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Procyclical Price-Rent Ratios: Theory and Implications

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November 2019

In this simiple structural model of a cyclical housing market, periods of expanding and contracting aggregate demand are interrupted by unpredictable reversals. During expansions recoveries with decreasing vacancies and no construction are followed by booms with construction and no vacanies. Risk-neutral investors have no behavioral biases, capital constraints, private information, or trading frictions. In the resulting equilibrium, procyclical changes in price-rent ratios and expected appreciation rates lead procyclical changes in housing prices, which lead procyclical changes in rental income. Forward-looking housing prices must have more procyclical volatility than short-term rents. With heterogenous owners of housing, procyclical speculation is necessary for equilibrium,

Key words: procyclical price-rent ratios.
JEL classifications: E32 and R31.

Acknowledgements: I am grateful to Peter Chinloy and Adam Nowak for helpful suggestions. All errors are mine. Cpmments are welcome. Please do not circulate this early draft.

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1. Introduction

Housing markets are commonly characterized as cyclical. Hot and cold periods with increasing and decreasing housing prices and rents are separated by infrequent, random reversals, the timing of which is difficult to predict: Krainer (2001) and Shiller (2008). Empirically, growth rates of housing prices and housing construction are serially correlated, positively over periods of one year and negatively over periods of five years: Glaeser and Nathanson (2015) and Ghysels et al (2017). Housing prices have greater procyclical volatility of housing prices than housing rents, especially during the housing bubble of the 2000s: Glaeser and Nathanson (2015). Also, procyclical price-rent ratios lead procyclical real housing prices, but not housing rents: Gallin (2008). Finally, procyclical speculation appears to be a persistent property of cyclical housing markets.[2]

Some of these properties are easily explained, but others are not. If increases and decreases in the driver of housing demand are separated by low-frequency, randomly-timed reversals, then rates of housing appreciation and construction should be serially correlated, positively between reversals and negatively across reversals. Under the same conditions, both prices and price-rent ratios should be procyclical if housing prices are forward-looking, but short-term rents are not. With procyclical price-rent ratios, the procyclical volatility of housing prices should exceed the procyclical volatility of housing rents. For the same reason prices and price-rent ratios should lead procyclical rents, but why should price-rent ratios also lead prices? Also, how can price-rent ratios lead real housing prices but not housing rents? More generally, in what equilibrium can forward-looking, procyclical housing prices be consistent with the sticky rents reported initially in Genesove (2003)? Finally, is procyclical speculation persistent because it is both a cause and a consequence of cyclical housing markets?

To clarify the above issues, consider an important example: the widely reported, greater procyclical volatility of housing prices than housing rents. Many markets are characterized by intermittent or variable flows of new information.[3] In housing markets important new information about future growth rates of aggregate demand for housing services arrives intermittently, separated by quiet periods with relatively little new information about the same growth rates. Because housing prices are forward-looking, while short-term rents are not, prices are much more volatile than rents during active periods. This contrasts with

[2] See Gao et al (2019) and the many references cited therein.

[3] The literature for financial markets is reviewed in Ang and Timmerman (2012).

quiet periods when prices and rents change at roughly the same rate. This combination produces procyclical prices with greater volatility than procyclical fundamentals and rents.

In the empirical literature the deviation of housing prices from fundamentals is frequently attributed to mispricing relative to a simple, standard user-cost model without cyclical states: Ghysels et al (2017). An important example is the large literature on housing bubbles: Mayer (2011). More generally, empirical investigations have focused on the relative contributions of fundamental factors versus both behavioral biases and capital constraints.[4] In turn, this mispricing of housing has motivated many modifications of the simple user-cost model, including investors with limited rationality: Glaeser and Nathanson (2015) and Defusio et al (2017). Can similar results follow from fully rational pricing in a frictionless market driven by cyclical changes in a fundamental factor? If so, a minimalist cyclic model focused on core issues could confirm intuition, clarify details, and reveal other implications.

The above observations and questions motivate this simple structural model of cyclic housing markets. Here, housing has two critical characteristics. First, its prices are forward-looking, while its spot rents are not. Instead, these short-term rents depend only on the current aggregate demand and supply of perishable housing services. Second, hot and cold housing markets are modeled as expansions and contractions separated by randomly-timed reversals. A summary statistic for fundamental factors driving the aggregate demand for housing changes through time at a rate that depends on the state of the housing market. It increases during expansions and decreases during contractions. Together with other assumptions, these two properties produce in a frictionless market the main results of the model: procyclical price-rent ratios and expected appreciation rates, followed first in equilibrium by procyclical prices and then by procyclical rental income. Additional results follow from the structural elements of the model: excess supply of housing only during contractions and their subsequent recoveries, followed by expansions with construction of new homes. Easy extensions with heterogeneous housing, households, and investors generate the remaining results.

Procyclical growth rates of fundamentals separated by infrequent, random reversals is a natural representation of housing cycles. Its surprising absence from the literature is likely due to the complexity of the subsequent analysis relative to that of similar models with procyclical levels of fundamentals. This complexity is unavoidable here because price-rent ratios and thereby expected appreciation rates are subsequently shown to be countercyclical when procyclical growth rates of the fundamental factor are replaced by procyclical levels.

For the above reason the model is stripped to its barest bones. A summary statistic for fundamentals that affect the aggregate demand for housing services has one of two exogenous growth rates: positive during expansions and negative during contractions. Expansions and contractions are separated by Poisson transitions. With this single source of uncertainty, market reversals are both abrupt and completely unpredictable. No information about the

[4] Recent examples include Cox and Ludvigson (2019), Liu et al (2019), and the references cited therein.

timing of the next reversal leaks into the market before that event. By assumption, all reversals are also observed immediately by all identical, fully rational, risk-neutral investors with constant discount rates. Their aggregate demand for perishable housing services is isoelastic. The model has no behavioral biases, capital constraints, or informational asymmetries.

