Likelihood Evaluation of Models with Occasionally Binding Constraints[1]

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Motivation

- Growing importance of nonlinear methods in applied macroeconomic research.

  - Regime changes and time-varying volatility.

  - Long-run risk.

  - Asymmetric adjustment costs of prices and wages.

  - Collateral and borrowing constraints.

  - Zero lower bound.

- Substantial progress in two important fronts:
  - Model solution.

  - Evaluation of likelihood function.

Motivation

- Common approach to tackle nonlinear DSGE models:

    - Accurate nonlinear solution: high-order perturbations, projection methods, time-iteration, value function iteration.
      Fernandez-Villaverde, Rubio-Ramirez and Schorfheide (2016).

    - Use of particle filter to approximate likelihood function. Herbst and Schorfheide (2015).

- Use of particle filter:

    - Good: Very general approach and relatively easy to implement.

    - Bad: Computationally demanding. Multiple "flavors".

    - Ugly: Auxiliary measurement error for practical implementation.
      Often fixed as constant fraction of variance of observables!

- Accuracy of likelihood approximation as important as accuracy of model solution. Interaction?

Some examples

"...we include measurement error in the observation equation (16). One reason for its presence is feasibility...we set me = 0.25 for the baseline parameter estimates of the model and me = 0.1 for a lower measurement error case."

Gust, C., E. Herbst, D. Lopez-Salido, and M. E. Smith. 2017. *"The Empirical Implications of the Interest-Rate Lower Bound."* American Economic Review.

"Regarding the tuning of the filter, I set $N = 100000$. The matrix Σ [of measurement errors] is diagonal, and the diagonal elements equal 25% of the variance of the observable variables."

L.Bocola. 2016.*"The Pass-Through of Sovereign Risk."* Journal of Political Economy.

▶ Literature

What we do

Focus on nonlinear DSGE model with occasionally binding constraints.

Main question: How does misspecification from using an incorrect model solution interact with misspecification from approximating the likelihood function to affect parameter inference?

- Consider three ways to solve the model:
    1. Global nonlinear solution using VFI/time-iteration.

    2. Piecewise linear method based on OccBin toolkit.

    3. First-order perturbation method.

- Consider two ways to approximate the likelihood function:
    1. Inversion filter

    2. Particle filter

Our strategy

1. We generate data from a model solved using the global nonlinear solution without measurement error.

2. We construct the posterior distribution assuming the correct model solution and using an inversion filter.

3. We compare this posterior distribution to the resulting posterior distributions from three cases:
    - **Solution error:** solve the model using piecewise linear or first-order perturbation and evaluate the likelihood using an inversion filter
    - **Likelihood approximation error:** solve the model using a global nonlinear solution and evaluate the likelihood using a particle filter assuming fixed measurement error
    - The interaction between solution and likelihood approximation error

Main results:

- Solution and likelihood approximation error bias parameter inference.
- Interaction of bias is nonlinear.

Conceptual Framework

- Consider equilibrium conditions of a generic model:

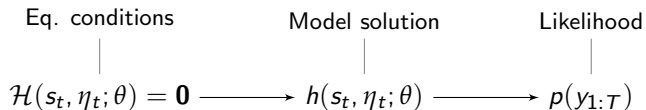$$\mathcal{H}(s_t, \eta_t; \theta) = \mathbf{0}$$

- Solution is a function $h(s_t, \eta_t; \theta)$, which can be used to express the dynamics of y_t and s_t as a nonlinear state space model:

$$s_{t+1} = h(s_t, \eta_t; \theta), \quad \text{State transition equation.} \tag{1}$$

$$y_t = g(s_t; \theta) + \zeta_t, \quad \text{Measurement equation.} \tag{2}$$

Conceptual Framework

- Framework:

Eq. conditions Model solution Likelihood

$$\mathcal{H}(s_t, \eta_t; \theta) = \mathbf{0} \longrightarrow h(s_t, \eta_t; \theta) \longrightarrow p(y_{1:T})$$

- We are interested in the following choices:

