does not depend on whether other players know about $j$'s shirking, he or she will not benefit from withholding such information. □

In Stage 1, each possible link forms with probability $p_r \equiv F(u^{\mathrm{B}})$. In Stage 2, players can form clusters by closing triples; each link forms with probability $p_n \equiv F(u^{\mathrm{T}})$. In Stage 3 they can search for additional links to form to support their relationships.

Let $m = p_r m_r + p_n m_n$ and $r = p_r m_r / p_n m_n$.

**Remark 1.** *When the population size is arbitrarily large, the growing network has*

- *Moderate clustering: The limiting probability that two neighbors of the same player are linked is strictly positive if $r > 1$, and it converges to $0$ as $m_r \to \infty$.*

- *High support: The limiting probability that two linked players share a common neighbor is strictly positive, and it converges to above $1 - e^{-\frac{1}{r}}$ as $m_r \to \infty$.*

**Support and clustering in Stage 2** By theorem 2 of Jackson and Rogers (2007), the clustering measure is

$$C_2 = \begin{cases} 0 & \text{if } r \leq 1; \\ \frac{6}{(1+r)[(3m-2)(r-1)+2mr]} & \text{if } r > 1. \end{cases} \tag{12}$$

We now calculate the support measure. All $p_n m_n$ links formed by network-based search must be supported, so we focus on the $p_r m_r$ links formed randomly. Say $ij$ is a random link. (While we consider undirected links, we use the order $ij$ to indicate the link is formed by $i$ when she is born to an existing agent $j$). There are three possibilities to support $ij$. First, it could be supported by another random link $ik$ such that $jk$ is linked. As the population increases, the probability of link $jk$ goes to zero. Second, it could be supported by a later agent $h$, who forms a random link $hi$ and then forms $hj$ by network-based search. Again, as the population increases, the probability of link $hi$ goes to zero. So, it left with the third case in which $ik$ is formed by network-based search through the link $jk$.

For each of $m_n$ agents, with the probability $\frac{1}{p_r m_r}$ it is search through agent $j$, say through link $jk$, and $i$ forms a link to $k$ with probability $p_n$. Thus the probability link $ij$ is supported

$$\beta_2 = 1 - \left(1 - \frac{p_n}{p_r m_r}\right)^{m_n} \tag{13}$$

In other words, the link $ij$ is not supported if none of her link to all $m_n$ agents supports $ij$. To sum up,

$$s_2 = \frac{p_r m_r \beta_2 + p_n m_n}{p_r m_r + p_n m_n} \tag{14}$$

**Support and clustering in Stage 3** We first calculate the probability a random-search link $ij$ is not supported in stage 2, but becomes supported in stage 3, denoted as $\beta_3$. Recall that $\alpha_{3L} = F(2u^T - u^B)$ and $\alpha_{3H} = F(3u^T - 3u^B)$. Using (13), the lower bound on $\beta_3$ is

$$\beta_{3L} = \left(1 - \frac{p_n}{p_r m_r}\right)^{m_n} - \left(1 - \frac{\alpha_{3L}}{p_r m_r}\right)^{m_n} \tag{15}$$

Recall that $K_3$ is the number of links $i$ forms in stage 3. Thus, $K_3 \geq \beta_{3L} p_r m_r$, which is on average the number of $i$'s random-search links that become supported in stage 3.

Putting them together, the lower bound on support in the final network is

$$s_3 \geq \frac{(\beta_2 + \beta_{3L}) p_r m_r + p_n m_n + \beta_{3L} p_r m_r}{p_r m_r + p_n m_n + \beta_{3L} p_r m_r} \tag{16}$$

Similarly, the upper bound on $\beta_3$ is

$$\beta_{3H} = \left(1 - \frac{p_n}{p_r m_r}\right)^{m_n} - \left(1 - \frac{\alpha_{3H}}{p_r m_r}\right)^{m_n} \tag{17}$$

The number of links $i$ forms in stage 3 is at most $K_3 \leq \beta_{3H} p_r m_r$. Thus, the upper bound on support in the final network is

$$s_3 \leq \frac{(\beta_2 + \beta_{3H}) p_r m_r + p_n m_n + \beta_{3H} p_r m_r}{p_r m_r + p_n m_n + \beta_{3H} p_r m_r} \tag{18}$$

