Predicting Long Dated Bond Returns

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Overview

- In an arbitrage-free setting, the yield at each term can be decomposed into:
  1. the $\mathbb{P}$ expectation of the time average of future short rates
  2. plus a risk premium for that term (AKA term premium), which can take either sign
  3. less a positive convexity effect.
- The goal of this talk is to theoretically explain yields at any term and to use a version of this decomposition at the long end of the yield curve to empirically explain short term returns from holding long term bonds.
- We present a simple alternative to both short-rate models, eg. Hull White/CIR, and to whole term-structure models, HJM/BGM. Both types of models have legitimate purposes eg market-making/hedging, but they are not as well suited to the task of forecasting real-world returns from holding long-dated bonds.

Our objective: Analyzing returns on long-dated bonds

- How can we forecast changes in a long-dated yield, based on the current observed shape of the yield curve, while accounting for the risk premium and for convexity effects?
- More importantly, how can we forecast excess returns on long-dated bonds?
- Forecasting changes in the 50 year rate using a short rate model requires the modeler to specify the $\mathbb{P}$ dynamics of the short rate for the next 50 years.
- When the speed of mean reversion is calibrated to time series or to the short end of the yield curve, the implied movement in long rates from a short rate model is much smaller than empirically observed.
- Empirically, long rates move randomly, and with *substantial volatility*.
- Can we say something useful about the one year return on a 50-year bond *without* making a 50-year projection of short rates?
- The divergence of the empirical behavior of long-dated bonds from that predicted by short-rate models asks for a distinct modeling approach.

A new modeling approach

We propose a new modeling framework that is particularly well suited for forecasting long bond returns:

- Our new framework links the pricing of long-dated bonds directly to the attribution of P&L from holding that long-dated bond.
  - The P&L attribution makes it clear what to bet on and what to hedge.
- We determine the arbitrage-free yield for any term based on its *own* $\mathbb{Q}$ dynamics, not the $\mathbb{Q}$ dynamics of the short rate.
  - This localization to a particular term is less ambitious than trying to simultaneously forecast changes in rates across all terms, but allows one to make more confident statements on the particular long rate whose changes one wishes to forecast.
  - The model can say something useful about an investment in a 50-yr bond without making a 50-year projection, especially if one just wants to hold the bond for the short term (say one year).
- We will show empirically that we can explain a sizable fraction of the variance in long bond *returns*, even though we will begin by assuming that the long rate is a $\mathbb{P}$ martingale.

Decomposing yield to maturity

- Yield to maturity is a *declining, but nonlinear, transformation* of the bond price.
- Fundamentally, in any arbitrage-free model, the yield to maturity can always be decomposed into,
  1. *Expectation*: market's $\mathbb{P}$ expectation of the time average of future short rates
  2. *+Risk Premium*: sign-indefinite compensation for bearing the risk of interest rate fluctuations about the $\mathbb{P}$ expectation,
  3. *-Convexity*: a negative effect on yield induced by the declining nonlinear relation between bond price and yield-to-maturity.
- We operationalize the decomposition via a driftless diffusion assumption and then separate the risk premium from the other two components in order to predict excess returns of long-dated bonds, without resorting to a forecasting regression.

Notation and the Classical Setting

- Let B_t be the time-t price of a default-free coupon bond (portfolio) with fixed future cash flows $\{C_j\}$ at times $\{t+\tau_j\} \geq t$ for $j = 1, 2, \cdots, N$.
- The classic valuation of this coupon bond can be represented as

$$B_t(T) = \sum_j C_j \mathbb{E}_t^{\mathbb{P}} \left[M_{t,t+\tau_j} \right] = \sum_j C_j \mathbb{E}_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) e^{-\int_t^{t+\tau_j} r_u du} \right]$$
$$= \sum_j C_j \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_u du} \right].$$

  - $\mathbb{E}_t[\cdot]$ — expectation under time-t filtration,
  - $M_{t,T}$ — the pricing kernel linking value at time t to value at time T
  - $\mathbb{P}$ — the real world probability measure,
  - $\mathbb{Q}$ — the so-called risk-neutral measure,
  - r_t — instantaneous short rate
  - $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defines the measure change from $\mathbb{P}$ to $\mathbb{Q}$. It is the martingale component of the pricing kernel that defines the pricing of various risks.
- *Yield-to-maturity* y_t of a bond is implicitly defined by this map to price:

$$B_t(T) \equiv \sum_j C_j e^{-y_t \tau_j}.$$

Yield to Maturity of a Zero Coupon Bond

- As a special case, let $y_t(T)$ be the continuously-compounded yield-to-maturity (YTM) at time $t \geq 0$ of a default-free zero-coupon bond (ZCB) paying \$1 at $T \geq t$.

