Information Disclosure in Contests: Private versus Public Signals

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Abstract

Two players compete for a prize in an all-pay auction where their private binary valuations are independent from each other. A contest organizer commits to disclose additional information about the opponent's valuation to each player – privately or publicly – to maximize either players' expected payoff or total expected effort. I characterize the unique equilibrium of the contest when the organizer discloses a public signal to all players and a symmetric equilibrium when he discloses a private signal to each. When the organizer discloses privately, I show that any partially informative private signals induce higher expected payoffs for players and lower total expected effort than when no signal is disclosed. When the organizer discloses publicly, I characterize a public disclosure policy that induces higher total expected effort than when no signal is disclosed. I also characterize optimal public signals that maximize players' expected payoff. Finally, the ranking between private and public signals in terms of maximizing players' expected payoff is indeterministic. In terms of revenue ranking, the all-pay auction with the public disclosure policy dominates the first- and the second-price auctions with binary independent private valuation regardless of whether private or public disclosure is used in these winner-pay auctions.

Keywords: All-pay auction, Information disclosure, Private and public signals, Information Design

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1 Introduction

When employees compete for a promotion, firms for market share, or lobbyists for policies, the outcomes of such contests, e.g., payoffs of contestants and their total effort invested, are largely determined by their beliefs about opponents' competitiveness. This notion implies that an informed contest organizer – whether an employer, regulator, or government – can manipulate players' beliefs through information disclosure to select favorable outcomes. This paper studies how the organizer should disclose information before the contests to obtain desirable outcomes.[1]

On the one hand, the contest organizer may disclose information to achieve one of two objectives. A natural objective is to maximize players' total effort invested in competition, e.g., in promotion contests. Alternatively, the organizer may also disclose in order to maximize players' expected payoff. For instance, an industrial association discloses information to improve the welfare of member firms who compete in the same market. On the other hand, information disclosure can usually take one of two forms. The organizer can either *whisper* information to each participant or *broadcast* it to all. The aim of the paper is to find the optimal signals that the organizer whispers or broadcasts in order to achieve a given objective.

I model the contest as a first-price all-pay auction with two players who have independent, private, and binary valuations $\{v_h, v_l\}$ which are players' "type". In addition, each of them also observes a binary signal $s \in \{h, l\}$ disclosed by the organizer. When the organizer *whispers* to each player, he in effect discloses to each player a private signal whose distribution is conditioning on the opponent's valuation. When the organizer *broadcasts* to all players, he in effect discloses a public signal whose distribution is conditioning on all players' valuations.

When signals are disclosed privately, both players' type and the signals they observe are private information. I characterize a symmetric equilibrium and show that the *increment* of the following likelihood ratio

$$\frac{\Pr(s_{-i}|v_i = v_h)}{\Pr(s_{-i}|v_i = v_l)} \tag{1}$$

determines the monotonicity of equilibrium. Specifically, when the *increment* of (1) (from $s_{-i} = l$ to $s_{-i} = h$) is larger than the threshold $v_h/v_l - v_l/v_h$, the equilibrium is nonmonotonic. The intuition is that such an increment measures the informativeness of the private signals. A sufficiently large increment features a sufficiently transparent environment in which players have

[1]It has been shown that the total effort in the contest can be boosted by concealing all players' private information in the all-pay auctions (Fu et al., 2014; Lu et al., 2018), partially revealing such information to the opponents in Tullock contests (Serena, 2017), providing reviews (Gershkov and Perry, 2009) of previous performance or publicly announcing it (Aoyagi, 2010) in multi-stage contests, disclosing opponent's previous performance (Sheremeta, 2010) or his expenditure (Fallucchi et al., 2013) in rent-seeking contests.

to randomize in overlapping intervals. When the *increment* of the likelihood ratio is less than the threshold, however, the equilibrium is monotonic such that the high type bids higher than low type with a probability one. Furthermore, the high (low) type's effort first order stochastically increases (decreases) in the signal it observes. Intuitively, the high type is motivated to exert more effort when the opponent is likely to be a high type as well, whereas the low type is discouraged under the same circumstances.

Given the equilibrium characterized, I show that partially informative private signals always induce a strictly higher payoff for players and a strictly lower total expected effort than when no signal is disclosed (i.e., in the IPV setting). Compared to other auction formats, the all-pay auction induces strictly lower revenue than the second-price auction in which bidding valuation is still the dominant strategy even when players observe the private signals (Fang and Morris, 2006). However, the revenue ranking between the all-pay and the first-price auction is indeterministic.

When the organizer discloses publicly, the unique equilibrium is symmetric and can be nonmonotonic in players' type. I show that the monotonicity of a likelihood ratio

$$\frac{\Pr(v_{-i}|v_i = v_h, s_i)}{\Pr(v_{-i}|v_i = v_l, s_i)} \tag{2}$$

is sufficient and necessary for nonmonotonicity of the unique equilibrium. Monotonically increasing likelihood ratio (2) (*MLRP* condition) features a highly competitive environment in which each player believes that her opponent is equally competitive as she does. In the resulting *Strong Competition Equilibrium* (SCE), each player earns zero expected payoff. Alternatively, a monotonically decreasing likelihood ratio (2) (*rev-MLRP* condition) features an uncompetitive environment in which each player believes that she is likely to compete against the opposite type. In the resulting *Weak Competition Equilibrium* (WCE), players earn a strictly positive ex ante expected payoff. Finally, if (2) is nonmonotonic in v_{-i}, the bidding strategy in the resulting *Monotonic Strategy Equilibrium* (MSE) is monotonically increasing in types.

The intuition behind the connection between (2) and the monotonicity of equilibrium is that the likelihood ratio (2) being monotonic indicates that the public signal is sufficiently informative. In that case, players either learn that they are very likely exposed to the same (increasing (2)) or the opposite (decreasing (2)) type of their opponent. This high level of transparency forces different types to bid in overlapping supports. If (2) is not monotonic so that the signal is not informative enough, then players can hide behind their private information and play the MSE similar to what they do in the IPV setting (Konrad, 2004; Amann and Leininger, 1996).

In terms of the optimal public signals that maximize players' payoff, Lu et al. (2018) analyze partial disclosure policies in the all-pay auction. According to Serena (2017), the partial disclosure policy is a mapping from a set of anonymous valuation profiles,[2] $\{(v_h, v_h), (v_h, v_l), (v_l, v_l)\}$, to a binary decision between *concealing* (C) or *disclosing* (D) each profile to both players.[3] Such partial disclosure policies are special cases of the public signals in the current paper. Lu et al. (2018) show that the disclosure policy $\{C, C, D\}$, i.e., the low type has complete information and the high type holds prior belief, maximizes players' expected payoff, and the maximum is $\min\{p_h(v_h - v_l), p_h p_l v_h\}$. I show that this maximum is the upper bound of payoffs for all public signals. However, I also show that many other public signals induce the same maximum and provide full characterization of these signals.

Lu et al. (2018) also show that fully concealing the valuation profile, i.e., $\{C, C, C\}$, induces maximum total expected effort which is equal to the effort in the IPV setting. In Section 4, however, I characterize a public disclosure policy that induces a total effort strictly higher than such a maximum. It is based on some public signals that induce an SCE with efficient allocation.[4] Azacis and Vida (2015) characterize the equilibrium bidding strategy in the first-price auction (FPA) with any symmetric, possibly correlated, binary signals. The authors show that the revenue of the FPA is at most equal to the revenue in the IPV setting. This result, in fact, implies that the all-pay auction with the public disclosure policy can raise higher revenue than first- and second-price auctions with any symmetric binary signals.

Since private disclosure can, at best, induce a total effort equal to the effort in the IPV setting, public disclosure thus dominates private disclosure in maximizing total effort. Alternatively, in terms of maximizing players' expected payoff numerical examples suggest that there is no general ranking between the two modes of disclosure. Another interesting observation is that the low type player always earns zero payoff in public disclosure but earns positive payoff in private disclosure. Consider a scenario in which both players are low type but only one player observes the correct private signal "l". Since she wins for sure in this scenario, which happens with strictly positive probability, she earns an information rent. An inequity averse organizer may take this into account when choosing disclosure policies. Finally, the large differences in the outcome of contests under the two modes of disclosure policies suggest that whether the signal disclosed is common knowledge or not is an important aspect of information design in contests.

[2] The disclosure policy is anonymous in the sense that the policy depends on the type profile that does not differentiate the identities of players. See Serena (2017) for more details.

[3] For example, $\{C, C, D\}$ corresponds to the disclosure policy which conceals the valuation profile (v_h, v_h), i.e., both players have the high valuation, and (v_h, v_l), i.e., players have different valuations, and discloses the profile only when it is (v_l, v_l), i.e., both players have the low valuation.

[4] Note that in the SCE players earn zero expected payoff and thus efficient allocation in such an equilibrium produces the highest possible total effort, which equals the social surplus with allocative efficiency. However, Proposition 4 shows the impossibility of inducing such an equilibrium with both signal realizations "h" and "l".

Related literature: Prior studies on first- and second-price auctions have shown that receiving private signals (Fang and Morris, 2006) or public signals (Azacis and Vida, 2015) enable bidders to earn a higher expected payoff and bid lower in winner-pay auctions with IPV than when no additional signal is observed. This paper shows that, in the all-pay auction, receiving private signals also enable players to earn higher payoff and bid lower. However, receiving public signals may induce players to bid higher and earn lower expected payoff than when no signal is disclosed. In the literature of information disclosure in contests, Lu et al. (2018) also focus on the all-pay auction but restricts attention to public disclosure policies, which are first studied by Serena (2017). This paper extends the analysis of disclosure policies in contests to include not only a larger set of public signals but also conditionally independent private signals.

A strand of rapidly developing literature on information design/Bayesian persuasion in contests has emerged in recent years.[5] On Tullock contests, the literature focuses on public disclosure and studies Bayesian persuasion in a one-sided private information setting (Zhang and Zhou, 2016) as well as partial disclosure policies in a two-sided private information setting (Serena, 2017). On all-pay auction contests, the emerging literature focuses on public Bayesian persuasion in the all-pay auction with either one-sided (Feng and Lu, 2016; Zheng et al., 2018) or two-sided private information (Zheng et al., 2017). Since all existing and ongoing studies focus on public disclosure in contests, comparison between public and private disclosure – as in the current paper – is not possible.[6]

Also related is the literature on the all-pay auction with affiliated values. It shows that the sufficient condition for the existence of monotonic strategy equilibrium is that players' valuations are not "too affiliated" (Chi et al., 2018; Liu and Chen, 2016; Siegel, 2014; Krishna and Morgan, 1997). Analogously, this paper shows that the sufficient condition for the existence of MSE is that the public signal disclosed is not too informative. When the public signal is sufficiently informative in the sense that a likelihood ratio is monotonically increasing, this paper characterizes a SCE based on a similar equilibrium characterized by Chi et al. (2018) for the case when players' valuations are sufficiently affiliated. Similar to Chi et al. (2018) but focusing on a two-player setting, Liu and Chen (2016) characterizes an equilibrium for the case when valuations are "negatively correlated", which is similar to the WCE given in this paper. In a

[5]Prior to studies on Bayesian persuasion and information design in contests, earlier studies focus on comparing effort and welfare between complete information and one-sided private information (Denter et al., 2018), complete information and two-sided private information (Kovenock et al., 2015; Morath and Münster, 2008; Fu et al., 2014).

[6]This paper is also broadly related to the Bayesian persuasion literature (Rayo and Segal, 2010; Kamenica and Gentzkow, 2011) which focuses on designing optimal signals to manipulate the belief of a receiver who then takes an action. The literature has also been extended to multiple-receiver settings (Mathevet et al., 2017; Bergemann and Morris, 2016; Alonso and Câmara, 2016a,b; Wang, 2013). The current paper differs from this literature in that it focuses on information design with a somewhat restricted set of signals but a rather general action space, i.e., the continuous action space in the all-pay auctions.

general setting, Rentschler and Turocy (2016) provide an algorithm to characterize symmetric equilibria for any distribution of discrete signals. Since I focus on designing optimal signals in the all-pay auction, I restrict attention to binary type and binary signal setting.

The remainder of this paper is structured as follows. Section 2 presents the model setup. Section 3 analyzes private disclosure in an all-pay auction and compares revenue across auction formats. Section 4 analyzes public disclosure and characterizes optimal public signals. Section 5 compares the two modes of disclosure. Section 6 discusses the main implications of results and concludes.

2 The model

The Contest: Two risk-neutral players compete for an indivisible prize in a contest. Player i's ($i \in \{1, 2\}$) private valuation is independently drawn from a binary distribution: $v_i = v_h$ with probability $p_h \in (0, 1)$, and $v_i = v_l$ with probability $p_l \in (0, 1)$, where $v_h > v_l > 0$ and $p_h + p_l = 1$. In addition, player i also observes a signal, $s_i \in \{h, l\}$, regarding her opponent's valuation v_{-i}. The distribution of the signal will be discussed in detail shortly.

Players choose their efforts, (b_i, b_{-i}), simultaneously in the contest, which is a first-price all-pay auction. That is, the player who chooses higher effort wins and both players incur the costs of their own efforts, and ties are broken with equal probability. Formally, the player with the valuation v_i chooses effort b_i earns the following payoff:

$$U_i(b_i, b_{-i}, v_i) = \begin{cases} -b_i, \text{ if } b_i < b_{-i} \\ v_i - b_i, \text{ if } b_i > b_{-i} \\ \frac{1}{2}v_i - b_i, \text{ if } b_i = b_{-i} \end{cases}$$

Equilibrium: I refer to a player with valuation v_h (v_l) as a "high (low) type player". I also refer to player i with v_i who observes signal s_i as "type (v_i, s_i) of player i". Denote by $G_{(v_i,s_i)}(b)$ the c.d.f of type (v_i, s_i)'s mixed strategy, and denote by $G_i(b)$ player i's c.d.f of effort. Formally, a Bayesian Nash Equilibrium of the contest is defined as the following.

Definition 1. *A Bayesian Nash Equilibrium (BNE) of the contest is a vector of strategies* $\boldsymbol{G} = (G_1, G_2)$ *such that for all* $b_i \in supp[G_{(v_i,s_i)}]$, *we have*

$$b_i \in \arg\max_b U_i(b, v_i, s_i; G_{-i})$$

Timing: For both private and public disclosure, the game starts when the organizer announces and commits to a disclosure policy. The policy specifies the distribution of signals to be disclosed to players. After each player has observed her valuation and signal realization, she chooses an effort in the contest.

Social surplus and revenue: The social surplus with efficient allocation is $p_l^2 v_l + (1 - p_l^2) v_h$ given that valuations are independent. The total expected effort of the contest without disclosure is equivalent to the seller's revenue in the all-pay auction with IPV, i.e., $(1 - p_h^2) v_l + p_h^2 v_h$.

3 Private signals

In the private signals setting, player i's signal is generated as the following:

$$\begin{aligned} Pr(s_i = l | v_{-i} = v_l) = Pr(s_i = h | v_{-i} = v_h) &= q \in [\frac{1}{2}, 1] \\ Pr(s_i = h | v_{-i} = v_l) = Pr(s_i = l | v_{-i} = v_h) &= 1 - q. \end{aligned}$$

When $q = 0.5$, the information structure corresponds to the IPV setting and $q = 1$ corresponds to complete information setting. The signal s_i is player i's private information. Thus, the type space is two dimensional with four types in total: $(v_i, s_i) \in \{v_h, v_l\} \times \{h, l\}$. Denote by $\Pr(v_{-i}|s_i)$ the probability that the opponent's valuation is v_{-i} conditional on player i's signal s_i. Upon receiving a signal s_i, player i updates her belief according to Baye's rule:

$$\begin{aligned} \Pr(v_h|h) &= \frac{p_h q}{p_h q + p_l (1-q)} \quad \text{and} \quad \Pr(v_l|h) = \frac{p_l (1-q)}{p_h q + p_l (1-q)} \\ \Pr(v_h|l) &= \frac{p_h (1-q)}{p_h (1-q) + p_l q} \quad \text{and} \quad \Pr(v_l|l) = \frac{p_l q}{p_h (1-q) + p_l q} \end{aligned}$$

It becomes clear in the next section that the following condition is sufficient and necessary to allocative efficiency of the contest:

$$\frac{\Pr(s_i = l | v_{-i} = v_h)}{\Pr(s_i = l | v_{-i} = v_l)} = \frac{\Pr(s_i = h | v_{-i} = v_l)}{\Pr(s_i = h | v_{-i} = v_h)} = \frac{1-q}{q} \geqslant \frac{v_l}{v_h}.$$

Such a condition can be rewritten in two (equivalent) versions as given in Condition 1 below.

Condition 1. $\Pr(s_i = l | v_{-i} = v_h) v_h \geqslant \Pr(s_i = l | v_{-i} = v_l) v_l$ *and* $\Pr(s_i = h | v_{-i} = v_h) v_l \leqslant \Pr(s_i = h | v_{-i} = v_l) v_h$.

I refer to the opposite of Condition 1, i.e., $(1-q)/q < v_l/v_h$, as "Condition $\neg 1$".

3.1 Equilibrium

The equilibrium of the contest with private signals is characterized formally in Proposition 7, which is given in Appendix A2. Here, I provide verbal descriptions and graphical illustrations.

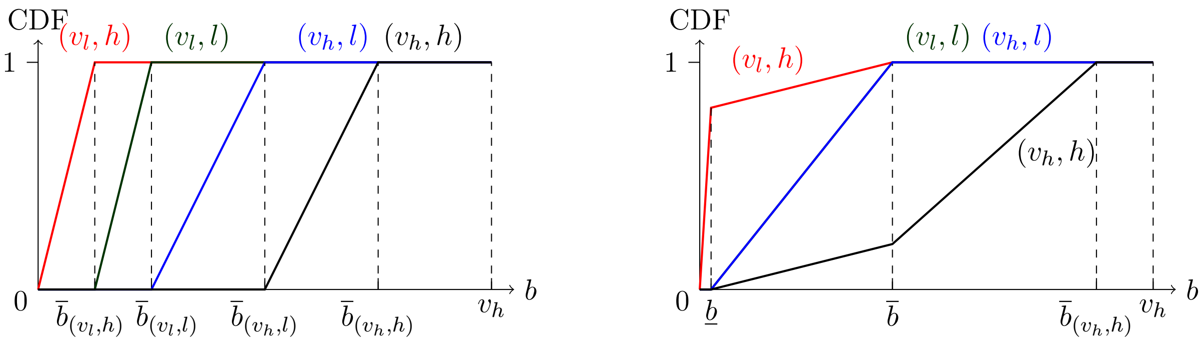

Figure 1: Left panel: MSE; Right panel: NMSE

- **Monotonic strategy equilibrium (MSE)**: see left panel of Figure 1. When Condition 1 is satisfied, there exists a symmetric equilibrium in which all types of players randomize uniformly in connected, nonoverlapping supports. Type (v_l, h) randomizes in the lowest support $[0, \bar{b}_{(v_l,h)}]$, then type (v_l, l) randomizes in a higher support $[\bar{b}_{(v_l,h)}, \bar{b}_{(v_l,l)}]$. Type (v_h, l) randomizes in the support $[\bar{b}_{(v_l,l)}, \bar{b}_{(v_h,l)}]$ and type (v_h, h) randomizes in the highest support $[\bar{b}_{(v_h,l)}, \bar{b}_{(v_h,h)}]$.