When $r \leq 1$, $C_3 = 0$. When $r > 1$, similarly to (35), the upper bound on clustering in stage 3 is

$$\begin{aligned} C_3 &\leq \frac{3m^2 \frac{1}{m(1+r)} + 3\beta_{3H} \frac{rm}{r+1}}{\frac{m(m-1)}{2} + m^2 + \frac{m(2mr+1-r)}{2(r-1)} + 2\beta_{3H} \frac{rm}{r+1}} \\ &= \frac{6 + 6\beta_{3H} r}{(1+r)[3m - 2 + \frac{2mr}{r-1}] + 4\beta_{3H} r} \end{aligned} \tag{19}$$

# 5 Concluding remarks

We introduce a class of multilateral restitution equilibria, which implement multilateral enforcement in an arbitrary network, and with players only knowing about their local neighborhood. The key component of multilateral restitution equilibria is that guilty players are not ostracized from the community, instead they work hard with their partners to preserve the stability of the network.

Then, anticipating playing the multilateral restitution equilibria, players particularly form links to support their existing links. While we show the network exhibits high support and relatively low clustering in a dynamic network formation process as in Jackson and Rogers (2007), the support and clustering patterns also hold under the static network formation process as shown in appendix B.1.

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# A Proofs

*Proof of Lemma 1.* Evaluating the integrals in equation (4) and rearranging, we have

$$c(y) \leq \frac{\lambda}{r}(x+y) - \frac{\lambda}{2(r+2\lambda)}(y+c(y))$$

Rearrange it,

$$\frac{r(2r+5\lambda)}{2\lambda(r+2\lambda)}c(y) - \frac{r+4\lambda}{2(r+2\lambda)}y \leq x$$

Take the cost of both sides of the inequality, we have

$$c\left(\frac{r(2r+5\lambda)}{2\lambda(r+2\lambda)}c(y) - \frac{r+4\lambda}{2(r+2\lambda)}y\right) \leq c(x) = y + 2c(y) \tag{20}$$

The last equality uses the observation that in the binding case $-c(y^{\mathrm{T}}) = -c(x^{\mathrm{T}}) + b(y^{\mathrm{T}})$.

By Assumption 1, $c'(0) = 0$, $c'(x)$ increases in $x$, and $\lim_{x\to\infty} c'(x) = \infty$. So, there exists a unique $\underline{y} > 0$ such that

$$\frac{r(2r+5\lambda)}{2\lambda(r+2\lambda)}c(\underline{y}) = \frac{r+4\lambda}{2(r+2\lambda)}\underline{y}$$

The LHS of (20) is zero when $y = \underline{y}$, while the RHS must be positive, so (20) holds. Then, as $y$ increases to infinity, the cost increases much faster than the value. As a result, when $y$ is sufficiently large, the LHS of (20) is always higher than the RHS. Specifically, we can first identify the threshold $y'$ such that when $y > y'$, $c(y) > y$. By convexity, $c(3y) \geq 3c(y) > y + 2c(y)$. Next we can find the threshold $y''$ such that when $y > y''$, $\frac{r(2r+5\lambda)}{2\lambda(r+2\lambda)}c(y) - \frac{r+4\lambda}{2(r+2\lambda)}y > 3y$. Then when $y > \max(y', y'')$, LHS of (20) is always higher than the RHS. Thus, there must exists a value $y^{\mathrm{T}}$ such that when $y = y^{\mathrm{T}}$ LHS is equal to RHS, and when $y > y^{\mathrm{T}}$ LHS is always higher than RHS. $x^{\mathrm{T}}$ can be calculated by $c(x^{\mathrm{T}}) = y^{\mathrm{T}} + 2c(y^{\mathrm{T}})$. □

*Proof of Lemma 2.* We prove $y^{\mathrm{P}} < y^{\mathrm{T}}$ by contradiction. Suppose $y^{\mathrm{P}} \geq y^{\mathrm{T}}$. Summing $b(y^{\mathrm{P}}) - c(x^{\mathrm{P}}) = 0$ and $b(x^{\mathrm{P}}) - c(y^{\mathrm{P}}) = \frac{1}{2}\left(x^{\mathrm{T}} + y^{\mathrm{T}}\right)$, it leads to

$$2(x^{\mathrm{P}} + y^{\mathrm{P}}) = x^{\mathrm{T}} + y^{\mathrm{T}} \leq x^{\mathrm{T}} + y^{\mathrm{P}}$$