In the resulting equilibrium all information about future demand and thereby future housing rents is fully reflected in forward-looking housing prices. During instantaneous market reversals, housing prices and price-rent ratios adjust fully in response to the shifting state of the market. Both jump up during random transitions from cold to hot markets and down during the reverse transitions. At the same time spot rents remain constant because aggregate demand does not change during instantaneous market reversals. Between these random reversals aggregate demand, rents, and thereby prices move together, continuously over time. If price-rent ratios change between reversals, they do so in response to information unrelated to the reversal. Changes in both the price-rent ratio and the expected appreciation rate of housing are procyclical. These necessary conditions for equilibrium have other empirical implications for housing, such as procyclical speculation and cross-sectional differences in pricing volatility.

Even this minimalist model has significant analytical complexity. To simplify further the subsequent analysis and its exposition, the model is presented in three, increasingly realistic versions. In its initial, introductory version, cold markets are stagnant with constant, unchanging aggregate demand for housing services. Also, the housing market has no construction. In this simplest case, the unique equilibrium satisfying a law of one price for all identical assets is easily identified. It has explicit price-rent ratios that are constant in each market, cold or hot, and procyclical across markets. This separates the procyclical volatility of housing prices into two procyclical components: discrete, proportional changes in prices and price-rent ratios during market reversals followed by continuous, proportional changes in prices and rents between reversals. In this situation procyclical changes in price-rent ratios lead procyclical changes in prices, which lead procyclical changes in rents. Also, the intertemporal volatility of housing prices exceeds the intertemporal volatility of both price-rent ratios and rents. All of these results have the same source. New information about future rents, which arrives only during market reversals, affects forward-looking housing prices, but not short-term rents. Rents remain constant during reversals because reversals alter only the growth rate of demand for housing services and not its current level.

The first modification of the initial model has contracting cold markets, again with no housing construction during expanding hot markets. During contractions the aggregate demand for housing services decreases and housing vacancies increase. As a result, expanding hot markets have two phases: initial recoveries during which vacancies disappear, followed by booms with no vacancies. To simplify the analysis, recoveries are, by assumption, always followed by booms, however brief. In the resulting equilibrium, price-rent ratios are constant during both booms and recoveries. During contractions the price-rent ratio changes continuously through time in response to the new information about the longer recovery that follows a longer contraction. It decreases during sufficiently slow contractions and increases otherwise. Also, sharper contractions produce higher price-rent ratios during subsequent recoveries and, to a lesser extent, booms. Otherwise, the principal properties of the previous equilibrium are unchanged. Most importantly, increments in price-rent ratios again lead increments in prices, which lead increments in rents.

Housing construction is introduced as a minor modification of the second model. In the resulting equilibrium homes are built only when demand is growing and vacancies are zero. This construction reduces the appreciation rate of housing during booms, which, in turn, reduces price-rent ratios at all times–most during booms and least during busts. Thereby, construction reduces the procyclical volatility of housing prices and price-rent ratios. Not surprisingly, all these effects are greater when unit construction costs increase less rapidly with aggregate construction.

Additional empirical implications follow from easy extensions of the model. In the first extension investors in housing are distinguished from owner-occupiers. For several reasons investors value more than owner-occupiers expected appreciation relative to rent. From the model the housing market must then have both counter-cyclical rent-price ratios and procyclical expected appreciation rates. Other things equal, the ratio of investors relative to owner-occupiers must be higher in expanding hot markets than contracting cold markets. As such, investors have shorter time horizons than owner-occupiers. In practice, this procyclical investment or speculation in housing has multiple manifestations. During expansions tenants purchase homes; owner-occupiers buy second homes; and investors buy rental housing. Simultaneously, speculators flip builders' contracts for future delivery of new homes. Because this activity is necessarily associated with equilibrium, more so with more procyclical price-rent ratios, speculation can be both a cause and a consequence of volatile housing prices.

In the second extension of the model heterogenous housing and households are distinguished along different dimensions: more customized versus less customized homes, primary versus secondary homes, and older versus younger households. These houses and households are then ordered by their price-rent ratios across segmented submarkets. As predicted by the model, homes with higher price-rent ratios commonly attract relatively more procyclical speculation, which generates more procyclical pricing volatility. This relationship then orders heterogenous houses and households by their procyclical pricing volatilities: less for more customized versus less customized homes, primary versus secondary homes, and homes occupied by older versus younger households. The latter predictions are consistent with the limited empirical evidence cited in Section 7.

The paper is organized as follows. In the second section the model is motivated in the context of the most relevant previous papers and its analysis is sketched. The formal model is presented in the third section and its initial equilibrium is identified in the fourth. The model's two modifications appear in the next two sections. Empirical implications and easy extensions are discussed in the seventh section; rigid rents are discussed in the eighth. Major results are summarized in the final section. All derivations appear in the Appendix.

2. Preliminaries

This section has two parts. The first focuses on the most closely related, recent theoretical literature on housing cycles. That discussion emphasizes the issues that motivate this minimalist model of housing cycles. Because this model has some unavoidable analytical complexity, the technique for its subsequent analysis is sketched in the second part.

Literature: The theoretical literature on housing cycles is surveyed in Glaeser and Nathanson (2015). Subsequent contributions include Burnside et al (2017). The theoretical literature on endogenous price-rent ratios is much thinner. Recent contributions include the assignment model of Landvoight et al (2015), the general equilibrium in Flavilkus et al (2017), and endogenous search across market segments in Williams (2018).

In empirical studies fundamental housing prices are frequently defined as the expected present value of future housing rents. This risk-neutral pricing model, commonly called the user-cost model, has a long history in the housing literature. The user-cost model is then paired with alternative stochastic processes for exogenous rents. In the recent survey of the literature, each stochastic process has a linear or additive specification in discrete time: Glaeser and Nathanson (2015). The resulting pricing equations for housing are stationary and linear in all state variables, including rents. As a result, the models do not emphasize the essential elements that drive the core results in this paper.