$$B_t(T) \equiv e^{-y_t(T)(T-t)}.$$

- Solving for $y_t(T)$ explicitly relates the yield to the bond's price $B_t(T)$:

$$y_t(T) \equiv -\frac{\ln B_t(T)}{T-t}, \qquad t \in [0, T].$$

- Since bond prices are positive, yields are real-valued and move in the opposite direction of bond prices.

- Substituting the bond pricing formula $B_t(T) = E_t^{\mathbb{Q}} e^{-\int_t^T r_u du}$ into the above yield equation reveals the link between the T maturity yield observed at time $t \in [0, T]$ and future short rates r_u realized at times $u \in [t, T]$:

$$y_t(T) \equiv -\frac{1}{T-t} \ln E_t^{\mathbb{Q}} e^{-\int_t^T r_u du}, \qquad t \in [0, T].$$

Decomposing Yield to Maturity of a Zero Coupon Bond

- By adding & subtracting the same term twice, the ZCB yield decomposes as:

$$y_t(T) = E_t^{\mathbb{P}} \frac{\int_t^T r_u du}{T-t} + E_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right) \frac{\int_t^T r_u du}{T-t} \right] - C,$$

where $C \equiv \frac{1}{T-t} \left[\ln E_t^{\mathbb{Q}} e^{-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du} \right] \geq 0$ is the convexity effect.

- The 1st term is the average short rate $\frac{\int_t^T r_u du}{T-t}$ over times and states. If future interest rates are sign-indefinite, then this term is also sign-indefinite.

- The second term is the risk premium as captured by the covariance under $\mathbb{P}$ of this average short rate with the random variable, $\frac{d\mathbb{Q}}{d\mathbb{P}} - 1$, which has zero mean under $\mathbb{P}$. This covariance can have either sign, so the middle term cannot be signed either. If interest rates are deterministic, then the covariance vanishes. If interest rates are stochastic and if bond returns are thought to have a positive risk premium, then the covariance in the second term is also positive. The introduction of both risky bond returns and positive risk aversion raises the yield and hence lowers the bond price.

Convexity Effect on Yield of a Zero Coupon Bond

- Recall $y_t(T) = E_t^{\mathbb{P}} \frac{\int_t^T r_u du}{T-t} + E_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right) \frac{\int_t^T r_u du}{T-t} \right] - C$, where $C \equiv \frac{1}{T-t} \left[\ln E_t^{\mathbb{Q}} e^{-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du} \right] \geq 0$ is the convexity effect.
- While the $\mathbb{P}$ mean of future rates and the risk premium can in theory have either sign, the convexity effect $C \geq 0$ can only lower yield.
- One can interpret C as a *nonstandard* deviation under $\mathbb{Q}$ of the zero mean random variable $-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du$. When compared to the standard deviation, the nonstandard deviation replaces the quadratic function with an exponential (which requires that its argument be dimensionless).
- The convexity of the exponential function and Jensen's inequality implies $C \geq 0$.
- Intuitively, bond prices are both convex in the time average of future short rates and declining in yield. As the uncertainty of future rates rise, bond prices rise and hence yields fall.

Term Structure Effects

- $y_t(T) = E_t^{\mathbb{P}} \frac{\int_t^T r_u du}{T-t} + E_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right) \frac{\int_t^T r_u du}{T-t} \right] - \frac{1}{T-t} \left[\ln E_t^{\mathbb{Q}} e^{-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du} \right]$.

- The relative importance of the three terms in this yield decomposition can differ across maturities.

- As the maturity date T approaches the current time, t, the last two terms vanish, so that the current yield-to-maturity approaches the short rate:

$$\lim_{T \downarrow t} y_t(T) = r_t, \qquad t \geq 0.$$

- As we raise the time to maturity, the 2nd and 3rd terms both start affecting the yield, but at different speeds.

- As the maturity date T becomes infinite, the first two terms will in general asymptote to finite constants, while the last term can either behave the same way or explode, taking the yield to negative infinity. An example of a short rate model in which the latter behavior occurs is the well known Vasicek model.