Then $x^{\mathrm{T}} > x^{\mathrm{P}} + y^{\mathrm{P}}$. On the other hand, by $c(x^{\mathrm{T}}) = c(y^{\mathrm{T}}) + b(y^{\mathrm{T}})$, we have $c(y^{\mathrm{P}}) + b(y^{\mathrm{P}}) > c(x^{\mathrm{T}})$. Together with $b(y^{\mathrm{P}}) - c(x^{\mathrm{P}}) = 0$, it implies $c(x^{\mathrm{P}}) + c(y^{\mathrm{P}}) > c(x^{\mathrm{T}})$. Since $c(x)$ is strictly convex, it must be that $x^{\mathrm{T}} < x^{\mathrm{P}} + y^{\mathrm{P}}$. It is a contradiction. So $y^{\mathrm{P}} < y^{\mathrm{T}}$ must hold.

Lastly, observing that $c(x^{\mathrm{P}}) = b(y^{\mathrm{P}}) < c(y^{\mathrm{T}}) + b(y^{\mathrm{T}}) = c(x^{\mathrm{T}})$, so $x^{\mathrm{P}} < x^{\mathrm{T}}$. □

# B Extensions and discussion

## B.1 Static network formation model

In a static network formation process, all agents are born at the same time and form their links prior to the repeated interaction game. The network formation game proceeds as follows:

1. Stage 0: An i.i.d. random cost $\gamma_{ij}$ is drawn for each pair of players $\{i, j\} \subset N$, from a distribution with CDF $F$. These costs are not revealed to the players.

2. Stage 1: Simultaneously, each pair $\{i, j\}$ is recognized to meet with i.i.d. probability $\frac{1}{n}\pi_1$. When they meet they learn their cost $\gamma_{ij}$, and then $i$ and $j$ simultaneously propose whether to form or not form a link. The link forms if both of them propose to form it; otherwise the link does not form.

3. Stage 2: Simultaneously, each triple $\{i, j, k\}$, such that $ij$ and $ik$ are linked but $jk$ is not, is recognized to meet with i.i.d. probability $\pi_2$. When they meet they learn the cost $\gamma_{jk}$, and then $j$ and $k$ simultaneously propose whether to form or not form a link. The link forms if both of them propose to form it; otherwise the link does not form.

4. Stage 3: In uniform random sequence, each triple $\{i, j, k\}$ (such that $ij$ and $ik$ are linked but $jk$ is not) that was recognized in Stage 2 but did not form a link is recognized again. When their triple is recognized, each member of the triple learns if $ij$ and $ik$ belong to any triangle.

   The three of them then simultaneously propose whether to form the $jk$ link, along with a vector of balanced transfers among them to be implemented if the link is formed. If all three proposals are identical, then the link forms and the transfers are implemented; otherwise no link forms and no transfers are implemented.

For any link $ij$ that is formed, both player $i$ and player $j$ incur the cost $\gamma_{ij}$. The network that arises at the end of Stage 1 is called the *backbone network* $G_1$, and its links are called "backbone links." The network that arises at the end of stage 2 is $G_2$, and the network that arises at the end of the network formation game is simply "the network", denoted $G$.

While a multilateral restitution equilibrium exists in the repeated interaction game for any network, in this section, we show that players who anticipate playing a multilateral restitution equilibrium will form a network with certain realistic characteristics in the network formation game. For tractability, in our analytic results we focus on an equilibrium that is somewhat naive from the players' collective perspective, due to some coordination failures. We denote $u^{\mathrm{B}} = \frac{1}{2}(x^{\mathrm{B}} + y^{\mathrm{B}})$ be the utility of a link with bilateral cooperation and $u^{\mathrm{T}} = \frac{1}{2}(x^{\mathrm{T}} + y^{\mathrm{T}})$ be the utility of a link with triangular cooperation.

- Stage 1: When a pair $\{i, j\}$ is recognized to meet, they each propose to form a link if and only if $\gamma_{ij} \leq u^{\mathrm{B}}$.
- Stage 2: When a triple $\{i, j, k\}$ is recognized to meet, the unlinked players $j$ and $k$ each propose to form the $jk$ link if and only if $\gamma_{jk} \leq u^{\mathrm{T}}$.
- Stage 3: When a triple $\{i, j, k\}$ is recognized to meet, they each propose to form the $jk$ link if and only if $\gamma_{jk} \leq u^{\mathrm{T}} + z_{ijk}(u^{\mathrm{T}} - u^{\mathrm{B}})$, where $z_{ijk} \in \{0, 1, 2\}$ is the number of unsupported links in the triple at the time they meet. If they propose to form the $jk$ link, they also propose transfers that allocate to each of them one third of their net gains, calculated as if the final network $G$ were to be the one that forms immediately from their decision.