This risk-neutral model has three important differences from standard user-cost models. First, the model is explicitly cyclic. The discrete state of the market switches stochastically between expansions and contractions in continuous time. As a result, the transition is both abrupt and randomly-timed. With Poisson transitions new information about future reversals is not revealed until the next reversal. Thereby, forward-looking housing prices change abruptly during market reversals, while short-term rents remain constant. Second, all changes are proportional rather than additive. This proportionality makes possible the simple, endogenous characterization of price-rent ratios. Finally, unlike standard user-cost models, this cyclical model also has a structural component. Contracting cold markets create excess capacity or vacancies in housing, which disappear during subsequent recoveries. As a result, recoveries with vacancies and no construction differ from expansions with construction and no vacancies.

Three cyclical, user-cost models are most closely related to this model. Each has search in housing markets with two states. In Krainer (2001) the state cycles stochastically between two levels of average housing rents: high in hot markets and low in cold markets. In the resulting equilibrium with costly search, hot markets have more liquidity and higher housing prices. In Besley and Mueller (2012), the state cycles between conflict and peace. At all times homeowners observe the level of violence that depends on the state and noise. In Ngai and Tenreyro (2014) the state cycles deterministically between two levels of housing turnover: high during hot markets, spring and summer, and low during cold markets, fall and winter. The analysis is then restricted to periodic steady states with two levels of housing inventory: high during hot markets and low during cold markets.

In each of the three models, exogenous and thereby endogenous variables are constant within markets or states and different across states. This specification greatly simplifies the Bellman equations that determine the endogenous variables and thereby facilitates the focus on other issues. The simplification is effectively acknowledged as a analytical convenience in both Krainer (2001) and Ngai and Tenreyro (2014). There, hot and cold markets are described as periods with rising and falling prices, rather than higher and lower prices. This description of hot and cold markets is further motivated by the empirical evidence provided in the latter paper.

Analysis: In this paper hot and cold housing markets are modeled as expansions and contractions. As previously indicated, this specification has several advantages. However, it also has one major disadvantage. Its analysis in the Appendix is much more complicated than the corresponding model with demand switching between two levels, high and low. For that reason the derivation is described briefly below.

The problem is solved in three steps of increasing difficulty. In the initial step, cold markets are stagnant and construction is precluded. Because aggregate demand does not decrease during cold markets, housing never has excess supply. With no excess supply and no construction, the cyclical user-cost model applies. In this case the expected present values of housing are calculated from a pair of Bellman equations, one for each market, linked by Poisson transitions between the two markets. The Bellman equations are first-order differential equations if the growth rates of housing demand are nonzero during both markets. With stagnant cold markets, the differential equation for cold markets disappears. It is replaced by simple proportional pricing relationship between the prices of housing in hot and cold markets that can be substituted into the differential equation for expanding hot markets. This yields a first-order differential equation for hot markets that does not depend on the price of housing during cold markets. That equation is easily solved. Its solution, combined with the above proportional relationship for cold markets, produces the price-rent ratios in Section 4.

Contrast this solution with the previous literature. If the two states are distinguished by levels of aggregate demand rather than growth rates, then the growth rates in both are zero and each differential equation above is replaced by a simple proportional relationship between the prices of housing in hot and cold markets. This simple solution can then be combined with other complications, such as search in Krainer (2001) or imperfectly observed states as in Besley and Mueller (2012).

With contracting cold and expanding hot markets, the two, linked differential equations must be solved simultaneously.[5] The solution is both complex and difficult to interpret economically. That solution is further complicated by the structural issues related to housing supply during expansions: excess capacity without construction versus construction without excess capacity. For this reason the problem is modified slightly. As indicated in the introduction, expanding hot markets are split into two phases: recoveries without construction, during which the vacancies from the previous contraction are absorbed, followed by expansions with no vacancies. By assumption, recoveries are always followed by booms, however brief. Specifically, the Poisson transition rate from recoveries to contractions is assumed to be zero.

With this simplifying assumption each contraction combined with its subsequent recovery has a critical property in common with a stagnant cold market. When either ends aggregate demand equals its historic maximum. The price-rent ratio during booms can then be calculated using a minor modification of the previous solution with stagnant cold markets. In essence, contractions and subsequent recoveries are replaced by synthetic, stagnant cold markets with two critical properties. Each synthetic stagnant market has the same expected duration as the corresponding contraction and subsequent recovery. Also, housing has the same expected present value at the start of either the synthetic market or the contraction followed by its recovery. This substitution generates a solution for housing prices during booms that matches the previous solution for expanding hot markets paired with stagnant cold markets. That solution delinks the differential equations for hot and cold markets and thereby permits a relatively simple recursive solution for housing prices during recoveries and busts. This modified problem is solved first without construction and then with construction.

3. Initial Model

The housing market is extremely simple. It has a fixed stock of atomistic, identical housing that never depreciates or otherwise obsolesces. Initially, the aggregate supply or stock of housing h is conveniently normalized at one: $h = 1$. Construction of new homes is deferred to Section 6. Each unit of housing produces per unit of time a unit of perishable housing services that its household immediately consumes. All atomistic households are identical. Households have an aggregate demand for perishable housing services that depends on an exogenous driver with the current value x. This single statistic x summarizes all attributes, other than the current rental rate, that affect the current demand for housing services.

The housing market has two completely observable states: cold and hot. The two states are distinguished only by the growth rate of the driver of aggregate demand for housing services x. In each state i this exogenous demand x changes at a constant rate through time: $\dot{x}/x = \rho_i$ for $i = 0, 1$. In the introductory model, exogenous demand x is constant

[5] See, for example, Zwillinger (1992), pages 360-363.

during cold markets: $\rho_0 = 0$. In the main model, exogenous demand decreases at a constant rate during cold markets: $\rho_0 < 0$. In both variants of the model, exogenous demand grows at a constant rate during hot markets: $\rho_1 > 0$. The initial model simplifies both the analysis and exposition of the main model.