- **Nonmonotonic strategy equilibrium (NMSE)**: see right panel of Figure 1. When Condition ¬1 is satisfied, there exists a symmetric equilibrium in which all types of players randomize uniformly in a common support $[\underline{b}, \bar{b}]$. In addition, type (v_l, h) also randomizes uniformly in support $[0, \underline{b}]$ with a positive probability mass and type (v_h, h) also randomizes uniformly in support $[\bar{b}, \bar{b}_{(v_h,h)}]$ with a positive probability mass.

In summary, the contest has a symmetric equilibrium which is monotonic when Condition 1 is satisfied and nonmonotonic when Condition ¬1 is satisfied instead. Note that when $q = 0.5$ Condition 1 is true and the former equilibrium replicates the equilibrium in the IPV setting (Konrad, 2004). Note also that when $q = 1$ Condition ¬1 is true and the latter equilibrium replicates the equilibrium in the complete information setting (Baye et al., 1996).

To understand the monotonicity of equilibrium, note that Condition 1, in fact, implies that

$$\frac{\Pr(s_i = h|v_{-i} = v_h)}{\Pr(s_i = h|v_{-i} = v_l)} - \frac{\Pr(s_i = l|v_{-i} = v_h)}{\Pr(s_i = l|v_{-i} = v_l)} \leqslant \frac{v_h}{v_l} - \frac{v_l}{v_h} \tag{3}$$

i.e., the *increment of likelihood ratio* $\Pr(s_i|v_{-i} = v_h)/\Pr(s_i|v_{-i} = v_l)$ from l to h is no larger than a cutoff $v_h/v_l - v_l/v_h$. The magnitude of such an increment reflects the signal's informativeness. An increment lower than the cutoff means that the signal contains little information about the opponent. Hence, players determine their strategies mostly according to the private valuation rather than the private signal and hence, they play MSE similar as they do in the IPV setting.

Alternatively, Condition ¬ 1 implies that the reverse of (3) is true with strict inequality, which means that the increment of the likelihood ratio is larger than the cutoff. A large increment suggests that receiving signal h indicates a significant higher chance that the opponent has v_h than receiving l. Such a high level of transparency leads high type players to bid similar efforts as low type players when they both believe that the opponent is likely to be low type and hence, the nonmonotonicity in NMSE.[7]

Analogous to Condition 1, a "monotonicity" condition is shown to be sufficient for the existence of MSE in the all-pay auction with affiliated values (Siegel, 2014; Krishna and Morgan, 1997) and a "highly competitive" environment arises where only nonmonotonic equilibria exist when such a condition is violated (Rentschler and Turocy, 2016; Chi et al., 2018). Condition 1 requires that the signals players observe not be "too informative", similar to the interpretation that monotonicity condition requires the signals players receive are not "too affiliated".

3.2 Expected payoff and total expected effort

Proposition 1 shows that both players benefit from partially informative private signals.

Proposition 1. *When the organizer discloses private signals, player i ($i = 1, 2$) earns strictly higher expected payoff when $q \in (\frac{1}{2}, 1)$ than when $q = \frac{1}{2}, 1$, and the total expected effort is strictly lower when $q \in (\frac{1}{2}, 1]$ than when $q = \frac{1}{2}$.*

Figure 2 presents two examples of player i's expected payoff which are always higher when $q \in (0.5, 1)$ than when $q = 0.5, 1$. The thick green curves correspond to $q \leqslant q^*$ is the expected payoff of each player in MSE and the thick red curves correspond to $q > q^*$ is the expected payoff of each player in NMSE.

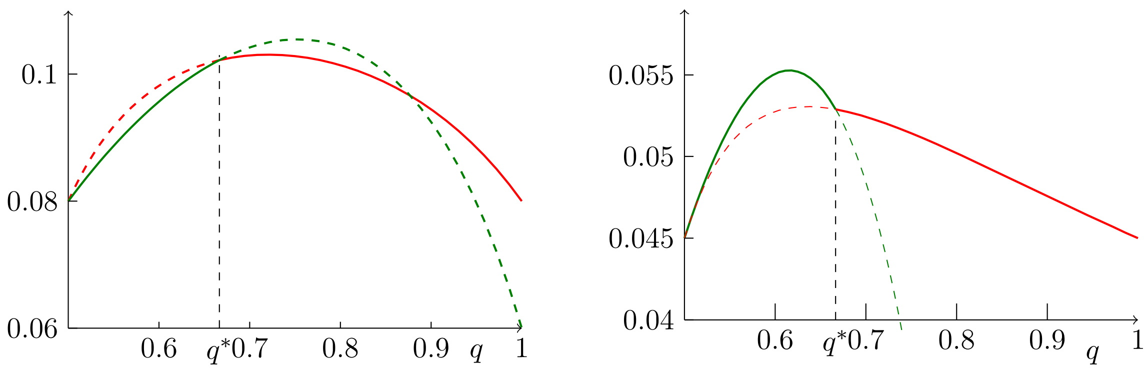

Figure 2: Thick green (red) curves for $q \leqslant q^*$ ($q > q^*$) is the expected payoff in MSE (NMSE) Left panel: $v_h = 1$, $v_l = 0.5$, and $p_h = 0.2$; Right panel: $v_h = 1$, $v_l = 0.5$, and $p_h = 0.9$.

Figure 3 shows the effort of each player corresponds to the same examples in Figure 2. The

[7]Note that type (v_h, l) and (v_l, l) play exactly the same mixed strategy in NMSE.

thick green curves are the expected effort of each player in MSE and the thick red ones are the effort in NMSE. As seen from the figures, the difference between NMSE and MSE has significant impact on the expected effort. The expected effort experiences a sudden drop at $q = q^* \equiv \frac{v_h}{v_h+v_l}$ where transition from MSE to NMSE takes place. These examples are consistent with Kovenock et al. (2015) and Morath and Münster (2008) in the sense that the total expected effort is lower when $q = 1$ (complete information) than when $q = \frac{1}{2}$ (IPV).

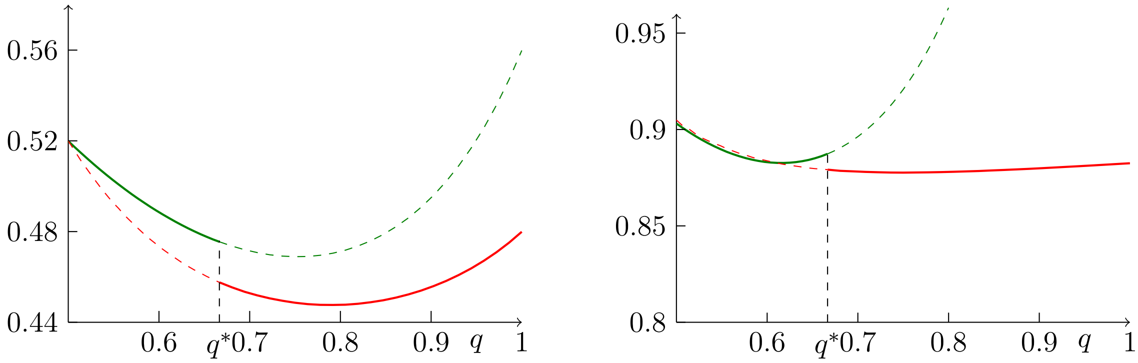

Figure 3: Thick green (red) curves for $q \leqslant q^*$ $(q > q^*)$ is the expected payoff in MSE (NMSE) Left panel: $v_h = 1$, $v_l = 0.5$, and $p_h = 0.2$; Right panel: $v_h = 1$, $v_l = 0.5$, and $p_h = 0.9$.

3.3 Revenue comparison across auction formats

Fang and Morris (2006) analyze the first- and the second-price winner-pay auctions with the same private signals setting in the current paper. This allows a revenue comparison across the all-pay auction and the winner-pay auctions with private signals.

Proposition 2 (Revenue Ranking)**.** *In terms of revenue ranking:*

- *The second price auction dominates the all-pay auction when* $q \in (\frac{1}{2}, 1]$.
- *The ranking between the first-price and the all-pay auction is indeterministic.*

When players observe private signals, bidding one's valuation is weakly dominant in the SPA (Fang and Morris, 2006). This implies that the revenue in the SPA is equal to the one in the IPV setting which, according to the revenue equivalence theorem, is the same as the revenue of the APA in the IPV setting. Thus, the fact that the total expected effort in the APA is strictly lower when $q \in (\frac{1}{2}, 1)$ implies that, with private signals, the SPA raises higher revenue than the APA. See Figure 4 for examples that illustrate the results.

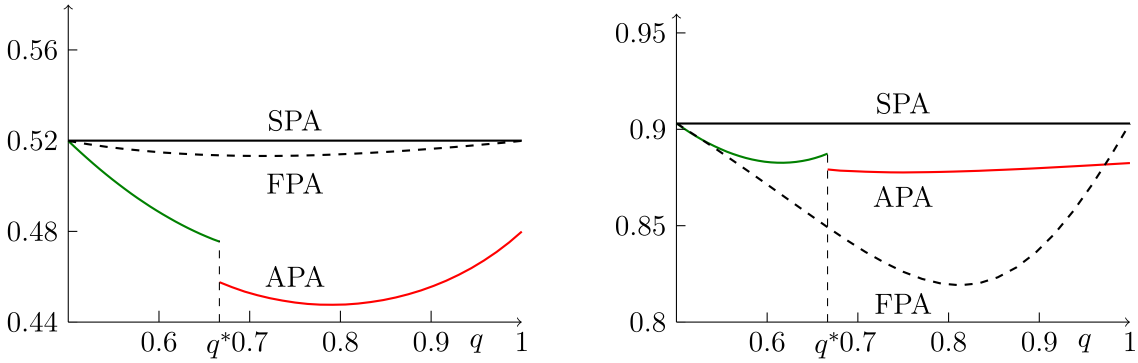

Figure 4: Revenue: FPA (dashed); APA (solid); SPA (horizontal)
Left panel: $v_h = 1, v_l = 0.5, p_h = 0.2$; Right panel: $v_h = 1, v_l = 0.5, p_h = 0.9$

4 Public signals

For public signals, it is always the case that $s_i = s_{-i} = s$ and that s is common knowledge. We focus only on distribution of signals that are symmetric across players. Specifically, the distribution of public signals is determined by the following parameters:

$$\begin{aligned} \alpha_h &= \Pr(s_1 = s_2 = h | v_1 = v_2 = v_h) & \alpha_l &= \Pr(s_1 = s_2 = l | v_1 = v_2 = v_h) \\ \beta_h &= \Pr(s_1 = s_2 = h | v_1 \neq v_2) & \beta_l &= \Pr(s_1 = s_2 = l | v_1 \neq v_2) \\ \gamma_h &= \Pr(s_1 = s_2 = h | v_1 = v_2 = v_l) & \gamma_l &= \Pr(s_1 = s_2 = l | v_1 = v_2 = v_l) \end{aligned}$$

where $\alpha_s, \beta_s, \gamma_s \in [0, 1]$ for $s \in \{h, l\}$ and $\alpha_h + \alpha_l = \beta_h + \beta_l = \gamma_h + \gamma_l = 1$. Here, α_s (γ_s) is the probability that players receive signal s when both players have high (low) valuation. Alternatively, β_s is the probability that players receive signal s when they have different valuations. We refer to a given public signal by its parameter vector $(\alpha_h, \beta_h, \gamma_h)$ since the rest of parameters $\alpha_l, \beta_l, \gamma_l$ can be uniquely determined. See Table 1 for the (joint) distribution of the class of public signals I study in this paper.

	(v_h, h)	(v_h, l)	(v_l, h)	(v_l, l)
(v_h, h)	$p_h^2 \alpha_h$	0	$p_h p_l \beta_h$	0
(v_h, l)	0	$p_h^2 \alpha_l$	0	$p_h p_l \beta_l$
(v_l, h)	$p_h p_l \beta_h$	0	$p_l^2 \gamma_h$	0
(v_l, l)	0	$p_h p_l \beta_l$	0	$p_l^2 \gamma_l$

Table 1: Public signal $(\alpha_h, \beta_h, \gamma_h)$

The above public signal captures the IPV setting when $(\alpha_h, \beta_h, \gamma_h) = (0, 0, 0)$, $(1, 1, 1)$, or

$(0.5, 0.5, 0.5)$, and captures complete information setting by $(\alpha_h, \beta_h, \gamma_h) = (1, 0, 1)$ or $(0, 1, 0)$. It also captures partial disclosure policy $\{C, C, D\}$ in Lu et al. (2018) and Serena (2017) by $(\alpha_h, \beta_h, \gamma_h) = (0, 0, 1)$ or $(1, 1, 0)$.[8]

Denote by $\Pr(v_{-i}|v_i, s)$ the probability that player $-i$ has value v_{-i} conditional on player i has value v_i and receives signal s. Thus, the conditional probabilities can be written as:

$$\Pr(v_h|v_h, h) = \frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h} \qquad \Pr(v_h|v_l, h) = \frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h}$$

$$\Pr(v_h|v_h, l) = \frac{p_h\alpha_l}{p_h\alpha_l + p_l\beta_l} \qquad \Pr(v_h|v_l, l) = \frac{p_h\beta_l}{p_h\beta_l + p_l\gamma_l}$$

When "h" is observed, Condition 2 and 3 determine monotonicity of equilibrium:

Condition 2. $\Pr(v_l|v_h, h)v_h \geqslant \Pr(v_l|v_l, h)v_l$,

Condition 3. $\Pr(v_h|v_h, h)v_h \geqslant \Pr(v_h|v_l, h)v_l$,

while Condition 4 and 5 determine monotonicity of equilibrium when "l" is observed:

Condition 4. $\Pr(v_l|v_h, l)v_h \geqslant \Pr(v_l|v_l, l)v_l$,

Condition 5. $\Pr(v_h|v_h, l)v_h \geqslant \Pr(v_h|v_l, l)v_l$.

We also define Condition ¬2, ¬3, ¬4, and ¬5 as the corresponding conditions that are the reverse of the above conditions. For instances, Condition ¬2 is $\Pr(v_l|v_h, h)v_h < \Pr(v_l|v_l, h)v_l$, and Condition ¬5 is $\Pr(v_h|v_h, l)v_h < \Pr(v_h|v_l, l)v_l$.

For signal realization h there are three possible combinations of conditions which determine equilibrium of the contest: (a) Condition 2 and 3 satisfy; (b) Condition ¬2 and 3 satisfy; (c) Condition 2 and ¬3.[9] For signal realization l there are three possible combinations analogously.

4.1 Equilibrium

The equilibrium of the contest with public signals is characterized in Proposition 8 given in Appendix A. Here, I illustrate the equilibrium with verbal descriptions and graphics.

For each signal realization "h" and "l", there are three types of equilibrium and each type corresponds to one of the combinations of conditions. When "h" is observed, there are three types of equilibrium as given below:

[8] See Appendix A for a complete summary of special cases in public signals.

[9] Conditions ¬2 and ¬3 cannot be both satisfied, as Condition ¬2 implies $\Pr(v_h|v_h, h)v_h = v_h - \Pr(v_l|v_h, h)v_h > v_h - \Pr(v_l|v_l, h)v_l > \Pr(v_h|v_l, h)v_l$ which contradicts Condition ¬3. Similarly, Conditions ¬4 and ¬5 cannot be both satisfied.

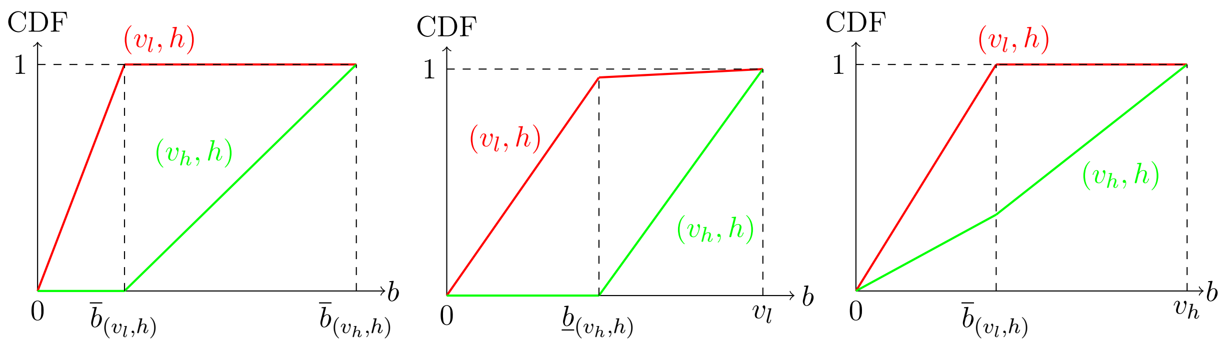

Figure 5: Equilibrium mixed strategies under different conditions
Left panel: MSE (2 & 3); Middle panel: WCE (2 & ¬3); Right panel: SCE (¬2 & 3)

- **Monotonic strategy equilibrium (MSE)**: see the left panel of Figure 5. Type (v_l, h) randomizes uniformly in the support $[0, \bar{b}_{(v_l,h)}]$, while type (v_h, h) randomizes uniformly in the support $[\bar{b}_{(v_l,h)}, \bar{b}_{(v_h,h)}]$. Monotonicity of strategy in this equilibrium ensures efficient allocation. The sufficient and necessary condition of the MSE is Condition 2 and Condition 3.

- **Weak competition equilibrium (WCE)**: see the middle panel of Figure 5. Type (v_l, h) randomizes uniformly in the support $[0, \underline{b}_{(v_h,h)}]$, while both type (v_h, h) and (v_l, h) randomize uniformly in the support $[\underline{b}_{(v_h,h)}, v_l]$. Such nonmonotonic equilibrium strategy causes some efficiency loss. The sufficient and necessary condition of WCE is Condition 2 and Condition ¬3. They jointly imply the following two (equivalent) versions of *"reversed" monotonic likelihood ratio property* (hereafter ***rev-MLRP***):

$$\frac{\Pr(v_h|v_h,h)}{\Pr(v_h|v_l,h)} < \frac{v_l}{v_h} \leqslant \frac{\Pr(v_l|v_h,h)}{\Pr(v_l|v_l,h)} \quad \Leftrightarrow \quad \frac{\Pr(v_l|v_l,h)}{\Pr(v_l|v_h,h)} \leqslant \frac{v_h}{v_l} < \frac{\Pr(v_h|v_l,h)}{\Pr(v_h|v_h,h)}$$

  The former (latter) indicates that, compared to the low (high) type, the high (low) type player is more likely to compete against a low (high) type opponent. Thus, players are likely to compete against the opposite type of the opponent. In such an uncompetitive environment, the high type of each player always earns a payoff of $p_h(v_h - v_l)$ since the upper bound of equilibrium support is v_l.