Players ignore past deviations, so the actions described are taken both on and off the equilibrium path.

It is clear that each link that forms is strictly beneficial to the players who form it. Since coordination is needed to form each link, there is no incentive for any player to deviate from this strategy profile. However, the players suffer from some coordination failure in Stages 1 and 3 that causes them to forego profitable linking opportunities. In Stage 1, they fail to anticipate links that may form in Stages 2 and 3; optimally anticipating later links would lead them to form some links in Stage 1 for which $\gamma_{ij} > u^{\mathrm{B}}$. In Stage 3, they fail to anticipate links that may form later in Stage 3: since supporting links are formed by a greedy algorithm rather than an optimal algorithm, a less beneficial link may be chosen early over a more beneficial one that is recognized later. Focusing on an equilibrium with these coordination failures aids in finding closed form expressions for network statistics like support and clustering.

This equilibrium generates a random network, whose properties depend on the population size $n$. Since in Stage 1 each possible link forms with probability $\alpha_1 \equiv \frac{1}{n}\pi_1 F(u^{\mathrm{B}})$, the backbone network $G_1$ is a uniform random graph. In Stage 2, players can form clusters by closing triples; an open triple in $G_1$ is closed in $G_2$ with probability $\alpha_2 \equiv \pi_2 F(u^{\mathrm{T}})$. In Stage 3 they can search for additional links to form to support their relationships.

We show that as the population size diverges to infinity, the following characteristics arise:[14]

**Remark 2.** *When the population size is arbitrarily large, the network has*

- *Moderate clustering: The limiting probability that two neighbors of the same player are linked is strictly positive, and it converges to* $0$ *as* $\pi_1 \to \infty$.
- *High support: The limiting probability that two linked players share a common neighbor is strictly positive, and it converges to* $1$ *as* $\pi_1 \to \infty$.

In particular, we can bound the limiting clustering and support measures when the population size is arbitrarily large, and the remark immediately follows from these bounds. We need a few notations to characterize the bounds. Let $\mathbb{E}_1(d)$ be the expected degree of a node in the backbone network $G_1$. Recall $\alpha_2$ is the probability of a link closing a triple in stage 2. Let $\alpha_{3L} \equiv F(2u^T - u^B)$ be the probability of a link with a cost below the value of both cooperating at the triangular level ($u^T$) and supporting one other link ($u^T - u^B$). And let $\alpha_{3H} \equiv F(3u^T - 2u^B)$ be the the probability of a link with a cost below the value of both cooperating at the triangular level and supporting two other links. Then, let

$$\beta_2 \equiv 1 - e^{-2\mathbb{E}_1(d)\alpha_2},$$
$$\beta_{3L} \equiv e^{-2\mathbb{E}_1(d)\alpha_2} - e^{-2\mathbb{E}_1(d)\alpha_{3L}},$$
$$\beta_{3H} \equiv e^{-2\mathbb{E}_1(d)\alpha_2} - e^{-2\mathbb{E}_1(d)\alpha_{3H}}.$$

We will show that with large population, $\beta_2$ is the probability that a backbone link $ij \in G_1$ is supported in stage 2. $\beta_{3L}$ and $\beta_{3H}$ are the lower and upper bounds on the *incremental*

[14] Clustering is one of main characteristics enumerated by Jackson and Rogers (2007), along with the high support characteristic identified by Jackson, Rodriguez-Barraquer, and Tan (2012).

probability that $ij$ is supported in stage 3.[15]

**Lemma 5.** *When the population size is arbitrarily large, the limiting clustering ($c_3$) and support ($s_3$) are bounded by*

$$c_3 \leq \frac{3\alpha_2 \mathbb{E}_1(d) + 3\beta_{3H}}{(1+2\alpha_2)\mathbb{E}_1(d) + \mathbb{E}_1(d)^2 2\alpha_2 + \mathbb{E}_1(d)^3\alpha_2^2 + 2\beta_{3H}}, \tag{21}$$