Over time the market switches randomly between the two states, cold and hot. During the short interval of time Δt, the market switches from state i to the alternative state, $j \neq i$, with the probability: $\alpha_i \Delta t + o(\Delta t)$ for $i, j = 0, 1$. The residual $o(\Delta t)$ represents all terms of smaller order than Δt. With these Poisson shifts between states, the remaining time in state i has at all times an independent negative exponential distribution with the mean $1/\alpha_i$. Consistent with recent empirical evidence on housing cycles, cold markets are no longer on average than hot markets: $0 < \alpha_1 \leq \alpha_0 < 1$.[6] All agents always observe the current state. The model has no other uncertainty.

Houses are both consumer durables and real assets. As consumer durables all identical houses produce perishable housing services at the same constant rate per unit of time. The aggregate demand for housing services depends on the exogenous variable x and the associated spot price for housing services. This spot price is the current rental rate for housing R. By assumption, the aggregate demand for rental services is isoelastic: $xR^{-\eta}$. The constant elasticity of aggregate demand with respect to rent $-\eta$ is negative and finite: $0 < \eta < \infty$. As indicated, the elasticity of demand with respect to the exogenous component of demand x is set equal to 1. No generality is lost because the variable x can be replaced by its power function without altering the subsequent results. This isoelastic aggregate demand for rental services is a familiar analytical convenience. It preserves the proportionality of the model that makes possible the relatively simple, explicit results for endogenous price-rent ratios.

The rental rate for housing is determined in the spot market for housing services. The rental rate R equates the above aggregate supply of housing services 1 with its aggregate demand $xR^{-\eta}$:

$$R(x) = x^{1/\eta}, \tag{1}$$

for $0 < x < \infty$. This spot rent has two important properties. It does not depend on the current state of the market i because it does not depend on future values of the exogenous demand x. Also, it does not change during stochastic transitions between markets because only the growth rate of the variable x changes during the instantaneous transitions.

Housing is real asset with net cash inflows in the form of rents or implicit rents. In this minimalist model, all expenses of ownership, including maintenance, repairs, and taxes, are ignored. Also, housing never depreciates. At all times the price of each home must equal

[6] Downturns have shorter average duration with the inclusion of the most recent upturn ending in 2006: Bracke (2011).

the expected present value of its future rents and appreciation over a short interval of time Δt:

$$P_i(x) = e^{-\delta\Delta t}\{R(x)\Delta t + P_i(x{+}\Delta x) + \alpha_i\Delta t[P_j(x{+}\Delta x) - P_i(x{+}\Delta x)]\} + o(\Delta t), \tag{2}$$

for $i \neq j \in \{0,1\}$ and $0 < x < \infty$. The present value of housing at time t is calculated from its expected future value at time $t{+}\Delta t$. That future value is discounted at the constant rate δ per unit of time. The first component of this future value is the rent $R(x)\Delta t$ over the short interval of time Δt. The second component is the future price conditional on the future exogenous demand $x + \Delta x$, both at the future time $t + \Delta t$. The remaining terms are the expected change in price of switching from state i to the alternative state over the interval Δt. This expectation reflects the sole source of uncertainty in the model: the probability $\alpha_i\Delta t + o(\Delta t)$ of switching from state i to state j during the same small interval of time Δt. The remaining terms $o(\Delta t)$ are of smaller order than Δt.

Equilibrium in the housing market has two components. The rental rate R clears in (1) the spot market for perishable housing services. Conditional on this rental rate, the price of housing P_i has the expected present value in (2). This produces the price-rent ratios, $p_i \equiv P_i/R$, and the associated rent-price ratios, $r_i \equiv 1/p_i$. These functions, R and P_i, are determined for all feasible values of the state variables: $i = 0, 1$ and $0 < x < \infty$.

4. Stagnant Cold Markets

The initial equilibrium with stagnant cold markets is characterized in this section. As such, it is an introduction to the more complicated, more realistic solution with contracting cold markets in Section 5. The initial equilibrium identified here is the unique solution to (1) and (2) with constant price-rent ratios, p_0 and p_1, during cold and hot markets, respectively. These constant price-rent ratios are motivated below the main result in this section as a law of one price consistent with the informational structure of the model: no new information between market reversals.

The pricing equation for housing (2) is now rewritten as follows. Expand the expected present value on the right side of (2) in Δt; subtract P_i from both sides of (2); divide by Δt; and let $\Delta t \to 0$. This generates two, linked differential equations that price housing as a real asset:

$$0 = \rho_i x P_i' - (\alpha_i{+}\delta{-}r_i)P_i + \alpha_i P_j, \tag{3}$$

for $i \neq j = 0, 1$. In (3) risk-neutral investors expect in each state i a total rate of return on housing equal to their common, constant discount rate δ. This expected return has two components: percentage rent equal to the rent-price ratio r_i and the residual expected appreciation, $m_i = \delta - r_i$. In turn, this expected appreciation m_i has two components: appreciation during market i at the rate, $\rho_i x P_i'/P_i$, and expected appreciation, positive or negative, during the random transition to the alternative market, $\alpha_i(P_j/P_i - 1)$.

The solution to the two linked differential equations in (3) must also satisfy two boundary conditions. By previous assumption, the growth rate ρ_i of exogenous demand x is constant in both states, $i = 0, 1$. In this case, the value, $x = 0$, is an absorbing state. If exogenous demand x reaches zero, then rent in (1) and thereby housing prices in (3) remain at zero forever. This produces the two initial conditions:

$$P_i(0) = 0, \tag{4}$$

for $i = 0, 1$.

Problem (3) and (4) is easily solved if exogenous aggregate demand is constant in one market, hot or cold. Here, cold markets are stagnant: $\rho_0 = 0$. In this special case, the differential equation in (3) for cold markets, $i = 0$, simplifies to a proportional relationship between housing prices in hot and cold markets. The housing price in a cold market $P_0(x)$ is the price in the next hot market $P_1(x)$ multiplied by a present value factor. This product replaces the price $P_0(x)$ in the remaining equation in (3) for hot markets, $i = 1$. That single, linear, first-order equation has a simple solution that satisfies the initial condition (4) if the unknown rent-price ratios r_i remain constant in both markets, $i = 0, 1$.

