

Frequency Dependent Risk*

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Abstract

We provide a nonparametric framework for studying state vector dynamics and its associated risk prices. In a setting where the stochastic discount factor (SDF) decomposes into permanent and transitory components, we analyze their contribution to the unconditional asset return premium using frequency domain techniques. We show analytically that the co-spectrum between returns and the SDF only displays frequency dependencies through the state vector. Moreover, we demonstrate that state vector dynamics and its risk prices can be uncovered by studying the covariance between asset returns. Empirically, we find low *and* high-frequency risk to be differentially priced for US equities.

Keywords: Asset Pricing, Factor Models, Nonparametric Measures, Spectral Analysis.

JEL classification: C1, G1, G11, G12

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1 Introduction

Asset pricing models commonly rely on a stochastic discount factor (SDF), whose dynamics is determined by a state vector, \mathbf{X} , comprised of either observable or unobservable components. Examples include the consumption CAPM and the intertemporal CAPM¹ (henceforth, CCAPM and ICAPM), where, in their original form, either consumption growth or a state vector without intertemporal persistence determine the dynamics of the SDF, key macroeconomic variables as well as the pricing of securities such as stocks, bonds, exchange rates, etc. When these models are confronted with the data, however, in different empirical settings, consensus has emerged that they generally fail to describe a range of key characteristics such as the aggregate equity premium and volatility (as well as risk premia in other asset classes), to fit the moments of consumption growth and to price the cross-section of stocks; see, e.g., the thorough reviews in Cochrane (2005) and Munk (2013).

Important extensions of the models have subsequently been developed, including the long-run risk model by Bansal & Yaron (2004), the dynamic rare disaster model by Gabaix (2012), and the regime-switching CCAPM by Lettau, Ludvigson & Wachter (2008). While these models differ greatly in their design, two simple commonalities between them exist; they specify the components of the state vector and they parameterize their dynamics. Specifically, the regime-switching CCAPM and the long-run risk model consider shocks to consumption growth and its volatility as state variables, but imposes different dynamics on such shocks. The disaster model, on the other hand, lets the state vector depend on systematic and stock specific resilience, capturing the recovery of wealth in the wake of a disaster. Despite being comprised of distinct economic drivers and their dynamics being parameterized differently, these models often let shocks to the state vector be very persistent and small in magnitude relative to the permanent innovations of the SDF, making them hard to detect empirically using standard statistical techniques.² These models calibrate and describe aggregate characteristics well, however, their performance in terms of cross-sectional pricing is mixed.³ This suggests that the state vector specifications of the three respective models fall short of fully capturing its dynamics as well as its pricing in the cross-section, raising two important questions: Which key features should asset pricing models have? And what are the dynamic properties of the state vector?

To address these questions, this paper provides a new framework for studying state vector dynamics

¹See, among others, Rubinstein (1976), Breeden & Litzenberger (1978) and Breeden & Litzenberger (1979) as well as Merton (1973) and Campbell (1993) for some early references on the two models.

²For example, and as will be detailed below, the shocks to consumption growth in the long-run risk model have a half-life of approximately 53 months; they are permanent in the regime-switching CCAPM; and the shocks to systematic and stock specific resilience have a half-life of 65 months in the dynamic rare disaster model. These half-lives are not observed in the autocorrelation functions of neither consumption growth nor asset returns.

³Whereas, among others, Parker & Julliard (2005), Bansal, Dittmar & Lundblad (2005), Bansal, Dittmar & Kiku (2007) and Gabaix (2012) provide support for either long-run risk in consumption growth or rare disasters, along with their recovery, as important determinants in matching the moments of key financial variables, Constantinides & Ghosh (2011) and Beeler & Campbell (2012) find reverse conclusions, e.g., that the long-run risk model fails to simultaneously match the moments of aggregate consumption and dividend growth, the unconditional volatility of the aggregate and risk-free return, and to price the cross-section of stock returns. They attribute the shortcoming of the model to the latent state variables, driving the intertemporal dynamics of the economic system, not being sufficiently persistent.

and its associated risk prices, letting, however, the components of the state vector remain unspecified and without parameterizing their dynamics. Specifically, in a discrete time setting where the log-SDF follows an affine jump diffusion (Duffie, Pan & Singleton 2000) with an unknown and unspecified state vector and, thus, decomposes into permanent and transitory components, see, e.g., Hansen, Heaton & Li (2008) and Hansen & Scheinkman (2009), we study the contribution of such components to the unconditional asset return premium using frequency domain techniques. First, we show analytically that the co-spectrum between asset returns and the SDF decomposes into three separate contributions: (1) one from permanent Gaussian shocks; (2) one from permanent non-Gaussian shocks (e.g., disasters); and (3) one from the state vector. This is labeled the *permanent-transitory spectrum decomposition*. Importantly, (3) is the only contribution that varies across frequencies, and whose magnitude and sign are determined by the intertemporal dynamics of the state vector and its associated risk prices. The remaining two contributions are constant across frequencies. Second, we utilize these insights and develop new frequency dependent covariance measures that allow us to study the dynamics of the state vector nonparametrically as well as the risk prices associated with its different components. Third, since the SDF is latent in our setting, we show analytically how to utilize these new covariance measures between asset returns, or portfolios of returns, to uncover such information. This allow us to map out a *frequency term structure of risk* that facilitates better identification of pricing factors. Importantly, even if we utilize known, and observable, risk factors in the literature, such as the market or Fama & French (1993) factors, as proxies for portfolio returns, our decomposition only requires that these factors have non-trivial loadings on components of the state vector for us to learn about their frequency dependencies and associated risk prices.⁴ Hence, conditional on non-zero loadings, our approach is fully nonparametric.⁵ This has a distinct advantage in that we are able to map out which features asset pricing models should have to comply with the implied state vector dynamics in a cross-section of assets and explain why some models are successful. As a by-product of our analysis, we develop spectrum results for four asset pricing models and new analytical results for log-linearized versions of the dynamic rare disaster model and the regime-switching CCAPM.

Our theoretical contributions and framework are related to others in the literature.⁶ First, our spectrum decomposition is related to the permanent-transitory decompositions proposed in Fama & French (1988), Lamoureux & Zhou (1996), Hansen et al. (2008), Hansen & Scheinkman (2009) and Borovicka, Hansen, Hendricks & Scheinkman (2011). They study the pricing of cash flow at different time horizons, with particular emphasis on the long-run, implicitly learning about both the permanent and transitory components of the SDF, which affect holding period returns at different horizons. Second, Calvet & Fisher (2007), Ortu, Tamoni & Tebaldi (2013) and Bandi & Tamoni (2017) provide dynamic decompositions of asset returns, consumption and the price-dividend ratio based on different time domain methods. Furthermore, they study the pricing implications of such decompositions, using

⁴This is ubiquitously satisfied for parametric asset pricing models, e.g., all the ones referenced above.

⁵Of course, nonparametric in this setting refers to the fact that the dynamic properties of the state vector as well as its dimension have been left unspecified. We are still working within the confines of models with a log-affine SDF.

⁶We will briefly explain the differences here and defer details to Sections 3.3 and 5.

both aggregate equity returns and the twenty-five Fama & French (1993) portfolios, demonstrating that by isolating shocks to consumption or the price-dividend ratio with different persistence, this generates to improved model fit. In the process, they provide analyses of persistence-specific betas in the CCAPM model. In a similar spirit, Bandi, Chaudhuri, Lo & Tamoni (2019) decompose the CAPM beta in the frequency domain and demonstrate that by measuring market betas at lower frequencies, this substantially reduces pricing errors for the twenty-five Fama-French portfolios.⁷ Moreover, they show that their spectral factor model reduces portfolio volatility in an asset allocation exercise.

Our framework differs from existing studies in that we use the frequency domain to study the pricing of different components of the state vector *without* working within the confines of a model (CAPM or CCAPM), and we refrain from imposing dynamics on the state vector.⁸ It is fully nonparametric, and it allows us to disentangle and infer the properties and pricing of state vector components by studying the permanent-transitory spectrum decomposition of the one-period equity premium using measures constructed solely from (portfolios of) asset returns. Specifically, we demonstrate how to use frequency decomposed covariance ratios, labelled *frequency risk*, between asset returns to identify dynamic features of the state vector and analyze how these are priced in the cross-section. Hence, our framework is complementary to those in Hansen et al. (2008), Hansen & Scheinkman (2009) and Borovicka et al. (2011) as well as the references above studying persistence-specific betas. Given its nonparametric nature, we may formulate stylized facts about the state vector and its risk prices that asset pricing models will have to comply with. For example, as detailed below, we use our carefully constructed measures to uncover frequency dependent factors related to components of the state vector in a large cross-section of US stocks, demonstrating that these contain important information not captured by standard risk factors and carry significant cross-sectional pricing implications.

Finally, our framework and analysis is related to prior studies in finance utilizing frequency domain methods: Berkowitz (2001), Cogley (2001) and Yu (2012) provide frequency-domain estimators of the CCAPM as well as the long-run risk and habit models and demonstrate that they fit (much) better at lower frequency ordinates; using wavelet techniques, Chincó & Ye (2018) uncover a significant relation between high-frequency movements in trading volume and stock returns, unrelated to firm characteristics; and Dew-Becker & Giglio (2016) decompose risk prices and the propagation of permanent shocks to asset pricing models in the frequency domain and use their framework to quantify the weight various asset pricing models place on certain frequency ranges. As explained above, our aim and framework differ from these important contributions, making our analyses complementary.

In addition to introducing a new theoretical framework for studying dynamics of the state vector and risk prices nonparametrically, we apply our methodology to the (entire) cross-section of US stocks and identify some new and interesting results. Specifically, we calculate the *frequency risk* associated with market returns, the Fama & French (1993) value and size factors, and with the Carhart (1997)

⁷Gençay, Selçuk & Whitcher (2003, 2005) similarly decompose betas at different time scales with, however, the focus of testing frequency-specific versions of the CAPM rather than study how dynamic decompositions aid the pricing of returns at the aggregate level (that is, not the individual components of the returns).

⁸We will, however, provide several examples of dynamic asset pricing models that are embedded in our framework.

momentum factor and analyze the measures using our permanent-transitory spectrum decomposition. First, we document systematic dispersion in excess returns (alphas) across all frequency dependent portfolios. Second, when sorting on frequency risk utilizing market returns, we find a striking *hockey stick pattern* in alphas, with a large positive risk premium commanded by stocks having excess risk at very low frequencies and a smaller, yet still significant, negative risk premium at high frequencies. The excess returns are unspanned by a battery of appraised risk factors and characteristics in the literature. Moreover, the alphas, with economically substantial magnitudes of approximately 6% and -2% per year, demonstrate that our frequency risk concept adds an important dimension to cross-sectional asset pricing. Third, when the frequency risk measure is constructed using an (orthogonalized) value factor, we similarly find a pronounced *hockey stick pattern*. For an orthogonalized size factor, we find a significant and positive risk premium at low frequencies, a negative premium at intermediate frequencies and no significant risk at high frequencies. Importantly, the LF factors extrapolated using the market, value and size portfolio returns are virtually uncorrelated. Hence, by applying our frequency domain framework, we have demonstrated that the state vector has two (or more) persistent transitory components, in addition to a risk factor operating at intermediate frequencies. Fourth, when constructing frequency risk using the momentum factor, we find a significant negative risk premium associated with HF risk and a flat term structure at lower frequencies. Hence, in combination with the mildly correlated HF factor extrapolated using market returns, this allows us to deduce that the state vector has, at least, one dynamic component operating at higher frequencies.

Throughout the empirical analysis, we discuss our findings in relation to the long-run risk model, the dynamic disaster model, the regime-switching CCAPM and our analytical results for the SDF and the equity premium. Whereas these models, indeed, can accommodate LF components in the state vector and their cross-sectional pricing, the models lack the flexibility to capture all dimensions of frequency risk. The nonparametric nature of our framework, thus, suggests to increase the number of transitory components in the SDF, allowing them to have risk prices of opposite sign.

Finally, we note that the importance of low-frequency risk is consistent with the empirical findings in Berkowitz (2001), Cogley (2001), Calvet & Fisher (2007), Yu (2012), Ortu et al. (2013), Dew-Becker & Giglio (2016), Bandi & Tamoni (2017) and Bandi et al. (2019), who document improved model- and/or utility-specification fit, lower pricing errors, tighter risk price estimation and better asset allocation from using low-frequency variation. However, we add to their findings by identifying important components of the state vector nonparametrically using a large cross-section of stocks and by quantifying their associated risk premia. Moreover, we also document significant dimensions of risk in other parts of the frequency domain, besides lower frequencies.

The outline of the paper is as follows. Section 2 introduces the pricing framework, definitions, assumptions and provide examples of asset pricing models. Section 3 introduces the theory of frequency dependent risk and establishes analytical results that lay the foundation for the nonparametric analysis. Section 4 discusses frequency domain implications for different asset pricing models. Section 5 operationalizes the frequency risk concept and provides the empirical analysis. The appendix in

Section A develops additional theory for the long-run-risk model, the dynamic rare disaster model and the regime-switching CAPM, and the one in Section B contains proofs. The following notation is used throughout. For some matrix \mathbf{A} and vector \mathbf{b} , $\|\mathbf{A}\| = \sqrt{\text{Tr}(\mathbf{A}\mathbf{A}')}$ denotes Frobenius norm of \mathbf{A} and $\text{Tr}(\mathbf{A})$ is its trace; $\mathbb{D}(\mathbf{b})$ is a diagonal matrix with elements \mathbf{b} ; and \circ denotes the Hadamard product. For some information filtration \mathcal{F}_t , $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ is the conditional expectation and $\mathbb{V}_t[\cdot] = \mathbb{V}[\cdot | \mathcal{F}_t]$ is the conditional variance. Lastly, we denote the imaginary unit by $i = \sqrt{-1}$ and remark that a standard i may still be used for enumeration.

2 Definitions, Dynamic Setting and Examples

Before introducing a new framework for studying the dynamics of the state vector and its associated risk prices, we describe a general discrete time stochastic discount factor (SDF) setting as well as introduce various frequency domain statistics that will be used throughout. The exposition is purposely given in discrete time since all the economic models, we consider as illustrative examples, were originally designed as such. Moreover, since the estimation is necessarily carried out using discretely sampled data, we avoid taking a stand on, and examine the effects of, how the discretization of a continuous time process affect the properties of our empirical risk measures.

2.1 The Stochastic Discount Factor

First, let \mathbf{X}_t be a d -dimensional vector Markov process and $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ define a filtered probability space, on which all stochastic processes are defined. Then, we adopt a discrete asset pricing framework that is similar to the affine setting of Duffie et al. (2000). Specifically, we stipulate that the discrete time SDF, denoted by S_t , is log-linear with Gaussian and, possibly, Poisson shocks,

$$\Delta \ln S_{t+1} = \mu + \mathbf{F}' \mathbf{X}_t + \mathbf{G}'_t \mathbf{W}_{t+1} + \mathbf{H}' \Delta \mathbf{J}_{t+1}, \quad (1)$$

where $\mathbf{W}_{t+1} \in \mathbb{R}^k$ is a multivariate sequence of standard Gaussian random variables, and $\Delta \mathbf{J}_{t+1} \in \mathbb{R}^k$ is a vector of non-Gaussian shocks. In particular, for the vector $\mathbf{J}_{t+1} = (J_{t+1,1}, \dots, J_{t+1,k})'$, we suppose that $J_{t+1,i} = \sum_{j=1}^{N_{t+1,i}} \psi_{j,i}$, where $N_{t+1,i}$ is a Poisson counting process and $\psi_{j,i}$ is the (random) jump size for the j th jump of the i th component. Moreover, the time-varying intensity of the vector counting process $\mathbf{N}_{t+1} = (N_{t+1,1}, \dots, N_{t+1,k})'$ is given by $\boldsymbol{\lambda}_t$. Before detailing our pricing framework, and relating it to the extant literature, let us impose some formal structure on (1):

Assumption 1 (Non-Gaussian Shocks). *Conditional on \mathcal{F}_t , we have for $i = 1, \dots, k$,*

- (i) $N_{t+1,i}$ is independent of $N_{t+1,j}$ when $i \neq j$;
- (ii) $\psi_{t+1,i}$ is independent with $\mathbb{E}_t[\psi_{t+1,i}] = \varpi_{t,i}$ and $\mathbb{V}_t[\psi_{t+1,i}^2] = \xi_{t,i}$;
- (iii) the components $N_{t+1,i}$ and $\psi_{t+1,j}$ are mutually independent for all i and j .

Assumption 2 (Affine Dependence). Define the finite and fixed parameter matrices $\mathbf{R} \in \mathbb{R}^{k \times k}$, $\mathbf{R}_i \in \mathbb{R}^{k \times k}$, $i = 1, \dots, d$, $\boldsymbol{\lambda}_0 \in \mathbb{R}^k$, $\boldsymbol{\lambda}_1 \in \mathbb{R}^{k \times d}$, $\boldsymbol{\varpi}_0 \in \mathbb{R}^k$, $\boldsymbol{\varpi}_1 \in \mathbb{R}^{k \times d}$, $\boldsymbol{\xi}_0 \in \mathbb{R}^k$, and $\boldsymbol{\xi}_1 \in \mathbb{R}^{k \times d}$. Moreover, let $\boldsymbol{\varpi}_t = (\varpi_{t,1}, \dots, \varpi_{t,k})'$ and $\boldsymbol{\xi}_t = (\xi_{t,1}, \dots, \xi_{t,k})'$, then \mathbf{G}_t , $\boldsymbol{\lambda}_t$, $\boldsymbol{\varpi}_t$ and $\boldsymbol{\xi}_t$ are assumed to be affine functions of the state vector \mathbf{X}_t , in particular,

$$\mathbf{G}_t \mathbf{G}_t' = \mathbf{R} + \sum_{i=1}^d \mathbf{R}_i X_{t,i}, \quad \boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \mathbf{X}_t, \quad \boldsymbol{\varpi}_t = \boldsymbol{\varpi}_0 + \boldsymbol{\varpi}_1 \mathbf{X}_t \quad \text{and} \quad \boldsymbol{\xi}_t = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 \mathbf{X}_t.$$

First, note that this framework generalizes the equivalent settings in Alvarez & Jermann (2005) and Hansen et al. (2008), who impose a similar structure on the SDF, but with several simplifying assumptions. For example, Hansen et al. (2008) impose coefficient restrictions $\boldsymbol{\lambda}_t = \mathbf{0}$ and $\mathbf{R}_i = \mathbf{0}$, for all t and $i = 1, \dots, d$, in addition to first-order vector autoregression (VAR) for \mathbf{X}_t ,

$$\mathbf{X}_{t+1} = \boldsymbol{\Phi}_x \mathbf{X}_t + \mathbf{G}_x \mathbf{W}_{t+1}, \tag{2}$$

where the $d \times d$ matrix $\boldsymbol{\Phi}_x$ has eigenvalues that are strictly smaller than one, thus making the state vector, \mathbf{X}_{t+1} , a stationary sequence. Hence, we generalize their framework by accommodating stochastic volatility in the SDF, non-Gaussian shocks as well as by avoiding parametric assumptions on the dynamics of state vector. As will be explicated below, we only impose mild regularity conditions on the spectral density of \mathbf{X}_{t+1} , allowing for state variables that have excess low, business cycle or high frequency variation. Second, the structure of the SDF in (1) has the familiar permanent-transitory decomposition from the corresponding continuous time settings of, e.g., Hansen & Scheinkman (2009) and Hansen (2012). Specifically, as in their case, the linear trend is captured by μ , the martingale component by $\mathbf{G}_t' \mathbf{W}_{t+1} + (\Delta \mathbf{J}_{t+1} - \mathbb{E}_t[\Delta \mathbf{J}_{t+1}])$, and the transitory “stationary” component by the composite term $\mathbf{F}' \mathbf{X}_t + \mathbb{E}_t[\Delta \mathbf{J}_{t+1}]$. It is important to note, however, that we do not necessarily require the latter to be a stationary process in the statistical sense, allowing, among others, \mathbf{X}_t to contain elements with dynamics described by Markov regime switching processes. This feature is important since, in conjunction with the non-Gaussian shocks, it allow us to include the dynamic rare disaster model and the regime-switching CCAPM, both described below, in our framework.

2.2 Frequency Domain Definitions

We rely on frequency domain techniques to decompose and study exposure to state vector, or factor, risk. Let \mathbf{y}_t and \mathbf{x}_t be compatible vectors, for which

$$\mathbf{C}_{yx}(h) = \text{Cov}(\mathbf{y}_t, \mathbf{x}_{t-h}), \quad \mathbf{f}_{yx}(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathbf{C}_{yx}(h) e^{-i\lambda h}, \tag{3}$$

for $i = \sqrt{-1}$ and frequency λ , denote their cross-autocovariance function and co-spectral density, respectively. Now, suppose \mathbf{y}_t and \mathbf{x}_t are observed at times $t = 1, \dots, T$ and let $\lambda_j = 2\pi j/T$ denote

the Fourier frequencies, then, it follows from Parseval's theorem that

$$\mathbf{C}_{yx}(0) = 2\pi \int_0^\infty \mathbf{f}_{yx}(\lambda) d\lambda \simeq \frac{2\pi}{T-1} \sum_{j=1}^{T-1} \Re(\mathbf{f}_{yx}(\lambda_j)) \quad (4)$$

with $\Re(\cdot)$ denoting the real part of its argument. The last approximation holds using Riemann integration, leaving an error that is generally of order $O(T^{-1})$. Importantly, and as will be detailed below, this decomposition provides a unique lens for studying dynamic contributions to an otherwise static covariance measure. Moreover, to ensure that our frequency dependent decompositions are well-defined, we require the following assumption on the state vector, \mathbf{X}_t .