*and*

$$\frac{\beta_2 + \beta_{3L} + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}}{1 + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}} \leq s_3 \leq \frac{\beta_2 + \beta_{3H} + \alpha_2\mathbb{E}_1(d) + \beta_{3H}}{1 + \alpha_2\mathbb{E}_1(d) + \beta_{3H}}. \tag{22}$$

Before proceeding to the details of calculating the support measure, we remark on the effect of seeking for support. In particular, it is the difference between the final support measure $s_3$ versus the support by the end of stage 2, $s_2$. Next section will show that

$$s_2 = \frac{\beta_2 + \alpha_2\mathbb{E}_1(d)}{1 + \alpha_2\mathbb{E}_1(d)}.$$

Thus, the effect of seeking for support is at least

$$\Delta s \geq \frac{\beta_2 + \beta_{3L} + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}}{1 + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}} - \frac{\beta_2 + \alpha_2\mathbb{E}_1(d)}{1 + \alpha_2\mathbb{E}_1(d)}.$$

Table 1 illustrate the effect of seeking for support which could increase the support measure by as high as 30%-40%, in the case when $\alpha_2 = 0.08$, $\alpha_{3L} = 0.15$ and $\alpha_{3H} = 0.2$.

**Support in Stage 2** First, we calculate the support for the stage-2 network. Let $d_i(g)$ be the degree of node $i$ in network $g$. Consider two separate cases for a link $ij \in G_2$: $ij \in G_2 \setminus G_1$ and $ij \in G_1$. If $ij \in G_2 \setminus G_1$, $ij$ is supported by construction. If $ij \in G_1$, the probability $ij$ is supported in $G_2$ is

$$1 - (1-\alpha_2)^{d_i(G_1\setminus\{ij\}) + d_j(G_1\setminus\{ij\})} \to \beta_2 \equiv 1 - e^{-2\mathbb{E}_1(d)\alpha_2} \tag{23}$$

[15] In other words, $\beta_2 + \beta_{3L}$ and $\beta_2 + \beta_{3H}$ are the lower and upper bounds on the probability that $ij$ is supported in the network $G$.

| $\mathbb{E}_1(d)$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $s_2$ | 0.50 | 0.60 | 0.68 | 0.74 | 0.79 | 0.83 |
| $s_{3L}$ | 0.69 | 0.78 | 0.85 | 0.89 | 0.93 | 0.95 |
| $\Delta s$ | 0.19 | 0.18 | 0.17 | 0.15 | 0.13 | 0.12 |
| $c_2$ | 0.14 | 0.13 | 0.11 | 0.10 | 0.09 | 0.08 |
| $c_{3H}$ | 0.29 | 0.23 | 0.19 | 0.16 | 0.13 | 0.11 |

Table 1: Support and clustering in stage 2 network ($s_2$ and $c_2$) and the lower bound of support and higher bound of clustering in the final network ($s_{3L}$ and $c_{3H}$), in the case when $\alpha_2 = 0.08$, $\alpha_{3L} = 0.15$ and $\alpha_{3H} = 0.2$.

The LHS says that $ij$ is not supported only if $i$ does not want to form links to anyone in $N_j^1$ and $j$ does not want to form links to anyone in $N_i^1$. The limit result is because $d_i(G_1 \setminus \{ij\})$ (or $d_j(G_1 \setminus \{ij\})$), that is the number of $i$'s (or $j$'s) neighbors in $G_1 \setminus \{ij\}$, converges to a Poisson distribution of mean $\mathbb{E}_1(d)$ as $n \to \infty$.

For a given agent $i$, the number of links $ij \in G_1$ is $d_i$, with the expectation $\mathbb{E}_1(d)$. Similarly, the expected number of links $ij \in G_2 \setminus G_1$ is

$$\alpha_2 \mathbb{E} \sum_{j \in N_i} d_j(G_1 \setminus \{ij\}) = \alpha_2 \mathbb{E}_1(d)^2 \tag{24}$$

Putting these two cases together, the limiting support coefficient, which is the expected fraction of supported links per player, is

$$s_2 = \frac{\beta_2 \mathbb{E}_1(d) + \alpha_2 \mathbb{E}_1(d)^2}{\mathbb{E}_1(d) + \alpha_2 \mathbb{E}_1(d)^2} = \frac{\beta_2 + \alpha_2 \mathbb{E}_1(d)}{1 + \alpha_2 \mathbb{E}_1(d)} \tag{25}$$