The initial equilibrium with stagnant cold markets is completely characterized by the rent in (1) and the prices from (3), and (4). The unique solution with constant rent-price ratios r_i in the two states, $i = 0, 1$, is derived in the Appendix and presented in the first proposition.

Proposition 1: *With stagnant cold markets, $\rho_0 = 0 < \rho_1$, the constant rent-price ratios,* r_0 and r_1, *that uniquely satisfy (1), (3), and (4) conditions have the respective values:*

$$r_0 = \delta - \frac{\alpha_0 \rho_1/\eta}{\alpha_0 + \alpha_1 + \delta - \rho_1/\eta}, \qquad r_1 = \delta - \frac{(\alpha_0 + \delta)\, \rho_1/\eta}{\alpha_0 + \alpha_1 + \delta}, \tag{5}$$

with $0 < r_1 < r_0 < \delta$.

The constant rent-price ratios in (5) are countercyclical across states: $r_0 > r_1$. In this case, the constant, state-dependent, price-rent ratios must be procyclical: $p_0 < p_1$. With the constant discount rate δ, the constant expected appreciation rates, $m_i = \delta - r_i$ from (3), must also be procyclical: $m_0 < m_1$. Not surprisingly, expected appreciation rates are less procyclical than appreciation rates within markets: $0 < m_0 < m_1 < \rho_1/\eta$. This follows from the cyclic, mean-reverting transitions between hot and cold markets. As those reversions become less frequent, the difference between expected appreciation rates and rates of appreciation within markets disappears: $m_i \to \rho_i/\eta$ as $\alpha_i \to 0$ for $i = 0, 1$.

Constant price-rent ratios between market reversals can be motivated as follows. In this model no new information arrives in either market, hot or cold, before its next random reversal to the alternative state. Specifically, the time to the next reversal has in each market

i an independent, negative exponential distribution with the constant mean $1/\alpha_i$. With the proportionality everywhere in this model and the constant, exogenous rates of return, the mean appreciation rate of housing during market i must then be an endogenous constant m_i, independent of the exogenous excess demand x. In turn, the rent-price ratio r_i must also be constant: $r_i = \delta - m_i$ for $i = 0, 1$. This requires a constant price-rent ratio p_i at all times during each market i. In short, price-rent ratios cannot change between reversals because no new information arrives in the market until the next reversal. Essentially, this is a law of one price for each market. In this stationary model it applies over time within each market–somewhat different than its standard application across locations within a market.

Constant spot rents during reversals combined with constant price-rent ratios between reversals have strong implications. To see the first set of implications, focus on a reversal and the subsequent market before the next reversal. During each such period price-rent ratios change only during the reversal; rents change only after the reversal; while prices change both during and after the reversal. In the model all these changes are procyclical. Therefore, changes in price-rent ratios must lead changes in prices, which must lead changes in rent. Because expected appreciation rates change concurrently with price-rent ratios, changes in expected appreciation rates must also lead changes in prices and thereby rents.

Constant spot rents during reversals combined with constant price-rent ratios between reversals also require that the change in prices be proportional to both the change in price-rent ratios during each reversal and the change in rents after each reversal. This proportionality has another implication. Consider transitions between stagnations and expansions and the subsequent markets between the next reversals. The percentage change in prices across these periods has a variance equal to the sum of three terms. The first is the variance across reversals of the percentage changes in price/rent ratios during reversals. The second is the corresponding variance of the percentage change in rents before the next reversal. The third is the positive covariance between these two procyclical percentage changes. By this metric, the variance of prices must exceed the sum of the first two terms: the variances of price-rent ratios and rents, correctly measured. The correct measurement of rental income is discussed in Section 7.

The price-rent ratios from (5) have additional properties. As shown in the Appendix, both ratios, p_0 and p_1, increase with both the growth rate of demand during hot markets ρ_1 and the expected duration of hot markets $1/\alpha_1$. Both changes are more rapid for hot markets than cold markets. By contrast, the same ratios decrease with the expected duration of cold markets $1/\alpha_0$, less rapidly so for hot markets. Thereby, both price-rent ratios, p_0 and p_1, and their ratio p_1/p_0 are higher with either sharper or longer expanding hot markets or shorter stagnant cold markets. With higher ratios p_1/p_0, price-rent ratios are more procyclical.

This cyclic model also has long-term mean reversion in the following sense. Denote by $\Pr(t)$ the probability at the current time 0 of an expanding hot market at the future time $t > 0$. The concurrent probability of a contracting or stagnant cold market is the residual:

$1-\Pr(t)$. This current probability of a future expansion has the value:[7]

$$\Pr(t)=\frac{\alpha_0}{\alpha_0+\alpha_1}+\left[\Pr(0)-\frac{\alpha_0}{\alpha_0+\alpha_1}\right]\exp[-(\alpha_0+\alpha_1)t]\,. \tag{6}$$

As indicated, the probability (6) converges to its limit $\alpha_0/(\alpha_0+\alpha_1)$ as the future time t becomes more remote, $t\to\infty$. The convergence is mean-reverting: negative if the probability is sufficiently large, $\Pr(t)>\alpha_0/(\alpha_0+\alpha_1)$, and positive if the inequality is reversed. This mean reversion also applies to expected, future price-rent ratios and appreciation rates of housing.

The above results require that hot and cold housing markets be distinguished by the growth rates of aggregate demand, rather than their levels. To see this, suppose that exogenous aggregate demand x does not change during either market: $\rho_0=0=\rho_1$. Imagine, instead, that demand jumps up from x to $\beta_0 x$ during transitions from cold to hot markets and down from x to $\beta_1 x$ during the reverse transitions from hot to cold: $\beta_0>1\geq\beta_1>0$. The resulting price-rent ratios, p_0 and p_1, have the values, (A.7) in the Appendix. These values are countercyclical: $p_0>p_1$. Price-rent ratios are countercyclical because forward-looking prices are smoothed relative to spot rents across mean-reverting market reversals, while both prices and rents are constant between reversals. With procyclical rent-price ratios $r_0<r_1$, expected housing appreciation, $m_t=\delta-r_t$, must then be countercyclical. Because p_0/p_1 is everywhere increasing in β_0 and decreasing in β_1, this countercyclically is greater with more procyclical changes in demand. Finally, changes in price-rent ratios cannot lead changes in either prices or rents because both change only during transitions between markets. All these results are counterfactual.