  1. Model Solution: Numerical Approximation of $h(.; \theta)$.

  2. Likelihood Function: Given sequence $y_{1:T}$, evaluate $p(y_{1:t}; \theta)$

Application

Consumption-Saving Model with Occasionally Binding Borrowing Constraint

- Household solves:

$$\max_{\{C_t, B_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1 - \gamma} \tag{3}$$

$$st:$$

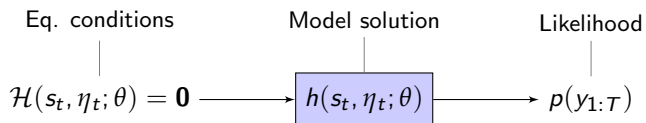$$C_t + RB_{t-1} = Y_t + B_t, \tag{4}$$

$$B_t \leq mY_t, \tag{5}$$

$$\ln Y_t = \rho \ln Y_{t-1} + \sigma \epsilon_t \tag{6}$$

- FOC:

$$C_t^{-\gamma} = \beta R E_t \left(C_{t+1}^{-\gamma} \right) + \lambda_t$$
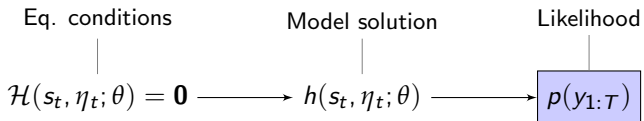
$$\lambda_t \left(B_t - mY_t \right) = 0.$$

Framework: Solution

Eq. conditions

Model solution

Likelihood

$$\mathcal{H}(s_t, \eta_t; \theta) = \mathbf{0} \longrightarrow \boxed{h(s_t, \eta_t; \theta)} \longrightarrow p(y_{1:T})$$

Solution Algorithms

- Three alternatives to approximate the function $h(.)$:
  - $h_{vfi}(.)$: value function iteration (Ljungqvist and Sargent, 2004).
  - $h_o(.)$: piecewise linear solution (Guerrieri and Iacoviello, 2015) .
  - $h_l(.)$: First-order perturbation (Judd, 1992).

- VFI: exploits dynamic programming structure.
  - **Pros:** highly accurate.
  - **Cons:** slow, course of dimensionality.

- OccBin: Approximate dynamics in **two** regimes: (i) binding and (ii) non-binding. Use shooting algorithm to obtain combined solution.
  - **Pros:** relatively fast, can be combined with inversion filter.
  - **Cons:** doesn't capture precautionary motive.

- Linear: Approximate linear dynamics in **one** regime: always binding.
  - **Pros:** Very fast, easy to implement.
  - **Cons:** Inaccurate, ignores possibility of switches across regimes.

▸ Decision rules   ▸ Accuracy

Framework: Likelihood

Eq. conditions   Model solution   Likelihood

$\mathcal{H}(s_t, \eta_t; \theta) = \mathbf{0} \longrightarrow h(s_t, \eta_t; \theta) \longrightarrow \boxed{p(y_{1:T})}$

Likelihood of DSGE model

- Likelihood function plays a crucial role in confronting DSGE model with data:

$$p(y_{1:T}; \theta) = \prod_{t=1}^{T} p(y_t; y_{1:t-1}, \theta) = \int p(y_{1:T}, s_{1:T}; \theta) ds_{1:T}$$

- Two approaches to compute likelihood function:

  - Exact likelihood: Inversion Filter (Hamilton (1994)). ▸ More

  - Approximated likelihood: Particle filter (Herbst and Schorfheide, (2015)). ▸ More

    - Bootstrap particle filter.

    - 2,000,000 particles.

Monte Carlo Experiment

- Calibrate parameters as follows:

γ	m	ρ	σ	R	β
1	1	0.9	0.01	1.05	0.945

- Simulate $T = 100, 500, 1000$ consumption observations from most accurate solution method and no measurement error (h_{vfi} and $\Omega = 0$) $\rightarrow y_{1:T}$.