**Bounds on support in the Stage 3** To begin with, we consider lower bound of forming additional links due to seeking for support. That is the link $ij$ is supported as long as either there is neighbor $k \in N_i$ with $\gamma_{jk} < 2u^{\mathrm{T}} - u^{\mathrm{B}}$ or there is neighbor $l \in N_j$ with $\gamma_{il} < 2u^{\mathrm{T}} - u^{\mathrm{B}}$. Let $\alpha_{3L} = F(2u^{\mathrm{T}} - u^{\mathrm{B}})$; this the social benefit of forming a link that brings support in $G_3$ to one link that was unsupported in $G_2$. Let $S_2$ be the set of links that are supported in $G_2$ and let $S_3$ be the set of links that are supported in $G_3$; let $U_2 = G_2 \setminus S_2$ and $U_3 = G_3 \setminus S_3$.

We calculate the support measure by considering two separate cases for a link $ij \in G_3$: $ij \in G_1$ and $ij \in G_3 \setminus G_1$. If $ij \in G_3 \setminus G_1$, $ij$ must be supported. If $ij \in G_1$, then $ij \in S_3$ if there is any neighbor $k \in N_i \setminus \{j\}$ such that $\gamma_{jk} \leq 2u^{\mathrm{T}} - u^{\mathrm{B}}$, or similarly if there is any neighbor $\ell \in N_j \setminus \{i\}$ such that $\gamma_{i\ell} \leq 2u^{\mathrm{T}} - u^{\mathrm{B}}$. Therefore

$$\Pr(ij \in S_3 | ij \in G_1) \geq 1 - (1 - \alpha_{3L})^{d_i(G_1 \setminus \{ij\}) + d_j(G_1 \setminus \{ij\})} \quad \to 1 - e^{-2\mathbb{E}_1(d)\alpha_{3L}} \tag{26}$$

Then, we calculate the relative weights of the two separate cases. For a given agent $i$, his number of links in $G_1$ is $d_i$, with expectation $\mathbb{E}_1(d)$. Similarly, the expected number of links that $i$ forms in stage 2 is $\alpha_2 \mathbb{E}_1(d)^2$. The limiting expected number of links $i$ forms in stage 3 is denoted as $K_3$. To compute a lower bound for $K_3$, we can start from the probability $ij$ is not supported in stage 2, but is supported in stage 3 because there is at least one potential supporting link not in $G_2$ that has cost below $2u^{\mathrm{T}} - u^{\mathrm{B}}$:

$$\begin{aligned}
&\Pr(ij \in S_3, ij \in U_2 | ij \in G_1) \\
&\geq (1 - \alpha_2)^{d_i(G_1 \setminus \{ij\}) + d_j(G_1 \setminus \{ij\})} \left(1 - \left(\frac{1 - \alpha_{3L}}{1 - \alpha_2}\right)^{d_i(G_1 \setminus \{ij\}) + d_j(G_1 \setminus \{ij\})}\right) \\
&\to \beta_{3L} \equiv e^{-2\mathbb{E}_1(d)\alpha_2} - e^{-2\mathbb{E}_1(d)\alpha_{3L}}
\end{aligned} \tag{27}$$

We now have a lower bound:

$$K_3 \geq \frac{1}{2}\beta_{3L}\mathbb{E}_1(d) \tag{28}$$

From right to left, this is because $i$ has in expectation $\mathbb{E}_1(d)$ links in $G_1$, each of which is unsupported in $G_2$ but supported in $G_3$ with probability at least $\beta_{3L}$, but the link that brings the support may also support a second link and thus for a given $i$ represents at least half of an additional link.

Thus, the lower bound on support in the final network is

$$\begin{aligned}
s_3 &\geq \frac{(\beta_2 + \beta_{3L})\mathbb{E}_1(d) + \alpha_2\mathbb{E}_1(d)^2 + \frac{1}{2}\beta_{3L}\mathbb{E}_1(d)}{\mathbb{E}_1(d) + \alpha_2\mathbb{E}_1(d)^2 + \frac{1}{2}\beta_{3L}\mathbb{E}_1(d)} \\
&= \frac{\beta_2 + \beta_{3L} + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}}{1 + \alpha_2\mathbb{E}_1(d) + \frac{1}{2}\beta_{3L}}
\end{aligned} \tag{29}$$