5. Contracting Cold Markets

The stagnant cold markets of the previous section are replaced in this section by contracting cold markets. Here, exogenous demand decreases at a constant rate during cold markets: $\rho_0<0$. In the initial model with stagnant cold markets and expanding hot markets, $\rho_0=0<\rho_1$, the exogenous aggregate demand for housing x never decreases. As a result, exogenous demand x equals at all times t its historical high or running maximum:

$$\bar{x}_t=\max\{x_\tau : 0\leq\tau\leq t\}\,.$$

During contracting cold markets, called busts in this section, exogenous demand x_t decreases from its historical high: $\bar{x}_t$. During the subsequent expansion, exogenous demand x_t then increases toward its historical high $\bar{x}_t$. With stochastic transitions from hot to cold markets, $\alpha_1>0$ in (3), expansions can end before exogenous demand returns to its historical high: $x_t=\bar{x}_t$. This possibility greatly complicates the analysis of equilibrium with contracting

[7] See, for example, Cox and Miller (1965), page 172, equation (64).

cold markets. That complexity motivates the modification below of the previous problem, (3) and (4). In the subsequent text the subscript for time t is deleted.

The previous problem is now simplified as follows. Recoveries never switch stochastically to busts. Instead, recoveries end when aggregate demand first returns to its historical high: $x = \bar{x}$. In this case, expanding hot markets have two phases: an initial recovery with $x < \bar{x}$ and a subsequent boom with $x = \bar{x}$. Housing has excess supply or vacancies during busts and recoveries, but none during booms. Vacancies accumulate during busts and diminish during recoveries. Booms begin when vacancies disappear. Construction of new housing during booms is added in the next section.

This modification of the previous problem is formalized as follows. In (3) contracting cold markets or busts are identified by the index, $i = 0$. By contrast, expanding hot markets, $i = 1$, are split into two phases: recoveries, $i = 1.0$, and booms, $i = 1.1$. Busts become recoveries with the previous probability per unit of time α_0 that stagnant cold markets switched to expanding hot markets. Also, booms become busts with the corresponding probability that expanding hot markets switched to stagnant cold markets: $\alpha_{1.1} = \alpha_1$. By contrast, recoveries never switch stochastically to busts: $\alpha_{1.0} = 0$. Instead, recoveries become booms when vacancies disappear. This modification of the previous problem, (3) and (4), appears in the Appendix as (A.8) and (A.9).

To solve the modified model, (1) with (A.8) and (A.9), consider a contracting cold market or bust combined with its subsequent recovery. This combination begins and ends with exogenous aggregate demand at its historical high $\bar{x}$. As such, the combination matches the initial model where stagnant cold markets also begin and end with the same exogenous demand $\bar{x}$. However, the combination has a longer expected duration than a bust, which, by assumption, has the same expected duration as a stagnant cold market $1/\alpha_0$. Also, the combination generates at all times after its beginning and before its end less rent than the stagnant cold market: $R(x) < R(\bar{x})$ for $x < \bar{x}$.

The above properties of the combined bust and recovery suggest the following solution technique. Construct a synthetic stagnant cold market with two properties. It has the same expected duration and the same expected present value of rents as the combined bust and recovery. In this case, the risk-neutral investors in the model must be indifferent at all times during booms between housing in either of two economies: one with booms followed by the combination of busts with recoveries and the second with the same booms followed by the above synthetic cold markets. Investors' indifference applies not only during booms but also at the start of the subsequent bust when exogenous aggregate demand x equals its historic maximum $\bar{x}$. Thereby, housing must have during booms and immediately thereafter the same prices in either of the two economies. This synthetic cold market is constructed in the Appendix and its principal properties are identified below.

Longer busts require longer subsequent recoveries under the assumption that recoveries end only with the absorption of all excess housing from the previous bust. In this case, the decrease in demand during a bust must be equal the increase in demand during the subsequent recovery. During a bust with the expected duration $1/\alpha_0$, demand decreases by $-\rho_0/\alpha_0$. During the subsequent recovery of expected duration d, demand must then increase by the equal amount $\rho_1 d$. This equality requires the expected duration: $d = -\rho_0/\alpha_0\rho_1$. At the beginning of a bust, the bust and its subsequent recovery then has the expected duration: $1/a_0 = 1/\alpha_0 - \rho_0/\alpha_0\rho_1$. The latter equality requires the new parameter:

$$a_0 \equiv \frac{\alpha_0}{1-\rho_0/\rho_1} < \alpha_0. \tag{7}$$

Details appear in the Appendix.

The second adjustment is also straightforward. With risk-neutral investors the value of rent received during a bust and subsequent recovery is measured by its expected present value at the beginning of a bust. Divide this expected present value by the expected present value of rents during stagnant cold markets with the same expected duration. The resulting ratio is calculated in the Appendix and represented here by the new parameter:

$$\lambda \equiv \frac{\alpha_0 + \delta}{\alpha_0 + \delta - \rho_0/\eta}, \tag{8}$$

with $0 < \lambda < 1$. The rent (1) during a bust and subsequent recovery has the same expected present value at the beginning of the bust as the constant rent, $\lambda\bar{x}^{1/\eta}$ from (8), during a stagnant cold market with the expected duration, $1/a_0$ from (7).