- Solve model for range of values of $\gamma^i \in (0, 4.5]$ using three methods: $h_{vfi}(.; \hat{\gamma}^i), h_o(.; \hat{\gamma}^i), h_l(.; \hat{\gamma}^i)$

- Evaluate likelihood
  - For each solution use: inversion filter (IF) and particle filter (PF).

  - Three choices of measurement error (Ω): 1%, 5%, 20%.

- Flat prior over γ

- **We repeat this exercise** 100 **times**

Benchmark: No solution, no filtering error

Benchmark: VFI + IF

- Negligible Euler equation errors
- Avoids misspecification in measurement equations

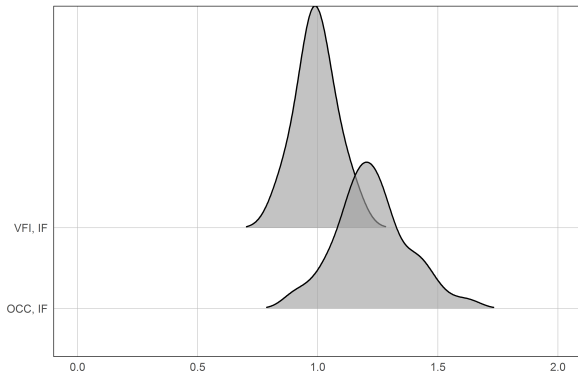Distribution of parameter estimates

Monte carlo, VFI, sample=500
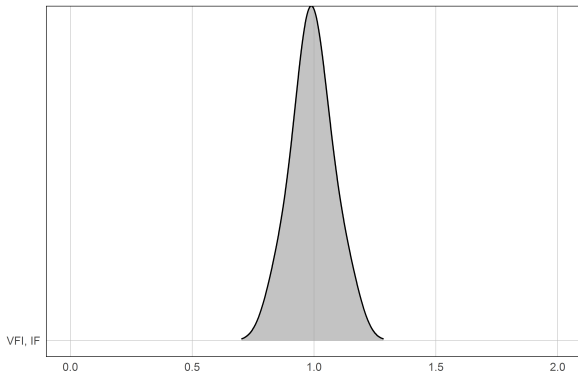
- Posterior mode estimate $\hat{\gamma}$ has a bias near 0
- 90% credible sets include $\gamma = 1$: 88% of the time

Result 1: Some solution error, no filtering error

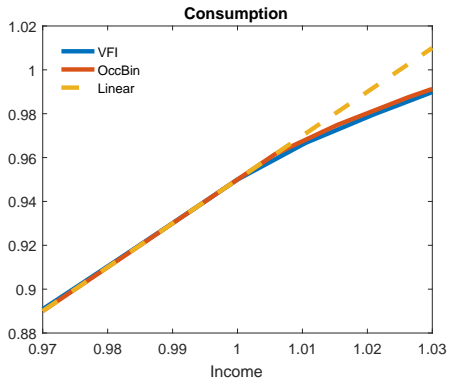**OccBin + IF:** Solution errors causes a bias in the parameter estimate

Distribution of parameter estimates
Monte carlo, VFI and OCC, sample=500

- Posterior mode estimate $\hat{\gamma}$ has a bias of 0.23 ▸ Bias in estimates
- Almost all 90% credible sets are shifted to the right relative to benchmark
  - Average shift lower bound: 0.17
  - Average shift upper bound: 0.29
- 90% credible sets include $\gamma = 1$: 32% of the time

Result 1: Intuition

- **Solution error biases the estimates of γ upwards.**
- OccBin ignores precautionary motive present in true DGP:
  - C_o more sensitive to income than C_{vfi} when constraint does not bind.
  - In order to match data, the likelihood function calls for higher γ/risk aversion.
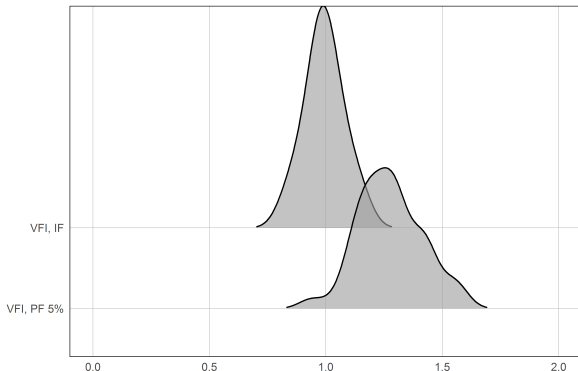

Consumption

(Figure: Consumption plotted against Income. Legend: VFI, OccBin, Linear.)