Bond P&L Attribution for Yield Diffusion

- The previous decomposition applies only to yields on zero coupon bonds. For coupon bearing bonds, one can similarly decompose yield by assuming that the yield moves continuously over time (no jumps):

$$dy_t = \mu^{\mathbb{P}}_{t,y} dt + \sigma_{t,y} dW_t$$

- For fixed payment times T_j, define $B(y,t) \equiv \sum_j C_j e^{-y\tau_j}$, where $\tau_j \equiv T_j - t$.

$$\text{By Itô: } dB_t = \frac{\partial B}{\partial t} dt + \frac{\partial B}{\partial y} dy_t + \frac{1}{2}\frac{\partial^2 B}{\partial y^2}(dy_t)^2.$$

- The ex ante expected return from the bond investment is

$$\mathbb{E}^{\mathbb{P}}_t \left[\frac{dB_t}{B_t dt}\right] = y_t - \mu^{\mathbb{P}}_{t,y}\tau + \frac{1}{2}\sigma^2_{t,y}\tau^2$$

  - $\mu^{\mathbb{P}}_{t,y}$ — the time-t level of the drift/direction of the yield.
  - $\sigma^2_{t,y}$ — the time-t level of its variance rate.
  - τ and τ^2 — value-weighted maturity (duration) and maturity squared:

$$\tau = \sum_j \frac{C_j e^{-y_t\tau_j}}{B_t}\tau_j, \quad \tau^2 = \sum_j \frac{C_j e^{-y_t\tau_j}}{B_t}\tau_j^2.$$

Decomposing expected return on bond investments

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{dB_t}{B_t dt}\right] = y_t - \mu_{t,y}^{\mathbb{P}}\tau + \frac{1}{2}\sigma_{t,y}^2\tau^2$$

- Decomposes expected bond return into three sources:
  1. *Carry*: Bonds with a higher yield have higher returns due to carry.
  2. *Prediction*: Expections of rate hikes reduce expected return.
  3. *Convexity*: Since a bond's price is convex in yield, random vibrations of the yield (without trend) leads to a positive return.
     - A duration neutral portfolio that is long longer-term bonds is analogous to a delta-neutral long options position.
- Implications
  - If one has no view on the expected change in future rates, form a duration-neutral portfolio (to neutralize out the second term).
  - Go long/short convexity based on your view of future realized volatility

Decomposing Yield of a Coupon Bond

- Recall: $\mathbb{E}_t^{\mathbb{P}}\left[\frac{dB_t}{B_t dt}\right] = y_t - \mu_{t,y}^{\mathbb{P}}\tau + \frac{1}{2}\sigma_{t,y}^2\tau^2$
- Assume that the bond market is free of arbitrage, so that there exists a risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.
- Replacing $\mathbb{P}$ with $\mathbb{Q}$ and further replacing $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{dB_t}{B_t dt}\right]$ with r_t implies:

$$r_t = y_t - \mu_{t,y}^{\mathbb{Q}}\tau + \frac{1}{2}\sigma_{t,y}^2\tau^2$$

- Solving for the yield y_t gives another 3 term decomposition:

$$y_t = r_t + \mu_{t,y}^{\mathbb{Q}}\tau - \frac{1}{2}\sigma_{t,y}^2\tau^2$$

- The sum of the first 2 terms correspond to the sum of mean future rates and the risk premium. The third term is the usual convexity effect.

Bond pricing based on *local, near-term dynamics*

$$y_t = r_t + \mu_{t,y}^{\mathbb{Q}}\tau - \frac{1}{2}\sigma_{t,y}^2\tau^2 \quad (1)$$

- *Local*: The fair valuation of the bond investment in (**??**) does not depend on short-rate dynamics, but only depend on the *behavior of its own yield*.
- *Near-term*: The pricing of the yield does not even depend on its own full dynamics, but only depends on the current level of the $\mathbb{P}$ drift and volatility.
  - The drift rate $\mu_{t,y}^{\mathbb{Q}}$ and volatility $\sigma_{t,y}$ can each follow some stochastic process, and/or depend on other rates/economic state variables ...
  - Unlike short rate models with stochastic drift or diffusion of short rates, the dynamics of the drift and diffusion of the yield do *NOT* enter into the pricing relation.
- *Views, not (much) dynamics*: One can bring in forecasts/estimates/opinions on volatility, risk premium, & rate prediction, and examine their implications on the yield (curve).
  - The estimates can come from any (other) model assumptions, algorithms, or information sources.