Assumption 3 (State vector). *The co-spectrum of \mathbf{X}_t satisfies $\| \int_0^\infty \mathbf{f}_{XX}(\lambda) d\lambda \| < \infty$.*

The high-level integrability condition on $\mathbf{f}_{XX}(\lambda)$ is rather general, allowing us to avoid taking a parametric stand on the exact dynamics of the state vector. For example, the condition is satisfied by autoregressions such as (2) as well as Markov regime switching processes, and by cases where the persistence of (a subset of) the state vector is described using persistent fractionally integrated or local-to-unity processes. We provide examples of dynamic processes and asset pricing models that satisfy Assumption 3 below. Importantly, by remaining agnostic about \mathbf{X}_t , it not only lets us embed many existing models in our framework, we establish nonparametric evidence of state vector features that can guide future developments of asset pricing models. Of course, the asset pricing setting still have to comply with the log-affine structure of the SDF as well as Assumptions 1 and 2.

2.3 Examples of Asset Pricing Models

The general theory is illustrated throughout using four different asset pricing models, which are widely appraised in the literature. Before proceeding, however, we fix some notation. Let δ be the subjective time preference rate, γ the relative risk aversion, ψ is the elasticity of intertemporal substitution, and $\theta = (1 - \gamma)/(1 - 1/\psi)$. Moreover, let c_t and r_t^w be the log-consumption and log-return on the wealth portfolio, respectively, then, for an individual with Epstein & Zin (1989) preferences, we can write the SDF as,

$$\Delta \ln S_{t+1} = -\delta\theta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{t+1}^w, \quad (5)$$

noting that corresponding SDF for time-additive power utility is recovered for $\theta = 1$. The stochasticity in (5) is, thus, driven by Δc_{t+1} and r_{t+1}^w , which, in turn, may be determined by a vector of state variables, Gaussian and non-Gaussian innovations. We give such examples next.⁹

Example 1. *Suppose that $\Delta c_{t+1} \sim N(\mu_c, \sigma_c^2)$, $r_{t+1}^w \sim N(\mu_w, \sigma_w^2)$ and $\text{Cov}[\Delta c_{t+1}, r_{t+1}^w] = \sigma_{c,w}$, then*

⁹The first two model representations are straightforward adaptations and re-parameterizations of existing results for the log-normal CCAPM and long-run risk model, respectively, and are, thus, stated without proof.

we may represent the SDF on the form (1), with

$$\mu = -\delta\theta - \frac{\theta}{\psi}\mu_c - (\theta - 1)\mu_w, \quad \mathbf{R} = \begin{pmatrix} \sigma_c^2(\theta/\psi)^2 & \sigma_{c,w}(\theta/\psi)(\theta - 1) \\ \sigma_{c,w}(\theta/\psi)(\theta - 1) & \sigma_w^2(\theta - 1)^2 \end{pmatrix},$$

the Brownian motion, \mathbf{W}_t , being 2-dimensional and the remaining parameter matrices as well as the state vector \mathbf{X}_t contain only zeros. In this case, Assumption 3 is trivially satisfied.

Example 2. In the long-run risk (LRR) model of Bansal & Yaron (2004), the aggregate return on the wealth portfolio is given by $r_{t+1}^w \simeq \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}$, where z_t is the log-price-consumption ratio, and, with \bar{z} being its average, $\kappa_1 = e^{\bar{z}}/(1 + e^{\bar{z}}) \in (0, 1)$ and $\kappa_0 = \ln(1 + e^{\bar{z}}) - \kappa_1 \bar{z}$.¹⁰ Next, the growth in (log-)consumption can be written as $\Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1}$, where

$$x_{t+1} = \rho_x x_t + \varphi_x \sigma_t e_{t+1}, \quad \sigma_{t+1}^2 = \sigma^2 + \nu(\sigma_t^2 - \sigma^2) + \sigma_v v_{t+1},$$

and the shocks η_{t+1} , e_{t+1} and v_{t+1} are independent and standard Gaussian. Moreover, the persistence parameters satisfy $0 \leq \rho_x < 1$ and $0 \leq \nu < 1$, albeit close to unity.¹¹ To solve the model, z_t is stipulated to be affine in the state variables $z_{t+1} = A_0 + A_1 x_{t+1} + A_2 \sigma_{t+1}^2$, such that we may write the SDF on the form (1) with state vector $\mathbf{X}_t = (x_t, \sigma_t^2)'$, Gaussian shocks $\mathbf{W}_t = (\eta_t, e_t, v_t)'$, and

$$\mu = (\theta - 1)q - \delta - \frac{\mu_c}{\psi}, \quad q = -\delta + (1 - 1/\psi)\mu_c + \kappa_0 + A_0(\kappa_1 - 1) + \kappa_1 A_2 \sigma^2(1 - \nu),$$

$$\mathbf{F} = \begin{pmatrix} -1/\psi \\ (\theta - 1)A_2(\kappa_1 \nu - 1) \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_v^2 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} \lambda_\eta^2 & 0 & 0 \\ 0 & \lambda_e^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda_v = (\theta - 1)\kappa_1 A_2 \sigma_v$, $\lambda_\eta = -\gamma$, and $\lambda_e = (\theta - 1)\kappa_1 A_1 \varphi_e$; the constants A_0 , A_1 and A_2 are provided in Appendix A.1; and where the remaining parameter matrices in (1) contain zeros.

Example 3. We further illustrate our frequency domain theory for modified, adapted to fit the SDF structure in (1), and log-linearized versions of the dynamic rare disaster model of Gabaix (2012) as well as the regime-switching CCAPM of Lettau et al. (2008). As these, however, require separate introductions, intuition and theoretical developments, we defer details to Appendices A.2 and A.3 for simplicity of exposition. We emphasize here, however, that all results in the appendix are new.

3 The Theory of Frequency Dependent Risk

This section introduces the notion of frequency dependent risk for the broad class of asset pricing models (1). Specifically, we will show that the study of the *unconditional equity premium*, including

¹⁰This log-linear approximation is known to be “very good”, even in the presence of price bubbles, see, e.g., Engsted, Quistgaard & Tanggaard (2012) and Munk (2013), and it is exact if $\psi = 1$, see Hansen et al. (2008)

¹¹Note that Bansal & Yaron (2004) report estimates of the persistence parameters $\rho_x = 0.979$ and $\nu = 0.987$, suggesting that shocks to consumption growth and its volatility have half-lives exceeding 52 months.

the relation of asset returns to various pricing factors, depend crucially on the stationary component of the respective models, $\mathbf{F}'\mathbf{X}_t$, whereas the permanent components of the SDF carry no temporal information about the state vector. We introduce a new frequency decomposition of pricing covariances, which will have implications for the estimation and interpretation of state vector dynamics, cross-sectional asset pricing, and risk-based investments, as will be shown in our empirical analysis below. Finally, we introduce two new measures to study the proposed frequency decomposition empirically. Despite providing examples from parametric models in Section 4, it is important to note that our proposed tools are nonparametric and apply generally.

3.1 A Frequency Decomposition of the Equity Premium

Let us denote by $r_{i,t}$ the time- t log-return on asset i and, similarly, the one-period risk-free interest rate by $r_{f,t}$, then we may generally write the unconditional excess return as

$$\mathbb{E}[r_{i,t+1} - r_{f,t}] = \mathbf{C}_{i\Delta \ln S}(0) + \text{Jensen's inequality correction.} \quad (6)$$

Motivated by this representation, and as alluded to in (4), we will introduce a new and general frequency decomposition of $\mathbf{C}_{i\Delta \ln S}(0)$ to study the proficiency and implications of different asset pricing models.¹² To this end, we first establish the unconditional spectrum properties for the SDF:

Theorem 1. *Suppose that the setup of Section 2.1 and the regularity conditions in Assumptions 1-3 hold. Moreover, let $\varrho_{t+1} = \Delta \ln S_{t+1}$, $g_{t+1} = \mathbf{G}'_t \mathbf{W}_{t+1}$, and $\ell_{t+1} = \mathbf{H}' \Delta \mathbf{J}_{t+1}$ then it follows that*

- (a) $\mathbf{C}_{gg}(0) = \mathbb{E}[\text{Tr}(\mathbf{G}_t \mathbf{G}'_t)] < \infty$ and $\mathbf{C}_{\ell\ell}(0) = \mathbf{H}' (\mathbb{E}[\mathbb{D}(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t \circ \boldsymbol{\lambda}_t) \circ \mathbb{D}(\boldsymbol{\xi}_t)]) \mathbf{H} < \infty$,
- (b) $\mathbf{f}_{\varrho\varrho}(\lambda) = (\mathbf{C}_{gg}(0) + \mathbf{C}_{\ell\ell}(0)) / (2\pi) + \mathbf{F}' \mathbf{f}_{XX}(\lambda) \mathbf{F}$,
- (c) $\mathbf{C}_{\varrho\varrho}(0) = \mathbf{C}_{gg}(0) + \mathbf{C}_{\ell\ell}(0) + \mathbf{F}' (2\pi \int_0^\infty \mathbf{f}_{XX}(\lambda) d\lambda) \mathbf{F}$.

Theorem 1 provides two key insights. First, spectrum contributions of g_{t+1} and ℓ_{t+1} to the log-innovations in the SDF, ϱ_{t+1} , are constants, implying that if the SDF only has permanent components, its spectrum will be constant across frequencies, λ . This follows directly from the Gaussian and non-Gaussian shocks being conditionally independent across time. However, if the SDF has a transitory part, then we expect to see differential variation in its spectral density $\mathbf{f}_{XX}(\lambda)$ at low and high frequencies, depending on the dynamics of the state vector \mathbf{X}_t .¹³ We shall refer to the representation in Theorem 1(b) as the *permanent-transitory spectrum decomposition*. Second, as seen in Theorem 1(c), this carries implications for the structure of the SDF variance, $\mathbf{C}_{\varrho\varrho}(0)$, which is similarly composed of contributions from factors that may be either persistent or short-lived (or both) as well as from permanent components of the SDF. The relative importance of each component conveys information about the underlying sources of economic uncertainty and, as a result, about how parametric asset pricing

¹²Note that we write $\mathbf{C}_{i\Delta \ln S}(0)$ instead of $\mathbf{C}_{r_i \Delta \ln S}(0)$ for notational simplicity.

¹³Of course, if \mathbf{X}_t is independent over time, then its spectrum will be constant across λ , too.

models should be designed. It is important to note that Theorem 1 applies to latent dependencies in the unconditional risk premium, which is different from modeling time-variation in the conditional risk premium, see, among others, Bollerslev, Engle & Wooldridge (1988), Harvey (1989), Jagannathan & Wang (1996) and Lettau & Ludvigson (2010). Our framework also accommodates time-variation in the conditional equity premium through the stochastic volatility matrix, \mathbf{G}_t , and the time-varying moments of the non-Gaussian shocks in Assumption 1.

Next, we study the co-spectrum $\mathbf{C}_{i\Delta \ln S}(0)$. However, before stating equivalent results, we impose mild structure on $r_{i,t}$, noting that this is satisfied by all four models in Section 2.3:

Assumption 4 (Asset Returns). $r_{i,t+1} = \mu_i + \mathbf{F}'_i \mathbf{X}_t + \mathbf{G}'_{i,t} \mathbf{W}_{t+1} + \mathbf{H}'_i \Delta \mathbf{J}_{t+1}$ where $\mathbf{G}_{i,t}$ satisfies regularity conditions that are similar to those stated in Assumption 2 for \mathbf{G}_t .

Theorem 2. Suppose that the setup of Theorem 1 and the regularity conditions in Assumptions 1-4 hold. Moreover, let $g_{i,t+1} = \mathbf{G}'_{i,t} \mathbf{W}_{t+1}$, and $\ell_{i,t+1} = \mathbf{H}'_i \Delta \mathbf{J}_{t+1}$ then it follows that

$$(a) \mathbf{C}_{g_i g}(0) = \mathbb{E}[\text{Tr}(\mathbf{G}_{i,t} \mathbf{G}'_t)] < \infty \text{ and } \mathbf{C}_{\ell_i \ell}(0) = \mathbf{H}'_i (\mathbb{E}[\mathbb{D}(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t \circ \boldsymbol{\lambda}_t) \circ \mathbb{D}(\boldsymbol{\xi}_t)]) \mathbf{H} < \infty,$$

$$(b) \mathbf{f}_{i\varrho}(\lambda) = (\mathbf{C}_{g_i g}(0) + \mathbf{C}_{\ell_i \ell}(0)) / (2\pi) + \mathbf{F}'_i \mathbf{f}_{XX}(\lambda) \mathbf{F},$$

$$(c) \mathbf{C}_{i\varrho}(0) = \mathbf{C}_{g_i g}(0) + \mathbf{C}_{\ell_i \ell}(0) + \mathbf{F}'_i (2\pi \int_0^\infty \mathbf{f}_{XX}(\lambda) d\lambda) \mathbf{F}.$$

Theorem 2 shows that the spectrum decomposition of the SDF readily carries over to the pricing of returns. That is, the unconditional pricing of $r_{i,t}$ will depend on permanent components, changing the level of returns proportional to the factor loadings $\mathbf{G}_{i,t}$ and \mathbf{H}_i , as well as on the transitory state vector, \mathbf{X}_t , whose impact materializes at different frequencies, e.g., long-term shocks to consumption growth generate a larger impact from $\mathbf{f}_{XX}(\lambda)$ at smaller λ in consumption-based asset pricing models. Hence, by decomposing $\mathbf{C}_{i\varrho}(0)$ across frequencies, we can study the properties of \mathbf{X}_t , thereby testing features of asset pricing models, without taking a parametric stand on their dynamics.

An important implication of Theorem 2 is that we can use the log-affine structure of the SDF in conjunction with Assumption 4 to decompose bivariate return covariances as,

$$\mathbf{C}_{ib}(0) = \mathbf{C}_{g_i g_b}(0) + \mathbf{C}_{\ell_i \ell_b}(0) + \mathbf{F}'_i \left(2\pi \int_0^\infty \mathbf{f}_{XX}(\lambda) d\lambda \right) \mathbf{F}_b, \quad (7)$$

for two assets i and b . Hence, without even knowing, or imposing assumptions and structure on, the dynamics of the SDF, we may utilize frequency dependent covariance measures between two assets to uncover features of the state vector, \mathbf{X}_t . A key example of such measures, which we take advantage of in the empirical analysis below, is to decompose covariances between returns from a cross-section of stocks and the return on the market portfolio, $r_{M,t}$, thereby using partitions of CAPM-style covariances to draw deductive inference about the implied dynamics of \mathbf{X}_t . That is, the combined use of spectrum techniques and (7) may reveal different dynamic components of the state vector, operating at different frequencies. Moreover, since all returns are affine functions of SDF components,

this principle generalizes to other factor portfolio returns, such as Fama & French (1993), since these are constructed as linear combinations of individual assets. If the risk factors in multi-factor models carry orthogonal state vector information, then frequency dependent decompositions of $\mathbf{C}_{ib}(0)$ using alternative baseline factor portfolio returns for asset b will reveal different features of \mathbf{X}_t , thus aiding in providing a detailed description of its structure. In order to identify the latter, it suffices to have a loading matrix for the selected baseline asset, \mathbf{F}_b , which has, at least, one nontrivial element. This is, indeed, the case for the market return in the CCAPM, the long-run risk model, the dynamic rare disaster model and the regime-switching CCAPM as shown by Bansal & Yaron (2004), Gabaix (2012), Lettau et al. (2008) and Propositions 1 and 2 in Appendix A. Conditional on nontrivial loadings, the recovery of state vector features from the decomposition in (7) is nonparametric. Moreover, as demonstrated below, if the dynamics driving the factors operate at different frequencies, then the utilization of frequency decompositions may give new insights into cross-sectional asset pricing.

The permanent-transitory spectrum decomposition in (7) provides the theoretical foundation behind our empirical framework for studying the implied state vector dynamics and its risk pricing. Specifically, we operationalize the former by proposing two frequency dependent covariance measures and a framework for making deductive inference about dynamic components of \mathbf{X}_t .

3.2 The Frequency Term Structure of Risk

First, based on the frequency decompositions in (4) and (7), we propose to isolate certain frequency ranges implicit in the unconditional covariance measure,

$$\begin{aligned} \mathbf{C}_{ib}(\vartheta_1, \vartheta_2) &\equiv \frac{2\pi}{T} \sum_{j=\vartheta_1}^{\vartheta_2} \Re(\mathbf{f}_{ib}(\lambda_j)) \\ &= (\mathbf{C}_{g_i g_b}(0) + \mathbf{C}_{\ell_i \ell_b}(0)) \left(\frac{\vartheta_2 - \vartheta_1}{T} \right) + \mathbf{F}'_i \left(\frac{2\pi}{T} \sum_{j=\vartheta_1}^{\vartheta_2} \Re(\mathbf{f}_{XX}(\lambda_j)) \right) \mathbf{F}_b, \end{aligned} \quad (8)$$

where $1 \leq \vartheta_1 < \vartheta_2 \leq T$, to study the properties of the state vector, \mathbf{X}_t , and its associated risk prices. To facilitate interpretation, and understanding, of this measure, let us consider the CAPM ($b = M$) and suppose \mathbf{X}_t is one-dimensional. Moreover, we let the factor loadings \mathbf{F}_i and \mathbf{F}_M have the same sign (+/-). Then, by letting $\vartheta_1 < \vartheta_2 < \vartheta_3 < \vartheta_4$ with $\vartheta_2 - \vartheta_1 = \vartheta_4 - \vartheta_3$, we can make nonparametric statements about the state variable dynamics based on the relative magnitudes of our frequency dependent covariance measures $\mathbf{C}_{iM}(\vartheta_1, \vartheta_2)$ and $\mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$:

$$\begin{cases} \mathbf{C}_{iM}(\vartheta_1, \vartheta_2) > \mathbf{C}_{iM}(\vartheta_3, \vartheta_4) & \text{LF component more important than HF,} \\ \mathbf{C}_{iM}(\vartheta_1, \vartheta_2) = \mathbf{C}_{iM}(\vartheta_3, \vartheta_4) & \text{LF component equally important to HF,} \\ \mathbf{C}_{iM}(\vartheta_1, \vartheta_2) < \mathbf{C}_{iM}(\vartheta_3, \vartheta_4) & \text{LF component less important than HF,} \end{cases} \quad (9)$$

where, in this simple setup, we assume that $(\vartheta_1, \vartheta_2)$ and $(\vartheta_3, \vartheta_4)$ correspond to Fourier frequencies capturing low-frequency (LF) and high-frequency (HF) movements in \mathbf{X}_t , respectively. These deductions follow since the permanent components in (8) cancel and the inequalities are solely determined by the mass of $\mathbf{f}_{XX}(\lambda_j)$ across different frequencies, $\lambda_j, j = \vartheta_1, \dots, \vartheta_2$.

If \mathbf{X}_t is multi-dimensional, similar deductions can be made, although one has to be careful with the interpretation. In this case, the relative magnitudes of $\mathbf{C}_{iM}(\vartheta_1, \vartheta_2)$ and $\mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$ will depend on the dynamics of \mathbf{X}_t , including the persistence of the components and the magnitudes of their innovations, as well as on the loadings \mathbf{F}_i and \mathbf{F}_M . For example, if $\mathbf{C}_{iM}(\vartheta_1, \vartheta_2) > \mathbf{C}_{iM}(\vartheta_3, \vartheta_4) > 0$, we can infer that there is, at least, one important LF component to the state vector, which has a loading of the same sign. If, on the other hand, $0 > \mathbf{C}_{iM}(\vartheta_1, \vartheta_2) > \mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$, then there is, at least, one important HF component with a loading of opposite sign. Let us write $\min \mathcal{F}_{iM} = \min(\mathbf{C}_{iM}(\vartheta_1, \vartheta_2), \mathbf{C}_{iM}(\vartheta_3, \vartheta_4))$ as well as $\max \mathcal{F}_{iM} = \max(\mathbf{C}_{iM}(\vartheta_1, \vartheta_2), \mathbf{C}_{iM}(\vartheta_3, \vartheta_4))$, then the indentifying deductions in the multi-dimensional state vector and two-frequency split case, using the market return as a single baseline asset, may be summarized by the following 2×2 table:

	$\mathbf{C}_{iM}(\vartheta_1, \vartheta_2) > \mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$	$\mathbf{C}_{iM}(\vartheta_1, \vartheta_2) < \mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$
$\min \mathcal{F}_{iM} > 0$	Important LF component with loading of the same sign	Important HF component with loading of the same sign
$\max \mathcal{F}_{iM} < 0$	Important HF component with loading of the opposite sign	Important LF component with loading of the opposite sign

This analysis may easily be extended to multiple frequency bins to map the dynamic behavior of the components of \mathbf{X}_t and their associated loadings \mathbf{F}_i and \mathbf{F}_M in greater detail. For example, one could choose to split the spectrum into HF, LF and business cycle frequencies.