- **Strong competition equilibrium (SCE)**: see the right panel of Figure 5. Both type (v_l, h) and (v_h, h) randomize uniformly in the support $[0, \bar{b}_{(v_l,h)}]$, while type (v_h, h) also randomizes uniformly in the support $[\bar{b}_{(v_l,h)}, v_h]$. Such nonmonotonic strategy causes some efficiency loss in the equilibrium. The sufficient and necessary condition of SCE is Condition ¬2 and Condition 3. They jointly imply the following two (equivalent) versions of *monotonic likelihood ratio property* (hereafter ***MLRP***):

$$\frac{\Pr(v_l|v_h,h)}{\Pr(v_l|v_l,h)} < \frac{v_l}{v_h} \leqslant \frac{\Pr(v_h|v_h,h)}{\Pr(v_h|v_l,h)} \quad \Leftrightarrow \quad \frac{\Pr(v_h|v_l,h)}{\Pr(v_h|v_h,h)} \leqslant \frac{v_h}{v_l} < \frac{\Pr(v_l|v_l,h)}{\Pr(v_l|v_h,h)}$$

The former (latter) indicates that high (low) type is more likely to compete against a high (low) type opponent. Thus, players compete in a very competitive environment in which they are likely to be evenly matched. In fact, such fierce competition drives all types of players' payoffs down to zero.

There are three similar equilibria for signal l: MSE/WCE/SCE when Condition 4/4/¬4 and 5/¬5/5 are satisfied, respectively. The analysis with regard to "h" applies equally to "l".

4.2 Optimal public signal: Payoff maximization

We now turn to the optimal public signals which maximize each player's expected payoff.

Proposition 3. *The maximum expected payoff of each player when the organizer discloses public signals is* $\min\{p_h(v_h - v_l), p_h p_l v_h\}$. *Furthermore, the following signals induce the maximum* $p_h(v_h - v_l)$ *when* $p_h v_h \leqslant v_l$*:*

- $\alpha_h = \beta_h = 1$*, and* $\gamma_h \leqslant \frac{v_l - p_h v_h}{p_l v_h}$*;*
- $\alpha_h = \beta_h = 0$*, and* $\gamma_h > 1 - \frac{v_l - p_h v_h}{p_l v_h}$.

and the following signals induce the maximum $p_h p_l v_h$ *when* $p_h v_h \geqslant v_l$*:*

- $\alpha_h \geqslant \frac{p_l}{p_h} \frac{v_l}{v_h - v_l}$, $\beta_h = 1$*, and* $\gamma_h = 0$*;*
- $\alpha_h \leqslant 1 - \frac{p_l}{p_h} \frac{v_l}{v_h - v_l}$, $\beta_h = 0$*, and* $\gamma_h = 1$.

The maximum payoff given in Proposition 3 is strictly larger than $p_h p_l(v_h - v_l)$ which is the expected payoff in both the IPV and the complete information settings. See Table 2 for an illustration of the first signal, which provides no information about the opponent to the high type of each player. The player thus holds prior belief. For the low type, it provides noisy information when $\gamma_h > 0$ and complete information when $\gamma_h = 0$. On the other hand, Table 3 illustrates the third signal. It provides the high type noisy information when $\alpha_h < 1$ and no information when $\alpha_h = 1$; however, it provides complete information to the low type of each player.

Lu et al. (2018) characterize the same maximum as given in Proposition 3 and show that the unique optimal disclosure policy satisfies two requirements: (a) the low type of each player

	(v_h,h)	(v_h,l)	(v_l,h)	(v_l,l)
(v_h,h)	p_h^2	0	$p_h p_l$	0
(v_h,l)	0	0	0	0
(v_l,h)	$p_h p_l$	0	$p_l^2\gamma_h$	0
(v_l,l)	0	0	0	$p_l^2\gamma_l$

Table 2: $\alpha_h = \beta_h = 1$, and $\gamma_h \leqslant \frac{v_l - p_h v_h}{p_l v_h}$.

	(v_h,h)	(v_h,l)	(v_l,h)	(v_l,l)
(v_h,h)	$p_h^2\alpha_h$	0	$p_h p_l$	0
(v_h,l)	0	$p_h^2\alpha_l$	0	0
(v_l,h)	$p_h p_l$	0	0	0
(v_l,l)	0	0	0	p_l^2

Table 3: $\alpha_h \geqslant \frac{p_l}{p_h}\frac{v_l}{v_h - v_l}$, $\beta_h = 1$, and $\gamma_h = 0$.

has complete information and (b) the high type of each player holds prior belief. The signals illustrated in Tables 2 and 3 imply that only one of the requirements is needed. In particular, only (b) is met in the former and only (a) is met in the latter. In fact, none of the requirements (a) and (b) is necessary to achieve the maximum payoff. The class of public signals that satisfy Conditions 2, ¬3, 4, and ¬5 does not have to meet either requirements; however, it ensures that WCE is played regardless of whether h or l is observed. Thus, each player earns an ex ante expected payoff of $p_h(v_h - v_l)$, which equals the maximum.[10]

4.3 Optimal public signal: Effort maximization

It is mathematically difficult to find optimal public signals that maximize total effort by solving constrained optimization problems. Instead, I take a different approach. Consider the case when Condition 2 is binding and players observe "h". It can be verified that type (v_h,h) randomizes in $[\bar{b}_{(v_l,h)}, v_h]$ and type (v_l,h) randomizes in $[0, \bar{b}_{(v_l,h)}]$ in the unique equilibrium from Proposition 8 in Appendix A3. A similar equilibrium is played when Condition 4 is binding and players observe "l". I refer to such an equilibrium as the "efficient SCE" since it features both allocative efficiency, which maximizes social surplus, and fierce competition, which drives players' expected payoffs down to zero.

However, Conditions 2 and 4 can never be both binding: see Proposition 4 below.

Proposition 4. *When the organizer discloses public signals, the total expected effort of the contest lies strictly below the social surplus when there is efficient allocation,* $p_l^2 v_l + (1 - p_l^2)v_h$.

Nevertheless, the organizer can increase total effort by choosing the largest possible parameters $(\alpha_h, \beta_h, \gamma_h)$ which ensure that Condition 2 is binding. In this case, whenever players observe signal h the efficient SCE is played. Meanwhile, choosing parameters as high as possible increases the probability that players observe signal h and thus, that SCE is played.

The signal that meets the above criteria is

$$(\alpha_h, \beta_h, \gamma_h) = \left(1, \hat{\beta}_h, 1\right) \tag{4}$$

[10] This corresponds to case 1 in the proof of Proposition 3 given in Appendix B.

where $\hat{\beta}$ is determined by solving for β in binding Condition 2 in which $\alpha_h = \gamma_h = 1$ is set.[11] This signal assigns the value of 1 to two of three parameters $(\alpha_h, \beta_h, \gamma_h)$ and ensures that Condition 2 is binding. It indeed induces an expected effort higher than in the IPV setting under some conditions.

Lemma 1. *The public signal (4) induces a total expected effort strictly higher than the expected effort in the IPV setting, i.e.,* $p_h^2 v_h + (1 - p_h^2) v_l$*, if and only if* $\hat{\beta}_h > B_1 \equiv \frac{v_l}{2v_h + v_l}$.

Another possible candidate signal that meets the requirements is given by

$$(\alpha_h, \beta_h, \gamma_h) = \left(1, \frac{p_h v_l}{v_h - p_l v_l}, \frac{p_h v_l}{v_h - p_l v_l}\right), \tag{5}$$

where the value of β_h and γ_h are obtained by letting $\alpha_h = 1$ and $\beta_h = \gamma_h$ in binding Condition 2. Such a public signal always induces an expected effort strictly higher than in the IPV setting.[12]

Lemma 2. *The public signal given by (5) induces a total expected effort strictly higher than the effort in the IPV setting, i.e.,* $p_h^2 v_h + (1 - p_h^2) v_l$.

The contest organizer can then apply a disclosure policy based on the two signals to increase total effort, see Proposition 5 below.

Proposition 5. *The following public disclosure policy induces a total expected effort strictly higher than in the IPV setting:*

- *Signal (4) if* $\hat{\beta}_h > B_2$;
- *Signal (5) if* $\hat{\beta}_h \leqslant B_2$,

where $B_2 \equiv B_1 \left[1 + p_h v_h \left(\frac{1}{v_h - p_l v_l} + \frac{p_h v_l}{(v_h - p_l v_l)^2}\right)\right]$.

Proposition 5 is illustrated in the following numerical examples.

Example 1. *Let* $v_h = 2, v_l = 1, p_h = 0.5$. *Then,* $\hat{\beta}_h = 0.5$ *in signal (4), and* $\hat{\beta}_h > \max\{B_1 = 0.2, B_2 = 0.3780\}$. *Thus, signal (4) induces higher expected effort,* 1.4375*, than (5),* 1.3611*, which are both greater than the effort in the IPV setting,* 1.25.

Example 2. *Let* $v_h = 2, v_l = 1, p_h = 0.25$. *Then,* $\hat{\beta}_h = 0.2808$ *in signal (4), and* $B_1 = 0.2 < \hat{\beta}_h < B_2 = 0.2960$. *Thus, signal (5) induces higher expected effort,* 1.1075*, than (4),* 1.1004*, which are both greater than the effort in the IPV setting,* 1.06.

[11] Two other possible signals fall in the same category: $(1, 1, p_h v_h/(v_l - p_l v_h))$ and $((v_h - p_l v_l)/p_h v_l, 1, 1)$. However, these are not feasible signals as $p_h v_h/(v_l - p_l v_h) > 1$ and $(v_h - p_l v_l)/p_h v_l > 1$.

[12] Two other possible signals share the same feature as signal (5) are: (a) $\alpha_h = \beta_h = v_l/p_h v_h - p_l/p_h$ and $\gamma_h = 1$; (b) $\alpha_h = \gamma_h = \left[\sqrt{p_l^2(v_h - v_l)^2 + 4p_h^2 v_h v_l} - p_l(v_h - v_l)\right]/2p_h v_l$ and $\beta_h = 1$. The problem with the former is that α_h and β_h do not always lie in $[0, 1]$ and when they do, our simulation suggests the total effort is no higher than (5). The latter signal is not feasible since α_h and γ_h do not lie in the interval $[0, 1]$.

Example 3. *Let* $v_h = 2, v_l = 1, p_h = 0.125$. *Then,* $\hat{\beta}_h = 0.1375$ *in signal (4), and* $\hat{\beta}_h < \min\{B_1 = 0.2, B_2 = 0.2494\}$. *Thus, signal (5) induces higher expected effort,* 1.0291*, than the effort in the IPV setting,* 1.02*, which are both greater than the effort induced by (4),* 0.9985.

4.4 Revenue comparison across auction formats

Azacis and Vida (2015) analyze private and public disclosure in the first-price winner-pay auction with independent private value. The authors show that the highest possible revenue in the first-price auction with either private or public signals is equal to the revenue in the IPV setting. For second price auction, it has been shown that bidding one's own valuation is still dominant under either public (Azacis and Vida, 2015) or private disclosure (Fang and Morris, 2006). Therefore, according to Proposition 5 we have the following result on revenue ranking among the three auction mechanisms when players receive public signals.

Proposition 6 (Revenue Ranking)**.** *Under the public disclosure policy given by Proposition 5, the all-pay auction dominates both the second- and the first-price auction with any private or public signals.*

5 Ranking private and public signals

In this section, we compare private and public signals in terms of maximizing players' expected payoffs or the total expected effort. We start by showing some numerical examples suggesting that there is no general ranking between the two signals in terms of maximizing players' expected payoffs. On the one hand, the following example suggests that the maximum expected payoff induced by public signals is greater than all the possible payoffs induced by private signals.

Example 4. *Suppose* $p_h = \frac{1}{2}$, $v_h = 2$ *and* $v_l = 1$. *According to Proposition 3, the maximum expected payoff for player* i *with public signals is* $p_h p_l v_h = p_h(v_h - v_l) = \frac{1}{2}$. *The expected payoff with private signals in MSE is* $-\frac{5}{2}\left(q - \frac{13}{20}\right)^2 + \frac{49}{160}$ *which reaches its maximum at* $\frac{49}{160} < \frac{1}{2}$ *when* $q = \frac{13}{20}$*, and in NMSE is* $\frac{23}{36} - \frac{1}{9(3q-1)} - \frac{1}{3}q$ *which reaches its maximum at* $\frac{11}{36} < \frac{1}{2}$ *when* $q = \frac{2}{3}$.

On the other hand, the following example shows that a private signal induces an expected payoff of each player greater than the maximum payoff induced by optimal public signals when $p_h v_h \leqslant v_l$, i.e., $p_h(v_h - v_l)$.

Example 5. *Suppose* $p_h = \frac{2}{10}$, $v_h = 2$, $v_l = 1$ *and* $q = 0.7$. *First, note that the cutoff value is* $q^* = \frac{2}{3}$*; thus, with private signals the set of parameters entails an NMSE, which makes the expected payoff* 0.2297. *Second, note also that* $p_h v_h - v_l = -0.6 < 0$*; thus, the maximum expected payoff with public signals is* $p_h(v_h - v_l) = 0.2 < 0.2297$.

Similarly, the following example shows that a private signal induces an expected payoff greater than the maximum expected payoff of each player induced by optimal public signals when $p_h v_h \geqslant v_l$, i.e., $p_h p_l v_h$.

Example 6. *Suppose $p_h = \frac{1}{2}$, $v_h = 100$, $v_l = 1$ and $q = 0.7$. First, note that the cutoff value is $q^* = \frac{100}{101}$; thus, with private signals the set of parameters entails an MSE, which makes the expected payoff* 27.844*. Second, note also that $p_h v_h - v_l = 49 > 0$; thus, the maximum expected payoff with public signals is $p_h p_l v_h = 25 < 27.844$.*

The above examples imply that there is no deterministic ranking between public and private signals in maximizing players' expected payoff.

Let us turn to the comparison of signals in terms of maximizing total expected effort. Recall from Proposition 1 that the total expected effort with private signals when $q \in (\frac{1}{2}, 1]$ is always lower than when $q = \frac{1}{2}$. This implies that the maximum effort that private signals can induce is equal to the effort in the IPV setting, i.e., $(1 - p_h^2)v_l + p_h^2 v_h$. Recall that Proposition 5 shows that a public disclosure policy can induce higher total expected effort than in the IPV setting. Thus, public disclosure dominates private disclosure in maximizing total expected effort.

Finally, there is another interesting observation in comparing the equilibrium strategy in the contest across private and public disclosure. Recall from the preceding section that the low type always earns zero expected payoff in all equilibria when signals are disclosed publicly. When signals are disclosed privately instead, the low type always earns positive expected payoff since type (v_l, l) of each player randomizes in a support strictly above zero in the symmetric equilibrium. To see the intuition behind this, consider a scenario in which both players turn out to be the low type: v_l. In public disclosure, the low type of each player has neither competitive advantage – since she is the weakest type – nor informational advantage – since the signal is public. In private disclosure, however, type (v_l, l) of each player has an informational advantage over type (v_l, h) of the opponent since the latter player receives a wrong signal. Such a possible scenario provides a special "information rent" to a player even if she is the weakest in competition.

6 Conclusion

When players receive additional information regarding the opponent's valuation, they are always better off if the information is disclosed through conditional independent private signals. They may be worse off if the information is disclosed through public signals. In terms of maximizing total expected effort, a public disclosure policy is shown to outperform any private signals.

The comparison between private and public disclosure in contests raises several policy implications. In practice, information is usually disclosed publicly in order to facilitate transparency and to restrict overexpenditure in rent-seeking activities.[13] However, this paper shows that public disclosure can also backfire since it can induce higher total expenditure than no disclosure. This is the case when, for example, a contest organizer who aims to maximize players' expected payoff miscalculates the informativeness of public signals, ends up increasing total expenditure. Private disclosure, however, always reduces expenditure. Intuitively, the distinction arises from the fact that, unlike in private disclosure, signals disclosed publicly are common knowledge and serve as a coordination device which can either soften the competition or intensify it.

There are multiple directions to generalize the current paper. First, a general disclosure policy with partially correlated signals may be able to induce an even higher expected payoff of players. Since none of the two modes of information disclosure in this paper dominates the other, it can then be expected that a combination of the two might perform better. [14] Second, following the literature on auctions with a general information structure (Bergemann et al., 2017), it is also interesting to consider the lower or upper bounds of players' expected payoff or total expected effort when there are no restrictions on information structure. Lastly, the number of players can be generalized to n players.

Appendix A

A1: Special cases in public signals

Table 4 below shows the partial disclosure polices in Serena (2017) and the corresponding public signals in the form of $(\alpha_h, \beta_h, \gamma_h)$.

Partial disclosure policies	$\{C, C, C\}$	$\{D, D, D\}$	$\{C, C, D\}$	$\{D, C, C\}$
Corresponding public signals	$(0, 0, 0)$ $(1, 1, 1)$	$(0, 1, 0)$ $(1, 0, 1)$	$(0, 0, 1)$ $(1, 1, 0)$	$(0, 1, 1)$ $(1, 0, 0)$

Table 4: Partial disclosure polices and corresponding public signals

A2: Equilibrium of the contest with private signals

Proposition 7. *If $q \in [\frac{1}{2}, 1]$, then there exists a symmetric equilibrium in which all types randomize over connected supports.*

When Condition 1 is satisfied, then

[13] Kovenock et al. (2015) show that total expenditure is lower under full disclosure than under no disclosure.

[14] Consistent with this conjecture, Mathevet et al. (2017) shows that the optimal signals in games with finite actions always consist of an optimal private signal and an optimal public signal.