The valuation equations for housing are solved recursively as follows. Begin with booms: $i = 1.1$. The pricing function for housing during booms $P_{1.1}$ is calculated in the Appendix from the corresponding pricing function during expansions with the above synthetic stagnation. That solution uses the new notation:

$$r_{0a} \equiv \delta - a_0 \frac{\lambda\rho_1/\eta + (1-\lambda)\delta}{a_0 + \lambda(\alpha_1 + \delta - \rho_1/\eta)} \tag{9}$$

and

$$r_{1a} \equiv \delta - \frac{(a_0 + \delta)\rho_1/\eta - (1-\lambda)(\alpha_1 + \delta/\lambda)}{a_0 + \lambda\alpha_1 + \delta}, \tag{10}$$

with the corresponding price-rent ratios, $p_{ia} = 1/r_{ia}$, for $i = 0, 1$. The rent-price ratios (9) and (10) are the previous rent-price ratios, r_0 and r_1 in (5), with the stagnant cold market replaced by the synthetic cold market in this section. The differences, $r_{0a} - r_0$ and $r_{1a} - r_1$, disappear with the parameter values, $a_0 = \alpha_0$ in (7) and $\lambda = 1$ in (8).

Conditional on the housing prices during booms, the remaining prices for housing during recoveries and busts are calculated recursively as follows. Because recoveries are always followed by booms, the pricing function for housing during recoveries $P_{1.0}$ is calculated as the present value from (3) for hot markets, $i = 1$, with no stochastic transitions to cold markets:

$\alpha_1 = 0$. This differential equation is solved subject to the boundary or terminal condition: $P_{1.0}(\bar{x}) = P_{1.1}(\bar{x})$. The latter equality holds because recoveries end and booms begin when exogenous aggregate demand x first reaches its historical maximum $\bar{x}$. Conditional on the price of housing during recoveries $P_{1.0}$, the pricing function during busts P_0 is then calculated as the present value from (3) with $i = 0$ and $\rho_0 < 0$. This solution is then shown to satisfy the initial conditions (4). Here, a recursive solution is possible because the simultaneity of the solution in (3) for hot and cold markets is broken by the separate derivation using synthetic cold markets of the pricing function during booms $P_{1.1}$. Details appear in the Appendix.

Because bigger busts require longer recoveries, the duration of a bust has valuable information about the subsequent recovery. That information is reflected in current prices but not current rent. For this reason, the rent-price ratio during busts $r_{0.1}$ cannot be constant, independent of the exogenous demand x. In this case, the previous analysis must be modified. That modification includes the new notation:

$$\phi \equiv \frac{a_0 + \delta - \rho_1/\eta}{(a_0 + \delta - \rho_0/\eta)(\delta - \rho_1/\eta)} > 0, \qquad \psi \equiv -\frac{a_0 + \delta}{\rho_0} > 0.$$

Also, the rent-price ratio during busts $r_{0.1}$ is assumed to be differentiable everywhere in the exogenous demand x. The remaining rent-price ratios during recoveries $r_{1.0}$ and booms $r_{1.1}$ are constant as before. The three endogenous rent-price ratios are calculated in the Appendix and displayed below in the second proposition.

Proposition 2: *With busts, $\rho_0 < 0 < \rho_1$, and no stochastic transitions from recoveries, the differentiable rent-price ratio during busts $r_{0.1}$ and constant rent-price ratios during recoveries $r_{1.0}$* and booms $r_{1.1}$ *that uniquely satisfy (1), (A.8), and (A.9) are $r_{0.1} = 1/p_{0.1}$ with*

$$p_{0.1}(x) = \phi + (\lambda p_{0a} - \phi)\left(\frac{x}{\bar{x}}\right)^{\psi} \tag{11}$$

and

$$r_{1.0} = \delta - \frac{\rho_1}{\eta}, \qquad r_{1.1} = r_{1a}. \tag{12}$$

The core properties of Propositions 1 and 2 are strikingly similar. Most importantly, prices again have more procyclical volatility than either price-rent ratios or rents. Also, changes in price-rent ratios lead changes in prices, which lead changes in rents. The latter result follows from two properties of the new equilibrium. First, the price-rent ratios during recoveries and booms, $p_{1.0}$ and $p_{1.1}$ from (12), are constant, much like the previous price-rent ratio p_0 during expansions with stagnant cold markets. Second, the change in the price-rent ratio during busts, $p_{0.1}$ in (11), is independent of its previous decrease $p_{0.1}(\bar{x})/p_{1.1}(\bar{x})$ during the transition from boom to bust. The latter independence follows immediately from two properties of the model. Stochastic transitions between markets are independent Poisson events. Also, those events are immediately observed by all investors and fully reflected in housing prices.

The time-varying price-rent ratio during busts (11) can be understood as follows. It changes with exogenous demand x because the lower demand x realized during a longer bust requires a longer recovery. Specifically, price-rent ratio $p_{0.1}(x)$ is a weighted average of two price-rent ratios: λp_{a0} and ϕ_a. The first is the price-rent ratio at the beginning of busts: $p_{0.1}(\bar{x}) = \lambda p_{a0}$. As indicated this ratio is calculated from booms or expansions paired with a synthetic cold market. In (11) It has the weight: $0 < (x/\bar{x})^{\psi} < 1$. As the contraction continues, exogenous demand x decreases toward 0 and the price-rent ratio $p_{0.1}(x)$ approaches the second ratio: $p_{0.1}(0) = \phi$. Depending on the parameter values, the price-rent ratio $p_{0.1}(x)$ either increases or decreases continuously with the duration of the contraction. With sharper contractions, represented here by more negative values ρ_0, the convergence is less rapid. It stops when the market switches stochastically from contraction to expansion.

Contracting cold markets have other effects. Several are striking. Contractions depress price-rent ratios during both busts and booms. Both price-rent ratios, $p_{0.1}$ and $p_{1.1}$ from (11) and (12), are less than the corresponding values with stagnant cold markets, p_0 and p_1 from (5). The same price-rent ratios, $p_{0.1}$ and $p_{1.1}$, are smaller with sharper contractions or shorter expansions, other things equal. Booms also have lower price-rent ratios $p_{1.1}$ with longer contractions or shorter expansions. Finally, price-rent ratios during recoveries and booms, $p_{1.0}$ and $p_{1.1}$ in (12), are higher during recoveries than booms: $p_{1.0} > p_{1.1}$. Surprisingly, the last property holds even without construction during booms. Comparative statics are summarized in Table 2.