In addition to nonparametric statements about the dynamic features of the state vector as well as factor loadings, (8) can reveal how different components of \mathbf{X}_t are priced cross-sectionally. Again, suppose for simplicity that \mathbf{X}_t is one-dimensional and we have used (9) to deduce that LF variation is more important than HF, then we can make further deductions of the form,

$$\mathbf{C}_{iM}(\vartheta_1, \vartheta_2) - \mathbf{C}_{iM}(\vartheta_3, \vartheta_4) > \mathbf{C}_{jM}(\vartheta_1, \vartheta_2) - \mathbf{C}_{jM}(\vartheta_3, \vartheta_4) \implies \mathbf{F}_i > \mathbf{F}_j, \quad (10)$$

suggesting that asset i assign a higher risk price than asset j to the transitory component. On the other, if there is excess HF variation, then the inequality in (10) implies $\mathbf{F}_i < \mathbf{F}_j$. Thus, for each deduction of the dynamic properties of \mathbf{X}_t , the relative magnitudes of the frequency dependent covariance measures across assets speak directly to the cross-sectional pricing implications of the state vector. The interpretation, as above, becomes more subtle if the state vector is multi-dimensional. In this case, comparisons such as (10) in conjunction with the 2×2 table may be utilized to deduce the cross-sectional pricing. Let $\mathbf{R}_{iM}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \mathbf{C}_{iM}(\vartheta_1, \vartheta_2) - \mathbf{C}_{iM}(\vartheta_3, \vartheta_4)$, and \mathcal{Q}_{hk} with $h, k = \{1, 2\}$ correspond to the four quadrants above, then the relative risk prices for the multi-dimensional state vector components and two-frequency split may be determined as follows:

	$\mathbf{R}_{iM}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) > \mathbf{R}_{jM}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$	$\mathbf{R}_{iM}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) < \mathbf{R}_{jM}(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$
\mathcal{Q}_{11}	$\mathbf{F}_i > \mathbf{F}_j$	$\mathbf{F}_i < \mathbf{F}_j$
\mathcal{Q}_{12}	$\mathbf{F}_i < \mathbf{F}_j$	$\mathbf{F}_i > \mathbf{F}_j$
\mathcal{Q}_{21}	$\mathbf{F}_i > \mathbf{F}_j$	$\mathbf{F}_i < \mathbf{F}_j$
\mathcal{Q}_{22}	$\mathbf{F}_i < \mathbf{F}_j$	$\mathbf{F}_i > \mathbf{F}_j$

The table demonstrates that the permanent-transitory frequency decomposition and the resulting covariance measure (8) may be used to unveil cross-sectional pricing of HF and LF components of the state vector. This information is lost when aggregating across all frequency ordinates.

Whereas the measure in (8), indeed, provide frequency-specific information about \mathbf{X}_t as well as the factor loadings \mathbf{F}_i and \mathbf{F}_j , one cannot readily compare cases $\vartheta_2 - \vartheta_1 \neq \vartheta_4 - \vartheta_3$ since they assign different weights on the constant contribution from the permanent components of the SDF. Hence, we propose a simple modification

$$\begin{aligned} \mathbf{T}_{ij}(\vartheta_1, \vartheta_2) &\equiv \frac{T}{\vartheta_2 - \vartheta_1} \mathbf{C}_{ij}(\vartheta_1, \vartheta_2) \\ &= \mathbf{C}_{g_i g_j}(0) + \mathbf{C}_{\ell_i \ell_j}(0) + \mathbf{F}'_i \left(\frac{2\pi}{\vartheta_2 - \vartheta_1} \sum_{j=\vartheta_1}^{\vartheta_2} \Re(\mathbf{f}_{XX}(\lambda_j)) \right) \mathbf{F}_j, \end{aligned} \quad (11)$$

which corrects such limitations. This rescaled covariance measure may readily be use to study frequency dependencies in the state vector, factor loadings and cross-sectional pricing over a flexible set of time horizons, which we label the *frequency term structure of risk*.

It is important to emphasize one limitation of our framework when utilizing only a single baseline portfolio to study state vector dynamics and its risk pricing. Suppose that there are two independent and very persistent AR(1) components in the state vector, but with different factor loadings, then our frequency decomposed methods (8) and (11) will, indeed, detect that there is excess LF dependence, but they will measure the collective pricing from the two components. In this case, we may utilize multiple “baseline” portfolios. If these load non-trivially on some components of the state vector and trivially on others, this will allow us to learn about multiple components with similar spectral behavior and their associated risk prices. In our empirical application, we consider the market, the Fama & French (1993) value and size, as well as the Carhart (1997) momentum factors.

3.3 Relation to the literature

The proposed framework for studying the unconditional equity premium is related to, but distinct from, the frequency domain decomposition in Dew-Becker & Giglio (2016) as well as the permanent-transitory decompositions of Alvarez & Jermann (2005) and Hansen et al. (2008). To simplify the exposition and highlight the relevant differences, suppose, in this section, that the setting of Hansen et al. (2008) apply, that is, where the state vector obey the dynamics in (2). The comparison readily extends to our generalized setting, at the expense of more complicated notation.

Let $\Delta\mathbb{E}_{t+1} = \mathbb{E}_{t+1} - \mathbb{E}_t$ denote changes in expectations, then Dew-Becker & Giglio (2016) study frequency decompositions of

$$\Delta\mathbb{E}_{t+1} \ln S_{t+1} = \mathbf{G}'\mathbf{W}_{t+1}, \quad \mathbf{G}\mathbf{G}' = \mathbf{R},$$

demonstrating that the preference structures of asset pricing models imply different decompositions of the risk prices, \mathbf{G} , for the vector of shocks, \mathbf{W}_{t+1} . In particular, for preferences that can be solved forward (e.g., habit formation or recursive utility), they show that $\mathbf{G} = \sum_{j=0}^{\infty} \mathbf{z}_j \mathbf{r}_j$, whose component matrices, \mathbf{z}_j and \mathbf{r}_j , of dimension $k \times k$ contain shock weights and impulse response functions of preferences, respectively, has an equivalent spectral representation. Moreover, the authors quantify the weight various asset pricing models place on frequency ranges and analyze the implications of frequency-dependent risk prices of shocks empirically. Rather than studying the intertemporal properties of risk prices for the martingale, or permanent, component of the SDF, our framework may be used to decompose the risk prices and exposures to transitory components of state vector, nonparametrically, and to study cross-sectional pricing implications. In our setting, the frequency decomposition is useful for obtaining identification of state vector risk that may not be visible in standard unconditional measures, using deductions as described in the previous section.

Hansen et al. (2008) study the pricing of cash flow at different horizons, with particular emphasis on long-run implications.¹⁴ In particular, let D_t^* be the cash flow growth process, modeled as a random walk with drift, and $f(\mathbf{X}_t)$ be a claim to (a subset of) the state vector. Then, they demonstrate that observed cash flow decomposes as $D_t = D_t^* f(\mathbf{X}_t)$, the long-run risk-return trade-off is invariant to the specification f , and that the transient component affect asset valuation, but not the implied risk prices in the (long-horizon) limit.¹⁵ However, when studying risk premia dynamics over alternative, and multiple, horizons, the expected return reflects both risk exposure and risk prices associated with the transitory component $f(\mathbf{X}_t)$ as well as the permanent component. Hence, by quantifying risk premia for several horizons, these decompositions demonstrate that the term structure convey information about the transitory state vector and its temporal impact on asset valuation; see also Borovicka et al. (2011). Factors that are short-lived will impact shorter holding horizon returns, and more persistent factors will affect both short- and long-horizon holding returns. Thus, the one-period expected return, whose composition is often studied within the confines of an asset pricing model, e.g., Bansal & Yaron (2004) and Campbell & Vuolteenaho (2004), will be some composite of risk factors and their pricing, which is not easily disentangled without parametric assumptions on the SDF. We provide a framework that facilitates the analysis of risk premia decompositions of expected holding-period returns for all horizons (not just for one-period returns, although this is our focus in (6)), in addition to conveying

¹⁴Using a similar decomposition of the SDF, Alvarez & Jermann (2005) derive a lower bound for the volatilities of its permanent and transitory components and show that the former is considerably more volatile than the latter, highlighting the importance of including a permanent component in the innovations to consumption-based models.

¹⁵The contributions of Hansen & Scheinkman (2009), Hansen (2012) and Qin & Linetsky (2017) show that such decompositions and long-run risk-return insights hold under very general conditions on the SDF. Lettau & Wachter (2007) provide similar results for the permanent component of the SDF and use them to explain the value premium.

information about state vector dynamics. In particular, we provide frequency domain procedures to nonparametrically identify the risk exposures associated with dynamic components of the state vector, utilizing their different persistence properties, as well as their pricing.

Finally, Calvet & Fisher (2007), Ortu et al. (2013), and Bandi & Tamoni (2017) provide dynamic decompositions of asset returns, consumption growth, and price-dividend ratios based on different time domain methods. In particular, whereas Calvet & Fisher (2007) stipulate that dividends and returns are priced using a Markov-Switching stochastic volatility model, Ortu et al. (2013), and Bandi & Tamoni (2017) analyze the dynamics of consumption and CCAPM betas using extended Wold decompositions, thereby providing improved inference for the asset pricing implications of different components of the state vector (consumption, in their setting). Similarly, Bandi et al. (2019) decompose the CAPM beta in the frequency domain and demonstrate that by measuring market betas at lower frequencies, this substantially reduces pricing errors for the twenty-five Fama-French portfolios.¹⁶ While the motivation and aim of their methodologies are similar in spirit to our frequency domain framework – improved identification of risk prices and exposures for different dynamic components of the state vector – there are important differences between our and their respective frameworks. First, whereas they provide dynamic decompositions within the confines of a model *and* under parametric assumptions on the economy, we refrain from imposing such structure. Instead, we demonstrate how to utilize returns, or portfolios of returns, to make nonparametric deductions about components of state vector and their pricing. Second, we work in the frequency domain and require only that the spectral density of the state vector satisfies the mild regularity conditions in Assumption 3. Hence, our framework is robust to misspecification of the state vector dynamics, nesting both the Markov-switching processes adopted in Calvet & Fisher (2007) and the weakly dependent dynamics in Ortu et al. (2013), Bandi & Tamoni (2017) and Bandi et al. (2019). Our framework requires only that the spectral density of the state vector is sufficiently well-behaved such that we can study shocks with differing periodicity. Third, due to the nonparametric nature of our framework, we need not take an a priori stand the dimension of the state vector, whereas the aforementioned procedures requires adaptation to comply with the assumptions at hand. Finally, and as detailed below, we utilize our framework to uncover frequency dependent pricing factors in a large cross-section of US stocks using *co-spectrum contribution ratios* labelled *frequency risk*, rather the decomposing pricing betas.¹⁷

4 Frequency Risk in Asset Pricing Models

It is instructive to illustrate our general frequency domain theory, presented in the previous section, within the confines of important parametric asset pricing models in the literature, namely the CCAPM

¹⁶Bandi, Perron, Tamoni & Tebaldi (2017) decomposes aggregate stock return predictions (betas) using techniques closely related to those in Bandi & Tamoni (2017), and Boons & Tamoni (2017) similarly decomposes cross-sectional return predictions (betas) from macroeconomic variables. As explained in our empirical analysis below, we shall be looking at frequency decomposed covariance contribution ratios, using our nonparametric frequency domain framework.

¹⁷Despite the differences, their important insights about how financial and macroeconomic time series may be decomposed into dynamic components of different persistence, in part, motivate our paper.

and LRR model, which are described in Examples 1 and 2, respectively, as well as our log-linearized versions of the dynamic disaster model and the regime-switching CCAPM, which are developed in Appendices A.2 and A.3. In addition, these models may aid the formation of expectations about dynamic patterns in risk exposures and risk prices obtained from asset returns, thereby providing guidance for our empirical study. Interestingly, by combining results from Bansal & Yaron (2004) for the LRR model, explicated in Appendix A.1, with our new results in the appendix, it follows that asset returns may be written on the same generic form in all four models,

$$r_{i,t+1} = \mu_i + \mathbf{F}_i' \mathbf{X}_t + \mathcal{I}_{i,t+1}, \quad (12)$$

with, however, the constants, μ_i and \mathbf{F}_i , the state vector, \mathbf{X}_t , as well as the composite of “permanent” innovations, $\mathcal{I}_{i,t+1}$, differing across models. Hence, despite the asset pricing models having different motivation, structure and assumptions, we may treat their implications for frequency domain decompositions of the unconditional covariance measure, $\mathbf{C}_{ij}(0)$, in a unified manner. Specifically, as conveyed by (6)-(11), the dynamic properties of \mathbf{X}_t , reflected by its spectral density, and its asset-specific loading \mathbf{F}_i determines how risk factors are priced. As a result, it is instructive to study the spectral properties of the respective state vectors in the four asset pricing models.

CCAPM. The spectral decomposition is trivial as returns only have permanent innovations. Hence, the model predicts the (co-)spectrum being constant across all frequencies.

LRR model. The state vector consists of two components, x_t and σ_t^2 , both of which obey first-order autoregressive dynamics and one with conditional heteroskedasticity. The composite innovation term, $\mathcal{I}_{i,t+1}$, is comprised of conditionally heteroskedastic Gaussian shocks. The spectral densities of the state variables follow from their stationary AR(1) structure as,

$$f_x(\lambda_j) = \frac{\varphi_x^2 \sigma^2}{(1-\nu)2\pi} \frac{1}{1 - 2\rho_x \cos(2\pi\lambda_j) + \rho_x^2}, \quad f_{\sigma^2}(\lambda_j) = \frac{\sigma_v^2}{2\pi} \frac{1}{1 - 2\nu \cos(2\pi\lambda_j) + \nu^2}. \quad (13)$$

Hence, apart from constant scaling factors, their respective contribution to $\mathbf{C}_{ij}(0)$ is determined by the persistence parameters ρ_x and ν , which, as mentioned earlier, are close to unity, generating shocks to consumption growth and its volatility with half-lives exceeding 52 months.

Disaster model. The state vector dynamics is driven by two components; systemic and stock-specific resilience in the recovery following a disaster, denoted by $\mathcal{B}_t(a)$ and $\mathcal{H}_{i,t}(a)$ with a being a generic constant. Both components obey AR(1) dynamics with homoskedastic Gaussian shocks, implying that their respective spectral densities may be written,

$$f_{\mathcal{B}}(\lambda_j) = \frac{\zeta_{\mathcal{B}}^2}{2\pi} \frac{1}{1 - 2\varphi_{\mathcal{B}} \cos(2\pi\lambda_j) + \varphi_{\mathcal{B}}^2}, \quad f_{i,\mathcal{H}}(\lambda_j) = \frac{\zeta_{i,\mathcal{H}}^2}{2\pi} \frac{1}{1 - 2\varphi_{i,\mathcal{H}} \cos(2\pi\lambda_j) + \varphi_{i,\mathcal{H}}^2}. \quad (14)$$

As for the LRR model, the frequency dependencies in $\mathbf{C}_{ij}(0)$ is determined by $\varphi_{\mathcal{B}}$ and $\varphi_{i,\mathcal{H}}$. The calibration in Gabaix (2012) fixes the persistence parameters to $\exp(-0.13) \simeq 0.88$, referencing studies

that utilizes annual data. This corresponds to $0.88^{1/12} \simeq 0.989$ for monthly observations and, thereby, a half-life of the recovery from disasters of more than 65 months.

Regime-switching CCAPM. The state vector consists of the time-varying mean and variance of consumption growth, $\mu_{c,t}$ and $\sigma_{c,t}^2$, respectively, which, however, are assumed to obey Markov regime switching processes with “small” probabilities of transitioning into “bad” regimes. Hence, using Christensen & Varneskov (2017, Lemmas 3-4), their spectral densities may be written as

$$f_{\mu}(\lambda_j) \simeq \frac{C_{\mu}}{2\pi} \frac{\lambda_j^{-1}}{j}, \quad f_{\sigma^2}(\lambda_j) \simeq \frac{C_{\sigma^2}}{2\pi} \frac{\lambda_j^{-1}}{j}. \quad (15)$$

At lower frequencies, that is, when $j \rightarrow 0$ and $\lambda_j \rightarrow 0$, we have spikes in $f_{\mu}(\lambda_j)$ and $f_{\sigma^2}(\lambda_j)$, whose shapes only differ by a constant scale factor. Hence, shocks to $\mu_{c,t}$ and $\sigma_{c,t}^2$ are permanent.

Despite the LRR model, dynamic disaster model and the regime-switching CCAPM either arising from very different theoretical starting points, consumption versus disaster modeling, or having different dynamics, persistent AR(1) versus regime-switching, it is clear from (13)-(15), that the spectral densities of the state variables in each model generate substantial *low-frequency variation*, which is illustrated in Figure 1. In fact, the spectral shapes of the components are almost identical across models, using the recommended calibration of the persistence parameters. This similarity suggests that we could expect to uncover an important low-frequency pricing component using the framework in (6)-(11), if our set of tests assets load non-trivially on said part of the state vector.¹⁸

A few additional remarks are in order. First, all state vector components of the three models are very persistent and, thus, fail to speak about the *high-frequency* contributions to $C_{ij}(0)$. Second, the LRR model does not necessarily need stochastic volatility to generate low-frequency variation; it will still arise through x_t . The weight on dynamic components, however, will change, and so will the aggregate weight on the lower frequency dependencies in the model.¹⁹ Third, the excess low-frequency variation does not depend on a baseline independence assumption between the different state variables in each model. If there is dependence between the latter, their co-spectrum can be bounded by the Cauchy-Schwartz inequality in conjunction with (13)-(15). Fourth, note that it is the resilience of the recovery, not the disasters themselves, that generate frequency dependencies in $C_{ij}(0)$. The disasters are non-Gaussian shocks, which generate skewness and a permanent impact on the co-spectrum, suggesting that the Barro (2006) disaster model does not generate frequency-dependent contributions to the unconditional risk premium; see Theorem 2 and Proposition 1 in Appendix A.2. Similarly, despite Drechsler & Yaron (2011) introducing non-Gaussian features in the LRR model to provide an explanation for the variance risk premium, this does not aid identification of state vector dynamics nor its pricing. However, it introduces important non-Gaussian shocks in conditional risk premium,

¹⁸Yu (2012) studies the co-spectrum between aggregate, or market, returns and consumption growth for the habit model of Campbell & Cochrane (1999) and the LRR model, finding that the habit model cannot reconcile increased correlation between returns and consumption growth at lower frequencies, whereas the LRR model can. In addition to studying different asset pricing models above, the differences between the present framework and Yu (2012) is similar to those described for the frameworks of Calvet & Fisher (2007), Ortú et al. (2013) and Bandi & Tamoni (2017) in Section 3.3.

¹⁹However, as shown by Bansal & Yaron (2004), stochastic volatility is important for generating reasonable asset prices.

similarly to those for the dynamic disaster model. Finally, note that all of these models rely on calibration and/or estimation of the model parameters for implementation, where, in particular, disasters and persistent components of consumption growth can be hard to detect empirically. Our framework sidesteps such issues, focusing exclusively on the shape of the spectral density and nonparametric deductions to infer features of the state vector and their pricing.

5 Empirical Application: Frequency Dependent Risk

We illustrate the usefulness of our nonparametric framework for studying factor dynamics and their pricing using data on a large cross-section of US stocks. Specifically, we collect daily US stock returns from the CRSP database between January 1964 and December 2018, including all US common stock, and use the CRSP value-weighted index as the market return. When computing monthly excess returns, the one-month US Treasury rate is used as a risk-free rate. Finally, when studying the pricing of frequency dependent risk factors, we compute their alphas against the Fama & French (1993), Carhart (1997) and Fama & French (2015) models using data obtained from Kenneth French’s data library as well as data related to observable stock characteristics from Compustat.