- *type* (v_l, h) *mixes over* $[0, \bar{b}_{(v_l,h)}]$ *uniformly according to*

$$G_{(v_l,h)}(b) = \frac{p_l(1-q) + p_h q}{p_l(1-q)^2 v_l} b,$$

- *type* (v_l, l) *mixes over* $[\bar{b}_{(v_l,h)}, \bar{b}_{(v_l,l)}]$ *uniformly according to*

$$G_{(v_l,l)}(b) = \frac{p_h(1-q) + p_l q}{p_l q^2 v_l} b,$$

- *type* (v_h, l) *mixes over* $[\bar{b}_{(v_l,l)}, \bar{b}_{(v_h,l)}]$ *uniformly according to*

$$G_{(v_h,l)}(b) = \frac{p_h(1-q) + p_l q}{p_h(1-q)^2 v_h} b,$$

- *type* (v_h, h) *mixes over* $[\bar{b}_{(v_h,l)}, \bar{b}_{(v_h,h)}]$ *uniformly according to*

$$G_{(v_h,h)}(b) = \frac{p_l(1-q) + p_h q}{p_h q^2 v_h} b,$$

where

$$\begin{aligned}
\bar{b}_{(v_l,h)} &= \frac{p_l(1-q)^2 v_l}{p_l(1-q) + p_h q}, \\
\bar{b}_{(v_l,l)} &= \bar{b}_{(v_l,h)} + \frac{p_l q^2 v_l}{p_h(1-q) + p_l q}, \\
\bar{b}_{(v_h,l)} &= \bar{b}_{(v_l,l)} + \frac{p_h(1-q)^2 v_h}{p_h(1-q) + p_l q}, \\
\bar{b}_{(v_h,h)} &= \bar{b}_{(v_h,l)} + \frac{p_h q^2 v_h}{p_l(1-q) + p_h q}.
\end{aligned}$$

When Condition $\neg 1$ *is satisfied, then*

- *type* (v_h, h) *mixes over* $[\bar{b}, \bar{b}_{(v_h,h)}]$ *uniformly according to*

$$G_{(v_h,h)}(b) = \frac{p_h q + p_l(1-q)}{p_h q^2 v_h} b - \frac{p_h q v_h + p_l v_l}{p_h q^2 v_h} (1-q),$$

  and mixes over $[\underline{b}, \bar{b}]$ *according to*

$$G_{(v_h,h)}(b) = \frac{1}{2q-1} \left(\frac{q}{v_h} - \frac{1-q}{v_l} \right) b - \frac{1-q}{2q-1} \frac{p_l (1-q)}{p_h q + p_l(1-q)} \frac{q v_l - (1-q) v_h}{q v_h - (1-q) v_l} \frac{v_h - v_l}{v_h},$$

- *type* (v_h, l) *and* (v_l, l) *mix over* $[\underline{b}, \overline{b}]$ *uniformly according to*

$$G_{(v_h,l)}(b) = G_{(v_l,l)}(b) = \frac{1}{2q-1}\left(\frac{q}{v_l} - \frac{1-q}{v_h}\right)b - \frac{1-q}{2q-1}\frac{p_l(1-q)}{p_h q + p_l(1-q)}\frac{v_h - v_l}{v_h},$$

- *type* (v_l, h) *mixes over* $[\underline{b}, \overline{b}]$ *uniformly according to*

$$\begin{aligned} G_{(v_l,h)}(b) &= \frac{1}{2q-1}\left(\frac{q}{v_h} - \frac{1-q}{v_l}\right)b \\ &\quad - \frac{1-q}{2q-1}\frac{p_l(1-q)}{p_h q + p_l(1-q)}\frac{q v_l - (1-q)v_h}{q v_h - (1-q)v_l}\frac{v_h - v_l}{v_h} - \frac{v_h - v_l}{(1-q)v_l - q v_h}, \end{aligned}$$

and mixes over $[0, \underline{b}]$ *according to*

$$G_{(v_l,h)}(b) = \frac{p_h q + p_l(1-q)}{p_l(1-q)^2 v_l}b,$$

where

$$\begin{aligned} \overline{b}_{(v_h,h)} &= \frac{p_h q v_h + p_l(1-q)v_l}{p_h q + p_l(1-q)}, \\ \overline{b} &= \frac{q\left[p_h q + (p_l - p_h)(1-q)\right]v_h - (1-q)^2 p_l v_l}{\left[p_h q + p_l(1-q)\right]\left[q v_h - (1-q)v_l\right]}v_l, \\ \underline{b} &= \frac{p_l(1-q)}{p_h q + p_l(1-q)}\frac{(1-q)v_l}{q v_h - (1-q)v_l}(v_h - v_l). \end{aligned}$$

A3: Equilibrium of the contest with public signals

Proposition 8. *When players receive the public signal* $(\alpha_h, \beta_h, \gamma_h)$*, the unique equilibrium is symmetric, and all types randomize over connected supports.*
Specifically, for type (v_h, h) *and* (v_l, h)*:*

- *If Condition 2 and 3 are satisfied, then type* (v_l, h) *mixes over* $[0, \overline{b}_{(v_l,h)}]$ *and* (v_h, h) *mixes over* $[\overline{b}_{(v_l,h)}, \overline{b}_{(v_h,h)}]$ *according to CDF* $G_{(v_l,h)}(b)$ *and* $G_{(v_h,h)}(b)$*, respectively:*

$$\begin{aligned} G_{(v_l,h)}(b) &= \frac{p_h\beta_h + p_l\gamma_h}{p_l\gamma_h v_l}b, \\ G_{(v_h,h)}(b) &= \frac{p_h\alpha_h + p_l\beta_h}{p_h\alpha_h v_h}b - \frac{v_l}{v_h}\frac{p_l\gamma_h}{p_h\alpha_h}\frac{p_h\alpha_h + p_l\beta_h}{p_h\beta_h + p_l\gamma_h}, \end{aligned}$$

where $\overline{b}_{(v_l,h)} = \frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h}v_l$ *and* $\overline{b}_{(v_h,h)} = \frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h}v_h + \overline{b}_{(v_l,h)}$.

- *If Condition 2 and* $\neg 3$ *are satisfied, then type* (v_h, h) *mixes over* $[\underline{b}_{(v_h,h)}, v_l]$ *according to*

CDF $G_{(v_h,h)}(b)$*:*

$$G_{(v_h,h)}(b) = \frac{\beta_h\left(p_h\beta_h + p_l\gamma_h\right)v_h - \gamma_h\left(p_h\alpha_h + p_l\beta_h\right)v_l}{p_h\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h v_l}b - \gamma_h\frac{p_l\beta_h + p_h\alpha_h}{p_h\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h}(v_h - v_l),$$

while type (v_l, h) *mixes over* $[\underline{b}_{(v_h,h)}, v_l]$ *according to CDF* $G_{(v_l,h)}(b)$*:*

$$G_{(v_l,h)}(b) = \frac{-\alpha_h\left(p_h\beta_h + p_l\gamma_h\right)v_h + \beta_h\left(p_l\beta_h + p_h\alpha_h\right)v_l}{p_l\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h v_l}b + \beta_h\frac{p_l\beta_h + p_h\alpha_h}{p_l\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h}(v_h - v_l),$$

and mixes over $[0, \underline{b}_{(v_h,h)}]$ *according to CDF* $G_{(v_l,h)}(b)$*:*

$$G_{(v_l,h)}(b) = \frac{p_h\beta_h + p_l\gamma_h}{p_l\gamma_h v_l}b,$$

where $\underline{b}_{(v_h,h)} = \frac{\gamma_h(p_l\beta_h + p_h\alpha_h)}{\beta_h(p_h\beta_h + p_l\gamma_h)v_h - \gamma_h(p_l\beta_h + p_h\alpha_h)v_l}\left(v_h - v_l\right)v_l$*.*

- *If Condition ¬2 and 3 are satisfied, then type* (v_h, h) *mixes over* $[0, \overline{b}_{(v_l,h)}]$ *according to CDF* $G_{(v_h,h)}(b)$*:*

$$G_{(v_h,h)}(b) = \frac{\gamma_h\left(p_h\alpha_h + p_l\beta_h\right)v_l - \beta_h\left(p_h\beta_h + p_l\gamma_h\right)v_h}{p_h\left(\alpha_h\gamma_h - \beta_h^2\right)v_l v_h}b,$$

and mixes over $[\overline{b}_{(v_l,h)}, v_h]$ *according to CDF* $G_{(v_h,h)}(b)$*:*

$$G_{(v_h,h)}(b) = \frac{p_h\alpha_h + p_l\beta_h}{p_h\alpha_h v_h}b - \frac{p_l\beta_h}{p_h\alpha_h},$$

while type (v_l, h) *mixes over* $[0, \overline{b}_{(v_l,h)}]$ *according to* $G_{(v_l,h)}(b)$*:*

$$G_{(v_l,h)}(b) = \frac{\alpha_h\left(p_l\gamma_h + p_h\beta_h\right)v_h - \beta_h\left(p_l\beta_h + p_h\alpha_h\right)v_l}{p_l\left(\alpha_h\gamma_h - \beta_h^2\right)v_h v_l}b,$$

where $\overline{b}_{(v_l,h)} = \frac{p_l\left(\alpha_h\gamma_h - \beta_h^2\right)v_h v_l}{\alpha_h(p_l\gamma_h + p_h\beta_h)v_h - \beta_h(p_l\beta_h + p_h\alpha_h)v_l}$*.*

Alternatively, for type (v_h, l) *and* (v_l, l)*:*

- *If Condition 4 and 5 are satisfied, then type* (v_l, l) *mixes over* $[0, \overline{b}_{(v_l,l)}]$ *and* (v_h, l) *mixes over* $[\overline{b}_{(v_l,l)}, \overline{b}_{(v_h,l)}]$ *according to CDF* $G_{(v_l,l)}(b)$ *and* $G_{(v_h,l)}(b)$*, respectively:*

$$G_{(v_l,l)}(b) = \frac{p_h\beta_l + p_l\gamma_l}{p_l\gamma_l v_l}b,$$
$$G_{(v_h,l)}(b) = \frac{p_h\alpha_l + p_l\beta_l}{p_h\alpha_l v_h}b - \frac{v_l}{v_h}\frac{p_l\gamma_l}{p_h\alpha_l}\frac{p_h\alpha_l + p_l\beta_l}{p_h\beta_l + p_l\gamma_l},$$

where $\overline{b}_{(v_l,l)} = \frac{p_l\gamma_l}{p_h\beta_l + p_l\gamma_l}v_l$ *and* $\overline{b}_{(v_h,l)} = \frac{p_h\alpha_l}{p_h\alpha_l + p_l\beta_l}v_h + \overline{b}_{(v_l,l)}$*.*

- *If Condition 4 and ¬5 are satisfied, then type* (v_h, l) *mixes over* $[\underline{b}_{(v_h,l)}, v_l]$ *according to CDF* $G_{(v_h,l)}(b)$*:*

$$G_{(v_h,l)}(b) = \frac{\beta_l (p_h\beta_l + p_l\gamma_l) v_h - \gamma_l (p_h\alpha_l + p_l\beta_l) v_l}{p_h \left(\beta_l^2 - \alpha_l\gamma_l\right) v_h v_l} b - \gamma_l \frac{p_l\beta_l + p_h\alpha_l}{p_h \left(\beta_l^2 - \alpha_l\gamma_l\right) v_h} (v_h - v_l),$$

while type (v_l, l) *mixes over* $[\underline{b}_{(v_h,l)}, v_l]$ *according to CDF* $G_{(v_l,l)}(b)$*:*

$$G_{(v_l,l)}(b) = \frac{-\alpha_l (p_h\beta_l + p_l\gamma_l) v_h + \beta_l (p_l\beta_l + p_h\alpha_l) v_l}{p_l \left(\beta_l^2 - \alpha_l\gamma_l\right) v_h v_l} b + \beta_l \frac{p_l\beta_l + p_h\alpha_l}{p_l \left(\beta_l^2 - \alpha_l\gamma_l\right) v_h} (v_h - v_l),$$

and mixes over $[0, \underline{b}_{(v_h,l)}]$ *according to CDF* $G_{(v_l,l)}(b)$*:*

$$G_{(v_l,l)}(b) = \frac{p_h\beta_l + p_l\gamma_l}{p_l\gamma_l v_l} b,$$

where $\underline{b}_{(v_h,l)} = \frac{\gamma_l (p_l\beta_l + p_h\alpha_l)}{\beta_l (p_h\beta_l + p_l\gamma_l) v_h - \gamma_l (p_l\beta_l + p_h\alpha_l) v_l} (v_h - v_l) v_l$.

- *If Condition ¬4 and 5 are satisfied, then type* (v_h, l) *mixes over* $[0, \overline{b}_{(v_l,l)}]$ *according to CDF* $G_{(v_h,l)}(b)$*:*

$$G_{(v_h,l)}(b) = \frac{\gamma_l (p_h\alpha_l + p_l\beta_l) v_l - \beta_l (p_h\beta_l + p_l\gamma_l) v_h}{p_h \left(\alpha_l\gamma_l - \beta_l^2\right) v_l v_h} b,$$

and mixes over $[\overline{b}_{(v_l,l)}, v_h]$ *according to CDF* $G_{(v_h,l)}(b)$*:*

$$G_{(v_h,l)}(b) = \frac{p_h\alpha_l + p_l\beta_l}{p_h\alpha_l v_h} b - \frac{p_l\beta_l}{p_h\alpha_l},$$

while type (v_l, l) *mixes over* $[0, \overline{b}_{(v_l,l)}]$ *according to* $G_{(v_l,l)}(b)$*:*

$$G_{(v_l,l)}(b) = \frac{\alpha_l (p_l\gamma_l + p_h\beta_l) v_h - \beta_l (p_l\beta_l + p_h\alpha_l) v_l}{p_l \left(\alpha_l\gamma_l - \beta_l^2\right) v_h v_l} b,$$

where $\overline{b}_{(v_l,l)} = \frac{p_l \left(\alpha_l\gamma_l - \beta_l^2\right) v_h v_l}{\alpha_l (p_l\gamma_l + p_h\beta_l) v_h - \beta_l (p_l\beta_l + p_h\alpha_l) v_l}$.

Appendix B: Proofs

Proof of Proposition 1

Proof. Note first that when $q = \frac{1}{2}$, the model is equivalent to the IPV setting and thus it is well known that the expected payoff of a player is $p_h p_l (v_h - v_l)$. Note also that when $q = 1$, the model is equivalent to the complete information setting, thus the expected payoff of a generic player is also $p_h p_l (v_h - v_l)$.

In the MSE where $(1-q)v_h \geqslant qv_l$, the expected payoffs of each type of players are:

$$\begin{aligned}
&(v_h, h)\text{'s payoff:} && \pi^M_{(v_h,h)}(q) = v_h - \frac{p_h q^2 v_h + p_l(1-q)^2 v_l}{p_h q + p_l(1-q)} - \frac{p_l q^2 v_l + p_h(1-q)^2 v_h}{p_h(1-q) + p_l q} \\
&(v_h, l)\text{'s payoff:} && \pi^M_{(v_h,l)}(q) = \frac{p_l q v_h - p_l q^2 v_l}{p_h(1-q) + p_l q} - \frac{p_l(1-q)^2 v_l}{p_h q + p_l(1-q)} \\
&(v_l, l)\text{'s payoff:} && \pi^M_{(v_l,l)}(q) = \frac{p_l q(1-q) v_l}{p_h(1-q) + p_l q} - \frac{p_l(1-q)^2 v_l}{p_h q + p_l(1-q)} \\
&(v_l, h)\text{'s payoff:} && \pi^M_{(v_l,h)}(q) = 0
\end{aligned}$$

Thus, the ex ante expected payoff of each player is given by

$$\begin{aligned}
\pi^M(q) &= p_h[(p_h q + p_l(1-q)]\pi^M_{(v_h,h)}(q) + p_h[p_h(1-q) + p_l q]\pi^M_{(v_h,l)}(q) + p_l[p_h(1-q) + p_l q]\pi^M_{(v_l,l)}(q) \\
&= p_h p_l(v_h - v_l) + p_h p_l(2q-1)\cdot \\
&\quad \left[\frac{p_h(1-q)(p_h q + p_l(1-q))v_h}{(p_h(1-q) + p_l q)(p_h q + p_l(1-q))} - \frac{(-p_h - q - 3p_h q^2 - p_h^2 q + 2p_h^2 q^2 + 4p_h q + q^2)v_l}{(p_h(1-q) + p_l q)(p_h q + p_l(1-q))}\right] \\
&\geqslant p_h p_l(v_h - v_l) + p_h(2q-1)\frac{p_l(1-q)v_l}{p_h q + p_l(1-q)} \\
&> p_h p_l(v_h - v_l)
\end{aligned}$$

The first inequality is due to the condition $(1-q)v_h \geqslant qv_l$.

In the NMSE where $(1-q)v_h \leqslant qv_l$, the expected payoffs of each type of players are the following:

$$\begin{aligned}
&(v_h, h)\text{'s payoff:} && \pi^N_{(v_h,h)}(q) = \frac{p_l(1-q)}{p_h q + p_l(1-q)}(v_h - v_l) \\
&(v_h, l)\text{'s payoff:} && \pi^N_{(v_h,l)}(q) = \frac{(v_h - v_l)}{qv_h - (1-q)v_l}\left(\frac{p_l q^2}{p_h(1-q) + p_l q}v_h - \frac{p_l(1-q)^2)}{p_h q + p_l(1-q)}v_l\right) \\
&(v_l, l)\text{'s payoff:} && \pi^N_{(v_l,l)}(q) = \frac{(1-q)v_l}{qv_h - (1-q)v_l}\left(\frac{q}{p_h(1-q) + p_l q} - \frac{(1-q)}{p_h q + p_l(1-q)})p_l(v_h - v_l\right) \\
&(v_l, h)\text{'s payoff:} && \pi^N_{(v_l,h)}(q) = 0.
\end{aligned}$$

Thus, each player's expected payoff is given by:

$$\begin{aligned}
\pi^N(q) &= p_h p_l(v_h - v_l)\frac{q(p_h q + p_l(1-q))v_h - (1-q)(-p_h - 3q + 2p_h q + 2)v_l}{(qv_h - (1-q)v_l)(p_h q + p_l(1-q))} \\
&= p_h p_l(v_h - v_l)\left[1 + \frac{(1-q)(2q-1)v_l}{(qv_h - (1-q)v_l)(p_h q + p_l(1-q))}\right] \\
&> p_h p_l(v_h - v_l)
\end{aligned}$$

Therefore, each player's expected payoff $\pi(q)$ is equal to $\pi^M(q)$ when $q \leqslant q^*$ and $\pi^N(q)$ when $q > q^*$. This completes the proof of the first part of the equilibrium.

Now we turn to the second part. The social surplus, SS, is $p_l^2 v_l + (1 - p_l^2)v_h$ in the MSE since the prize is efficiently allocated in such an equilibrium. However, SS is less than that in the NMSE. Thus, the total expected effort can be calculated by letting the social surplus to minus the total expected payoff of both players:

$$\begin{aligned} R(q) &= SS - 2\pi(q) \\ &\leqslant p_l^2 v_l + (1 - p_l^2)v_h - 2 \cdot \pi(q) \\ &< p_l^2 v_l + (1 - p_l^2)v_h - 2p_h p_l(v_h - v_l) \\ &= p_h^2 v_h + (1 - p_h^2)v_l \end{aligned}$$

This completes the proof. □

Proof of Proposition 2

Proof. In the SPA, it is still dominant to bid one's own valuation even when players observe private signals. Thus, the prize is efficiently allocated and the revenue of the SPA is $p_h^2 v_h + (1 - p_h^2)v_l$. Proposition 1 then immediately implies the first part of the proposition.