Different frameworks serve different purposes

Classical Short Rate Models

- *Full* short rate dynamics prices bond of *all* maturities.
- Maintain cross-sectional consistency across the whole curve.
- Hard to reconcile the empirical long rate behavior with the *actual* short rate dynamics.
- Better suited to construct smooth curves with cross-sectional consistency.

New framework

- Each yield is priced according to *its own near-term* predictions.
- Ask for the most relevant predictions for the pricing.
- Hard to maintain cross-sectional consistency across all bonds.
- Better suited to analyze specific bond (portfolios) and connect to (views on) their own, current behaviors.

Decomposing long-bond returns via driftless diffusion

- It is difficult to predict long rate movements. So we start by assuming a driftless diffusion on the floating (constant time to maturity) long rate:

$$dy_t(\tau) = \sigma_t(\tau) dW_t^{\mathbb{P}}.$$

- A risk-neutral drift in yields is induced when bond price risk ($-dW_t$) is priced:

$$dy_t(\tau) = \lambda_t \sigma_t dt + \sigma_t(\tau) dW_t^{\mathbb{Q}}.$$

  - Empirically, the market price of interest rate risk tends to be negative, which leads to a positive market price λ_t of bond price risk.

- The risk-neutral drift of a *fixed-τ* rate is further adjusted by the local shape of the yield curve ("sliding"):

$$\mu_t^{\mathbb{Q}} = \lambda_t \sigma_t - y_t'(\tau).$$

  - Our previous pricing relation was based on the yield dynamics of a fixed maturity date coupon bond, but it is easier to model/estimate floating rate dynamics (e.g., 30-yr rate).

Decomposing long-bond returns

- Plugging the driftless diffusion assumption into the pricing relation leads to

$$\frac{\partial [y_t \tau]}{\partial \tau} = r + \lambda_t \sigma_t(\tau)\tau - \frac{1}{2}\sigma_t^2(\tau)\tau^2.$$

  - For zeros, $\frac{\partial [y_t \tau]}{\partial \tau} = f(\tau)$ is the instantaneous forward rate.
  - Define instantaneous volatility weighted duration and convexity as

$$d_t = \sigma_t(\tau)\tau, \quad c_t = \sigma_t^2(\tau)\tau^2.$$

- Integrate

$$y_t = r + \lambda_t D_t - \frac{1}{2} C_t,$$

  with D and C denoting the **integrated duration and convexity**:

$$D_t \equiv \left[\frac{1}{\tau}\int_0^\tau d_t(s)ds\right], \quad C_t \equiv \left[\frac{1}{\tau}\int_0^\tau c_t(s)ds\right]$$

  What matters is not just sensitivity (τ), but also volatility.
- In the absence of any $\mathbb{P}$ drift of yields, a positive risk premium drives long rates up, while the convexity effect drives long rates down.

Extract bond risk premium from long rates

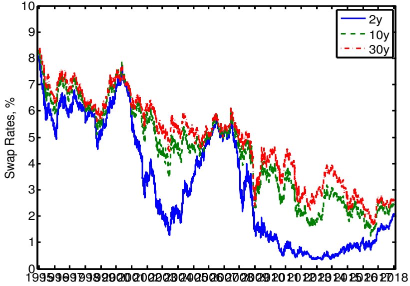
- We can extract a long-dated bond's risk premium from its yield and its yield volatility:

$$\lambda_t = \frac{y_t - r_t + \frac{1}{2}C_t}{D_t}$$

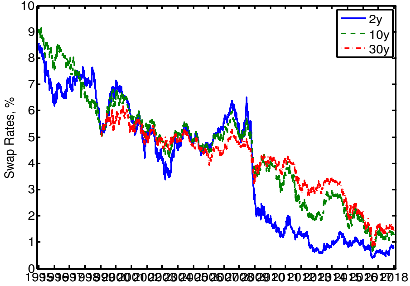
  - The long-dated rate (y_t) and the financing cost (r_t) are directly observed.
  - Variance term structure $\sigma_t(\tau)$ can be estimated using recent history (e.g., via GARCH, from options, curve)
- We can then examine whether the ex-ante risk premium predicts ex post bond excess return, without ever the need to fit a predictive regression.