5.1 Estimation and Frequency Decompositions

First, we estimate frequency dependent covariances between individual stocks and the market factor, returning to non-market factors in Section 5.3. Throughout, we eliminate concerns due to nonsynchronous trading by using three day overlapping log-returns, $\tilde{r}_{i,t} = \sum_{k=0}^2 \ln(1 + r_{i,t+k})$. Moreover, we operationalize $\mathbf{C}_{iM}(\vartheta_1, \vartheta_2)$ in (8) using the discrete Fourier transform and co-periodograms,

$$\mathbf{C}_{iM}(\vartheta_1, \vartheta_2) \simeq \frac{2\pi}{T} \sum_{j=\vartheta_1}^{\vartheta_2} \Re(\mathbf{I}_{iM}(\lambda_j)), \quad \text{with} \quad (16)$$

$$\mathbf{I}_{iM}(\lambda_j) = \mathbf{w}_i(\lambda_j) \bar{\mathbf{w}}_M(\lambda_j), \quad \mathbf{w}_i(\lambda_j) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^T \tilde{r}_{i,t} e^{it\lambda},$$

and the “bar” denoting the complex conjugate. Now, rather than studying the *market beta* at different frequencies, we motivate our empirical first sorting measure by applying a decomposition the unconditional risk premium in the standard CAPM, which isolates the contributions due to *frequency dependent risk* embedded in the market factor, that is, we consider

$$\begin{aligned} \mathbb{E}[r_{i,t+1} - r_{f,t}] &\simeq \gamma \times \mathbf{C}_{iM}(1, T) \\ &= \gamma \times \underbrace{\mathbf{C}_M(1, T) \boldsymbol{\beta}_{i,M}}_{\text{beta risk}} \times \underbrace{\sum_{j=1}^q \frac{\mathbf{C}_{iM}(\vartheta_j, \vartheta_{j+1})}{\mathbf{C}_{iM}(1, T)}}_{\text{frequency risk}} \end{aligned} \quad (17)$$

where $q > 1$ is the number of partitions of $\mathbf{C}_{iM}(1, T)$.²⁰ This decomposition may be seen as arising from a CCAPM where the representative agent has power utility ($\theta = 1$ and $\gamma = 1/\psi$) and all aggregate dividends are consumed, see Munk (2013, Chapters 8 and 10). Hence, by utilizing the frequency domain framework in (6)-(11), standardized by the total covariance $\mathbf{C}_{iM}(1, T)$, we can make nonparametric deductions about the dynamic behavior of state vector components, with which the market factor loads non-trivially on, and how they are priced in the cross-section. Specifically, this is achieved by answering whether $\mathbf{C}_{iM}(\vartheta_j, \vartheta_{j+1})/\mathbf{C}_{iM}(1, T)$ is constant across $j = 1, \dots, q$ for all stocks and, if there is frequency-dependent dispersion in the ratio, by studying whether stocks that have larger fraction of its variation stemming from certain parts of the spectrum command a different risk premium than stocks, whose variation mainly comes from movements in the state vector at other frequencies. Importantly, the motivational decomposition (17) illustrates that this issue is different from the study of *beta risk*. However, as the pricing of market risk has a long history in finance, dating back to the contributions by Sharpe (1964) and Lintner (1965), we will, of course, control for this along with a battery of other appraised risk factors in our pricing exercise.²¹

We carry out the analysis and construction of frequency-risk portfolios in a *real time* manner to avoid issues with in-sample overfitting. Specifically, we use a rolling window of 5 years to estimate frequency-partitioned covariance ratios and require at least 2.5 years (375 observations) to remain in the sample. Hence, our panel of stocks is unbalanced and we synthesize the cross-sectional information using portfolio sorts. First, at the beginning of each month, rolling estimates of *frequency risk* are constructed for ten equally spaced intervals from low to high frequencies:

$$\left[0, \frac{\pi}{10}\right), \left[\frac{\pi}{10}, \frac{2\pi}{10}\right), \dots, \left[\frac{8\pi}{10}, \frac{9\pi}{10}\right), \left[\frac{9\pi}{10}, \pi\right]. \quad (18)$$

$\xleftarrow{\text{lower frequencies}} \qquad \qquad \qquad \xrightarrow{\text{higher frequencies}}$

In this simple partition, low-frequency risk in daily returns is conveyed by covariance ratios over the leftmost interval $[0, \frac{\pi}{10})$, corresponding to state vector variation in cycles that take, at least, 20 trading days (approximately one month) to complete. Similarly, the next frequency bin captures cycles that take between 10 and 20 days to complete, and the highest frequency bin contains cycles that take at most $20/9 \simeq 2.2$ days to complete.²² At the beginning of each calendar month, we sort stocks into ten

²⁰As mentioned in Section 3.3, Bandi & Tamoni (2017) and Bandi et al. (2019) study decompositions of consumption growth and market betas for 25 portfolios formed on size and the book-to-market ratio. There are three main differences between their and our cross-sectional pricing exercise. First, the framework is different. They study consumption or market betas at different frequencies, i.e., a measure of the form $\mathbf{C}_{iM}(\vartheta_j, \vartheta_{j+1})/\mathbf{C}_M(\vartheta_j, \vartheta_{j+1})$, adapted to our notation, whereas we use our *frequency risk* measure and nonparametric framework. Second, we carry out a real time analysis to unveil components of the state vector and their pricing, whereas they conduct specification tests a decomposed CAPM or CCAPM. Finally, we focus on the entire cross-section of US stocks, not on factor portfolios.

²¹We emphasize that the decomposition in (17) only motivates our frequency dependent covariance ratios as a relevant sorting statistic. We refrain from imposing CAPM assumptions and will, as a result, not apply (17) to test its validity. Rather, we will use the market return and *frequency risk* to make nonparametric deductions about state vector components and their pricing, assuming that the market return loads nontrivial on (a subset of) the latter.

²²We have experimented with different partitions of the covariance measure frequencies, e.g., using 3 or 5 equally spaced intervals. Similarly, we have carried out robustness checks with respect to the rolling window length and the overlapping

equally weighted portfolios corresponding to the ten frequency dependent risk measures based on the past 5 years of data. Portfolios are, then, rebalanced every month to maintain equal weighting.

5.2 Frequency Risk Dispersion and Cross-sectional Pricing

We proceed by assessing the dynamic behavior of state vector components and their cross-sectional pricing using portfolio sorts, as described above. Specifically, Table 1 shows the average excess returns earned by the ten portfolio sorted on our covariance ratio measure using the lowest, respectively, highest frequency bins.²³ For the LF sort, we observe that stocks with a higher fraction of their market covariance materializing at LFs (P10) earn a higher return than stocks having a lower fraction of LF variation (P1), with only a slight increase in volatility. Moreover, the returns are monotonically increasing for stocks with low to high LF risk. As a result, the long-short P10-P1 portfolio averages a significant positive return and earns a Sharpe ratio of 0.48, with little-to-no market beta (-0.10). On the other hand, stocks with a high fraction of HF risk earn lower average returns than stocks with a low fraction, generating a negative Sharpe ratio on the corresponding long-short P10-P1 portfolio. The difference between the average (risk-adjusted) returns on the two long-short portfolios is particularly interesting. To facilitate a better understanding of why, let us, for simplicity of argument, imagine a setting in which we only utilize a LF and a HF bin, splitting the frequency range in two. Since the risk measure (17) is a ratio, this would imply $HF = 1 - LF$ by symmetry. However, the risk-adjusted return to the LF long-short portfolio does not mirror this symmetry, suggesting that separate dynamic components of the state vector, one LF and one HF, as well as their respective factor loadings drive the differences in returns. To corroborate this finding, we compute correlations between frequency dependent long-short portfolios in Figure 6 (which will be described in detail later), and observe that the HF and LF portfolios only have a mild negative correlation of -0.49 .

To evaluate the significance of this frequency-dependent return pattern in greater detail, and to summarize information for all frequency ranges in (17) and (18), we focus on P10-P1 portfolios. In doing so, and to simplify the nomenclature, we shall henceforth refer to the portfolio that is long stocks with high LF risk and short stocks with low LF risk as the *low frequency portfolio*, the corresponding portfolio for HF risk as the *high frequency portfolio*, and define portfolios for the remaining bins analogously.²⁴ The results in Table 1 indicate that the state vector is comprised of more than one component with differential dynamics and risk prices. However, to discern whether this finding reflect identification of new risk factors (or state vector components) or simply reshuffles existing factors, we test the risk-adjusted returns (alphas) of the long-short portfolios against a battery of appraised risk factors in the literature. In particular, Tables 2 and 3 show significance tests of the alphas for the LF and HF portfolios relative to the CAPM, the Fama-French three and five factor models, and the three and five factor model augmented with the momentum factor (Carhart 1997). First, from Table 2, we

log-return correction. The results are similar to those presented below and are, thus, left out for brevity.

²³Specifically, for the high frequency (HF) sort, we take the average over the results from the ninth and tenth frequency bins, which both have cycles that complete in less than 2.5 trading days.

²⁴Again, note that HF risk spans frequency bins nine and ten.

observe that the alpha for the LF portfolio is strongly statistically significant as well as economically substantial, being approximately 0.5% per month. Moreover, while market returns, small-minus-big (smb) and robust-minus-weak (rmw) all appear significantly in the regressions, the explanatory power is low, indicating that LF risk is an important dimension of cross-sectional pricing, which is unspanned by existing factors. Second, from Table 3, we observe that the HF portfolio has a significant negative risk premium of approximately -0.20% per month. Moreover, as for LF risk, only the market, smb and rmw factors appear significantly in the regressions and the R^2 is low. Interestingly, whereas the loadings on the market and rmw factors are of opposite sign, both HF and LF maintain a small positive exposure to smb, illustrating, again, the lack of perfect symmetry between the two.

We generalize and visualize our results by depicting the long-short portfolio alphas as a function of frequency bins in Figure 2, that is, by mapping out the *frequency term structure of risk*. The pattern is striking. We observe a large and positive risk premium only for the very LF bin. For all other frequency ranges, the alphas are negative and similar in magnitude to the alpha described for the HF portfolio, resulting in a “hockey stick pattern” when moving from LF to HF sorts. Now, by utilizing deductions, as described in Section 3.2, we can make two nonparametric statements about the state vector: (1) The state vector has a dynamic component operating at very low frequencies, which command a large positive risk premium; (2) There is a less persistent component operating at higher frequencies, requiring a smaller negative risk premium. Importantly, these components are not spanned by standard risk factors, illustrating the usefulness of our frequency risk concept and frequency domain approach. If we relate these findings to the four asset pricing models discussed earlier, it is interesting to see that the LF results are consistent with the LRR model, the dynamic disaster model and the regime-switching CCAPM, whose spectral contributions from transitory components of the SDF to the unconditional risk-premium, as captured by our risk measure, are predominantly materializing at low frequencies, as illustrated by Figure 1. Hence, if we make the (extremely plausible) argument that market returns load non-trivially on either consumption growth and/or recovery resilience, then our results exactly capture the pricing of shocks to consumption growth and/or disasters; demonstrating that the price of such risks is large, amounting to a risk-premium of 6% per year. Importantly, we arrive at these results without estimating a parametric asset pricing model. On the other hand, our results are directly at odds with the standard CCAPM, which fails to generate frequency-specific variation in the unconditional risk premium.²⁵ These results are consistent with Dew-Becker & Giglio (2016), who, by quantifying the propagation of *permanent* shocks in various asset pricing models and, thus, analyzing the implications of frequency-dependent risk prices, find that asset pricing models with preferences that load significantly on infrequent shocks, such as Epstein-Zin or a bandpass utility specification, fit characteristics and industry-sorted portfolios better than asset pricing models, whose preferences load more heavily on recurring frequent shocks such as the Habit model (Campbell & Cochrane 1999). Moreover, Dew-Becker & Giglio (2016) find that low-frequency macroeconomic shocks carry a positive risk price in their test portfolios. Hence, despite our focus on the contribution of the transitory part

²⁵As discussed in Section 4, the same holds for the Barro (2006) disaster model.

of the SDF to the unconditional risk-premium, using nonparametric techniques, rather than the price and propagation of permanent shocks, we also find LF risk to carry a positive risk price and we demonstrate that it commands a large risk premium in the full cross section of US stocks.

Finally, while the LF results are consistent with the LRR model, the dynamic disaster model and the regime-switching CCAPM, none of the three models are able to explain the significant, albeit smaller negative risk premium associated with HF risk. Which dynamic feature of the state vector are these models missing? As an exploratory gauge, we simulate a state vector with two autoregressive components, having persistence parameters 0.97 and -0.2 as well as Gaussian innovations with variances multiplied by 0.1 and 4, respectively, and opposite risk prices of 1 and -1 . These features are consistent with a difficult-to-detect highly persistent mean component of either consumption growth, its volatility or recovery resilience, yet this is augmented by a separate component, displaying short-lived dependencies. In the context of the LRR model, such a feature would arise if the stochastic consumption volatility had a long-run *and* a short-run component, which has a risk price with an opposite sign compared to either of the persistent mean or volatility components. Similarly, the consumption, or dividend, growth dynamics may have two components.²⁶ Alternatively, if considering the dynamic disaster model, as described in Appendix A.2, this would arise if, for example, stock-specific and systemic resilience featured different persistence and loadings. Now, by depicting the individual and joint spectral densities of the two dynamic components in Figure 3, we observe that such dynamic features are consistent with the frequency term-structure of risk in Figure 2. The model includes a significant LF component, which is detectable only at the very lowest frequencies, not at higher frequencies, and a HF component, whose significance is much weaker, yet detectable in the frequency domain. Whereas this suggests that all three dynamic asset pricing models can match the LF variation in the unconditional risk premium, they lack a short-lived component with opposite signed risk price to fully match the frequency term structure of risk embedded in market returns. Importantly, the joint results from Figures 2 and 3 highlights the importance of utilizing the frequency domain to disentangle dynamic components of the state vector and their cross-sectional pricing implications.

5.3 Frequency Dependent Risk embedded in Non-Market Factors

We have, so far, utilized the market portfolio to unveil two features of the SDF; that it contains a very persistent component that carries a positive risk premium and a less persistent component that earns a smaller negative risk premium. However, as described in Section 3.1, we may, in principle, use all portfolios of returns to unveil features of the SDF, provided the selected “baseline” portfolio loads nontrivially on these. In this section, we thus expand our empirical analysis by determining whether there are frequency dependent risk premia embedded in other popular factor portfolios as well as by examining how these relate to our LF and HF factors. Specifically, we carry out a similar portfolio sorting exercise based on the individual stocks’ frequency dependent covariance ratios with the value (hml), size (smb) and momentum (umd) factors, summarizing the results (alphas) by their

²⁶Zhou & Zhu (2015) and Li & Zhang (2016) provide such extensions of the LRR model.

frequency term structures of risk in Figure 4 and numerical alphas in Table 4. Interestingly, for the frequency dependent value alphas, we observe a *hockey stick pattern*, very similar to the market factor, albeit with smaller LF alphas in the 0.10-0.24 range and slightly weaker statistical significance overall. For the size factor, we similarly find a statistically significant alphas at LFs, but also a sharpe reversal with negative and significant alphas at frequency bins two and three, that is, at intermediate frequencies (IFs), whereas the HF alphas are largely insignificant. As discussed for the market factor, this suggests that size factor loads nontrivially, and with opposite sign, on two separate state vector components, one LF and one operating at IFs. The frequency dependent momentum risk portfolios, on the other hand, carry a negative risk premia for HF sorts and no statistically significant risk premia for LF sorts. Generally, the statistical significance of the alphas across the various frequency-range and factor combinations is lower compared to the corresponding market factor results.

Since the frequency term structures of risk for the value and size factors, among others, unveils a positive premium associated with LF risk and momentum a negative premium for HF risk, similarly to the market factor, this, naturally, renders concerns about whether these results are driven by the Fama-French and momentum factors being correlated with the market factor. Hence, we orthogonalize the value, size and momentum factors with respect to the market and remaining factors, that is, by constructing

$$r_t^{\text{hml}} = \beta_0 + \beta_1 r_t^{\text{mktf}} + \beta_2 r_t^{\text{smb}} + \beta_3 r_t^{\text{umd}} + \varepsilon_t, \quad r_t^{\text{hml}^\perp} \equiv \hat{\varepsilon}_t, \quad (19)$$

and similarly for size and momentum. We then repeat the sorting and pricing analysis using the orthogonalized factors, summarizing the results in Figure 5. The main difference between the results in Figures 4 and 5 is that the statistical significance drops for the frequency dependent value portfolio alphas, whose qualitative pattern, however, remains intact. The results for the frequency dependent size and momentum factors, on the other hand, are almost identical to those without orthogonalization. This corroborates the existence of a significantly positive risk premia associated with LF value and size, and significant negative risk premia associated with HF momentum and IF size. The risk premia are smaller in magnitude than the corresponding for our LF market factor in Table 2, however, being (absolutely) above 0.1% per month in all cases, they are economically meaningful.

To further investigate the coherence between frequency dependent factors, we compute correlations between the HF and LF factor portfolio returns based on covariance sorts with market, value, size and momentum returns in Figure 6. In addition, we include correlations to the original four factors. There are several interesting observations from Figure 6. First, both the LF and HF market factors display low correlation with the four original pricing factors, confirming the results in Tables 2-3. Second, the correlation between the frequency dependent factors are low, being less than $\pm 1/2$ in all cases.²⁷ In particular, our LF market factor, commanding a large positive risk premium, is almost uncorrelated with the LF size and HF momentum factors, also commanding significant risk premia, and only mildly correlated with the LF value factor. Moreover, the HF market factor is only mildly correlated with the HF momentum factor. Third, the low correlation between the LF market, value and size

²⁷The only exceptions being some frequency dependent factors and their own orthogonalized version.

factors suggests the existence of more than one persistent component in the SDF, receiving a nontrivial loading. Interestingly, this observation is, again, consistent with the three dynamic asset pricing models discussed in Section 4, which can each feature two persistent components; the mean and volatility of consumption growth and/or systemic- and stock-specific recovery resilience. Finally, it is important to, once again, emphasize that the unveiling of multiple persistent and shorter-run components in the state vector using our frequency domain approach only requires that (some) factors load nontrivially on these components, not on any parametric asset pricing model. Hence, our approach and nonparametric evidence demonstrate that asset pricing models should feature two, or more, persistent components and, at least, one short-lived component, with risk prices of opposite sign and magnitude.

5.4 Properties of Frequency Dependent Risk Factors

This section proceeds by dissecting the properties of the LF market factor and its risk premium. Specifically, we open “the black box” and study the firm characteristics of companies in the long-short portfolio. Moreover, we discuss persistence of the sorting statistic, transaction costs as well as the time series properties of the low-frequency risk premium. We repeat this exercise for the HF market factor in the Online Appendix, which shows implications similar to those described below.

5.4.1 Firm Characteristics

We dissect the properties of the stocks included in the LF market portfolio by providing a cross-sectional comparison of nine firm characteristics for the companies belonging to the long and short leg separately in Figure 7.²⁸ All of the characteristics are well-known cross-sectional return predictors; the book-to-market ratio (Fama & French 1992), the debt-to-price ratio (Litzenberger & Ramaswamy 1979), market equity (often referred to as “size”, e.g., Banz (1981)), profitability (recently re-examined by Ball, Gerakos, Linnainmaa & Nikolaev (2015)), investment (Fama & French 2015), operating accruals (Sloan 1996), the trailing three-year volatility (Freyberger et al. 2019), short-term reversal as well as (standard) momentum, both of which are documented in Jegadeesh & Titman (1993).

From Figure 7, we observe that the low-frequency portfolio is, typically, long value firms and short growth firms, although the differences book-to-market ratios are small. This is consistent with a positive, but insignificant coefficient on the value (hml) factor in Table 2. Second, the portfolio is, on average, long smaller firms with periodically higher volatility, again, consistent with the positive and significant coefficient on size (smb) in Table 2. Remarkably, however, there is no noticeable pattern for the debt-to-price ratio, profitability, investment, operating accruals, short term reversal (r_{2-1}) and standard momentum (r_{12-2}). Except for profitability (rmw), these findings corroborate the loadings on the corresponding factors in Table 2, which are, indeed, small and insignificant. Generally, the analysis of firm characteristics underscores that the large and positive LF risk premium, uncovered using our frequency domain approach, cannot be attributed to standard risk characteristics.

²⁸This exercise is based on the methodology and data in Freyberger, Neuhierl & Weber (2019).

5.4.2 Persistence of the sorting statistics and transaction costs

From the volatility modeling literature, it is well-established that asset return volatility and cross-asset covariances are very persistent, see, among others, Andersen, Bollerslev, Diebold & Labys (2003) and Andersen, Bollerslev, Diebold & Wu (2006). However, this literature is not concerned with covariance ratios that have been decomposed into different frequency ranges. In our context, the persistence properties of the frequency dependent covariance ratios are important for, at least, two reasons. First, if these measures are extremely volatile over time, it renders a risk-based interpretation difficult since it would imply that the noise-to-signal ratios of different components of the state vector are changing rapidly, as well, making the implied dynamics difficult to reconcile with standard, and more sophisticated, asset pricing models.²⁹ Second, and from a more practical perspective, it would imply that LF factor investments are associated with high transaction costs. Hence, to assess the dynamic properties of the sorting statistic, we first estimate a first-order panel autoregression, $c_{it} = \alpha + \beta c_{it-1} + \varepsilon_{it}$, for the LF and HF market covariance ratios. The results are presented in Table 5. Consistent with the volatility modeling literature, we find both the LF and HF sorting measures to be very persistent, having autocorrelation coefficients around 0.8. To corroborate and expand on these findings, we plot the transition frequencies between the ten LF sorted portfolios from time $t - 1$ to time t in Figure 8. Consistent with Table 5, this documents very low transitioning out of the lowest and highest frequency portfolios. In fact, 88.53% and 90.13% of the stocks in the two portfolios, respectively, remain between two consecutive periods, on average. Hence, not only is our results compatible with a risk-based interpretation of the alphas associated with the frequency-dependent factors, they are also useful from a more practical investment perspective.