Result 2: No solution error, some filtering error

VFI + PF with 5% **measurement error:** bias in the estimate
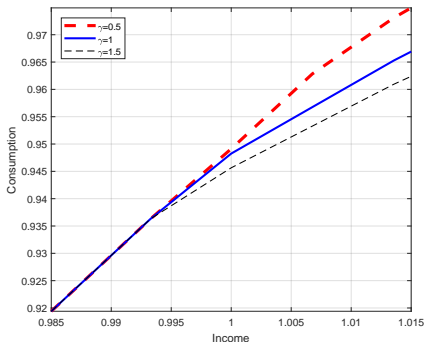
Distribution of parameter estimates

- Posterior mode estimate $\hat{\gamma}$ has a bias of 0.28   ▸ Estimate more parameters
- Almost all 90% credible sets are shifted to the right relative to benchmark
  - Average shift lower bound: 0.20
  - Average shift upper bound: 0.38
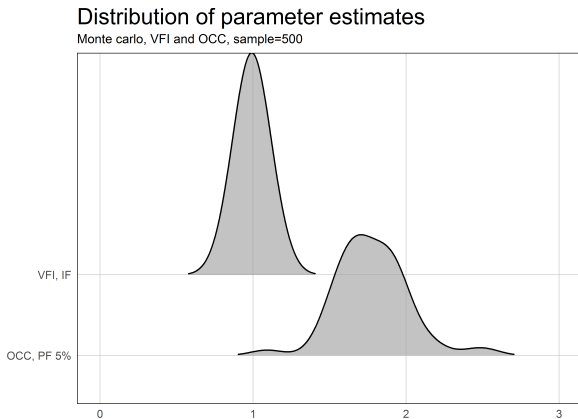- 90% credible sets include $\gamma = 1$: 26% of the time

Result 2: Intuition

- **Estimation measurement error biases γ upwards.**
- Lower γ makes C more sensitive to Y, and less skewed. (lower variance and less left-skewness)
- Measurement error increases variance of C explained by model
- Hence estimation calls for higher γ to compensate.

Result 3: Interaction of solution and filtering error

OccBin + PF with 5% **ME**: bias in the parameter estimate is amplified

Distribution of parameter estimates

Monte carlo, VFI and OCC, sample=500


- Posterior mode estimate $\hat{\gamma}$ has a bias of 0.78
- All 90% credible sets are shifted to the right relative to benchmark
  - Average shift lower bound: 0.60
  - Average shift upper bound: 1.00
- 90% credible sets include $\gamma = 1$: 2% of the time

Conclusion

- Explore interaction of solution approximation and likelihood approximation error.

- Use a simple nonlinear model of consumption-savings choice subject to occasionally binding constraints.

- Illustrate that both solution and likelihood approximation error bias parameter inference.

- Bias is increasing in size of measurement error in particle filter approximation.

- Solution and likelihood approximation error interact nonlinearly.

Decision Rules

- VFI captures precautionary motive.
- OccBin misses precautionary motive but captures anticipatory effect.
- Linear solution disregards the constraint.
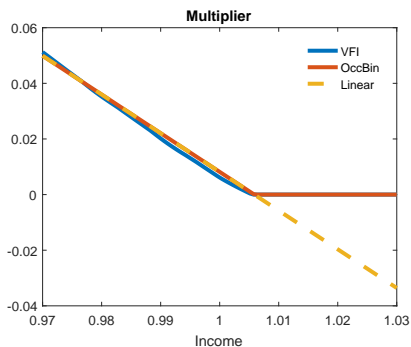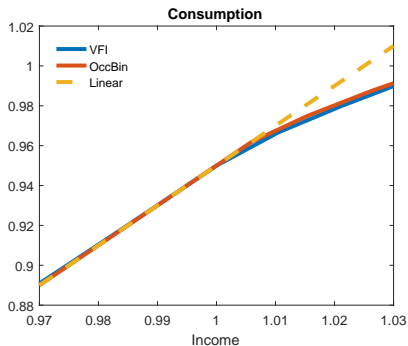
Accuracy of solutions

- Use Euler Equation Errors expressed in terms of consumption.