In terms of the FPA, examples in Figure 4 show that, with private signals, the FPA can either raise higher or lower revenue than the APA. The second part of the proposition follows. □

Proof of Proposition 3

Proof. Note first that there are 3×3 possible cases as it is not possible to have either $\neg 2$ and $\neg 3$ satisfied simultaneously or $\neg 4$ and $\neg 5$ satisfied simultaneously. In this proof, we denote by $V_{(v_i,s_i)}(\alpha_h, \beta_h, \gamma_h)$ as the expected payoff of a type (v_i, s_i) player when the public signal players observe has the parameters $(\alpha_h, \beta_h, \gamma_h)$. Furthermore, we denote by $V(\alpha_h, \beta_h, \gamma_h)$ a player's ex ante expected payoff when the public signal is $(\alpha_h, \beta_h, \gamma_h)$.

Case 1: When Condition 2, $\neg 3$, 4 and $\neg 5$ are satisfied, then by the equilibrium strategy given in Proposition 8, the expected payoffs of each type of players are: Type (v_h, h)'s expected payoff: $V_{(v_h,h)}(\alpha_h, \beta_h, \gamma_h) = v_h - v_l$; type (v_l, h)'s expected payoff: $V_{(v_l,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; and type (v_h, l)'s expected payoff: $V_{(v_h,l)}(\alpha_h, \beta_h, \gamma_h) = v_h - v_l$; type (v_l, l)'s expected payoff: $V_{(v_l,l)}(\alpha_h, \beta_h, \gamma_h) = 0$.

Thus, player i's expected payoff is $p_h(v_h - v_l)$ for all values of $(\alpha_h, \beta_h, \gamma_h)$ satisfying Condition 2, $\neg 3$, 4 and $\neg 5$. Suppose $p_h v_h > v_l$ then according to Condition $\neg 3$

$$\frac{p_h \beta_h}{p_h \beta_h + p_l \gamma_h} v_l \geqslant \frac{p_h \alpha_h}{p_h \alpha_h + p_l \beta_h} v_h > \frac{\alpha_h}{p_h \alpha_h + p_l \beta_h} v_l.$$

And according to Condition ¬5

$$\frac{p_h\beta_l}{p_h\beta_l+p_l\gamma_l}v_l \geqslant \frac{p_h\alpha_l}{p_h\alpha_l+p_l\beta_l}v_h > \frac{\alpha_l}{p_h\alpha_l+p_l\beta_l}v_l.$$

After rearrange, we have $\gamma_h\alpha_h < p_h\beta_h(\beta_h-\alpha_h)$ and $\gamma_l\alpha_l < p_h(1-\beta_h)(\alpha_h-\beta_h)$ which then implies $\alpha_h=\beta_h$. This is because $\alpha_h \neq \beta_h$ would imply that there is a negative parameter among $\alpha_h, \alpha_l, \gamma_l$, and γ_h, which contradicts to the assumption that all parameters are nonnegative. But then $\alpha_h=\beta_h$ implies that there is a negative parameter among α_h and γ_h, and another negative parameter among α_l and γ_l. This is also a contradiction. Therefore, it must be true that $p_hv_h \leqslant v_l$ and hence, the maximum payoff in Case 1 is $p_h(v_h-v_l) \leqslant p_hp_lv_h$.

Case 2: When Condition 2, 3, 4 and 5 are satisfied, then the expected payoffs of each type of players are: Type (v_h,h)'s expected payoff: $V_{(v_h,h)}(\alpha_h,\beta_h,\gamma_h)=\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}v_h-\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}v_l$; type (v_l,h)'s expected: $V_{(v_l,h)}(\alpha_h,\beta_h,\gamma_h)=0$; type (v_h,l)'s expected payoff: $V_{(v_h,l)}(\alpha_h,\beta_h,\gamma_h)=\frac{p_l\beta_l}{p_h\alpha_l+p_l\beta_l}v_h-\frac{p_l\gamma_l}{p_h\beta_l+p_l\gamma_l}v_l$; type (v_l,l)'s expected: $V_{(v_l,l)}(\alpha_h,\beta_h,\gamma_h)=0$

Thus, player i's expected payoff is

$$\begin{aligned} V(\alpha_h,\beta_h,\gamma_h) &= p_h(p_h\alpha_h+p_l\beta_h)V_{(v_h,h)}+p_h(p_h\alpha_l+p_l\beta_l)V_{(v_h,l)} \\ &= p_hp_lv_h-\left(\gamma_h\frac{p_h\alpha_h+p_l\beta_h}{p_h\beta_h+p_l\gamma_h}+\gamma_l\frac{p_h\alpha_l+p_l\beta_l}{p_h\beta_l+p_l\gamma_l}\right)p_hp_lv_l. \end{aligned}$$

Note that the expected payoff is maximized if the second term is zero. There are two cases which could let this happen. First, $\alpha_h=\gamma_l=\beta_h=1$. In this case, Condition 2 and 3 now become $p_lv_h \geqslant 0$ and $p_hv_h-v_l \geqslant 0$. Condition 4 and 5 are irrelevant as the probability of receiving a signal l is zero for players with v_h. Second, $\alpha_l=\beta_l=\gamma_h=1$. In this case, Condition 4 and 5 become $p_lv_h \geqslant 0$ and $p_hv_h-v_l \geqslant 0$. Thus, $p_hv_h-v_l \geqslant 0$ is true in both cases and thus, it is true that the maximum satisfies $p_hp_lv_h \leqslant p_h(v_h-v_l)$.

In fact, under these two sets of parameter values, (v_h,h) randomizes in an interval with zero as the lower bound while (v_l,h) bids zero with probability one in the unique equilibrium. This is similar as Case 6 below.

Case 3: When Condition ¬2, 3, 4 and 5 are satisfied, then the expected payoffs of each type of players are: Type (v_h,h)'s expected payoff: $V_{(v_h,h)}(\alpha_h,\beta_h,\gamma_h)=0$; type (v_l,h)'s expected payoff: $V_{(v_l,h)}(\alpha_h,\beta_h,\gamma_h)=0$; type (v_h,l)'s expected payoff: $V_{(v_h,l)}(\alpha_h,\beta_h,\gamma_h)=\frac{p_l\beta_l}{p_h\alpha_l+p_l\beta_l}v_h-\frac{p_l\gamma_l}{p_h\beta_l+p_l\gamma_l}v_l$; type (v_l,l)'s expected payoff: $V_{(v_l,l)}(\alpha_h,\beta_h,\gamma_h)=0$.

Thus, player i's expected payoff is

$$\begin{aligned} V(\alpha_h, \beta_h, \gamma_h) &= p_h \left(p_h \alpha_l + p_l \beta_l\right) \left(\frac{p_l \beta_l}{p_h \alpha_l + p_l \beta_l} v_h - \frac{p_l \gamma_l}{p_h \beta_l + p_l \gamma_l} v_l\right) \\ &= p_h p_l \beta_l v_h - \frac{p_h \alpha_l + p_l \beta_l}{p_h \beta_l + p_l \gamma_l} p_h p_l \gamma_l v_l, \end{aligned}$$

and when $\gamma_l = \beta_h = 0$, $V(\alpha_h, \beta_h, \gamma_h)$ reaches its maximum $p_h p_l v_h$. Check the conditions when $\gamma_l = \beta_h = 0$ in the order of Condition ¬2 and 3, 4 and 5:

$$\begin{aligned} -v_l &< 0 \text{ and } v_h \geqslant 0 \\ \frac{p_l}{p_h \alpha_l + p_l} v_h &\geqslant 0 \text{ and } \frac{p_h \alpha_l}{p_h \alpha_l + p_l} v_h - v_l \geqslant 0. \end{aligned}$$

Thus, Condition ¬2 and 3 are satisfied, but Condition 5 imposes a restriction on α_h: $\alpha_h \in [0, 1 - \frac{p_l}{p_h} \frac{v_l}{v_h - v_l}]$. Since α_h is restricted to be between zero and one, we need $1 - \frac{p_l}{p_h} \frac{v_l}{v_h - v_l} \geqslant 0$ which then implies $p_h v_h \geqslant v_l$, thus, $p_h(v_h - v_l) \geqslant p_h p_l v_h = V(\alpha_h, \beta_h, \gamma_h)$.

Case 4: When Condition ¬2, 3, ¬4 and 5 are satisfied, then the expected payoffs of each type of players are: Type (v_h, h)'s expected payoff: $V_{(v_h,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_l, h)'s expected payoff: $V_{(v_l,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_h, l)'s expected payoff: $V_{(v_h,l)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_l, l)'s expected payoff: $V_{(v_l,l)}(\alpha_h, \beta_h, \gamma_h) = 0$. Thus, player i's ex ante expected payoff is 0.

Case 5: When Condition 2, 3, ¬4 and 5 are satisfied, then the expected payoffs of each type of players are: type (v_h, h)'s expected payoff: $V_{(v_h,h)}(\alpha_h, \beta_h, \gamma_h) = \frac{p_l \beta_h}{p_h \alpha_h + p_l \beta_h} v_h - \frac{p_l \gamma_h}{p_h \beta_h + p_l \gamma_h} v_l$; type (v_l, h)'s expected payoff: $V_{(v_l,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_h, l)'s expected payoff: $V_{(v_h,l)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_l, l)'s expected: $V_{(v_l,l)}(\alpha_h, \beta_h, \gamma_h) = 0$

Thus, player i's ex ante expected payoff is:

$$V(\alpha_h, \beta_h, \gamma_h) = p_h p_l \beta_h v_h - \frac{p_h \alpha_h + p_l \beta_h}{p_h \beta_h + p_l \gamma_h} p_h p_l \gamma_h v_l$$

It is maximized at $p_h p_l v_h$ when $\gamma_l = \beta_h = 1$. See below (in the order of 2 and 3, ¬4 and 3) that Condition ¬4, 5, and 2 are satisfied, whereas Condition 3 imposes a restriction on α_h: $\alpha_h \geqslant \frac{p_l}{p_h} \frac{v_l}{v_h - v_l}$:

$$\begin{aligned} \frac{p_l}{p_h \alpha_h + p_l} v_h &\geqslant 0 \text{ and } \frac{p_h \alpha_h}{p_h \alpha_h + p_l} v_h - v_l \geqslant 0 \\ -v_l &< 0 \text{ and } v_h \geqslant 0 \end{aligned}$$

Since α_h has to be between zero and one, we need $\frac{p_l}{p_h} \frac{v_l}{v_h - v_l} \leqslant 1$ which then implies $v_l \leqslant p_h v_h$, and thus $p_h(v_h - v_l) \geqslant p_h p_l v_h$.

Case 6: When Condition ¬2, 3, 4 and ¬5 are satisfied, then the expected payoffs of each type of players are: Type (v_h, h)'s expected payoff: $V_{(v_h,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_l, h)'s expected payoff: $V_{(v_l,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_h, l)'s expected payoff: $V_{(v_h,l)}(\alpha_h, \beta_h, \gamma_h) = v_h - v_l$; type (v_l, l)'s expected payoff: $V_{(v_l,l)}(\alpha_h, \beta_h, \gamma_h) = 0$.

Thus, player i's ex ante expected payoff is:

$$V(\alpha_h, \beta_h, \gamma_h) = p_h \left(p_h \alpha_l + p_l \beta_l\right) \left(v_h - v_l\right)$$

which is maximized at $p_h \left(v_h - v_l\right)$ when $\alpha_h = \beta_h = 0$. In this case ¬5 implies $p_h v_h < \frac{p_h}{p_h + p_l \gamma_l} v_l < v_l$, then $p_h \left(v_h - v_l\right) \leqslant p_h p_l v_h$. Furthermore, ¬5 also implies that $\gamma_l < \frac{v_l - p_h v_h}{p_l v_h} < 1$.

Condition 4 does not impose restrictions, as it implies either that $\gamma_l \leqslant \frac{p_h v_h}{v_l - p_l v_h}$ when $v_l > p_l v_h$ or that $\gamma_l \geqslant \frac{p_h v_h}{v_l - p_l v_h}$ when $v_l < p_l v_h$. In the latter, any $\gamma_l \geqslant 0$ satisfies the condition. In the former, since it is always true that $v_l - p_l v_h < p_h v_h$, thus any $\gamma_l \leqslant 1$ satisfies the condition. Therefore, the optimal signal is $\alpha_h = \beta_h = 0$ and any $\gamma_l \in [0, \frac{v_l - p_h v_h}{p_l v_h})$ and the maximum is $p_h(v_h - v_l) \leqslant p_h p_l v_h$.

Case 7: When Condition 2, 3, 4 and ¬5 are satisfied, then the expected payoffs of each type of players are: Type (v_h, h)'s expected payoff: $V_{(v_h,h)}(\alpha_h, \beta_h, \gamma_h) = \frac{p_l \beta_h}{p_h \alpha_h + p_l \beta_h} v_h - \frac{p_l \gamma_h}{p_h \beta_h + p_l \gamma_h} v_l$; type (v_l, h)'s expected payoff: $V_{(v_l,h)}(\alpha_h, \beta_h, \gamma_h) = 0$; type (v_h, l)'s expected payoff: $V_{(v_h,l)}(\alpha_h, \beta_h, \gamma_h) = v_h - v_l$; type (v_l, l)'s expected payoff: $V_{(v_l,l)}(\alpha_h, \beta_h, \gamma_h) = 0$.

Thus, player i's ex ante expected payoff is:

$$\begin{aligned}
V(\alpha_h, \beta_h, \gamma_h) &= p_h \left(p_h \alpha_h + p_l \beta_h\right) V_{(v_h,h)} + p_h \left(p_h \alpha_l + p_l \beta_l\right) V_{(v_h,l)} \\
&= p_h p_l v_h - p_h v_l + p_h^2 \alpha_l v_h + \frac{p_h^2 \beta_h}{p_h \beta_h + p_l \gamma_h} \left(p_h \alpha_h + p_l \beta_h\right) v_l \\
&\leqslant p_h p_l v_h - p_h v_l + p_h^2 \alpha_l v_h + p_h^2 \alpha_h v_h \\
&= p_h (v_h - v_l)
\end{aligned}$$

The inequality is due to Condition 3 which is equivalent of $p_h \alpha_h v_h - \frac{p_h \beta_h}{p_h \beta_h + p_l \gamma_h} \left(p_h \alpha_h + p_l \beta_h\right) v_l \geqslant 0$. In other words, when Condition 3 is binding, the expected payoff reaches its maximum of $p_h \left(v_h - v_l\right)$. Thus, any public signal $(\alpha_h, \beta_h, \gamma_h)$ which satisfies the binding Condition 3 and satisfies Condition 2, 4, ¬5 maximizes the expected payoff.

Suppose $p_h v_h > v_l$, then ¬5 implies

$$0 > \frac{p_h \alpha_l}{p_h \alpha_l + p_l \beta_l} v_h - \frac{p_h \beta_l}{p_h \beta_l + p_l \gamma_l} v_l > \left(\frac{\alpha_l}{p_h \alpha_l + p_l \beta_l} - \frac{p_h \beta_l}{p_h \beta_l + p_l \gamma_l}\right) v_l$$

and thus, $\frac{\alpha_l}{p_h \alpha_l + p_l \beta_l} < \frac{p_h \beta_l}{p_h \beta_l + p_l \gamma_l} \Leftarrow \gamma_l \alpha_l < p_h \beta_l \left(\alpha_h - \beta_h\right)$. Similarly, Condition 3 implies

$\frac{\alpha_h}{p_h\alpha_h+p_l\beta_h} < \frac{p_h\beta_h}{p_h\beta_h+p_l\gamma_h}$ thus $\gamma_h\alpha_h < p_h\beta_h\left(\beta_h-\alpha_h\right)$. Hence, it must be true that $\alpha_h = \beta_h$. But then there is a negative parameter among γ_l and α_l, and another negative parameter among γ_h and α_h. This is a contradiction. Therefore, it must be true that $p_h v_h \leqslant v_l$ and thus, $p_h\left(v_h-v_l\right) \leqslant p_h p_l v_h = V(\alpha_h,\beta_h,\gamma_h)$.

Case 8: When Condition 2, ¬3, 4 and 5 are satisfied, then the expected payoffs of each type of players are: Type (v_h,h)'s expected payoff: $V_{(v_h,h)}(\alpha_h,\beta_h,\gamma_h) = v_h - v_l$; type (v_l,h)'s expected payoff: $V_{(v_l,h)}(\alpha_h,\beta_h,\gamma_h) = 0$; type (v_h,l)'s expected payoff: $V_{(v_h,l)}(\alpha_h,\beta_h,\gamma_h) = \frac{p_l\beta_l}{p_h\alpha_l+p_l\beta_l}v_h - \frac{p_l\gamma_l}{p_h\beta_l+p_l\gamma_l}v_l$; type (v_l,l)'s expected payoff: $V_{(v_l,l)}(\alpha_h,\beta_h,\gamma_h) = 0$.

Thus, player i's ex ante expected payoff is:

$$\begin{aligned} V(\alpha_h,\beta_h,\gamma_h) &= p_h\left(p_h\alpha_h+p_l\beta_h\right)V_{(v_h,h)}(\alpha_h,\beta_h,\gamma_h) + p_h\left(p_h\alpha_l+p_l\beta_l\right)V_{(v_h,l)}(\alpha_h,\beta_h,\gamma_h) \\ &= p_h\left(p_h\alpha_h+p_l\right)v_h - p_h\left(p_h\alpha_h+p_l\beta_h+\frac{p_h\alpha_l+p_l\beta_l}{p_h\beta_l+p_l\gamma_l}p_l\gamma_l\right)v_l \end{aligned}$$

It is decreasing in γ_l, thus let $\gamma_l = 0$, and we have

$$V(\alpha_h,\beta_h,\gamma_h) = p_h\left(p_h\alpha_h\left(v_h-v_l\right) - p_l\beta_h v_l\right) + p_h p_l v_h$$

which is increasing in α_h and decreasing in β_h. Now, given that $\gamma_l = 0$, the conditions become the following (in the order of Condition 2 and ¬3, 4 and 5)

$$\begin{aligned} \frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}v_h - \frac{p_l}{p_h\beta_h+p_l}v_l &\geqslant 0 \text{ and } \frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h}v_h - \frac{p_h\beta_h}{p_h\beta_h+p_l}v_l < 0 \\ \frac{p_l\beta_l}{p_h\alpha_l+p_l\beta_l}v_h &\geqslant 0 \text{ and } \frac{p_h\alpha_l}{p_h\alpha_l+p_l\beta_l}v_h - v_l \geqslant 0 \end{aligned}$$

Condition 5 then implies: $p_h\left(v_h-v_l\right) - p_l v_l \geqslant p_h\alpha_h\left(v_h-v_l\right) - p_l\beta_h v_l$ and thus, $V(\alpha_h,\beta_h,\gamma_h) \leqslant p_h\left(v_h-v_l\right)$. To reach the maximum, Condition 5 must be binding, i.e., $\frac{p_h\alpha_l}{p_h\alpha_l+p_l\beta_l}v_h = v_l$.