5.4.3 Time Series Properties of the Low-Frequency Premium

The unconditional risk premium commanded by the LF factor is substantial. However, to further develop intuition about the time series evolution of long-short portfolio alpha, we plot its logarithmic price (cumulative returns) in Figure 9, with the superimposed shaded grey areas in the graph depicting NBER recessions. While the returns are negative during the peak of the most recent global financial crisis in 2008, the portfolio more than recovers during 2009. In fact, the LF portfolio does not appear to have systematically different returns during any of the NBER recessions. We formally test this claim by running a simple regression of the returns to the LF portfolio on a constant and an NBER recession indicator, i.e., $r_t^{\text{LF}} = \alpha + \beta \times \text{NBER}_t + \varepsilon_t$. This results in point estimates of $\hat{\alpha} = 0.3415$ and $\hat{\beta} = 0.3345$, where, interestingly, the coefficient on the NBER recession indicator is not statistically significant, having a p -value of 0.29. Hence, this strongly indicates that the LF risk premium is invariant to business cycles. In the second part of Figure 9, we also plot the yearly returns of the low frequency portfolio. Apart from the extreme positive return during 2009, there seems to be no systematic changes in the yearly mean nor volatility over the 50 year period, clearly demonstrating

²⁹This essentially implies that all components of the state vector have volatilities that change rapidly over time.

the LF risk is a pervasive and significantly priced component of US stocks.

6 Conclusion

This paper provides a new nonparametric framework for studying the dynamics of the state vector and its associated risk prices. Specifically, in a general setting where the SDF decomposes into permanent and transitory components, we analyze their contribution to the unconditional asset return premium using frequency domain techniques. We show analytically that the co-spectrum between returns and the SDF only displays frequency dependencies through its transitory component, that is, through the state vector. The contribution from permanent components, on the other hand, such as Gaussian and non-Gaussian shocks (e.g., disasters) are constant across frequencies. We label this the *permanent-transitory spectrum decomposition*. Moreover, we demonstrate that state vector dynamics and its risk prices can be uncovered by studying (transformations of) the covariance between (portfolios of) asset returns. To this end, we introduce two new frequency risk measures. Since the SDF is latent in our setting, we utilize these measures to map out a *frequency term structure of risk*. Our nonparametric identification of state vector components relies on having one, or more, baseline portfolios with a non-trivial loading on it. Conditional on this, our approach is fully nonparametric, thus allowing us to answer questions about which key features a theoretical asset pricing model should possess to be consistent with the implied dynamics from a cross-section of assets.

We apply our framework to study frequency risk in the full cross-section of US stocks, utilizing the market, value, size and momentum factors as baseline portfolios to operationalize the risk measures. Our analysis uncovers the existence of, at least, two significantly priced low-frequency risk factors, one of which commands a large positive risk premium of 6% per year. Moreover, we document, at least, one high-frequency component in the state vector that carries a significant risk price and whose premium amounts to, approximately, -2% per year. Importantly, we show that these frequency dependent risk factors are unspanned by a battery of appraised risk factors and characteristics in the empirical asset pricing literature. Interestingly, when estimating the frequency term structures of risk for the market, value and, to some extent, size baseline portfolios, we document a pronounced *hockey stick pattern* in the risk premium when going from low to high frequencies. Hence, only by utilizing the frequency domain, can we uncover state vector components and their associated risk premia. Our analysis demonstrates that multiple factors with varying persistence and risk prices are needed to be consistent with asset pricing dynamics implied by the cross-section of US stocks.

Throughout, we contrast our findings with the implications of the long-run risk model, the dynamic disaster model as well as a regime-switching CCAPM, and, in the process, provide new analytical results for such models. We argue that these asset pricing models, indeed, capture important aspects of low-frequency risk. However, the models lack the flexibility to capture all dimensions of frequency risk. The nonparametric nature of our framework, thus, suggests to increase the number of transitory components in the SDF, thereby allowing them to contain components with different persistence and

risk prices of opposite sign. Besides providing guidance for how to extend existing asset pricing models, our framework may improve the estimation, or calibration, of asset pricing models by being able to disentangle risk factors, or priced state vector components, that need to be matched by model implied parameters, or moments, that speak directly to their persistence properties. However, we leave the development of new asset pricing models and estimation techniques for future research.

A Additional Definitions and Theory

This section provides definitions and return representation for the long-run risk model. Moreover, it introduces log-linearized versions of the dynamic disaster model of Gabaix (2012) as well as the regime-switching CCAPM model of Lettau et al. (2008).

A.1 Long Run Risk Constants and Return Representation

First, the three constants in the SDF of the long-run-risk model may be written as

$$\begin{aligned} A_1 &= \frac{1 - 1/\psi}{1 - \rho\kappa_1}, & A_2 &= \frac{\theta(1 - 1/\psi)^2}{2(1 - \kappa_1\nu)} \left[1 + \left(\frac{\kappa_1\varphi_x}{1 - \rho_x\kappa_1} \right)^2 \right], & \text{and} \\ A_0 &= \frac{-\delta + (1 - 1/\psi)\mu_c + \kappa_0 + \kappa_1 A_2 \sigma^2 (1 - \nu) + \theta\kappa_1^2 A_2^2 \sigma_v^2 / 2}{(1 - \kappa_1)}. \end{aligned}$$

Next, to achieve a representation for the log-return on asset $i = 1, \dots, n$, with $i = M$ denoting the market portfolio, we follow Bansal & Yaron (2004), see also Munk (2013, Chapter 9.3) for a textbook treatment, and use a first-order Taylor approximation,

$$r_{i,t+1} \simeq \kappa_{i,0} + \kappa_{i,1} z_{i,t+1} - z_{i,t} + \Delta d_{i,t+1}, \quad (\text{A.1})$$

whose constants $\kappa_{i,0}$ and $\kappa_{i,1}$ are defined as κ_0 and κ_1 in Example 2, $z_{i,t+1}$ is the asset-specific log-price-dividend ratio and, lastly, $\Delta d_{i,t+1}$ is the log-dividend growth. As for consumption growth, the latter is assumed to obey $\Delta d_{i,t+1} = \mu_{i,d} + \varphi_i x_t + \varphi_{i,d} \sigma_t u_{i,t+1}$, where $\varphi_i, \varphi_{i,d} > 0$ and $u_{i,t+1}$ is another standard Gaussian shock, independent of the other shocks in the system. Independence can be relaxed at the expense of more complicated notation. Let $\mu, \lambda_v, \lambda_e$ and λ_η be defined as in Example 2, then the one-period risk-free rate of return is determined as,

$$r_{f,t} = -\mu + \frac{x_t}{\psi} - \frac{\lambda_v^2 \sigma_v^2}{2} - \left((1 - \theta) A_2 (1 - \kappa_1 \nu) + (\lambda_\eta^2 + \lambda_e^2) / 2 \right) \sigma_t^2. \quad (\text{A.2})$$

To represent $r_{i,t+1}$ as a function of the state vector, and following Munk (2013, Chapter 9.3), define the additional constants $\zeta_{i,v} = \kappa_{i,1} A_{i,2}$ and $\zeta_{i,e} = \kappa_{i,1} A_{i,1} \varphi_x$ with

$$A_{i,1} = \frac{\varphi_i - 1/\psi}{1 - \rho_x \kappa_{i,1}}, \quad A_{i,2} = \frac{(1 - \theta) A_2 (1 - \kappa_1 \nu) + \gamma^2 / 2 + \varphi_{i,d}^2 / 2 + (\zeta_{i,e} - \lambda_e)^2 / 2}{1 - \nu \kappa_{i,1}}.$$

Moreover, define the ‘‘regression parameters’’,

$$\begin{aligned} \beta_{i,0} &= -\mu - \frac{\lambda_v^2 \sigma_v^2}{2} + \zeta_{i,v} \lambda_v \sigma_v^2 - \frac{\zeta_{i,v}^2 \sigma_v^2}{2}, & \beta_{i,1} &= 1/\psi, \\ \beta_{i,2} &= - \left((1 - \theta) A_2 (1 - \kappa_1 \nu) + (\lambda_\eta^2 + \lambda_e^2) / 2 \right) + \zeta_{i,e} \lambda_e - \left(\zeta_{i,e}^2 + \varphi_{i,d}^2 \right) / 2. \end{aligned}$$

Then, we may write

$$r_{i,t+1} = \beta_{i,0} + \beta_{i,1}x_t + \beta_{i,2}\sigma_t^2 + \zeta_{i,e}\sigma_t e_{t+1} + \zeta_{i,v}\sigma_t v_{t+1} + \varphi_{i,d}\sigma_t u_{t+1}, \quad (\text{A.3})$$

that is, returns are a linear function of the state variables, x_t and σ_t^2 , with conditionally heteroskedastic innovations. Moreover, the state vector $\mathbf{X}_t = (c_t, \sigma_t)^2$ follows a first-order VAR. The spectral density in (13), thus, follows using (3), the autoregressive structure and

$$\mathbf{C}_{xx}(h) = \mathbb{E}[(\rho_x^h x_t + \varphi_x \sum_{i=1}^h \rho_x^{h-1} \sigma_{t+h-i} e_{t+h+1-i}) x_t] = \rho_x^h \mathbb{E}[x_t^2] = \rho_x^h \frac{\varphi_x^2 \sigma^2}{1 - \nu}.$$

A.2 Dynamic Rare Disaster Risk Model

For direct comparability with the long run risk model, we introduce a log-linearized version of the disaster risk model in Gabaix (2012).³⁰ Specifically, we consider an investor with time-additive power utility $\theta = 1$, who, in each time period $t = 1, \dots, T$, faces the possibility that consumption growth is hit by a disaster, that is, $\Delta c_{t+1} = \mu_c + v_{t+1}$, where

$$v_{t+1} = B_{t+1} \times \mathcal{S}_{t+1}, \quad \mathcal{S}_{t+1} \sim \text{Bernoulli}(p_t),$$

with some time-varying disaster probability $0 < p_t < 1$, and where B_{t+1} measures the random drop in consumption. In a similar setup, Gabaix (2012) prices the claims to assets such as equities, bonds, options, and credit spreads, and Farhi & Gabaix (2016) consider exchange rates. We will illustrate our spectrum risk pricing theory throughout for the equity case. As for the LRR model, we adopt the first-order Taylor approximation in (A.1) and, consistent with Gabaix (2012), define the asset-specific log-dividend growth rate,

$$\Delta d_{i,t+1} = \mu_{i,d} + \sigma_{i,d} \epsilon_{i,t+1} + \eta_{i,t+1}, \quad \eta_{i,t+1} = F_{i,t+1} \times \mathcal{S}_{t+1},$$

where $\mu_{i,d}$ and $\sigma_{i,d} > 0$ are its mean and volatility, and $\epsilon_{i,t+1} \sim N(0, 1)$ are independent shocks. Here, $F_{i,t+1}$ captures the random recovery rate in the event of a disaster, allowing individual assets to “partially default” without affecting aggregate consumption. As in Gabaix (2012), we impose some simplifying assumptions on B_{t+1} , $F_{i,t+1}$ and \mathcal{S}_{t+1} . First, conditional on \mathcal{F}_t , the three components are independent. Second, let $B_{t+1} \sim N(\mu_{t,B}, \sigma_{t,B}^2)$ and $F_{i,t+1} \sim N(\mu_{t,i}, \sigma_{t,i}^2)$, then we model time-variation in the disaster shock and recovery rate through systemic and stock specific “resilience”,

$$\text{Systemic resilience: } \mathcal{B}_t(a) = p_t \left(a \mu_{t,B} + a^2 \sigma_{t,B}^2 / 2 \right),$$

³⁰Importantly, Gabaix (2012) and Wachter (2013) generalize the constant rare disaster risk model in Barro (2006) by allowing for a time-varying probability of disasters in discrete and continuous time settings, respectively, and demonstrate that this feature alleviates several asset pricing puzzles. Moreover, they also provide different model extensions such as treating Epstein-Zin preferences and time-variation in the expected recovery rate.

$$\text{Stock specific resilience: } \mathcal{H}_{i,t}(a) = p_t \left(a\mu_{t,i} + a^2\sigma_{t,i}^2/2 \right).$$

These quantities are comprised of two components; (1) the probability that a disaster occurs multiplied by (2) the time-varying moment generating function (MGF) of either a disaster shock or a recovery rate shock. Moreover, we stipulate that $\mathcal{B}_t(a)$ and $\mathcal{H}_{i,t}(a)$ obey AR(1) dynamics,

$$\mathcal{B}_{t+1}(a) = \varphi_{\mathcal{B}}\mathcal{B}_t(a) + \varsigma_{\mathcal{B}}v_{t+1}, \quad \mathcal{H}_{i,t+1}(a) = \varphi_{i,\mathcal{H}}\mathcal{H}_{i,t}(a) + \varsigma_{i,\mathcal{H}}u_{i,t+1}$$

for which $|\varphi_{\mathcal{B}}| < 1$ and $|\varphi_{i,\mathcal{H}}| < 1$, and the innovations $v_{t+1} \sim N(0,1)$ and $u_{i,t+1} \sim N(0,1)$ are independent of each other and $\epsilon_{i,t+1}$, B_{t+1} , $F_{i,t+1}$ and \mathcal{S}_{t+1} conditional on \mathcal{F}_t . Note that, we can allow for \mathcal{F}_t -adapted stochastic volatility and correlation between the various shocks of the system at the expense of more convoluted pricing expression (cf. Proposition 1 below).³¹ However, to highlight the importance of resilience dynamics, we refrain from making such generalizations here.

Our log-linearized disaster model differs from the framework in Gabaix (2012) in two ways. First, we have a linearized impact from B_{t+1} and $F_{i,t+1}$ through their MGF and the resilience terms, which allows us to decouple and specify differential dynamics for $\mathcal{B}_{t+1}(a)$ and $\mathcal{H}_{i,t+1}(a)$. For comparison, Gabaix (2012) captures resilience through $p_t\mathbb{E}_t[B_{t+1}^{-\gamma}F_{i,t+1} - 1]$. Second, to obtain closed form pricing expressions in the presence of such non-linearity, Gabaix (2012) assumes the coupled resilience follow a “twisted ” AR(1) process, whereas we stipulate standard AR(1) dynamics. Our use of a log-linearized system rather linearity-generating processes simplify our frequency domain analysis below, with no impact on the qualitative features of the model as it is first-order equivalent to that in Gabaix (2012), see, e.g., discussions in Gabaix (2009) and Filipovic, Larsson & Trolle (2018).

We end this section by showing that the log-linear dynamic disaster model maps into the SDF in (1) as well as a provide return representation results that are equivalent to (A.2) and (A.3). For the former, we have $\mu = -\delta - \gamma\mu_c$, a subset of the state vector is given by $\mathbf{X}_t^S = (p_t, \mu_{t,B}, \sigma_{t,B}^2)'$, the non-Gaussian innovation is $\Delta J_{t+1,1} = \mathcal{S}_{t+1}$, implying that for the jump intensity $\lambda_{t,1} = p_t$, and, as for the jump sizes, $\psi_{t+1,1} = B_{t+1}$, $\varpi_{t,1} = \mu_{t,B}$, $\xi_t = \sigma_{t,B}^2$. Moreover, the first entry of \mathbf{F} is $-\gamma$. Finally, the following proposition provides return representation results for the disaster model.

Proposition 1. *Suppose the conditions of Section A.2 hold. Moreover, define the constants $\beta_{i,0} = \kappa_{i,0} + (\kappa_{i,1} - 1)C_{i,0} + \mu_{i,d}$, $\beta_{i,1} = C_{i,1}(\kappa_{i,1}\varphi_{i,\mathcal{H}} - 1)$, $\beta_{i,2} = C_{i,2}(\kappa_{i,1}\varphi_{\mathcal{B}} - 1)$, with $C_{i,0}$, $C_{i,1}$ and $C_{i,2}$ defined in (B.9)-(B.10) of Appendix B.3, then, up to a first-order Taylor expansion,*

$$(a) \quad r_{f,t} = \delta + \gamma\mu_c - \mathcal{B}_t(-\gamma),$$

$$(b) \quad r_{i,t+1} = \beta_{i,0} + \beta_{i,1}\mathcal{H}_{i,t}(1) + \beta_{i,1}\mathcal{B}_t(-\gamma) + \sigma_{i,d}\epsilon_{i,t+1} + \eta_{i,t+1} + C_{i,2}\kappa_{i,1}\varsigma_{\mathcal{B}}v_{t+1} + C_{i,1}\kappa_{i,1}\varsigma_{i,\mathcal{H}}u_{i,t+1}.$$

Proposition 1 shows that the conditional mean implications of the dynamic disaster model are similar to those for the LRR model; both are driven by two first-order autoregressive processes.

³¹This is easily seen by noting that all components are conditionally Gaussian, implying that similar conditional heteroskedastic volatility and/or correlation terms will enter linearly in the pricing expressions.

The main difference between the models arises from the former allowing both Gaussian and non-Gaussian innovations, whereas the latter accommodates conditional heteroskedasticity. As conveyed by Theorems 1 and 2, however, the frequency term structure is determined by the temporal properties of the conditional mean, implying that the two models will generate equivalent implications.

A.3 A Regime Switching CCAPM

As a final example, we introduce a simple regime switching CCAPM, similar in spirit to the model in Lettau et al. (2008), who shows that regime shifts in consumption volatility explain rare changes in “unconditional” equity premium, cf. Fama & French (2002).³² Specifically, we consider an investor with $\theta = 1$, and let $\Delta c_{t+1} = \mu_{c,t} + \sigma_{c,t}\epsilon_{t+1}$, where

$$\mu_{c,t} = \mu_h \mathbf{1}\{s_t^\mu = 0\} + \mu_l \mathbf{1}\{s_t^\mu = 1\}, \quad \sigma_{c,t}^2 = \sigma_h^2 \mathbf{1}\{s_t^\sigma = 0\} + \sigma_l^2 \mathbf{1}\{s_t^\sigma = 1\},$$

and with the innovations $\epsilon_{t+1} \sim N(0, 1)$ being independent and mutually independent of the state variables $s_t^\mu \in \{0, 1\}$ and $s_t^\sigma \in \{0, 1\}$, which capture abrupt transitions between high and low consumption growth and volatility regimes, respectively. In this setting, the constants μ_h and μ_l indicate the consumption growth rates in *high* and *low* regimes, and similarly for σ_h^2 and σ_l^2 . The state variables are assumed to behave according to the transition matrices,

$$\mathcal{P}_T^\iota \equiv \begin{pmatrix} \mathbb{P}[s_t^\iota = 0 | s_{t-1}^\iota = 0] & \mathbb{P}[s_t^\iota = 1 | s_{t-1}^\iota = 0] \\ \mathbb{P}[s_t^\iota = 0 | s_{t-1}^\iota = 1] & \mathbb{P}[s_t^\iota = 1 | s_{t-1}^\iota = 1] \end{pmatrix} = \begin{pmatrix} p_{00,T}^\iota & p_{01,T}^\iota \\ p_{10,T}^\iota & p_{11,T}^\iota \end{pmatrix}, \quad \iota \in \{\mu, \sigma\},$$

where the sample size dependent switching probabilities $p_{01,T}^\iota$ and $p_{10,T}^\iota$ satisfy $p_{01,T}^\iota = p_{01}^\iota/T$ and $p_{10,T}^\iota = p_{10}^\iota/T$ with $0 < p_{01}^\iota, p_{10}^\iota < T$ fixed as well as $p_{00,T}^\iota + p_{01,T}^\iota = 1$ and $p_{10,T}^\iota + p_{11,T}^\iota = 1$. The switching probabilities are made dependent on the sample size, T , to ensure that, for a given sample realization $t = 1, \dots, T$, the processes generate a finite number of transitions between states. Finally, and as in Lettau et al. (2008), we assume that the log-dividend growth are leveraged consumption innovations, i.e., $\Delta d_{i,t} = \chi_i \Delta c_t$ and use the log-return approximation (A.1).