- VFI is highly accurate. OccBin accuracy is modest errors ($1 per $1000 of consumption). Linear approximation has worst accuracy.

Euler equation errors as a function of debt
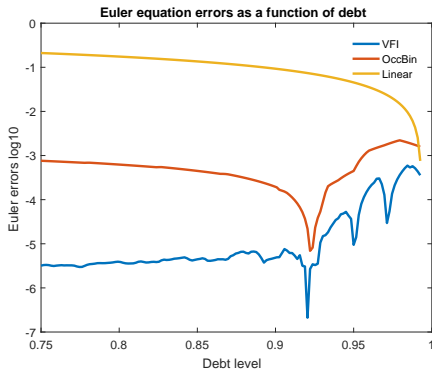
Exact likelihood: Inversion

- In the case of no error $\Omega = 0$ in measurement equation:

$$s_t = h(s_{t-1}, \eta_t; \theta), \quad y_t = g(s_t; \theta)$$

Combining:

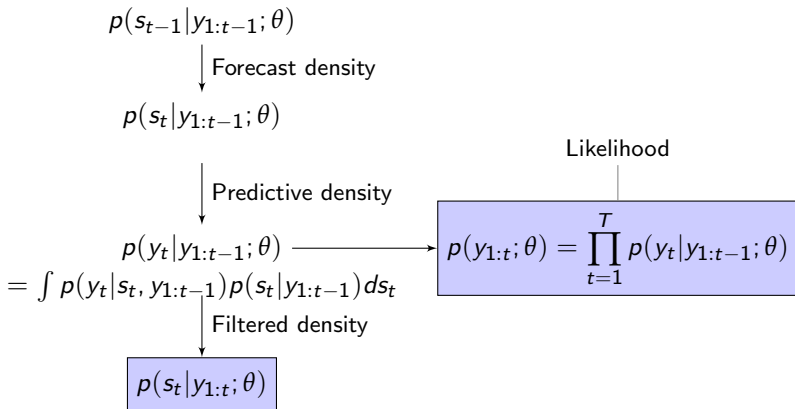$$y_t = f(s_{t-1}, \eta_t; \theta), \quad \eta_t \sim N(0, \Sigma)$$

- Change of variable and prediction-error decomposition yields exact expression for the log-likelihood function:

$$\log(p(y_{1:T}; \theta)) \propto -\frac{T}{2}\log(\det(\Sigma)) - \frac{1}{2}\sum_{t=1}^{T}\eta_t(\theta)'\Sigma^{-1}\eta_t(\theta)$$

$$+ \sum_{t=1}^{T}\log\left(|\det\frac{\partial\eta_t(\theta)}{\partial y_t}|\right)$$

- Number of structural shocks = number of observed variables.

▸ Back

Approximate Likelihood: Particle Filter

A generic filtering problem (Herbst and Schorfheide, 2015):

$$p(s_{t-1}|y_{1:t-1}; \theta)$$

$\Big\downarrow$ Forecast density

$$p(s_t|y_{1:t-1}; \theta)$$

$\Big\downarrow$ Predictive density

Likelihood

$$p(y_t|y_{1:t-1}; \theta) \longrightarrow \boxed{p(y_{1:t}; \theta) = \prod_{t=1}^{T} p(y_t|y_{1:t-1}; \theta)}$$

$$= \int p(y_t|s_t, y_{1:t-1})p(s_t|y_{1:t-1})ds_t$$

$\Big\downarrow$ Filtered density

$$\boxed{p(s_t|y_{1:t}; \theta)}$$

- Implementation of particle filter usually assumes normally distributed measurement error to compute $p(y_t|s_t, y_{1:t-1})$ in predictive density step.