- Data: US and UK swap rates 1995.1.3-2016.5.11, 5378 business days
  - Based on 6-month LIBOR Maturity, 2,3,4,5,7,10,15,20,30
  - Extended maturity since
    US: 2004/11/12 for 40 & 50 years
    UK: 1999/1/19 for 20 & 30 years, 2003/08/08 for 40 & 50 years

US



UK

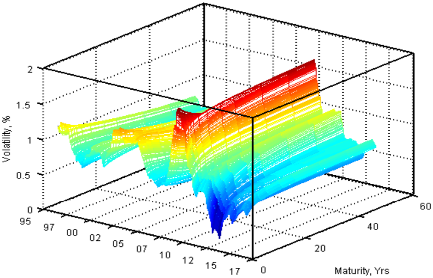


- Stripped Treasury zero rates for robustness check

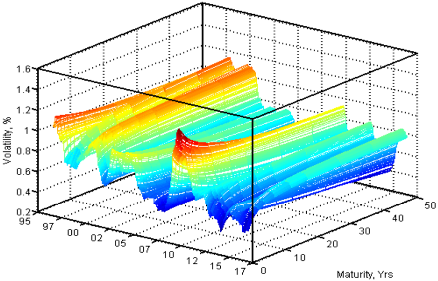
Swap rate variance term structure estimators

Estimate variance σ_t^2 on each floating swap rate series with a 1y rolling window.

US

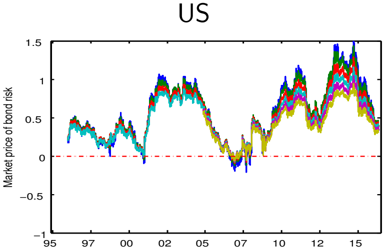


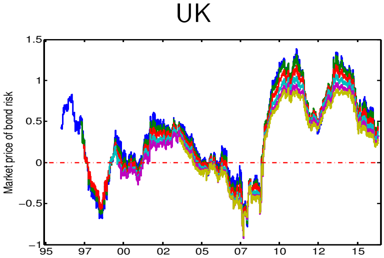
UK



- *Long rates vary as much as, if not more than, short rates.*

Market price of bond risk





- The market price of bond risk extracted from different rates are similar in magnitude and move together:
  - Over the common sample, the cross-correlation estimates among the different λ_t series average 99.67% for US, and 98.76% for UK.
  - The evidence supports a one-factor structure for the bond risk premium, as in Cochrane & Piazessi.
- In the US, market price of risk approached zero in late 1998, 2000, and 2007, but tended to be high during recessions.
- In the UK, the market price became quite negative during 1998 and 2007-2008.

Predicting ex post excess returns with ex ante risk premium

Correlation between ex ante risk premium ($\gamma_t \sigma_t^m$) and ex post excess returns on each par bond, with the average denoting the correlation between the average risk premium and the average bond excess return over the common sample period

Maturity	10	15	20	30	40	50	Average
				Horizon: 6-month			
US	0.31	0.28	0.26	0.24	0.27	0.26	0.29
UK	0.18	0.22	0.22	0.18	0.19	0.22	0.23
				Horizon: One year			
US	0.36	0.36	0.34	0.31	0.31	0.30	0.34
UK	0.36	0.42	0.40	0.32	0.33	0.37	0.39

- The assumption of no prediction on long-dated swap rates leads to significant prediction on bond excess returns.
- The predictors (risk premium) are generated based purely on a variance estimator and the current slope of the yield curve, without estimating predictive regressions.

Concluding remarks

- We proposed a new modeling framework that is particularly well suited for analyzing returns on a particular bond or bond portfolio.
- The framework does not try to model the full dynamics of an instantaneous short rate, but focus squarely on the behavior of the bond yield in question.
- It does not even ask for the full dynamical specification of this bond yield, but only needs estimates of its current expectation, risk premium, and volatility.
    - This new modeling framework can readily accommodate findings from other models, algorithms, information sources.
- The new modelling framework operationalizes a decomposition of yield into three components: expectation, risk premium, and convexity.
    - One can estimate the yield volatility from historical time series, or infer it from the curvature of the yield curve, or from caps/floors/swaptions.
    - Separating the risk premium from expectations of future rates can be a very challenging, but very fruitful endeavor.
- We showed that we can predict bond excess returns, without running predictive regressions, even by assuming a driftless diffusion on interest rates.