To describe the SDF as well as the frequency-dependent risk properties in the regime switching CCAPM model, let us define the (latent) transition processes

$$\begin{aligned} \bar{\mu}_t &= (\mu_l - \mu_h) \mathbf{1}\{s_t^\mu = 0\} + (\mu_h - \mu_l) \mathbf{1}\{s_t^\mu = 1\}, \\ \bar{\sigma}_t^2 &= (\sigma_l^2 - \sigma_h^2) \mathbf{1}\{s_t^\sigma = 0\} + (\sigma_h^2 - \sigma_l^2) \mathbf{1}\{s_t^\sigma = 1\}, \end{aligned}$$

and $p_{t,T}^\iota = p_{01,T}^\iota \mathbf{1}\{s_t^\iota = 0\} + p_{10,T}^\iota \mathbf{1}\{s_t^\iota = 1\}$. Then, we can write $\mu_{c,t}$ and $\sigma_{c,t}^2$ as

$$\mu_{c,t+1} = \mu_{c,t} + \bar{\mu}_t \times \mathcal{S}_{t+1}^\mu, \quad \sigma_{c,t+1}^2 = \sigma_{c,t}^2 + \bar{\sigma}_t^2 \times \mathcal{S}_{t+1}^\sigma,$$

³²The work in Lettau et al. (2008) builds on a large literature documenting structural breaks in various macroeconomic indicators, e.g., Hamilton (1989), Perron (1989), Kim & Nelson (1999) and McConnell & Perez-Quiros (2000).

where $\mathcal{S}_{t+1}^t \sim \text{Bernoulli}(p_{t,T}^t)$. This setting maps directly into (1) by writing the state, innovation and parameter vectors; $\mathbf{X} = (\mu_{c,t}, \sigma_{c,t}^2)'$, $\mathbf{W}_t = \epsilon_t$, $\mathbf{F} = (\gamma, 0)'$, and $\mathbf{R}_2 = \{(0, 0)', (0, \gamma^2)'\}$. Finally, the following proposition provides return representations for the regime switching CCAPM.

Proposition 2. *Suppose the conditions of Section A.3 hold. Moreover, define the constants $\beta_{i,0} = \kappa_{i,0} + (\kappa_{i,1} - 1)D_{i,0}$, $\beta_{i,1} = D_{i,1}(\kappa_{i,1} - 1) + \chi_i$, $\beta_{i,2} = D_{i,2}(\kappa_{i,1} - 1)$, with $D_{i,0}$, $D_{i,1}$ and $D_{i,2}$ defined in (B.12) of Appendix B.4, then, up to a first-order Taylor expansion and as $T \rightarrow \infty$,*

$$(a) \quad r_t^f = \delta + \gamma\mu_{c,t} - \gamma^2/2\sigma_{c,t}^2,$$

$$(b) \quad r_{i,t+1} = \beta_{i,0} + \beta_{i,1}\mu_{c,t} + \beta_{i,2}\sigma_{c,t}^2 + \kappa_{i,1}D_{i,1}\bar{\mu}_t\mathcal{S}_{t+1}^\mu + \kappa_{i,1}D_{i,2}\bar{\sigma}_t^2\mathcal{S}_{t+1}^\sigma + \chi_i\sigma_{c,t}\epsilon_{t+1}.$$

Proposition 2 demonstrates that our log-linear regime switching CCAPM generate return representations, which, similarly to the LRR and the dynamic disaster models, are affine functions of the state variables $\mathbf{X}_t = (\mu_{c,t}, \sigma_{c,t}^2)'$ and displays Gaussian as well as non-Gaussian innovations. An important difference between the models, however, is the $\mu_{c,t}$ and $\sigma_{c,t}^2$ are non-stationary regime switching processes, in contrast to the other two models whose state variables obey stationary first-order autoregressive processes. Moreover, note that we have simplified the model structure relative to the model studied by Lettau et al. (2008), who also include a learning mechanism. Our results may be generalized at the expense of an increasingly tedious exposition. Finally, given the assumptions on \mathbf{X}_t , the spectral densities in (15) follow by Christensen & Varneskov (2017, Lemmas 3-4).

B Proofs

For some generic process X_t , $t = 1, \dots, T$, define $\mathbb{M}_{t,X}(u) \equiv \mathbb{E}_t[e^{uX}]$ as the conditional moment generating function (CMGF). Then, if $X_{t+1} \sim N(\mu_{t,X}, \sigma_{t,X}^2)$, we have for some constant a ,

$$\mathbb{M}_{t,aX}(u) = \exp\left(a\mu_{t,X}u + \frac{a^2u^2}{2}\sigma_{t,X}^2\right). \quad (\text{B.4})$$

Lemma 1. *Suppose that $X_{t+1} \sim N(\mu_{t,X}, \sigma_{t,X}^2)$ and $\mathcal{Y}_{t+1} \sim \text{Bernoulli}(y_t)$, where $\mu_{t,X}$, $\sigma_{t,X} > 0$, and the probability $y_t \in (0, 1)$ are locally bounded and càdlàg processes, and that X_{t+1} and \mathcal{Y}_{t+1} are independent conditional on \mathcal{F}_t , then the following results for the CMGF hold,*

$$(a) \quad \mathbb{M}_{t,aX\mathcal{Y}}(u) = 1 + y_t(\mathbb{M}_{t,aX}(u) - 1).$$

(b) *Up to a first-order Taylor expansion, the following relation hold*

$$\ln \mathbb{M}_{t,aX\mathcal{Y}}(u) = y_t \left(a\mu_{t,X}u + \frac{a^2u^2}{2}\sigma_{t,X}^2 \right).$$

Proof. **For (a).** First, make the following expansion of $\mathbb{M}_{t,aXS}(u)$,

$$\mathbb{M}_{t,aX\mathcal{Y}}(u) = \mathbb{E}_t[e^{uaX} | \mathcal{Y}_{t+1} = 1] \times \mathbb{P}_t[\mathcal{Y}_{t+1} = 1] + (1 - \mathbb{P}_t[\mathcal{Y}_{t+1} = 1]),$$

readily providing the result, as $\mathbb{E}_t[e^{uaX} | \mathcal{Y}_{t+1} = 1] = \mathbb{M}_{t,aX}(u)$.

For (b). Note that we have the following Taylor expansions for exp and ln functions:

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}, \quad (\text{B.5})$$

which, together with **(a)**, readily provides the result. \square

B.1 Proof of Theorem 1

For (a). First, $\mathbb{E}[\mathbf{G}'_t \mathbf{W}_{t+1}] = \mathbb{E}[\mathbb{E}_t[\mathbf{G}'_t \mathbf{W}_{t+1}]] = \mathbf{0}$ by the law of iterated expectations (LIE). Hence, we may write

$$\mathbf{C}_{gg}(0) = \mathbb{E}[(\mathbf{G}'_t \mathbf{W}_{t+1})^2] = \mathbb{E}[\mathbf{G}'_t \mathbb{E}_t[\mathbf{W}_{t+1} \mathbf{W}'_{t+1}] \mathbf{G}_t] = \mathbb{E}[\text{Tr}(\mathbf{G}_t \mathbf{G}'_t)].$$

Next, using conditional independence of the components in Assumptions 1, Assumption 2 and the LIE,

$$\mathbf{C}_{\ell\ell}(0) = \mathbf{H}' (\mathbb{E}[\mathbb{V}_t[\Delta \mathbf{N}_{t+1}] \circ \mathbb{V}_t[\boldsymbol{\psi}_{t+1}]]) \mathbf{H} = \mathbf{H}' (\mathbb{E}[\mathbb{D}(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t \circ \boldsymbol{\lambda}_t) \circ \mathbb{D}(\boldsymbol{\xi}_t)]) \mathbf{H}.$$

Finally, $\mathbf{C}_{gg}(0) < \infty$ and $\mathbf{C}_{\ell\ell}(0) < \infty$ follow immediately by Assumption 3.

For (b). By applying the LIE and the same arguments as in **(a)**, we readily have $\mathbf{C}_{gg}(h) = 0$ and $\mathbf{C}_{\ell\ell}(h) = 0$ when $h \neq 0$. This implies $\mathbf{f}_{gg}(\lambda) = \mathbf{C}_{gg}(0)/(2\pi)$ and $\mathbf{f}_{\ell\ell}(\lambda) = \mathbf{C}_{\ell\ell}(0)/(2\pi)$. Hence, since we have that \mathbf{X}_t , g_{t+1} and ℓ_{t+1} are conditionally independent, the result follows by sequential use of the LIE, as above, to eliminate cross-component covariance terms.

For (c). Follows by **(a)** and **(b)** in conjunction with Parseval's theorem. \square

B.2 Proof of Theorem 2

The result follows by the same arguments provided for Theorem 1. \square

B.3 Proof of Proposition 1

For (a). First, for power utility $\theta = 1$, the SDF is $\Delta \ln S_{t+1} = -\delta - \gamma \Delta c_{t+1}$, and for the risk free rate, we have $\mathbb{E}_t[e^{\Delta \ln S_{t+1}} R_t^f] = 1$ or, by re-arranging, $r_{f,t} = \ln(\mathbb{E}_t[e^{\Delta \ln S_{t+1}}]^{-1})$. Now, since

$$\mathbb{E}_t[e^{\Delta \ln S_{t+1}}] = e^{-\delta - \gamma \mu c} \mathbb{M}_{t,-\gamma BS}(1),$$

the result follows by applying Lemma 1(b).

For (b). As for the long run risk model in Bansal & Yaron (2004), we stipulate that the log-price-dividend ratio is linear in stock specific and systemic resilience, i.e.,

$$z_{i,t} = C_{i,0} + C_{i,1}\mathcal{H}_{i,t}(1) + C_{i,2}\mathcal{B}_t(-\gamma), \quad (\text{B.6})$$

and subsequently verify this relation. Next, rewrite Euler pricing relation,

$$\begin{aligned} 1 &= \mathbb{E}_t \left[e^{-\delta - \gamma \Delta c_{t+1} + r_{i,t+1}} \right] \\ &= e^{-\delta - \gamma \mu_c + \kappa_{i,0} - C_{i,0} - C_{i,1}\mathcal{H}_{i,t}(1) - C_{i,2}\mathcal{B}_t(-\gamma) + \kappa_{i,1}C_{i,0} + \mu_{i,d}} \\ &\quad \times \mathbb{E}_t \left[e^{(F_{i,t+1} - \gamma B_{t+1})\mathcal{S}_{t+1} + \kappa_{i,1}(C_{i,1}\mathcal{H}_{i,t+1}(1) + C_{i,2}\mathcal{B}_{t+1}(-\gamma)) + \sigma_{i,d}\epsilon_{i,t+1}} \right], \\ &= e^{-\delta - \gamma \mu_c + \kappa_{i,0} + (\kappa_{i,1} - 1)C_{i,0} - C_{i,1}\mathcal{H}_{i,t}(1) - C_{i,2}\mathcal{B}_t(-\gamma) + \mu_{i,d} + \sigma_{i,d}^2/2} \\ &\quad \times \mathbb{E}_t \left[e^{(F_{i,t+1} - \gamma B_{t+1})\mathcal{S}_{t+1}} \right] \times \mathbb{E}_t \left[e^{\kappa_{i,1}(C_{i,1}\mathcal{H}_{i,t+1}(1) + C_{i,2}\mathcal{B}_{t+1}(-\gamma))} \right], \end{aligned}$$

with the last equality following using conditional independence of the stochastic components in the pricing relation. By the same argument and Lemma 1, we have, up to a first-order Taylor expansion,

$$\ln \mathbb{E}_t \left[e^{(F_{i,t+1} - \gamma B_{t+1})\mathcal{S}_{t+1}} \right] = \mathcal{H}_{i,t}(1) + \mathcal{B}_t(-\gamma) \quad (\text{B.7})$$

Similarly, we have exactly that

$$\begin{aligned} \mathbb{E}_t \left[e^{\kappa_{i,1}(C_{i,1}\mathcal{H}_{i,t+1}(1) + C_{i,2}\mathcal{B}_{t+1}(-\gamma))} \right] &= e^{\kappa_{i,1}C_{i,1}\varphi_{i,\mathcal{H}}\mathcal{H}_{i,t}(1) + \kappa_{i,1}^2 C_{i,1}^2 \varsigma_{i,\mathcal{H}}^2/2} \\ &\quad \times e^{\kappa_{i,1}C_{i,2}\varphi_{i,\mathcal{B}}\mathcal{B}_t(-\gamma) + \kappa_{i,1}^2 C_{i,1}^2 \varsigma_{i,\mathcal{B}}^2/2}, \end{aligned} \quad (\text{B.8})$$

implying that, by taking logs on both sides of the Euler condition and collecting terms, we have, up to a first-order Taylor expansion,

$$\begin{aligned} 0 &= q_i + q_{i,\mathcal{H}}\mathcal{H}_{i,t}(1) + q_{i,\mathcal{B}}\mathcal{B}_t(-\gamma), \quad \text{where} \\ q_i &= -\delta - \gamma \mu_c + \kappa_{i,0} + (\kappa_{i,1} - 1)C_{i,0} + \mu_{i,d} + \sigma_{i,d}^2 + \kappa_{i,1}^2/2 \left(C_{i,1}^2 \varsigma_{i,\mathcal{H}}^2 + C_{i,2}^2 \varsigma_{i,\mathcal{B}}^2 \right) \\ q_{i,\mathcal{H}} &= -C_{i,1} + 1 + \kappa_{i,1}C_{i,1}\varphi_{i,\mathcal{H}}, \quad q_{i,\mathcal{B}} = -C_{i,2} + 1 + \kappa_{i,1}C_{i,2}\varphi_{i,\mathcal{B}}. \end{aligned}$$

Since this relation is to be satisfied for all values of $\mathcal{H}_{i,t}(1)$ and $\mathcal{B}_t(-\gamma)$, we need $q_i = q_{i,\mathcal{H}} = q_{i,\mathcal{B}} = 0$, and, thus, solve for the coefficients $C_{i,0}$, $C_{i,1}$, and $C_{i,2}$,

$$C_{i,1} = \frac{1}{1 - \kappa_{i,1}\varphi_{i,\mathcal{H}}}, \quad C_{i,2} = \frac{1}{1 - \kappa_{i,1}\varphi_{i,\mathcal{B}}}, \quad (\text{B.9})$$

$$C_{i,0} = \frac{-\delta - \gamma \mu_c + \kappa_{i,0} + \mu_{i,d} + \sigma_{i,d}^2 + \kappa_{i,1}^2/2 \left(C_{i,1}^2 \varsigma_{i,\mathcal{H}}^2 + C_{i,2}^2 \varsigma_{i,\mathcal{B}}^2 \right)}{1 - \kappa_{i,1}}. \quad (\text{B.10})$$

This confirms the conjectured form (B.6), up to a first-order Taylor expansion, and, together with the log-linear return approximation in (A.1) and the dynamics of $\Delta d_{i,t+1}$, gives the final result. \square

B.4 Proof of Proposition 2

For (a). As in Appendix (B.3), we have $r_{f,t} = \ln(\mathbb{E}_t[e^{\Delta \ln S_{t+1}}]^{-1})$ and $\Delta \ln S_{t+1} = -\delta - \gamma \Delta c_{t+1}$. Now, since

$$\mathbb{E}_t[e^{\Delta \ln S_{t+1}}] = e^{-\delta} \times \mathbb{E}_t \left[e^{-\gamma(\mu_{c,t} + \sigma_{c,t} \epsilon_{t+1})} \right] = e^{-\delta - \gamma \mu_{c,t}} \mathbb{E}_t \left[e^{-\gamma \sigma_{c,t} \epsilon_{t+1}} \right],$$

we may use conditional log-normality of ϵ_{t+1} to establish the result.

For (b). As for the disaster risk model in Section B.3, we stipulate that the log-price-dividend ratio is linear in the regime switching mean and variance of consumption growth, i.e.,

$$z_{i,t} = D_{i,0} + D_{i,1} \mu_{c,t} + D_{i,2} \sigma_{c,t}^2 \tag{B.11}$$

and subsequently verify this. Hence, the Euler condition may be rewritten as

$$\begin{aligned} 1 &= \mathbb{E}_t[e^{-\delta - \gamma \Delta c_{t+1} + r_{i,t+1}}] \\ &= e^{\kappa_{i,0} - \delta + (\kappa_{i,1} - 1) D_{i,0}} \times e^{(\chi_i - \gamma - D_{i,1}) \mu_{c,t} - D_{i,2} \sigma_{c,t}^2} \times \mathbb{E}_t \left[e^{\kappa_{i,1} (D_{i,1} \mu_{c,t+1} + D_{i,2} \sigma_{c,t+1}^2) + (\chi_i - \gamma) \sigma_{c,t} \epsilon_{t+1}} \right], \end{aligned}$$

for which, using conditional independence among ϵ_{t+1} and the state variables \mathcal{S}_{t+1}^μ and \mathcal{S}_{t+1}^σ , the last term may be further decomposed as into three components,

$$\begin{aligned} \mathbb{E}_t \left[e^{\kappa_{i,1} (D_{i,1} \mu_{c,t+1} + D_{i,2} \sigma_{c,t+1}^2) + (\chi_i - \gamma) \sigma_{c,t} \epsilon_{t+1}} \right] &= \mathbb{E}_t \left[e^{\kappa_{i,1} D_{i,1} \mu_{c,t+1}} \right] \times \mathbb{E}_t \left[e^{\kappa_{i,1} D_{i,2} \sigma_{c,t+1}^2} \right] \\ &\quad \times \mathbb{E}_t \left[e^{(\chi_i - \gamma) \sigma_{c,t} \epsilon_{t+1}} \right]. \end{aligned}$$

Now, by applying Lemma 1, we have, up to a first-order Taylor expansion,

$$\begin{aligned} \ln \mathbb{E}_t \left[e^{\kappa_{i,1} D_{i,1} \mu_{c,t+1}} \right] &= \kappa_{i,1} D_{i,1} \left(\mu_{c,t} + \bar{\mu}_t p_{t,T}^\mu \right), \\ \ln \mathbb{E}_t \left[e^{(\chi_i - \gamma) \sigma_{c,t} \epsilon_{t+1}} \right] + \ln \mathbb{E}_t \left[e^{\kappa_{i,1} D_{i,2} \sigma_{c,t+1}^2} \right] &= \left((\chi_i - \gamma)^2 / 2 + \kappa_{i,1} D_{i,2} \right) \sigma_{c,t}^2 + \kappa_{i,1} D_{i,2} \bar{\sigma}_t^2 p_{t,T}^\sigma, \end{aligned}$$

implying that, by taking logs, we may rewrite the Euler condition as

$$\begin{aligned} 0 &= q_i + q_{i,\mu} \mu_{c,t} + q_{i,\sigma} \sigma_{c,t}^2 + \kappa_{i,1} D_{i,1} \bar{\mu}_t p_{t,T}^\mu + \kappa_{i,1} D_{i,2} \bar{\sigma}_t^2 p_{t,T}^\sigma, \quad \text{with} \\ q_i &= \kappa_{i,0} - \delta + (\kappa_{i,1} - 1) D_{i,0}, \\ q_{i,\mu} &= (\chi_i - \gamma + (\kappa_{i,1} - 1) D_{i,1}), \\ q_{i,\sigma} &= \left((\chi_i - \gamma)^2 / 2 + (\kappa_{i,1} - 1) D_{i,2} \right). \end{aligned}$$

Next, by model design, $p_{t,T}^\mu = O(1/T)$ and $p_{t,T}^\sigma = O(1/T)$ and all constants are bounded, implying that the limiting Euler condition may be written as $0 = q_i + q_{i,\mu} \mu_{c,t} + q_{i,\sigma} \sigma_{c,t}^2$ when $T \rightarrow \infty$. Since this

is satisfied for all $t = 1, \dots, T$, we need $q_i = q_{i,\mu} = q_{i,\sigma} = 0$, thus providing the solutions

$$D_{i,0} = \frac{\kappa_{i,0} - \delta}{1 - \kappa_{i,1}}, \quad D_{i,1} = \frac{\chi_i - \gamma}{1 - \kappa_{i,1}}, \quad D_{i,2} = \frac{(\chi_i - \gamma)^2/2}{1 - \kappa_{i,1}}, \quad (\text{B.12})$$

confirming the conjectured form (B.11), up to a first-order Taylor expansion as $T \rightarrow \infty$, and, together with the log-linear return approximation, this delivers the final result. \square