▶ Back

Bias in estimates

	$T = 100$		$T = 500$		$T = 1000$	
	VFI	OCC	VFI	OCC	VFI	OCC
IF	-0.01	0.29	-0.01	0.23	-0.01	0.21
ME1	0.05	0.46	0.04	0.39	0.03	0.35
ME5	0.28	0.91	0.28	0.78	0.27	0.76
ME20	1.47	2.37	1.67	2.55	1.68	2.56

▸ Back

Estimate more parameters

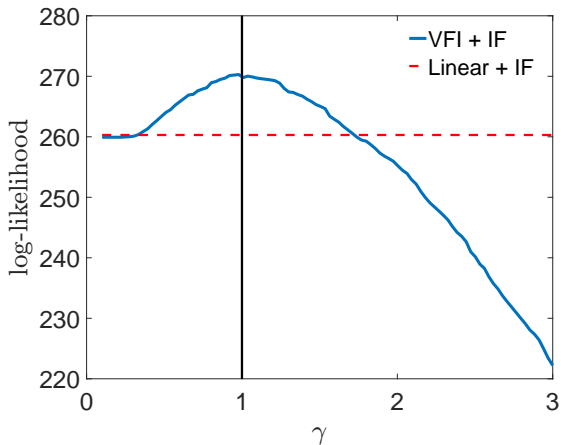- Estimate γ, ρ, σ

Estimation Exercise
MCMC 10,000 draws. VFI, sample=500
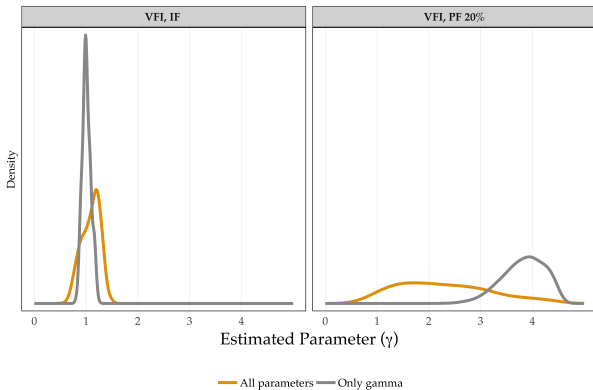
Result 1: More solution error, no filtering error

- Linear + IF

Back

Can the bias go the other way?

- Consider DGP with measurement error: $y_t = g(s_t; \theta) + \zeta_t$

- Particle filter with correctly specified ME gives the correct approximation to the likelihood.

- Econometrician ignores measurement error and uses inversion filter.

- Too little ME relative to the DGP generates a downward bias in γ.

► Back
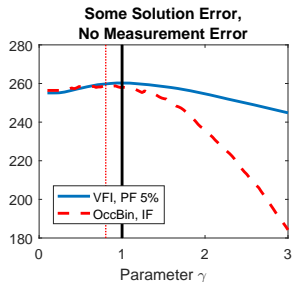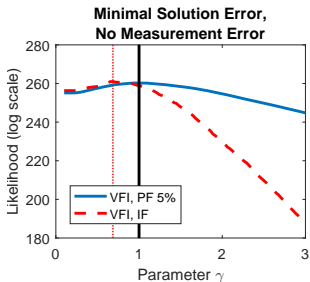
Related Literature

- **Nonlinear DSGE models without occasionally binding constraints.**
  - Fernandez-Villaverde and Rubio-Ramirez(2005, 2007). Andreasen(2008), Aruoba, Bocola and Schorfheide (2017). Farmer (2017).

- **Nonlinear DSGE models with occasionally binding constraints.**
  - Bocola(2016). Gust, Herbst, Lopez-Salido and Smith(2017). Guerrieri-Iacoviello(2015,2017), Aruoba, Cuba-Borda, Schorfheide(2018), Atkinson, Richter and Throckmorton (2018).

- **Measurement error in linearized DSGE models**
  - Canova(2009). Canova, Ferroni, and Matthes (2014). Canova, Ferroni and Matthes (2017).