Suppose $p_h v_h > v_l$, then by binding Condition 5 we have $v_l = \frac{p_h\alpha_l}{p_h\alpha_l+p_l\beta_l}v_h > \frac{\alpha_l}{p_h\alpha_l+p_l\beta_l}v_l$ thus $\alpha_h > \beta_h$. This then violates Condition ¬3 as $\frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h}v_h - \frac{p_h\beta_h}{p_h\beta_h+p_l}v_l > \left(\frac{\alpha_h}{p_h\alpha_h+p_l\beta_h} - \frac{p_h\beta_h}{p_h\beta_h+p_l}\right)v_l > \left(1-\frac{p_h\beta_h}{p_h\beta_h+p_l}\right)v_l > 0$. Contradiction. Thus, it must be true that $p_h\left(v_h-v_l\right) \leqslant p_h p_l v_h$.

Case 9: When Condition 2, ¬3, ¬4 and 5 are satisfied, then the expected payoffs of each type of players are: Type (v_h,h)'s expected payoff: $V_{(v_h,h)}(\alpha_h,\beta_h,\gamma_h) = v_h - v_l$; type (v_l,h)'s expected payoff: $V_{(v_l,h)}(\alpha_h,\beta_h,\gamma_h) = 0$; type (v_h,l)'s expected payoff: $V_{(v_h,l)}(\alpha_h,\beta_h,\gamma_h) = 0$; type (v_l,l)'s expected payoff: $V_{(v_l,l)}(\alpha_h,\beta_h,\gamma_h) = 0$.

Thus, player i's ex ante expected payoff is:

$$V(\alpha_h,\beta_h,\gamma_h) = p_h\left(p_h\alpha_h+p_l\beta_h\right)\left(v_h-v_l\right)$$

which is maximized at $p_h(v_h - v_l)$ when $\alpha_h = \beta_h = 1$. In this case, Condition ¬3 implies $p_h v_h < \frac{p_h}{p_h + p_l \gamma_h} v_l < v_l$ which then implies that $p_h(v_h - v_l) < p_h p_l v_h$. The condition also implies that $\gamma_h < \frac{v_l - p_h v_h}{p_l v_h} < 1$.

Condition 2, in fact, does not impose a constraint. It implies that $\gamma_h \leqslant \frac{p_h v_h}{v_l - p_l v_h}$ when $v_l > p_l v_h$ and $\gamma_h \geqslant \frac{p_h v_h}{v_l - p_l v_h}$ when $v_l < p_l v_h$. In the latter, any $\gamma_h \geqslant 0$ satisfies the condition. In the former, since it is always true that $v_l - p_l v_h < p_h v_h$, any $\gamma_h \leqslant 1$ satisfies the condition. Therefore, the optimal signal in case 9 is $\alpha_h = \beta_h = 1$ and any $\gamma_h \in [0, \frac{v_l - p_h v_h}{p_l v_h}]$.

We can now conclude that the maximum expected payoff is $\min\{p_h p_l v_h, p_h(v_h - v_l)\}$.

As for the optimal public signals given in the proposition, the first signal corresponds to case 9, the second to case 6, the third to case 5, and the last to case 3, respectively. □

Proof of Proposition 4

Proof. Suppose Condition 2 and 4 are both binding. Denote by $r_1 = \alpha_h/\beta_h, r_2 = \beta_h/\gamma_h, r_3 = \alpha_l/\beta_l$, and $r_4 = \beta_l/\gamma_l$. Thus, $r_1\beta_h = \alpha_h$ and $r_3\beta_l = \alpha_l$, and hence $\beta_l = \frac{1-r_1}{r_3-r_1}$, $\beta_h = \frac{r_3-1}{r_3-r_1}$. Similarly, by $r_2\gamma_h = \beta_h$ and $r_4\gamma_l = \beta_l$, and hence $\gamma_l = \frac{r_2-1}{r_2-r_4}$, $\gamma_h = \frac{1-r_4}{r_2-r_4}$. Then, according to $\alpha_h = r_1\beta_h = r_1 r_2 \gamma_h$ and $\alpha_l = r_3\beta_l = r_3 r_4 \gamma_l$, we have

$$\frac{r_1(r_3-1)}{r_3-r_1} = \frac{r_1 r_2(1-r_4)}{r_2-r_4} \tag{6}$$

$$\frac{r_3(1-r_1)}{r_3-r_1} = \frac{r_3 r_4(r_2-1)}{r_2-r_4} \tag{7}$$

Condition 2 and 4 are binding also implies that

$$\frac{\frac{p_h}{p_l} r_1 + 1}{\frac{p_h}{p_l} r_2 + 1} = \frac{v_h}{v_l}$$

$$\frac{\frac{p_h}{p_l} r_3 + 1}{\frac{p_h}{p_l} r_4 + 1} = \frac{v_h}{v_l}$$

(6) divide by (7):

$$\frac{r_3-1}{1-r_1} = \frac{r_2(1-r_4)}{r_4(r_2-1)} \tag{8}$$

Represent r_1 by $r_1 = \frac{v_h}{v_l} r_2 + \frac{p_l(v_h-v_l)}{p_h v_l}$ and r_3 by $r_3 = \frac{v_h}{v_l} r_4 + \frac{p_l(v_h-v_l)}{p_h v_l}$. Plug in (8) and rearrange, we have

$$\frac{v_h-v_l}{p_h v_l}\frac{r_2}{1-r_2} - \frac{v_h}{v_l} r_2 = \frac{v_h-v_l}{p_h v_l}\frac{r_4}{1-r_4} - \frac{v_h}{v_l} r_4 \tag{9}$$

However, due to the shape of the function

$$\frac{v_h - v_l}{p_h v_l} \frac{x}{1-x} - \frac{v_h}{v_l} x,$$

there do not exist any $r_2 > 1 > r_4$ or $r_4 > 1 > r_2$ satisfy (9). Contradiction. $\square$

Proof of Lemma 1

With the public signal $\left(1, \hat{\beta}_h, 1\right)$, Condition 2 is binding, and Condition 3, 4, and ¬5 are satisfied. Furthermore, $\underline{b}_{(v_h,l)} = 0$ since $\gamma_l = 0$.

In this case, the expected effort of type (v_h, h) is $\frac{1}{2}\left(1 + \frac{p_l \hat{\beta}_h}{p_h + p_l \hat{\beta}_h}\right) v_h$, the expected effort of type (v_l, h) is $\frac{p_l}{p_h \hat{\beta}_h + p_l} v_l$, the expected effort of type (v_h, l) is $\frac{1}{2} v_l$ and the expected effort of type (v_l, l) is $\frac{v_l^2}{2v_h}$. The total expected effort is two times the sum of these expected efforts weighted by their corresponding probabilities. We can then calculate the difference between such total expected effort and the effort in the IPV setting, $p_h^2 v_h + (1 - p_h^2) v_l$:

$$\frac{p_h p_l (v_h - v_l)}{v_h} \left[(2v_h + v_l)\hat{\beta}_h - v_l\right]. \tag{10}$$

Therefore, $\hat{\beta}_h > \frac{v_l}{2v_h + v_l}$ is sufficient and necessary for (10) to be positive.

Proof of Lemma 2

With the public signal $\left(1, \frac{p_h v_l}{v_h - p_l v_l}, \frac{p_h v_l}{v_h - p_l v_l}\right)$, Condition 2 is binding, and Condition 3, 4, and ¬5 are satisfied. We can then calculate the expected effort for each type:

- Type (v_l, h) : The expected effort of this type is $\frac{1}{2} p_l v_l$ and a player is such a type with probability $p_l(p_h \beta_h + p_l \gamma_h)$.

- Type (v_h, h) : The expected effort of this type is $p_l v_l + \frac{1}{2}(v_h - p_l v_l)$ and a player is such a type with probability $p_h(p_h \alpha_h + p_l \beta_h)$.

- Type (v_l, l) : The expected effort of this type is

  $$\frac{v_h - v_l}{v_h - p_l v_l} \frac{1}{2} \frac{p_l (v_h - v_l) v_l}{v_h - p_l v_l} + \left(1 - \frac{v_h - v_l}{v_h - p_l v_l}\right) \left[\frac{p_l (v_h - v_l) v_l}{v_h - p_l v_l} + \frac{1}{2}\left(v_l - \frac{p_l (v_h - v_l) v_l}{v_h - p_l v_l}\right)\right]$$

  and a player is such a type with probability $p_h(p_h \beta_l + p_l \gamma_l)$. So the probability weighted expected effort of (v_l, l) is

  $$\frac{\left[p_h \left((p_l + 1) v_h v_l - 2 p_l v_l^2\right) + p_l (v_h - v_l)^2\right] p_l (v_h - v_l) v_l}{2(v_h - p_l v_l)^3}.$$

- Type (v_h, l) : The expected effort of this type is $\left[\frac{p_l(v_h-v_l)v_l}{v_h-p_lv_l} + \frac{1}{2}\left(v_l - \frac{p_l(v_h-v_l)v_l}{v_h-p_lv_l}\right)\right]$ and a player is such a type with probability $p_l(p_h\alpha_l + p_l\beta_l)$. So the probability weighted expected effort of (v_h, l) is

$$\frac{p_hp_l(v_h-v_l)\left[2p_l(v_h-v_l)v_l + p_hv_hv_l\right]}{2(v_h-p_lv_l)^2}. \tag{11}$$

Hence, the expected effort of type (v_h, h) and (v_l, h) weighted by their corresponding probability is given by

$$2 * \left[p_l(p_h\beta_h + p_l\gamma_h)\cdot\frac{1}{2}p_lv_l + p_h(p_h\alpha_h + p_l\beta_h)\cdot\left(p_lv_l + \frac{1}{2}(v_h - p_lv_l)\right)\right] \tag{12}$$

$$= \frac{p_hp_l^2v_l^2 + p_h^2v_h(v_h + p_lv_l)}{v_h - p_lv_l} \tag{13}$$

Difference between (13) and the effort under the IPV setting, $p_h^2v_h + (1-p_h^2)v_l$, is given by

$$\frac{p_hp_l^2v_l^2 + p_h^2v_h(v_h + p_lv_l)}{v_h - p_lv_l} - \left[p_h^2v_h + (1-p_h^2)v_l\right] = \frac{p_l^2(2p_h+1)v_l(v_l-v_h)}{v_h-p_lv_l}$$

Adds up the probability weighted effort of type (v_h, l) given by (11):

$$\frac{p_l^2(2p_h+1)v_l(v_l-v_h)}{v_h-p_lv_l} + 2p_l(p_h\alpha_l + p_l\beta_l)\left[\frac{p_l(v_h-v_l)v_l}{v_h-p_lv_l} + \frac{1}{2}\left(v_l - \frac{p_l(v_h-v_l)v_l}{v_h-p_lv_l}\right)\right]$$

$$= \frac{p_l(v_h-v_l)v_l\left[(p_h^2-p_l)v_h + (p_l^2 - 2p_h^2p_l)v_l\right]}{(v_h-p_lv_l)^2}$$

Adds up the probability weighted effort of type (v_h, h) given at the beginning of this proof:

$$\frac{p_l(v_h-v_l)v_l(v_h-p_lv_l)\left[(p_h^2-p_l)v_h + (p_l^2-2p_h^2p_l)v_l\right]}{(v_h-p_lv_l)^3}$$

$$+ \frac{p_l(v_h-v_l)v_l\left[p_h\left((p_l+1)v_hv_l - 2p_lv_l^2\right) + p_l(v_h-v_l)^2\right]}{(v_h-p_lv_l)^3}$$

$$= \frac{p_h^2p_l(v_h-v_l)v_l\left[(v_h-p_lv_l)^2 + (v_h-p_lv_l)p_hv_l\right]}{(v_h-p_lv_l)^3} \tag{14}$$

$$> 0$$

This completes the proof. □

Proof of Proposition 5

The signal $(1, \hat{\beta}_h, 1)$ induces higher expected effort if the difference between (10) and (14) is positive. This is equivalent of

$$(2v_h + v_l)\hat{\beta}_h > \frac{(1+p_h)v_h^2 v_l + p_l^2 v_l^3 + (p_h^2 - p_h p_l - 2p_l)v_h v_l^2}{(v_h - p_l v_l)^2}.$$

By separating $\hat{\beta}_h$, it can be shown that the above inequality is true if and only if $\hat{\beta}_h > B_2$.

Proof of Proposition 7

Proof. The MSE part is proven by showing that each type of players is indifferent in their equilibrium supports and there is no profitable deviation exists. Here, I only show the prove that type (v_h, h) is indifferent in its equilibrium support and there is no profitable deviation. The proof for other types can be done in the same fashion and thus is omitted to save space.

The expected payoff of type (v_h, h) when choosing an effort within her own equilibrium support, $(\bar{b}_{(v_h,l)}, \bar{b}_{(v_h,h)})$, is given by:

$$\frac{p_l(1-q) + p_h q(1-q)}{p_l(1-q) + p_h q} + \frac{p_h q^2}{p_l(1-q) + p_h q} G_{(v_h,h)}(b) v_h - b$$

Plug in (v_h, h)'s mixed strategy $G_{(v_h,h)}(b)$, the expected payoff is $v_h - \bar{b}_{(v_h,h)}$, which is exactly her equilibrium payoff. Now we check whether type (v_h, h) wants to deviate to the supports of other players.

If type (v_h, h) deviate to (v_h, l) 's support, the expected payoff becomes

$$\left\{ \frac{p_l(1-q)}{p_l(1-q) + p_h q} + \frac{p_h q(1-q)}{p_l(1-q) + p_h q} G_{(v_h,l)}(b) \right\} v_h - b$$

plug in the equilibrium mixed strategy of (v_h, l), $G_{(v_h,l)}$, and rearrange. Then, the coefficient of the effort b becomes

$$\frac{p_l q + p_h(1-q)}{p_l(1-q) + p_h q} \frac{q}{1-q} - 1 \tag{15}$$

which is also the first order derivative of the above expected payoff function w.r.t b. If $p_h \leqslant p_l$, that is, expression (15) is positive, then type (v_h, h) can increase her payoff by increasing b, until it reaches the upper bound of (v_h, l)'s support, $\bar{b}_{(v_h,l)}$, which is also the lower bound of (v_h, h)'s own equilibrium support. This suggests deviating to (v_h, l)'s support is not profitable. If, however, $p_h > p_l$ and thus (15) is negative, type (v_h, h) should choose the lower bound of (v_h, l)'s support, $\bar{b}_{(v_l,l)}$, instead of any effort higher. Thus we need to check whether the expected payoff of choosing $\overline{b}_{(v_l,l)}$ is higher than (v_h, h)'s equilibrium expected payoff.

Let (v_h, h)'s equilibrium expected payoff be $\pi^*_{(v_h,h)}$ and her payoff from choosing $\overline{b}_{(v_l,l)}$ be $\pi_{(v_h,h)}(\overline{b}_{(v_l,l)})$, then the difference between the two:

$$\pi^*_{(v_h,h)} - \pi_{(v_h,h)}(\overline{b}_{(v_l,l)}) = p_h(1-q)\left[\frac{q}{p_h q + p_l(1-q)} - \frac{1-q}{p_h(1-q)+p_l q}\right] v_h > 0$$

Thus, we have shown that type (v_h, h) do not want to deviate to (v_h, l)'s support. An important observation is that the expected payoff from a type deviate to another type's support is always a linear function of b, due to the all-pay rule. This fact ensures that it is impossible that the optimal deviating effort lies in the interior of others' supports unless the player finds it indifferent across all efforts in each support. This means a simpler way of checking the equilibrium is to compare the equilibrium payoffs of each type with the payoffs from choosing each types' upper bounds of their equilibrium supports.

Since I have shown that $\overline{b}_{(v)l,l)}$ is not profitable to deviate to, the only things left to check are the profitability of choosing $\overline{b}_{(v_l,h)}$ and zero. When (v_h, h) chooses $\overline{b}_{(v_l,h)}$, the expected payoff is

$$\frac{(1-p)(1-q)q}{pq+(1-p)(1-q)} v_h - \frac{(1-p)(1-q)^2}{pq+(1-p)(1-q)} v_l \tag{16}$$

The gap between her equilibrium expected payoff and (16) is:

$$\frac{(1-p)(1-q)}{pq+(1-p)(1-q)}[(1-q)v_h - qv_l]$$

which is positive when $(1-q)v_h > qv_l$. It is trivial to show that choosing zero cannot be more profitable. Thus, it is not profitable for (v_h, h) to choose outside of her equilibrium support.

When $qv_l \geqslant (1-q)v_h$, the proof, again, consists of showing that players find all efforts in the equilibrium support indifferent and that there is no profitable deviation exists. It is easy to check that all types are indifferent when choosing an effort in $[\overline{b}, \underline{b}]$, thus it is omitted. Here, we show that type (v_h, h) doesn't find it profitable to deviate to $[0, \underline{b}]$ and that type (v_h, l) doesn't want to deviate to $[\overline{b}, \overline{b}_{(v_h,h)}]$.

If type (v_h, h) deviate to $(0, \underline{b})$, then the expected payoff is

$$\frac{p_l(1-q)q}{p_h q + p_l(1-q)} G_{(v_l,h)}(b) v_h - b = \frac{qv_h - (1-q)v_l}{(1-q)\, v_l} b$$

which is increasing in b since $qv_h > (1-q)v_l$. Hence, it is not a profitable deviation.

If type (v_h, l) deviate to $(0, \underline{b})$, then the expected payoff is given by:

$$\frac{p_l q^2}{p_h(1-q)+p_h q} G_{(v_l,h)}(b) v_h - b = \frac{q^2((1-q)p_l + p_h q)v_h - (1-q)^2 (p_h(1-q)+p_l q) v_l}{v_l (q-1)^2 (p_h(1-q)+p_l q)} b$$

It's increasing in b because

$$\begin{aligned} & q^2(p_h q + (1-q)p_l)v_h - (1-q)^2 (p_h(1-q)+p_l q) v_l \\ > \; & q^2(p_h q + (1-q)p_l)v_l - (1-q)^2 (p_h(1-q)+q p_l) v_l \\ = \; & (2q-1) \big(p_h(1-q) + p_h q^2 + q p_l (1-q) \big) v_l > 0. \end{aligned}$$

Hence, it is not a profitable deviation.

If type (v_l, l) deviates to $(0, \underline{b})$, the expected payoff:

$$\frac{p_h(2q-1)}{(1-q)(p_h(1-q)+p_l q)} b$$

which is increasing in b. Hence, it is not a profitable deviation.