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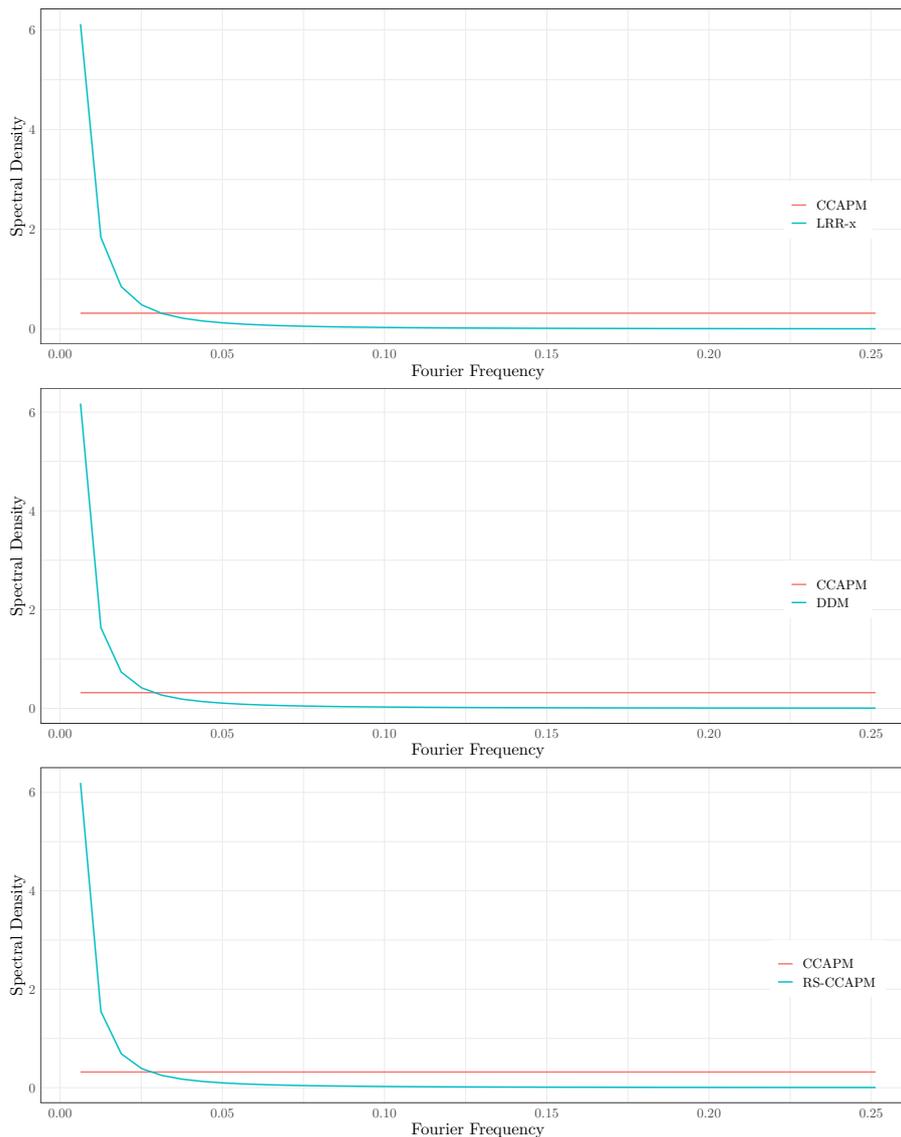
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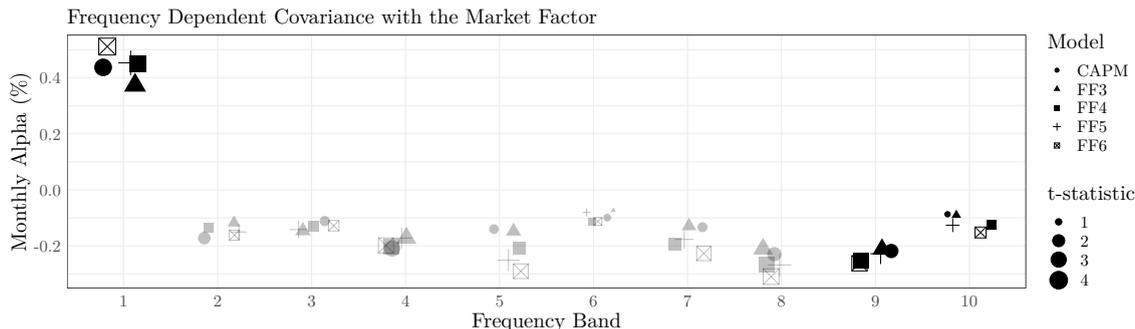
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Figure 1: Spectral Density Examples



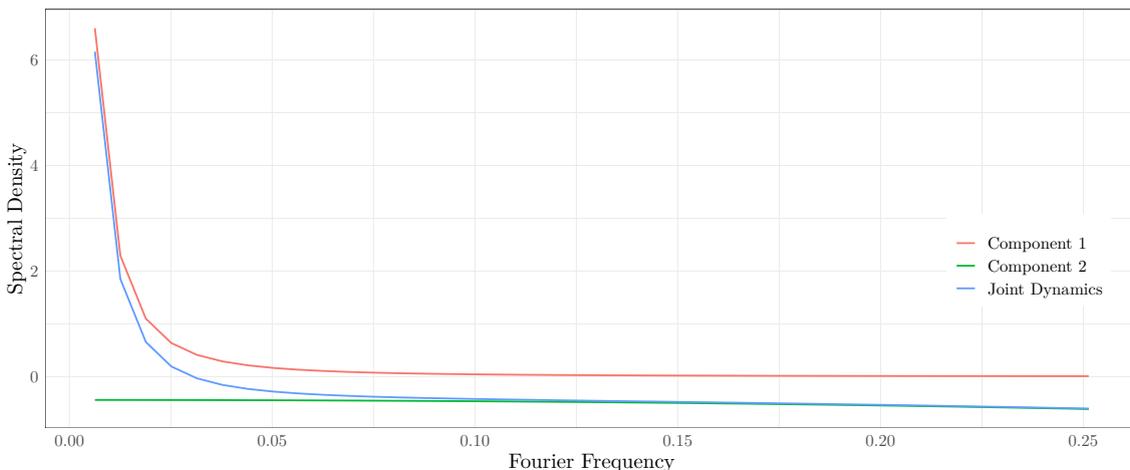
This figure shows spectral densities for persistent components of the state vector for the long-run risk model, the dynamic disaster model and the regime-switching CCAPM. Specifically, for each of the three models, the spectral density of one persistent component is simulated and benchmarked against the constant spectrum of the CCAPM, see Section 4 for a discussion. For each model, we calibrate the persistence to that recommended by the original authors, *i.e.*, Bansal & Yaron (2004), Gabaix (2012), and Lettau et al. (2008). For the long-risk model, we implement the AR(1) dynamics of x_t with persistence parameter 0.979, the dynamic disaster model with 0.989 and the regime-switching CCAPM as in (15). All constants are normalized to have unit long-run variance.

Figure 2: Monthly Alphas - Frequency Dependent Covariance Ratios with the Market Factor



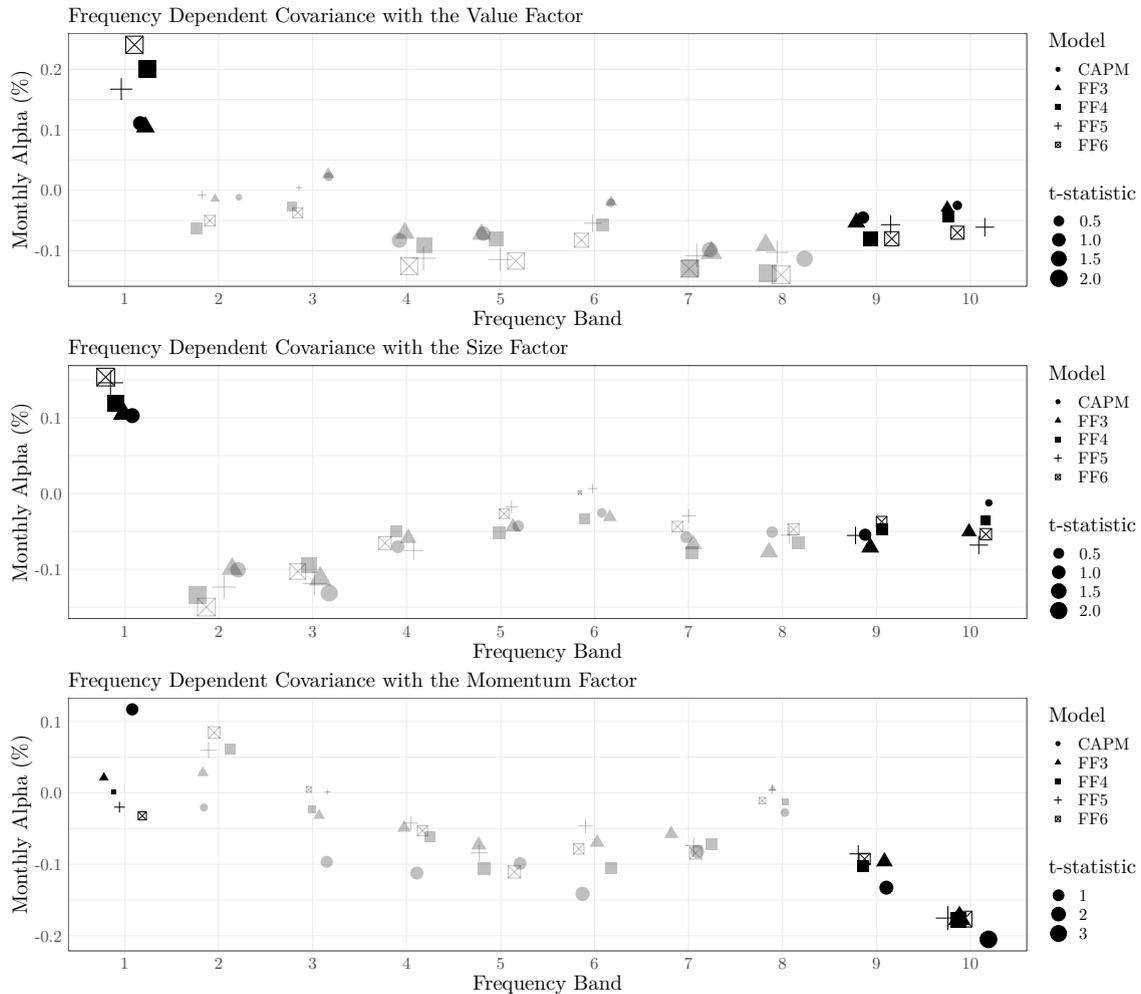
This figure plots monthly alphas (in percentage) against the Fama & French (1993) (FF3), Carhart (1997) (FF4), Fama & French (2015) model (FF5) and the five factor model augment with the momentum factor (FF6). At the beginning of each calendar month, stocks are ranked according to their frequency dependent covariance ratios and then assigned to one of the ten portfolios. Frequency dependent covariance is estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest frequency dependent covariance ratio, in a given frequency bin, and short the 10% stocks with lowest covariance ratio. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. The size of the marker is proportional to the (absolute value) of the t-statistic for alpha using the Newey & West (1987) procedure to estimate standard errors. The sample period is January 1964 to December 2018.

Figure 3: Spectral Density Examples



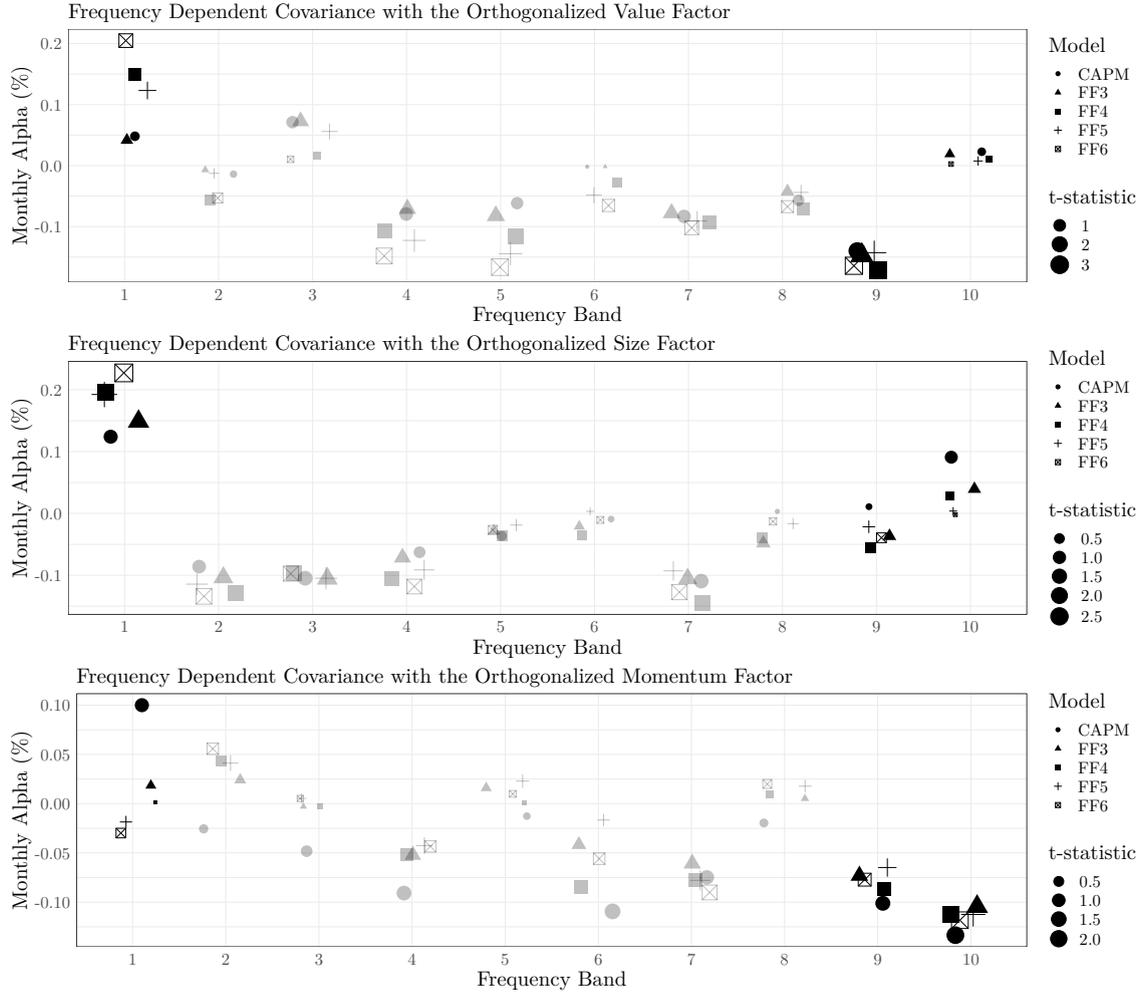
This figure illustrates the spectral density of a state vector with two independent and transitory components. In particular, the individual and joint spectral densities are depicted, assuming that the two components obey first-order autoregressive dynamics with persistence parameters of 0.97 and -0.2 as well as standard Gaussian innovations whose variances have been scaled by 0.1 and 4, respectively. Finally, the two components have been implemented with opposite risk prices of +1 and -1 .

Figure 4: Monthly Alphas - Frequency Dependent Covariance Ratios with the Value, Size and Momentum Factor



This figure plots monthly alphas (in percentage) against the Fama & French (1993) (FF3), Carhart (1997) (FF4), Fama & French (2015) model (FF5) and the five factor model augment with the momentum factor (FF6). At the beginning of each calendar month, stocks are ranked according to their frequency dependent covariance ratios and then assigned to one of the ten portfolios. Frequency dependent covariance is estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest frequency dependent covariance ratio, in a given frequency bin, and short the 10% stocks with lowest covariance ratio. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. The size of the marker is proportional to the (absolute value) of the t-statistic for alpha using the Newey & West (1987) procedure to estimate standard errors. The sample period is January 1964 to December 2018.

Figure 5: Monthly Alphas - Frequency Dependent Covariance Ratios with the Orthogonalized Value, Size and Momentum Factor



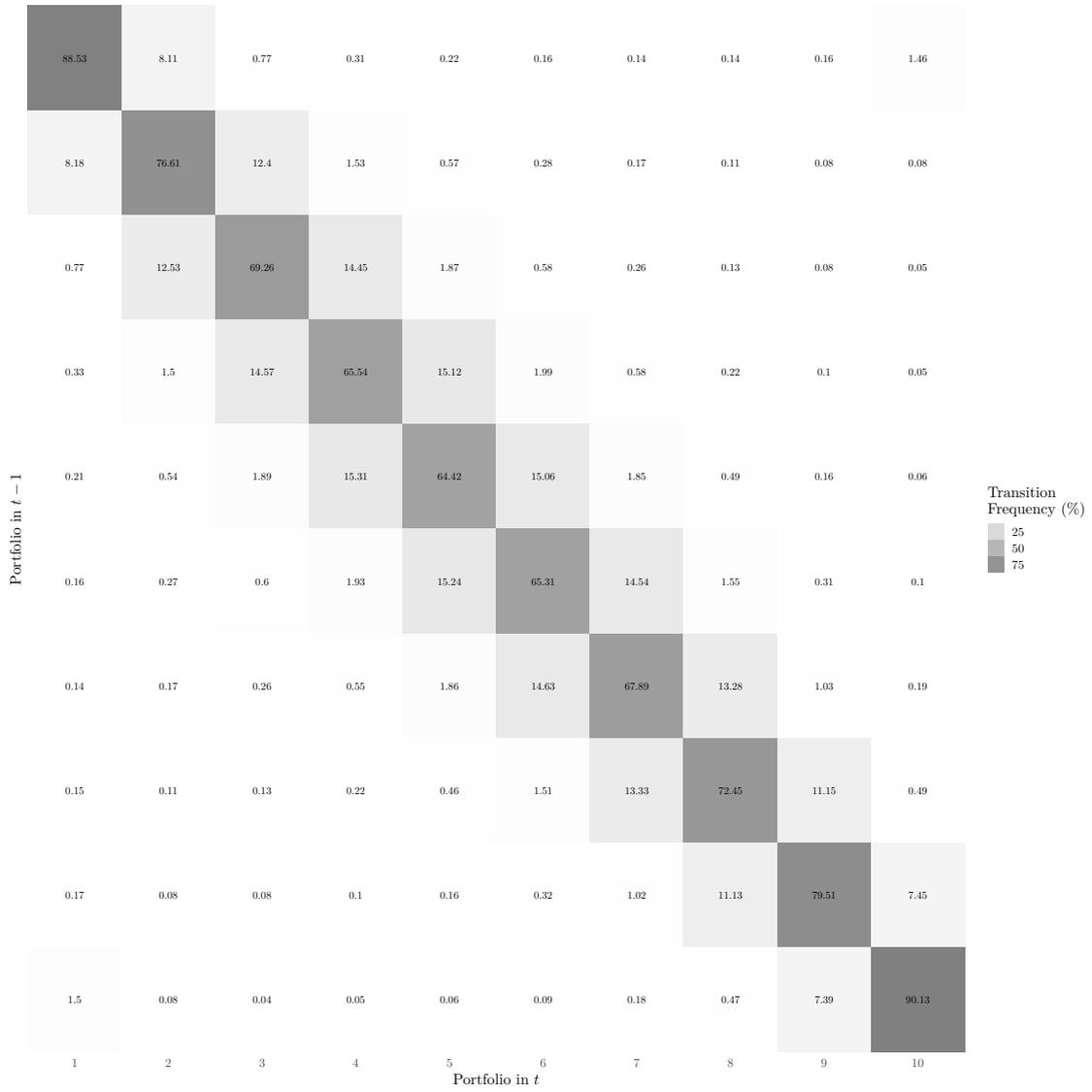
This figure plots monthly alphas (in percentage) against the Fama & French (1993) (FF3), Carhart (1997) (FF4), Fama & French (2015) model (FF5) and the five factor model augment with the momentum factor (FF6). At the beginning of each calendar month, stocks are ranked according to their frequency dependent covariance ratios and then assigned to one of the ten portfolios. Frequency dependent covariance is estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest frequency dependent covariance ratio, in a given frequency bin, and short the 10% stocks with lowest covariance ratio. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. The size of the marker is proportional to the (absolute value) of the t-statistic for alpha using the Newey & West (1987) procedure to estimate standard errors. The sample period is January 1964 to December 2018.