Then, we consider an upper bound based on adding links that bring support to two links at the same time. The benefit brought by such a link is $3u^{\mathrm{T}} - 2u^{\mathrm{B}}$; let $\alpha_{3H} = F(3u^{\mathrm{T}} - 2u^{\mathrm{B}})$. The probability $ij$ is not supported in $G_2$, but would be supported in $G_3$ if it were recognized first in the sequence is at most:

$$
\begin{aligned}
&(1-\alpha_2)^{d_i(G_1\setminus\{ij\})+d_j(G_1\setminus\{ij\})}\left(1-\left(\frac{1-\alpha_{3H}}{1-\alpha_2}\right)^{d_i(G_1\setminus\{ij\})+d_j(G_1\setminus\{ij\})}\right) \\
&\to \beta_{3H} \equiv e^{-2\mathbb{E}_1(d)\alpha_2} - e^{-2\mathbb{E}_1(d)\alpha_{3H}}
\end{aligned}
\tag{30}
$$

The number of links (thus triangles) added in stage 3 is at most the expected number of links that are unsupported in stage 2 but would supported in stage 3 if they were first in the sequence, because adding a link supports at least one such link. Thus we have an upper bound:

$$
K_3 \leq \beta_{3H}\mathbb{E}_1(d) \tag{31}
$$

Thus, the upper bound on support in the final network is

$$
\begin{aligned}
s_3 &\leq \frac{(\beta_2+\beta_{3H})\mathbb{E}_1(d)+\alpha_2\mathbb{E}_1(d)^2+\beta_{3H}\mathbb{E}_1(d)}{\mathbb{E}_1(d)+\alpha_2\mathbb{E}_1(d)^2+\beta_{3H}\mathbb{E}_1(d)} \\
&= \frac{\beta_2+\beta_{3H}+\alpha_2\mathbb{E}_1(d)+\beta_{3H}}{1+\alpha_2\mathbb{E}_1(d)+\beta_{3H}}
\end{aligned}
\tag{32}
$$

**Clustering in Stage 2** The clustering coefficient of a network is the fraction of transitive triples (i.e., paths of length 2) that are contained in triangles. The limiting clustering coefficient of the backbone network $G_1$ (as $n \to \infty$) is zero, since in the limit the network is a forest (a collection of trees) with probability one. We calculate the limiting clustering coefficient of network $G_2$ as follows.

1. Consider a 2-path comprising two backbone links $ij, ik \in G_1$. In the backbone network this path is "open" (i.e., $j$ and $k$ are unlinked). In network $G_2$, there are two subcases:

   (a) With probability $\alpha_2$ it is "closed" by link $jk$ being formed in $G_2$.

   (b) With probability $1-\alpha_2$ it is "open" because link $jk$ does not form in $G_2$.

2. Consider a 2-path comprising one backbone link $ij \in G_1$ and one newly formed link $i\ell \in G_2 \setminus G_1$. There are two subcases:

   (a) If $j\ell \in G_1$, then $i\ell$ completes the $ij\ell$ triangle. In this case the 2-path is closed.

(b) If $j\ell \notin G_1$, then for $i\ell$ to form in Stage 2, there must exist $ik, k\ell \in G_1$. Since $G_1$ is approximately a forest, $\ell$ and $j$ are almost surely distance 3 apart in $G_1$, so they cannot form a link in $G_2$. Therefore such a 2-path must be open.

3. Consider a 2-path comprising two newly formed links $i\ell, im \in G_2 \setminus G_1$. For these to form in Stage 2, there must exist $ij, j\ell, ik, km \in G_1$. Since $G_1$ is a tree, $\ell$ and $m$ are distance 4 apart in $G_1$, so they cannot form a link in $G_2$. Therefore such a 2-path must be open.

Observe that for any 2-path $ij, ik \in G_1$, with probability $1 - \alpha_2$ it is open in $G_2$ (i.e., $jk \notin G_2$), in which case it accounts for one open 2-path in $G_2$ of type 1(b) above. With complementary probability $\alpha_2$ it is closed in $G_2$, in which case it accounts for three closed 2-paths in $G_2$—one of type 1(a) and two of type 2(a). Moreover, for any given node $i$, the expected number of 2-paths in $G_1$ that originate with $i$ is $\mathbb{E}_1(d)^2$.

Similarly, any 3-path in $G_1$ can generate two instances of open 2-paths of type 2(b), each with independent probability $\alpha_2$. For a given node $i$, the expected number of 3-paths in $G_1$ that originate with $i$ is $\mathbb{E}_1(d)^3$.