If type (v_h, l) deviates to $(\bar{b}, \bar{b}_{(v_h,h)})$, the coefficient of b in the corresponding expected payoff

$$\frac{-p_l(2q-1)}{q(p_h(1-q)+p_l q)}$$

which is negative. Hence, it is not a profitable deviation.

If type (v_l, l) deviates to $(\bar{b}, \bar{b}_{(v_h,h)})$, the coefficient of b in the corresponding expected payoff:

$$\frac{(1-q)^2 (p_h q + p_l(1-q)) v_l - q^2 (p_h(1-q)+p_l q) v_h}{q^2 v_h (p_h(1-q)+p_l q)}$$

$$\begin{aligned} < \; & \frac{(1-q)^2 (p_h q + p_l(1-q)) - q^2 (p_h(1-q)+p_l q)}{q^2 v_h (p_h(1-q)+p_l q)} v_l \\ = \; & -\frac{(2q-1)(q(p_h(1-q)+p_l q) + (1-q)p_l)}{q^2 v_h (p_h(1-q)+p_l q)} v_l < 0. \end{aligned}$$

Hence, it is not a profitable deviation.

If type (v_l, h) deviates to $(\bar{b}, \bar{b}_{(v_h,h)})$, the coefficient of b in the expected payoff:

$$\frac{(1-q)v_l - q v_h)}{q v_h} = -\frac{q v_h - (1-q)v_l}{q v_h} < 0.$$

Hence, it is not a profitable deviation. Thus, in general, there is no type has profitable deviation.

□

Proof of Proposition 8

In this proof, Lemma 3, 4, 5 prove that the strategy profile corresponding to Condition 2 and 3 is the unique equilibrium. In particular, Lemma 3 shows that type (v_h, h) must randomize in a higher, nonoverlapping, support than type (v_l, h). Lemma 4 shows that each type of players is indifferent for any bid in their equilibrium support and none of the types find it profitable to deviate to any bid outside of the equilibrium support. Lemma 5 shows that there does not exist any asymmetric equilibria when condition 2 and 3 are satisfied.

Similarly, Lemma 6, 7, 8 prove that the strategy profile corresponding to Condition 2 and ¬3 is the unique equilibrium. In particular, Lemma 6 shows that type (v_h, h) and (v_l, h) must randomize in overlapping supports and the expected payoff of type (v_l, h) must be 0. Lemma 7 shows that each type of players is indifferent for any bid in their equilibrium support and none of the types find it profitable to deviate to any bid outside of the equilibrium support. Lemma 8 shows that there does not exist any asymmetric equilibria when condition 2 and ¬3 are satisfied.

Finally, Lemma 9, 10, 11 prove that the strategy profile corresponding to Condition ¬2 and 3 is the unique equilibrium. In particular, Lemma 9 shows that type (v_h, h) and (v_l, h) must randomize in overlapping supports and the expected payoff of both types must be 0. Lemma 10 shows that each type of players is indifferent for any bid in their equilibrium support and none of the types find it profitable to deviate to any bid outside of the equilibrium support. Lemma 11 shows that there does not exist any asymmetric equilibria when condition ¬2 and 3 are satisfied.

The proofs for the case when players observe "l" is exactly the same as the case when they observe "h" and hence, are omitted. The rest of the proof are shown in the following order: Lemma 3, 4, and 5; 6, 7, and 8; 9, 10, and 11.

Lemma 3. *When Condition 2 and 3 are satisfied, then in any symmetric equilibrium types (v_h, h) and (v_l, h) randomize in nonoverlapping supports. Furthermore, the support of type (v_h, h) is higher than the support of (v_l, h).*

Proof. For the first part of the lemma, suppose both the two types randomize in a same interval (b_1, b_2), then for any $b \in (b_1, b_2)$ it must be true that

$$\begin{aligned}
\left(\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h}G_{(v_h,h)}(b) + \frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}G_{(v_l,h)}(b)\right)v_h - b &= K_{(v_h,h)} \\
\left(\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h}G_{(v_h,h)}(b) + \frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h}G_{(v_l,h)}(b)\right)v_l - b &= K_{(v_l,h)}
\end{aligned}$$

where $K_{(v_h,h)}, K_{(v_l,h)}$ are constants. Thus, solve for $G_{(v_h,h)}(b)$ and $G_{(v_l,h)}(b)$:

$$
\begin{aligned}
G_{(v_h,h)}(b) &= \frac{\beta_h (p_h\beta_h + p_l\gamma_h) v_h - \gamma_h (p_h\alpha_h + p_l\beta_h) v_l}{p_h \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l} b \\
&\quad -\gamma_h \frac{p_l\beta_h + p_h\alpha_h}{p_h \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h}(v_h - v_l) \\
G_{(v_l,h)}(b) &= \begin{cases} \frac{-\alpha_h(p_h\beta_h+p_l\gamma_h)v_h+\beta_h(p_l\beta_h+p_h\alpha_h)v_l}{p_l\left(\beta_h^2-\alpha_h\gamma_h\right)v_h v_l} b + \frac{\beta_h(p_l\beta_h+p_h\alpha_h)(v_h-v_l)}{p_l\left(\beta_h^2-\alpha_h\gamma_h\right)v_h}, & \text{for } b \in [\underline{b}_{(v_h,h)}, v_l] \\ \frac{p_h\beta_h+p_l\gamma_h}{p_l\gamma_h v_l} b, & \text{for } b \in [0, \underline{b}_{(v_h,h)}] \end{cases}
\end{aligned}
$$

Thus, the slop of $G_{(v_h,h)}(b)$ is

$$
\begin{aligned}
& \frac{r (p_h\beta_h + p_l\gamma_h) v_h - \gamma_h (p_h\alpha_h + p_l\beta_h) v_l}{p_h \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l} \\
= \; & (p_h\beta_h + p_l\gamma_h)(p_h\alpha_h + p_l\beta_h) \frac{\frac{p_l\beta_h}{(p_h\alpha_h+p_l\beta_h)} v_h - \frac{p_l\gamma_h}{(p_h\beta_h+p_l\gamma_h)} v_l}{p_l p_h \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l},
\end{aligned}
$$

and the slop of $G_{(v_l,h)}(b)$ is

$$
\begin{aligned}
& \frac{-\alpha_h (p_h\beta_h + p_l\gamma_h) v_h + \beta_h (p_l\beta_h + p_h\alpha_h) v_l}{p_l \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l} \\
= \; & (p_h\beta_h + p_l\gamma_h)(p_l\beta_h + p_h\alpha_h) \frac{\frac{p_h\beta_h}{(p_h\beta_h+p_l\gamma_h)} v_l - \frac{p_h\alpha_h}{(p_l\beta_h+p_h\alpha_h)} v_h}{p_h p_l \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l}
\end{aligned}
$$

For the slop of $G_{(v_h,h)}(b)$ to be positive and Condition 2 to be satisfied, it must be true that $\left(\beta_h^2 - \alpha_h\gamma_h\right) > 0$, for the slop of $G_{(v_l,h)}(b)$ to be positive and Condition 3 to be satisfied, it must be true that $\left(\beta_h^2 - \alpha_h\gamma_h\right) < 0$. Thus, when Condition 2 and 3 both satisfied, type (v_h, h) and (v_l, h)'s support cannot be overlapping.

Now we prove that the support of (v_h, h) must be higher than the support of (v_l, h). Suppose instead that the type (v_l, h) mixes over the interval $[\widehat{b}, \widetilde{b}]$, but the type (v_h, h) randomizes in the interval $[0, \widehat{b}]$. Note that the lowest possible effort for each player must be 0. However, this then implies type (v_h, h) must earn an expected payoff of 0, which cannot be true in any equilibrium as she can also deviate by choosing v_l to earn positive payoff. This is because any effort above v_l is strictly dominated for type (v_l, h). □

Lemma 4. *When Condition 2 and 3 are satisfied, the mixed strategies given in the proposition form a symmetric equilibrium.*

Proof. When Condition 2 and 3 are satisfied, we first show that a player with type (v_l, h) is indifferent in the equilibrium support. By plugging in the mixed strategy $G_{(v_l,h)}(b)$ in equilibrium, the expected payoff indeed equals zero:

$$G_{(v_l,h)}(b)\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}v_l-b=0$$

For type (v_h,h), we plug in $G_{(v_h,h)}(b)$ and the expected payoff is also a constant:

$$\left(\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}+G_{(v_h,h)}(b)\frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h}\right)v_h-b=\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}v_h-\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}v_l$$

Note that Condition 2 guarantees the above expected payoff of type (v_h,h) to be nonnegative.

Now we check for profitable deviations when each type deviates to effort levels that are outside of her equilibrium support. When type (v_h,h) deviates to the support of (v_l,h), the expected payoff becomes

$$G_{(v_l,h)}(b)\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}v_h-b \quad = \quad \left(\frac{\beta_h}{\gamma_h}\frac{p_h\beta_h+p_l\gamma_h}{p_h\alpha_h+p_l\beta_h}\frac{v_h}{v_l}-1\right)b.$$

This expected payoff is increasing in b given that Condition 2 is satisfied. Thus, type (v_h,h) does not want to deviate to the support of (v_l,h). When type (v_l,h) deviates to the support of (v_h,h), the expected payoff is:

$$\begin{aligned}&\left(\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}+G_{(v_h,h)}(b)\frac{p_h\beta_h}{p_h\beta_h+p_l\gamma_h}\right)v_l-b\\ =\quad&\left(\frac{v_l}{v_h}\frac{\beta_h}{\alpha_h}\frac{p_h\alpha_h+p_l\beta_h}{p_h\beta_h+p_l\gamma_h}-1\right)b+\left(\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}-p_l\beta_h\frac{v_l}{v_h}\frac{\gamma_h}{\alpha_h}\frac{\beta_h p_l+\alpha_h p_h}{(\beta_h p_h+\gamma_h p_l)^2}\right)v_l.\end{aligned}$$

This expected payoff is decreasing in b given that Condition 3 is satisfied. Thus, this is not a profitable deviation. □

Lemma 5. *When Condition 2 and 3 are satisfied, then there is no asymmetric equilibrium.*

Proof. Denote by $\bar{b}_{1(v_h,h)}$ and $\bar{b}_{2(v_h,h)}$ the upper bound of equilibrium support of player 1 and 2, respectively. Based on a similar argument from the proof of Lemma 3, type (v_h,h) of both players must choose efforts no less than (v_l,h). Thus, it must be true that in any equilibrium, we have $\bar{b}_{1(v_h,h)}=\bar{b}_{2(v_h,h)}$. If there exists an asymmetric equilibrium, then it must be true that $\bar{b}_{1(v_l,h)}\neq\bar{b}_{2(v_l,h)}$. Suppose without loss that $\bar{b}_{1(v_l,h)}>\bar{b}_{2(v_l,h)}$. Since type (v_l,h) of player 1 is indifferent between any effort in $[0,\bar{b}_{2(v_l,h)}]$, thus her expected payoff being zero indicates that:

$$\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}G_{2(v_l,h)}(b)v_l-b=0 \;\Leftrightarrow\; G_{2(v_l,h)}(b)=\frac{p_h\beta_h+p_l\gamma_h}{p_l\gamma_h v_l}b$$

and $\bar{b}_{2(v_l,h)}=\frac{p_l\gamma_h v_l}{p_h\beta_h+p_l\gamma_h}=\bar{b}_{(v_l,h)}$ as given in the proposition. Since type (v_l,h) of player 2 is also indifferent between any effort in $[0, \bar{b}_{2(v_l,h)}]$, thus her expected payoff being zero indicates that:

$$\frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h} G_{1(v_l,h)}(b)v_l - b = 0 \;\Leftrightarrow\; G_{1(v_l,h)}(b) = \frac{p_h\beta_h + p_l\gamma_h}{p_l\gamma_h v_l} b$$

and it can be shown that $G_{1(v_l,h)}(\bar{b}_{(v_l,h)}) = 1$. Thus, it must be true that $\bar{b}_{1(v_l,h)} = \bar{b}_{2(v_l,h)}$, which is then a contradiction. □

Lemma 6. *When Condition 2 and ¬3 are satisfied, in any symmetric equilibrium types (v_h, h) and (v_l, h) randomize in overlapping supports. Furthermore, the upper bound of supports $\bar{b}_{(v_h,h)} = \bar{b}_{(v_l,h)}$. Finally, the expected payoff of (v_l, h) is 0 and the expected payoff of (v_h, h) is $v_h - v_l$.*

Proof. Suppose types (v_h, h) and (v_l, h) randomize in nonoverlapping supports in an equilibrium. Then the support of (v_h, h) must be higher than (v_l, h), and the mixed strategy of these types must be equivalent to those given in the proposition which correspond to the case when Condition 2 and 3 are satisfied. In that case, type (v_l, h)'s expected payoff must be zero. However, by choosing $\bar{b}_{(v_h,h)}$ the type (v_l, h)'s expected payoff must be $v_l - \bar{b}_{(v_h,h)} = \frac{p_h\beta_h}{p_h\beta_h+p_l\gamma_h} v_l - \frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h} v_h > 0$, as Condition ¬3 is instead satisfied. Thus, in any symmetric equilibrium it cannot be true that the supports are nonoverlapping.

In any symmetric equilibrium, it cannot be true that both types earn positive payoff, as one of the types must have the lower bound of support equals 0. Suppose both types have lower bound equals 0, then both earn a payoff of 0. In that case, the indifference conditions in the overlapping part of their supports are

$$\begin{aligned}
\left(\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h} G_{(v_h,h)}(b) + \frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h} G_{(v_l,h)}(b)\right) v_h - b &= 0 \\
\left(\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h} G_{(v_h,h)}(b) + \frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h} G_{(v_l,h)}(b)\right) v_l - b &= 0
\end{aligned}$$

and thus

$$\begin{aligned}
G_{(v_h,h)}(b) &= \frac{\beta_h (p_h\beta_h + p_l\gamma_h) v_h - \gamma_h (p_h\alpha_h + p_l\beta_h) v_l}{p_h \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l} b \\
G_{(v_l,h)}(b) &= \frac{-\alpha_h (p_h\beta_h + p_l\gamma_h) v_h + \beta_h (p_h\alpha_h + p_l\beta_h) v_l}{p_l \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l} b
\end{aligned}$$

By letting $G_{(v_h,h)}(b) = G_{(v_l,h)}(b) = 1$, we have

$$\bar{b}_{(v_h,h)} = \frac{p_h p_l \left(\beta_h^2 - \alpha_h\gamma_h\right) v_h v_l}{(p_h\beta_h + p_l\gamma_h)(p_h\alpha_h + p_l\beta_h)} \frac{1}{\frac{p_l\beta_h}{(p_h\alpha_h+p_l\beta_h)} v_h - \frac{p_l\gamma_h}{(p_h\beta_h+p_l\gamma_h)} v_l}$$

and

$$\bar{b}_{(v_l,h)} = \frac{p_h p_l \left(\beta_h^2 - \alpha_h \gamma_h\right) v_h v_l}{\left(p_h \beta_h + p_l \gamma_h\right)\left(p_l \beta_h + p_h \alpha_h\right)} \frac{1}{-\frac{p_h \alpha_h}{\left(p_l \beta_h + p_h \alpha_h\right)} v_h + \frac{p_h \beta_h}{\left(p_h \beta_h + p_l \gamma_h\right)} v_l}$$

thus their difference is

$$\begin{aligned}
& \bar{b}_{(v_h,h)} - \bar{b}_{(v_l,h)} \\
= \; & \frac{p_h p_l \left(\beta_h^2 - \alpha_h \gamma_h\right) v_h v_l}{\left(p_h \beta_h + p_l \gamma_h\right)\left(p_h \alpha_h + p_l \beta_h\right)} \left(\frac{1}{\frac{p_l \beta_h v_h}{\left(p_h \alpha_h + p_l \beta_h\right)} - \frac{p_l \gamma_h v_l}{\left(p_h \beta_h + p_l \gamma_h\right)}} - \frac{1}{\frac{p_h \beta_h v_l}{\left(p_h \beta_h + p_l \gamma_h\right)} - \frac{p_h \alpha_h v_h}{\left(p_l \beta_h + p_h \alpha_h\right)}} \right) \\
< \; & \frac{p_h p_l \left(\beta_h^2 - \alpha_h \gamma_h\right) v_h}{\left(p_h \beta_h + p_l \gamma_h\right)\left(p_h \alpha_h + p_l \beta_h\right)} \left(\frac{1}{\frac{p_l \beta_h}{\left(p_h \alpha_h + p_l \beta_h\right)} - \frac{p_l \gamma_h}{\left(p_h \beta_h + p_l \gamma_h\right)}} - \frac{1}{\frac{p_l \beta_h}{\left(p_l \beta_h + p_h \alpha_h\right)} - \frac{p_l \gamma_h}{\left(p_h \beta_h + p_l \gamma_h\right)}} \right) \\
= \; & 0.
\end{aligned}$$

This means $\bar{b}_{(v_h,h)} < \bar{b}_{(v_l,h)}$. Note that for type (v_l, h), $\bar{b}_{(v_l,h)} \leqslant v_l$ must be true as any effort above v_l is strictly dominated. But then type (v_h, h) has an incentive to choose $\bar{b}_{(v_l,h)}$ to earn $v_h - \bar{b}_{(v_l,h)} \geqslant v_h - v_l > 0$. Thus, there is a contradiction.

So the only case left is type (v_h, h) earns positive expected payoff whereas type (v_l, h) earns 0. Thus, the lower bound of (v_h, h)'s support must be positive. It cannot be true that $\bar{b}_{(v_h,h)} < \bar{b}_{(v_l,h)} < v_l$, as then the expected payoff of (v_l, h) would be positive. It cannot be true that $\bar{b}_{(v_h,h)} < \bar{b}_{(v_l,h)} = v_l$, as type (v_h, h) prefers $\bar{b}_{(v_l,h)}$ over any efforts in the interval $(\bar{b}_{(v_h,h)}, \bar{b}_{(v_l,h)})$. In particular, we have for type (v_l, h) in the interval $[\bar{b}_{(v_h,h)}, \bar{b}_{(v_l,h)}]$ that:

$$\left(\frac{p_l \gamma_h}{p_h \beta_h + p_l \gamma_h} G_{(v_l,h)}(b) + \frac{p_h \beta_h}{p_h \beta_h + p_l \gamma_h} \right) v_l - b = 0$$

thus

$$G_{(v_l,h)}(b) = \frac{p_h \beta_h + p_l \gamma_h}{p_l \gamma_h} \frac{b}{v_l} - \frac{p_h \beta_h}{p_l \gamma_h}$$

Now if type (v_h, h) increase the effort from $\bar{b}_{(v_h,h)}$ to $b \in (\bar{b}_{(v_h,h)}, \bar{b}_{(v_l,h)})$ the expected payoff increases by

$$\begin{aligned}
& \frac{p_l \beta_h}{p_h \alpha_h + p_l \beta_h} \left[G_{(v_l,h)}(b) - G_{(v_l,h)}(\bar{b}_{(v_h,h)}) \right] v_h - \left[b - \bar{b}_{(v_h,h)} \right] \\
= \; & \left[\frac{p_l \beta_h}{p_h \alpha_h + p_l \beta_h} \frac{p_h \beta_h + p_l \gamma_h}{p_l \gamma_h} \frac{v_h}{v_l} - 1 \right] \left[b - \bar{b}_{(v_h,h)} \right] > 0
\end{aligned}$$

This is positive, according to Condition 2. Contradiction. Thus, we must have $\bar{b}_{(v_h,h)} \geqslant \bar{b}_{(v_l,h)}$.