Figure 7: Characteristics of the Low Frequency Portfolio



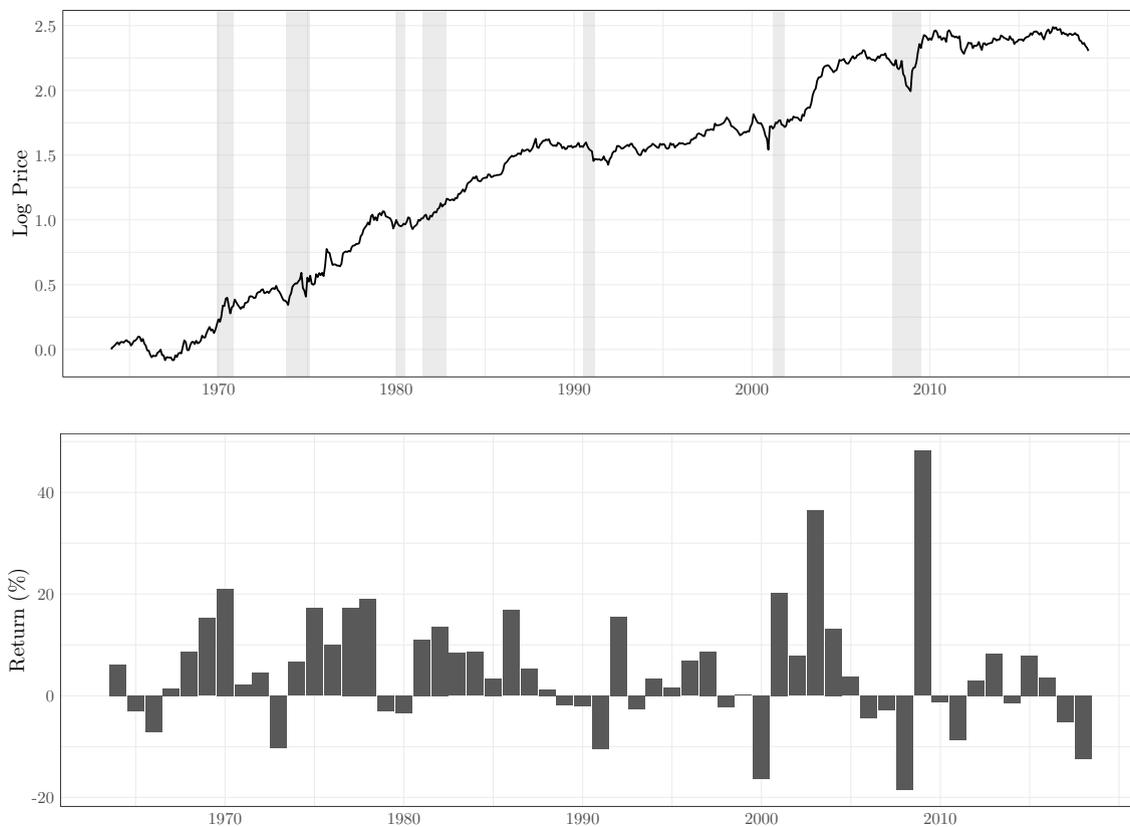
This figure shows the normalized rank of nine cross-sectional return characteristics for the long and short leg of the LF portfolio (constructed using market returns). These are the book-to-market ratio, debt-to-price ratio, market equity (size), profitability, investment, operating accruals, the trailing three-year volatility, return one month before portfolio formation (r_{2-1}) and the return from 12 to 2 months before prediction (r_{12-2}). Each month, the characteristics are normalized as in Freyberger et al. (2019), i.e., $\tilde{c}_{it} = \frac{\text{rank}(c_{it})}{N_t + 1}$, where c_{it} denotes the “raw” characteristic value and N_t denotes the number of firms in month t , \tilde{c}_{it} denotes the rank normalized characteristic, which is always between zero and one. The sample period is January 1964 to December 2018.

Figure 8: Transition Frequencies for the Portfolios Sorted on Low Frequency Risk



The figure shows the unconditional transition frequencies between the ten portfolio sorted on LF risk, constructed from market returns, from one month to the next. The sample period is January 1964 to December 2018.

Figure 9: Price Path and Yearly Return of the Low Frequency Portfolio



The top panel of the figure show the logarithmic price path (cumulative returns) of the LF long-short portfolio. At the beginning of each calendar month, stocks are ranked according to their LF covariance ratio and then assigned to one of ten portfolios. The frequency dependent covariance ratios are estimated as described in 5.1. The LF portfolio is, then, long the 10% stocks with highest LF covariance ratio and short the 10% stocks with lowest LF covariance ratio. All stocks are equally weighted within a portfolio. The gray areas depict NBER recessions. The lower panel of the figure shows the yearly returns of the LF portfolio. Portfolios are rebalanced every month. The sample period is January 1964 to December 2018.

Table 1: Returns of 10 Portfolios Sorted on Frequency Dependent Factor Risk

This table reports equally-weighted excess returns for ten portfolio sorted on frequency dependent covariance ratios with the market return. At the beginning of each calendar month, stocks are ranked according to their LF and HF covariance ratios and then assigned to one of the ten portfolios depending on the magnitude of the ratio, with P10 having the highest and P1 having the lowest. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. Newey & West (1987) standard errors are given in parentheses. The sample period is January 1964 to December 2018.

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P10-P1
Low Frequency:											
Excess Return	0.74	0.72	0.74	0.80	0.87	0.89	0.93	0.97	0.97	1.13	0.39
Standard Error	0.26	0.25	0.25	0.26	0.28	0.28	0.29	0.29	0.30	0.28	0.11
Volatility (annualized)	18.54	19.61	20.03	20.94	21.07	21.11	21.56	21.47	21.16	19.38	9.64
Sharpe Ratio (annualized)	0.48	0.44	0.45	0.46	0.49	0.50	0.52	0.54	0.55	0.70	0.48
Beta (ex. post)	0.99	1.14	1.18	1.22	1.21	1.18	1.19	1.15	1.07	0.89	-0.10
High Frequency:											
Excess Return	1.09	0.98	0.96	0.91	0.81	0.78	0.74	0.71	0.84	0.93	-0.15
Standard Error	0.28	0.28	0.27	0.28	0.27	0.27	0.27	0.26	0.26	0.27	0.05
Volatility (annualized)	19.44	19.78	20.13	20.48	20.93	21.15	21.29	20.52	19.86	19.22	4.05
Sharpe Ratio (annualized)	0.67	0.60	0.57	0.53	0.46	0.44	0.42	0.41	0.51	0.58	-0.45
Beta (ex. post)	0.97	1.05	1.12	1.17	1.21	1.22	1.22	1.17	1.10	1.00	0.03

Table 2: **Portfolios Sorted on Low Frequency Risk**

This table reports alphas and factor loadings based on the Fama & French (1993), Carhart (1997) and Fama & French (2015) models. At the beginning of each calendar month, stocks are ranked according to their low-frequency covariance ratio with the market return and then assigned to one of the ten portfolios. Low-frequency covariance ratios are estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest LF covariance ratios and short the 10% stocks with lowest LF covariance ratios. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. Newey & West (1987) standard errors are given in parentheses. The sample period is January 1964 to December 2018.

	CAPM	FF3	FF3 + UMD	FF5	FF5 + UMD
(Intercept)	0.44*** (0.11)	0.37*** (0.11)	0.45*** (0.13)	0.45*** (0.12)	0.51*** (0.14)
mktrf	-0.10** (0.04)	-0.13*** (0.03)	-0.14*** (0.04)	-0.14*** (0.04)	-0.15*** (0.04)
hml		0.10 (0.10)	0.06 (0.10)	0.08 (0.12)	0.04 (0.12)
smb		0.22*** (0.07)	0.22*** (0.06)	0.16** (0.06)	0.17** (0.07)
umd			-0.09 (0.06)		-0.08 (0.06)
rmw				-0.23*** (0.08)	-0.21*** (0.07)
cma				0.01 (0.15)	0.04 (0.14)
Adj. R ²	0.02	0.08	0.09	0.10	0.12
Num. obs.	660	660	660	660	660

*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

Table 3: Portfolios Sorted on High Frequency Risk

This table reports alphas and factor loadings based on the Fama & French (1993), Carhart (1997) and Fama & French (2015) models. At the beginning of each calendar month, stocks are ranked according to their high-frequency covariance ratio with the market return and then assigned to one of the ten portfolios. High-frequency covariance ratios are estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest HF covariance ratios and short the 10% stocks with lowest HF covariance ratios. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. Newey & West (1987) standard errors are given in parentheses. The sample period is January 1964 to December 2018.

	CAPM	FF3	FF3 + UMD	FF5	FF5 + UMD
(Intercept)	-0.15** (0.07)	-0.15* (0.08)	-0.19** (0.08)	-0.18** (0.08)	-0.21*** (0.08)
mktrf	0.08*** (0.02)	0.07*** (0.02)	0.08*** (0.02)	0.07*** (0.02)	0.08*** (0.02)
hml		-0.02 (0.04)	-0.00 (0.04)	-0.00 (0.05)	0.02 (0.05)
smb		0.05* (0.03)	0.05* (0.03)	0.07** (0.03)	0.07** (0.03)
umd			0.04** (0.02)		0.04* (0.02)
rmw				0.09** (0.04)	0.08* (0.04)
cma				-0.02 (0.07)	-0.04 (0.06)
Adj. R ²	0.05	0.05	0.07	0.06	0.07
Num. obs.	660	660	660	660	660

*** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$

Table 4: Estimated Alphas For Different Frequency Band and Factors

This table reports monthly alphas (in percentage) against the Fama & French (1993), Carhart (1997) and Fama & French (2015) models. At the beginning of each calendar month, stocks are ranked according to their frequency dependent covariance ratios and then assigned to one of the ten portfolios. Frequency dependent covariance is estimated as detailed in Section 5.1. The portfolio is then long the 10% stocks with highest frequency dependent covariance ratio, in a given frequency bin, and short the 10% stocks with lowest covariance ratio. All stocks are equally weighted within a portfolio. Portfolios are rebalanced every month. *t*-statistic are calculated using Newey & West (1987) standard errors. The sample period is January 1964 to December 2018.

	Frequency Band										
	1	2	3	4	5	6	7	8	9	10	HF
Market Factor											
$\hat{\alpha}_{CAPM}$	0.44***	-0.17**	-0.11	-0.21***	-0.14	-0.10	-0.13	-0.23***	-0.22***	-0.09	-0.15**
$\hat{\alpha}_{FF3}$	0.37***	-0.12	-0.15*	-0.17***	-0.15*	-0.07	-0.13*	-0.21***	-0.21***	-0.09	-0.15*
$\hat{\alpha}_{FF3 + UMD}$	0.45***	-0.13	-0.13	-0.20***	-0.21**	-0.11	-0.20**	-0.27***	-0.25***	-0.13	-0.19**
$\hat{\alpha}_{FF5}$	0.45***	-0.15	-0.14*	-0.17***	-0.25***	-0.08	-0.18**	-0.27***	-0.23***	-0.13	-0.18**
$\hat{\alpha}_{FF5 + UMD}$	0.51***	-0.16	-0.13	-0.20***	-0.29***	-0.11	-0.23***	-0.31***	-0.26***	-0.15*	-0.21***
Value Factor											
$\hat{\alpha}_{CAPM}$	0.11	-0.01	0.02	-0.08	-0.07	-0.02	-0.10*	-0.11**	-0.05	-0.03	-0.04
$\hat{\alpha}_{FF3}$	0.10	-0.01	0.03	-0.07	-0.07	-0.02	-0.10*	-0.09*	-0.05	-0.03	-0.04
$\hat{\alpha}_{FF3 + UMD}$	0.20**	-0.06	-0.03	-0.09*	-0.08	-0.06	-0.13**	-0.14**	-0.08	-0.04	-0.06
$\hat{\alpha}_{FF5}$	0.17*	-0.01	0.00	-0.11**	-0.11**	-0.05	-0.11*	-0.10*	-0.06	-0.06	-0.06
$\hat{\alpha}_{FF5 + UMD}$	0.24**	-0.05	-0.04	-0.13**	-0.12**	-0.08	-0.13**	-0.14**	-0.08	-0.07	-0.08
Size Factor											
$\hat{\alpha}_{CAPM}$	0.10	-0.10*	-0.13**	-0.07	-0.04	-0.03	-0.06	-0.05	-0.05	-0.01	-0.03
$\hat{\alpha}_{FF3}$	0.11*	-0.10*	-0.11**	-0.06	-0.04	-0.03	-0.07	-0.08	-0.07	-0.05	-0.06
$\hat{\alpha}_{FF3 + UMD}$	0.12**	-0.13**	-0.09*	-0.05	-0.05	-0.03	-0.08	-0.07	-0.05	-0.03	-0.04
$\hat{\alpha}_{FF5}$	0.15**	-0.12**	-0.12**	-0.08	-0.02	0.01	-0.03	-0.05	-0.06	-0.07	-0.06
$\hat{\alpha}_{FF5 + UMD}$	0.15**	-0.15**	-0.10*	-0.07	-0.03	0.00	-0.04	-0.05	-0.04	-0.05	-0.05
Momentum Factor											
$\hat{\alpha}_{CAPM}$	0.12	-0.02	-0.10	-0.11*	-0.10	-0.14**	-0.08**	-0.03	-0.13**	-0.21***	-0.17***
$\hat{\alpha}_{FF3}$	0.02	0.03	-0.03	-0.05	-0.07	-0.07	-0.06	0.01	-0.10	-0.18***	-0.14***
$\hat{\alpha}_{FF3 + UMD}$	0.00	0.06	-0.02	-0.06	-0.11	-0.10	-0.07	-0.01	-0.10	-0.18***	-0.14**
$\hat{\alpha}_{FF5}$	-0.02	0.06	0.00	-0.04	-0.08	-0.05	-0.07	0.00	-0.09	-0.18***	-0.13***
$\hat{\alpha}_{FF5 + UMD}$	-0.03	0.08	0.00	-0.05	-0.11	-0.08	-0.08	-0.01	-0.09	-0.18***	-0.13**
Orthogonalized Value Factor											
$\hat{\alpha}_{CAPM}$	0.05	-0.01	0.07	-0.08	-0.06	0.00	-0.08	-0.06	-0.14***	0.02	-0.06
$\hat{\alpha}_{FF3}$	0.04	-0.01	0.07	-0.07	-0.08	0.00	-0.08	-0.04	-0.15***	0.02	-0.06
$\hat{\alpha}_{FF3 + UMD}$	0.15	-0.06	0.02	-0.11*	-0.12**	-0.03	-0.09	-0.07	-0.17***	0.01	-0.08
$\hat{\alpha}_{FF5}$	0.12	-0.01	0.06	-0.12**	-0.14**	-0.05	-0.09	-0.04	-0.14**	0.01	-0.07
$\hat{\alpha}_{FF5 + UMD}$	0.20	-0.05	0.01	-0.15**	-0.17***	-0.07	-0.10	-0.07	-0.16***	0.00	-0.08
Orthogonalized Size Factor											
$\hat{\alpha}_{CAPM}$	0.12	-0.09	-0.10*	-0.06	-0.04	-0.01	-0.11	0.00	0.01	0.09	0.05
$\hat{\alpha}_{FF3}$	0.15**	-0.10*	-0.11*	-0.07	-0.03	-0.02	-0.11	-0.05	-0.04	0.04	0.00
$\hat{\alpha}_{FF3 + UMD}$	0.20**	-0.13*	-0.10*	-0.11	-0.04	-0.03	-0.14**	-0.04	-0.05	0.03	-0.01
$\hat{\alpha}_{FF5}$	0.19**	-0.11*	-0.10*	-0.09	-0.02	0.00	-0.09	-0.02	-0.02	0.00	-0.01
$\hat{\alpha}_{FF5 + UMD}$	0.23***	-0.13*	-0.10*	-0.12	-0.03	-0.01	-0.13*	-0.01	-0.04	0.00	-0.02
Orthogonalized Momentum Factor											
$\hat{\alpha}_{CAPM}$	0.10	-0.03	-0.05	-0.09	-0.01	-0.11*	-0.07	-0.02	-0.10	-0.13**	-0.12**
$\hat{\alpha}_{FF3}$	0.02	0.02	0.00	-0.05	0.02	-0.04	-0.06	0.01	-0.07	-0.10**	-0.09**
$\hat{\alpha}_{FF3 + UMD}$	0.00	0.04	0.00	-0.05	0.00	-0.09	-0.08	0.01	-0.09	-0.11**	-0.10*
$\hat{\alpha}_{FF5}$	-0.02	0.04	0.01	-0.04	0.02	-0.02	-0.08	0.02	-0.06	-0.11**	-0.09*
$\hat{\alpha}_{FF5 + UMD}$	-0.03	0.06	0.01	-0.04	0.01	-0.06	-0.09	0.02	-0.08	-0.12**	-0.10*

Table 5: Persistence of Spectral Covariance Ratios

This table presents the first-order autoregressive coefficient of the LF and HF portfolios' covariance ratios with the market return, i.e., the estimated slope from the model $c_{it} = \alpha + \beta c_{it-1} + \varepsilon_{it}$. The LF and HF covariance ratios are estimated as described in Section 5.1.

	Low Frequency	High Frequency
Autocorrelation	0.79	0.77
Adj. R^2	0.62	0.59

Online Appendix: Frequency Dependent Risk

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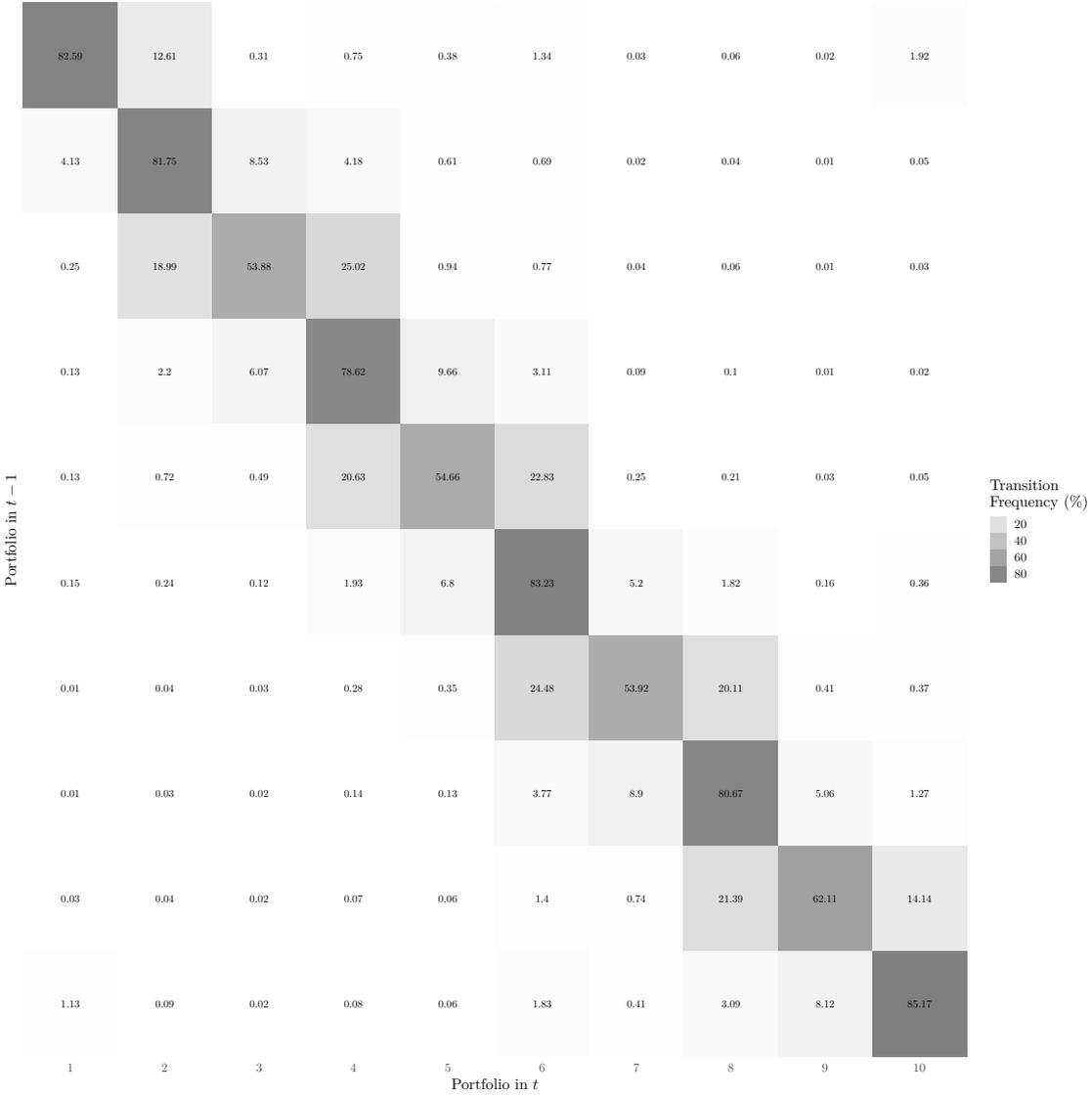
Not for Publication

Figure A.1: Characteristics of the High Frequency Portfolio



This figure shows the normalized rank of nine cross-sectional return characteristics for the long and short leg of the LF portfolio (constructed using market returns). These are the book-to-market ratio, debt-to-price ratio, market equity (size), profitability, investment, operating accruals, the trailing three-year volatility, return one month before portfolio formation (r_{2-1}) and the return from 12 to 2 months before prediction (r_{12-2}). Each month, the characteristics are normalized as in Freyberger et al. (2019), i.e., $\tilde{c}_{it} = \frac{\text{rank}(c_{it})}{N_t + 1}$, where c_{it} denotes the “raw” characteristic value and N_t denotes the number of firms in month t , \tilde{c}_{it} denotes the rank normalized characteristic, which is always between zero and one. The sample period is January 1964 to December 2018.

Figure A.2: Transition Frequencies for the Portfolios Sorted on High Frequency Risk



The figure shows the unconditional transition frequencies between the ten portfolio sorted on LF risk, constructed from market returns, from one month to the next. The sample period is January 1964 to December 2018.

Figure A.3: Price Path and Yearly Return of the High Frequency Portfolio



The top panel of the figure shows the logarithmic price path (cumulative returns) of the HF long-short portfolio. At the beginning of each calendar month, stocks are ranked according to their HF covariance ratio and then assigned to one of ten portfolios. The frequency dependent covariance ratios are estimated as described in 5.1. The LF portfolio is, then, long the 10% stocks with highest HF covariance ratio and short the 10% stocks with lowest HF covariance ratio. All stocks are equally weighted within a portfolio. The gray areas depict NBER recessions. The lower panel of the figure shows the yearly returns of the HF portfolio. Portfolios are rebalanced every month. The sample period is January 1964 to December 2018.