Finally, any 4-path in $G_1$ can generate one instance of an open 2-path of type 3, with probability $\alpha_2^2$. For a given node $i$, the expected number of 4-paths in $G_1$ that originate with $i$ is $\mathbb{E}_1(d)^4$.

Putting together these facts, the expected fraction of 2-paths in $G_2$ that are closed is

$$\begin{aligned} C_2 &= \frac{\mathbb{E}_1(d)^2 3\alpha_2}{\mathbb{E}_1(d)^2(3\alpha_2 + 1 - \alpha_2) + \mathbb{E}_1(d)^3 2\alpha_2 + \mathbb{E}_1(d)^4\alpha_2^2} \\ &= \frac{3\alpha_2}{1 + 2\alpha_2 + \mathbb{E}_1(d)2\alpha_2 + \mathbb{E}_1(d)^2\alpha_2^2} \end{aligned} \tag{33}$$

**Bounds on clustering in the Stage 3** As shown in the analysis of clustering in Stage 2, the clustering measure is not monotone as agents form links to close triangles. While the closure of one triangle increases clustering, the new link also creates more unclosed triangles that decrease clustering. To make the measure tractable, we let $T$ be the measure of average number of triangles an agent belongs to, which captures triangle closure as clustering but is monotone as more links are formed in Stage 3. Then in stage 2, following the analysis of clustering above, for $G_2$ we have

$$T_2 = \mathbb{E}_1(d)^2 3\alpha_2 \tag{34}$$

Then, the number of additional triangles per agent formed in stage 3 is $3K_3$. Its lower bound and upper bound are calculated above. So we have an upper bound on the clustering measure.

$$\begin{aligned} C_3 &\leq \frac{\mathbb{E}_1(d)^2 3\alpha_2 + 3\beta_{3H}\mathbb{E}_1(d)}{\mathbb{E}_1(d)^2(3\alpha_2 + 1 - \alpha_2) + \mathbb{E}_1(d)^3 2\alpha_2 + \mathbb{E}_1(d)^4\alpha_2^2 + 2\beta_{3H}\mathbb{E}_1(d)} \\ &= \frac{3\alpha_2\mathbb{E}_1(d) + 3\beta_{3H}}{(1+2\alpha_2)\mathbb{E}_1(d) + \mathbb{E}_1(d)^2 2\alpha_2 + \mathbb{E}_1(d)^3\alpha_2^2 + 2\beta_{3H}} \end{aligned} \tag{35}$$

Note that each link formed in stage 3 would create two additional pairs of connected links and complete three of such pairs including an original one.

## B.2 Sequential vs. simultaneous moves

We show that agents earn strictly higher expected payoffs from sequential moves. We prove this result for the bilateral cooperation, and it is straightforward to extend it to the triangular cooperation. Recall that in a bilateral cooperation, agents use $x^{\mathrm{B}}$ and $y^{\mathrm{B}}$ when they move sequentially, and let them both use $z^B$ when they move simultaneously.

**Remark 3.** *Consider bilateral cooperation:* $x^{\mathrm{B}} > z^B$ *and* $y^{\mathrm{B}} > z^B$.

*Proof.* If agents move simultaneously, their incentive constraint is

$$0 \leq -c(z) + \int_0^\infty e^{-rt}\lambda z dt = -c(z) + \frac{\lambda}{r}z$$

So $z^B$ satisfies $c(z^B) - \frac{\lambda}{r}z^B = 0$. By assumption 1, there is a unique solution of $z^B > 0$.

If agents move sequentially, $x^{\mathrm{B}}$ and $y^{\mathrm{B}}$ bind the constraints (1) and (2). In particular, summing up (1) and (2), we have $c(x^{\mathrm{B}}) - \frac{\lambda}{r}x^{\mathrm{B}} = (1+\frac{\lambda}{r})y^{\mathrm{B}}$. Since $y^{\mathrm{B}} > 0$, it is clear that $x^{\mathrm{B}} > z^B$. Next, (2) implies $c(y^{\mathrm{B}}) - \frac{\lambda}{r}\frac{1}{2}(x^{\mathrm{B}} + y^{\mathrm{B}}) = 0$. When $y^{\mathrm{B}} = z^B$, $c(z^B) - \frac{\lambda}{r}\frac{1}{2}(x^{\mathrm{B}} + z^B) < 0$ because $x^{\mathrm{B}} > z^B$. Thus, $y^{\mathrm{B}} > z^B$. □