Suppose $\bar{b}_{(v_h,h)} > \bar{b}_{(v_l,h)}$, then in the interval $[\bar{b}_{(v_l,h)}, \bar{b}_{(v_h,h)}]$, type (v_h, h) must be indifferent

$$\left(\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h}G_{(v_h,h)}(b) + \frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}\right)v_h - b = \widehat{\pi}_{(v_h,h)}$$

nd thus the mixed strategy is

$$G_{(v_h,h)}(b) = \frac{p_h\alpha_h + p_l\beta_h}{p_h\alpha_h}\frac{b + \widehat{\pi}_{(v_h,h)}}{v_h} - \frac{p_l\beta_h}{p_h\alpha_h}$$

If type (v_l, l) increases her effort from $\bar{b}_{(v_l,h)}$ to $b \in (\bar{b}_{(v_l,h)}, \bar{b}_{(v_h,h)})$ then the expected payoff increases by

$$\begin{aligned}
&\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h}\left[G_{(v_h,h)}(b) - G_{(v_h,h)}(\bar{b}_{(v_l,h)})\right]v_l - \left[b - \bar{b}_{(v_l,h)}\right] \\
= &\left[\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h}\frac{p_h\alpha_h + p_l\beta_h}{p_h\alpha_h}\frac{v_l}{v_h} - 1\right]\left[b - \bar{b}_{(v_l,h)}\right] > 0
\end{aligned}$$

This is positive, according to Condition ¬3. Contradiction. Thus, it must be true that $\bar{b}_{(v_h,h)} = \bar{b}_{(v_l,h)}$.

Consider the interval $[\underline{b}_{(v_h,h)}, \bar{b}_{(v_h,h)}]$, where type (v_h, h) and (v_l, h)'s indifference conditions must be

$$\begin{aligned}
\left(\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h}G_{(v_h,h)}(b) + \frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}G_{(v_l,h)}(b)\right)v_h - b &= \widehat{\pi}_{(v_h,h)} \\
\left(\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h}G_{(v_h,h)}(b) + \frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h}G_{(v_l,h)}(b)\right)v_l - b &= 0
\end{aligned}$$

and thus

$$\begin{aligned}
G_{(v_h,h)}(b) &= \frac{\beta_h\left(p_h\beta_h + p_l\gamma_h\right)v_h - \gamma_h\left(p_h\alpha_h + p_l\beta_h\right)v_l}{p_h\left(\beta_h^2 - \alpha_h\gamma_h\right)v_hv_l}b \\
&\quad -\gamma_h\frac{p_l\beta_h + p_h\alpha_h}{p_h\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h}\widehat{\pi}_{(v_h,h)} \\
G_{(v_l,h)}(b) &= \frac{-\alpha_h\left(p_h\beta_h + p_l\gamma_h\right)v_h + \beta_h\left(p_l\beta_h + p_h\alpha_h\right)v_l}{p_l\left(\beta_h^2 - \alpha_h\gamma_h\right)v_hv_l}b + \beta_h\frac{p_l\beta_h + p_h\alpha_h}{p_l\left(\beta_h^2 - \alpha_h\gamma_h\right)v_h}\widehat{\pi}_{(v_h,h)}
\end{aligned}$$

Given that $\bar{b}_{(v_h,h)} = \bar{b}_{(v_l,h)} = \beta$, we have $G_{(v_h,h)}(\beta) = G_{(v_l,h)}(\beta) = 1$, this implies $\widehat{\pi}_{(v_h,h)} = v_h - v_l$ and $\beta = v_l$. □

Lemma 7. *When Condition 2 and ¬3 are satisfied, the mixed strategies given in the proposition form a symmetric equilibrium.*

Proof. When Condition 2 and ¬3 are satisfied, we show that type (v_l, h) and (v_h, h) are indifferent in their equilibrium support. After plugging in the expression of $G_{(v_h,h)}(b)$ and $G_{(v_l,h)}(b)$

as given in the proposition, the expected payoff of type (v_l, h) when choosing an effort in the interval $[\underline{b}_{(v_h,h)}, v_l]$ is indeed a constant: zero.

$$\left(G_{(v_h,h)}(b)\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h} + G_{(v_l,h)}(b)\frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h} \right) v_l - b = 0$$

Similarly, type (v_h, h)'s expected payoff is a constant $v_h - v_l$:

$$\left(G_{(v_h,h)}(b)\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h} + G_{(v_l,h)}(b)\frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h} \right) v_h - b = v_h - v_l$$

Since only type (v_l, h) is choosing an effort in the interval $[0, \underline{b}_{(v_h,h)}]$, by plugging the $G_{(v_l,h)}(b)$ given in the proposition in, her expected payoff in this interval is

$$G_{(v_l,h)}(b)\frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h} v_l - b = 0$$

Now we prove that both types do not want to deviate to any effort outside of their equilibrium support. When type (v_h, h) deviates to $[0, \underline{b}_{(v_h,h)}]$, then her expected payoff would be

$$G_{(v_l,h)}(b)\frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h} v_h - b \quad = \quad \frac{\beta_h\left(p_h\beta_h + p_l\gamma_h\right)v_h - \gamma_h\left(p_h\alpha_h + p_l\beta_h\right)v_l}{v_l\gamma_h\left(\beta_h p_l + \alpha_h p_h\right)} b$$

This is increasing in b. Hence, type (v_h, h) does not want to deviate. □

Lemma 8. *When Condition 2 and ¬3 are satisfied, then there is no asymmetric equilibrium.*

Proof. If player 1 has $\bar{b}_{1(v_h,h)} = \bar{b}_{1(v_l,h)} = v_l$, then player 2 must have $\bar{b}_{2(v_h,h)} = \bar{b}_{2(v_l,h)} = v_l$. To see why, suppose $\bar{b}_{2(v_h,h)} < \bar{b}_{2(v_l,h)} = v_l$, then by the same argument in the previous lemma, type (v_h, h) of player 1 is strictly better off by reallocating probability mass from the interval $(\bar{b}_{2(v_h,h)}, \bar{b}_{2(v_l,h)})$ to v_l. Similarly, if $\bar{b}_{2(v_h,h)} = v_l > \bar{b}_{2(v_l,h)}$, then the previous lemma indicates that type (v_l, h) is strictly better off by reallocating probability mass from the interval $(\bar{b}_{2(v_l,h)}, \bar{b}_{2(v_h,h)})$ to v_l. Thus, it must be true that $\bar{b}_{1(v_h,h)} = \bar{b}_{1(v_l,h)} = \bar{b}_{2(v_h,h)} = \bar{b}_{2(v_l,h)} = v_l$, which means the expected payoff are the same as in the unique symmetric equilibrium.

By $G_{1(v_h.h)}(\underline{b}_{1(v_h,h)}) = 0$, it can be verified that $\underline{b}_{1(v_h,h)} = \underline{b}_{(v_h,h)}$ as in the symmetric equilibrium. Similarly, it can be verified that $\underline{b}_{2(v_h,h)} = \underline{b}_{(v_h,h)}$ by $G_{2(v_h.h)}(\underline{b}_{2(v_h,h)}) = 0$. Therefore, $\underline{b}_{1(v_h,h)} = \underline{b}_{2(v_h,h)} = \underline{b}_{(v_h,h)}$ in any equilibrium. □

Lemma 9. *When Condition ¬2 and 3 are satisfied, in any symmetric equilibrium types (v_h, h) and (v_l, h) randomize in overlapping supports. Furthermore, both the two types earn an expected payoff of* 0.

Proof. Suppose (v_h, h) and (v_l, h) randomize in nonoverlapping supports in a symmetric equilibrium, then again it must be true that (v_h, h)'s support is higher than (v_l, h). This implies type (v_h, h)'s expected payoff must be $v_h - \overline{b}_{(v_h,h)} = \frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}v_h - \frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}v_l \leqslant 0$, as Condition ¬2 is satisfied. Therefore, the two types must have overlapping supports.

Again, it cannot be true that both types earn positive payoff. Suppose type (v_h, h) earns positive payoff and type (v_l, h) earns zero. Thus $\underline{b}_{(v_h,h)} > \underline{b}_{(v_l,h)} = 0$. In the interval $[0, \underline{b}_{(v_h,h)}]$, type (v_l, h)'s indifference condition is

$$\frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h}G_{(v_l,h)}(b)v_l - b = 0$$

thus

$$G_{(v_l,h)}(b) = \frac{p_h\beta_h + p_l\gamma_h}{p_l\gamma_h}\frac{b}{v_l}$$

Now if type (v_h, h) decreases her effort from $\underline{b}_{(v_h,h)}$ to $b \in \left(0, \underline{b}_{(v_h,h)}\right)$, her expected payoff increases by

$$\begin{aligned} & \frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}\left[G_{(v_l,h)}(b) - G_{(v_l,h)}(\underline{b}_{(v_h,h)})\right]v_h - \left[b - \underline{b}_{(v_h,h)}\right] \\ = & \left[\frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}\frac{p_h\beta_h + p_l\gamma_h}{p_l\gamma_h}\frac{v_h}{v_l} - 1\right]\left[b - \underline{b}_{(v_h,h)}\right] > 0 \end{aligned}$$

According to Condition ¬2, the above is positive. Thus, it is profitable for type (v_h, h) to decrease the effort until 0. This implies the expected payoff of type (v_h, h) must also be 0, which implies $\overline{b}_{(v_h,h)} = v_h$, as any $v_h > \overline{b}_{(v_h,h)}$ suggests type (v_h, h) earns positive expected payoff. □

Lemma 10. *When Condition ¬2 and 3 are satisfied, the mixed strategies given in the proposition form a symmetric equilibrium.*

Proof. When Condition ¬2 and 3 are satisfied, both type (v_h, h) and (v_l, h) are indeed indifferent in the equilibrium support $[0, \overline{b}_{(v_l,h)}]$. This is because, after plugging in the mixed strategies given in the proposition, they both get zero expected payoff:

$$\begin{aligned} \left(G_{(v_h,h)}(b)\frac{p_h\beta_h}{p_h\beta_h + p_l\gamma_h} + G_{(v_l,h)}(b)\frac{p_l\gamma_h}{p_h\beta_h + p_l\gamma_h}\right)v_l - b &= 0 \\ \left(G_{(v_h,h)}(b)\frac{p_h\alpha_h}{p_h\alpha_h + p_l\beta_h} + G_{(v_l,h)}(b)\frac{p_l\beta_h}{p_h\alpha_h + p_l\beta_h}\right)v_h - b &= 0 \end{aligned}$$

Type (v_h, h) also get zero when choosing an effort in $[\overline{b}_{(v_l,h)}, v_h]$, since the expected payoff is zero after plugging in $G_{(v_h,h)}(b)$:

$$\left(\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}+G_{(v_h,h)}(b)\frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h}\right)v_h-b=0$$

Type (v_l,h) does not want to deviate to $[\overline{b}_{(v_l,h)},v_h]$ as

$$\begin{aligned}&\left(G_{(v_h,h)}(b)\frac{p_h\beta_h}{p_h\beta_h+p_l\gamma_h}+\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}\right)v_l-b\\ =&\left(\frac{\beta_h\left(p_l\beta_h+p_h\alpha_h\right)}{\alpha_h\left(p_l\gamma_h+p_h\beta_h\right)}\frac{v_l}{v_h}-1\right)b+v_l\left(\frac{p_l\gamma_h}{p_l\gamma_h+p_h\beta_h}+\frac{\beta_h}{\alpha_h}\frac{p_l\beta_h}{p_l\gamma_h+p_h\beta_h}\right)\end{aligned}$$

which is decreasing in b. Thus, none of the two types want to deviate. □

Lemma 11. *When Condition ¬2 and 3 are satisfied, there does not exist an asymmetric equilibrium.*

Proof. The only thing to check is that $\overline{b}_{1(v_l,h)}=\overline{b}_{2(v_l,h)}$. Suppose $\overline{b}_{1(v_l,h)}>\overline{b}_{2(v_l,h)}$. Type (v_h,h) and (v_l,h) of player 2's expected payoff when choosing a effort in the interval $(0,\overline{b}_{2(v_l,h)})$.

$$\begin{aligned}\left(\frac{p_h\alpha_h}{p_h\alpha_h+p_l\beta_h}G_{1(v_h,h)}(b)+\frac{p_l\beta_h}{p_h\alpha_h+p_l\beta_h}G_{1(v_l,h)}(b)\right)v_h-b&=0\\ \left(\frac{p_h\beta_h}{p_h\beta_h+p_l\gamma_h}G_{1(v_h,h)}(b)+\frac{p_l\gamma_h}{p_h\beta_h+p_l\gamma_h}G_{1(v_l,h)}(b)\right)v_l-b&=0\end{aligned}$$

when means

$$\begin{aligned}G_{1(v_h,h)}(b)&=\frac{\beta_h\left(p_h\beta_h+p_l\gamma_h\right)v_h-\gamma_h\left(p_h\alpha_h+p_l\beta_h\right)v_l}{p_h\left(\beta_h^2-\alpha_h\gamma_h\right)v_hv_l}b\\ G_{1(v_l,h)}(b)&=\frac{-\alpha_h\left(p_h\beta_h+p_l\gamma_h\right)v_h+\beta_h\left(p_l\beta_h+p_h\alpha_h\right)v_l}{p_l\left(\beta_h^2-\alpha_h\gamma_h\right)v_hv_l}b\end{aligned}$$

Then according to $G_{1(v_l,h)}(\overline{b}_{1(v_l,h)})=1$, we have $\overline{b}_{1(v_l,h)}=\overline{b}_{(v_l,h)}$. Similarly we can find $G_{2(v_l,h)}(b)$, and according to $G_{2(v_l,h)}(\overline{b}_{2(v_l,h)})=1$, we have $\overline{b}_{2(v_l,h)}=\overline{b}_{(v_l,h)}$. This is in contradiction to $\overline{b}_{1(v_l,h)}>\overline{b}_{2(v_l,h)}$. □

References

Alonso, R. and O. Câmara (2016a). Persuading voters. *American Economic Review 106*(11), 3590–3605.

Alonso, R. and O. Câmara (2016b). Political disagreement and information in elections. *Games and Economic Behavior 100*, 390–412.

Amann, E. and W. Leininger (1996). Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior 14*(1), 1–18.

Aoyagi, M. (2010). Information feedback in a dynamic tournament. *Games and Economic Behavior 70*(2), 242–260.

Azacis, H. and P. Vida (2015). Collusive communication schemes in a first-price auction. *Economic Theory 58*(1), 125–160.

Baye, M. R., D. Kovenock, and C. G. De Vries (1996). The all-pay auction with complete information. *Economic Theory 8*(2), 291–305.

Bergemann, D., B. Brooks, and S. Morris (2017). First-price auctions with general information structures: Implications for bidding and revenue. *Econometrica 85*(1), 107–143.

Bergemann, D. and S. Morris (2016). Information design, bayesian persuasion, and bayes correlated equilibrium. *American Economic Review 106*(5), 586–91.

Chi, C.-K., P. Murto, and J. Valimaki (2018). All-pay auctions with affiliated values. *Journal of Economic Theory (forthcoming).*

Denter, P., J. Morgan, and D. Sisak (2018). Showing off or laying low? the economics of psych-outs.

Fallucchi, F., E. Renner, and M. Sefton (2013). Information feedback and contest structure in rent-seeking games. *European Economic Review 64*, 223–240.

Fang, H. and S. Morris (2006). Multidimensional private value auctions. *Journal of Economic Theory 126*(1), 1–30.

Feng, X. and J. Lu (2016). The optimal disclosure policy in contests with stochastic entry: A bayesian persuasion perspective. *Economics Letters 147*, 103–107.

Fu, Q., Q. Jiao, and J. Lu (2014). Disclosure policy in a multi-prize all-pay auction with stochastic abilities. *Economics Letters 125*(3), 376–380.

Gershkov, A. and M. Perry (2009). Tournaments with midterm reviews. *Games and Economic Behavior 66*(1), 162–190.

Kamenica, E. and M. Gentzkow (2011). Bayesian persuasion. *American Economic Review 101*(6), 2590–2615.

Konrad, K. A. (2004). Altruism and envy in contests: an evolutionarily stable symbiosis. *Social Choice and Welfare 22*(3), 479–490.

Kovenock, D., F. Morath, and J. Munster (2015). Information sharing in contests. *Journal of Economics & Management Strategy 24*(3), 570–596.

Krishna, V. and J. Morgan (1997). An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory 72*(2), 343–362.

Liu, Z. and B. Chen (2016). A symmetric two-player all-pay contest with correlated information. *Economics Letters 145*, 6–10.

Lu, J., H. Ma, and Z. Wang (2018). Ranking Disclosure Policies in All-Pay Auctions with Incomplete Information. *Economic Inquiry 56*, 1464–1485.

Mathevet, L., J. Perego, and I. Taneva (2017). On information design in games.

Morath, F. and J. Münster (2008). Private versus complete information in auctions. *Economics Letters 101*(3), 214–216.

Rayo, L. and I. Segal (2010). Optimal information disclosure. *Journal of Political Economy 118*(5), 949–987.

Rentschler, L. and T. L. Turocy (2016). Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case. *Journal of Economic Theory 163*, 435–466.

Serena, M. (2017). Harnessing beliefs to stimulate efforts.

Sheremeta, R. M. (2010). Expenditures and information disclosure in two-stage political contests. *Journal of Conflict Resolution 54*(5), 771–798.

Siegel, R. (2014). Asymmetric all-pay auctions with interdependent valuations. *Journal of Economic Theory 153*, 684–702.

Wang, Y. (2013). Bayesian persuasion with multiple receivers. *Available at SSRN 2625399*.

Zhang, J. and J. Zhou (2016). Information disclosure in contests: A bayesian persuasion approach. *Economic Journal 126*(597), 2197–2217.

Zheng, J., J. Chen, and Z. Kuang (2018). Bayesian persuasion in all-pay auction contests.

Zheng, J., H. Zhao, and Z. Kuang (2017). Information design in simultaneous all-pay auction contests.