6. Construction

Construction of new homes is added in this section. To simplify the analysis, all builders are perfectly competitive and all development of housing is instantaneous once started. By assumption the unit cost of construction c depends only on aggregate current construction q. Also, unit costs and aggregate construction grow at proportional rates: $\dot{c}/c = \theta \dot{q}/q$. These assumptions generate an isoelastic unit cost function: $c \propto q^{\theta}$. The constant cost elasticity, $\theta \geq 0$, is exogenous to the model. It reflects an isoelastic aggregate supply of local inputs, like labor and land, in the production function for housing. In this minimalist model land is treated like other factors of production.

With instantaneous development and the above unit costs, new homes are built and sold only during booms when aggregate demand is growing and the excess supply of homes is zero. In all other aspects the previous model is preserved. Previously, the housing stock h was constant. Now, it increases with aggregate construction only during booms. Again, depreciation of existing housing is ignored. With instantaneous development, aggregate construction q then equals the change in the housing stock: $q = \dot{h}$. Previously, the equality of demand and supply with the fixed housing stock, $h = 1$, produced the rental rate (1) dependent on the relative exogenous demand x. Now, the same equality of supply and demand with the variable housing stock, $h \geq 1$, generates the rental rate (1) dependent on the relative exogenous demand x/h. In other words, x is replaced by x/h. This difference in notation is easily eliminated without loss of generality by transforming variables. Henceforth, the variable x represents the relative demand x/h. During booms this relative demand x grows at the net rate: $\dot{x}/x = \rho_1 - \dot{h}/h$. During busts and recoveries when the housing stock does not change, $\dot{h} = 0$, the relative demand x grows at its previous rates, ρ_0 and ρ_1.

In this model construction occurs only during booms. Again, the change in the housing stock always equals aggregate construction: $\dot{h} = q$. Focus on steady state in which the housing stock h grows during booms at a constant rate: $\dot{h} = \nu h$ for some constant, $\nu > 0$. In this case, aggregate construction and the housing stock must grow at the same rate: $\dot{q}/q = \ddot{h}/\dot{h} = \dot{h}/h$. In turn, the isoelastic unit costs c grow during booms at the constant rate: $\dot{c}/c = \theta\dot{q}/q = \theta\dot{h}/h$.

During booms the common price of identical homes must always equal the common cost of construction. This requires that housing costs and prices always grow at the same rate. Given a constant rent-price ratio during booms $r_{1.1}$, the unit cost c must then grow during booms at the same rate as housing rents in (1): $\dot{c}/c = \dot{R}/R = \dot{x}/\eta x$. With the above transformation of variables, relative demand x grows during booms at the rate: $\dot{x}/x = \rho_1 - \dot{h}/h$. As a result, unit construction costs must also grow during booms at the rate: $\dot{c}/c = (\rho_1 - \dot{h}/h)/\eta$.

In the previous two paragraphs, the growth rate of construction costs during booms must satisfy two constraints. Both growth rates of unit costs c during booms must be equal in equilibrium. This equality determines the growth rates during booms of the housing stock h and thereby relative demand x:

$$\frac{\dot{h}}{h} = (1-\zeta)\rho_1 \qquad \text{and} \qquad \frac{\dot{x}}{x} = \zeta\rho_1 \qquad \text{with} \qquad \zeta \equiv \frac{\eta\theta}{\eta\theta + 1}. \tag{13}$$

Not surprisingly, relative demand x grows less rapidly with construction: $0 \leq \zeta < 1$. Because it does not decrease during booms, relative demand always equals its historical maximum, $x = \bar{x}$, throughout booms. The corresponding growth rates during busts and recoveries are unchanged because homes are built only during booms.

The reduced growth rate during booms with construction of the relative demand in (13) alters the rent-price ratios in the previous proposition. The new, reduced growth rate $\zeta\rho_1$ replaces the corresponding growth rate ρ_1 of exogenous demand x during booms paired with the synthetic stagnations in the previous version of the model. This simple change in one constant produces the rent-price ratios in the third proposition. That proposition uses the new notation:

$$r_{0b} \equiv \delta - a_0 \frac{\lambda\zeta\rho_1/\eta + (1-\lambda)\delta}{a_0 + \lambda(\alpha_1 + \delta - \zeta\rho_1/\eta)} \tag{14}$$

and

$$r_{1b} \equiv \delta - \frac{(a_0 + \delta)\zeta\rho_1/\eta - (1-\lambda)(\alpha_1 + \delta/\lambda)}{a_0 + \lambda\alpha_1 + \delta}. \tag{15}$$

These new composite constants are the corresponding constants (9) and (10) from the previous proposition with the single substitution: $\zeta\rho_1$ for ρ_1. Again, these rent-price ratios have the corresponding price-rent ratios: $p_{bi} = 1/r_{bi}$ for $i = 0, 1$.

Proposition 3: *With busts, $\rho_0 < 0 < \rho_1$, and no stochastic transitions from recoveries, $\alpha_{1.0} = 0$, the differentiable rent-price ratio $r_{0.1}$ and the constant rent-price ratios, $r_{1.0}$ and $r_{1.1}$, that uniquely satisfy (1), (A.8), and (A.9) are $r_{0.1} = 1/p_{0.1}$ with*

$$p_{0.1}(x) = \phi + (\lambda p_{0b} - \phi)\left(\frac{x}{\bar{x}}\right)^{\psi} \tag{16}$$

and

$$r_{1.0} = \delta - \frac{\rho_1}{\eta}, \qquad r_{1.1} = r_{b1}. \tag{17}$$

Not surprisingly, housing construction reduces the appreciation rate of housing during booms. With the isoelastic costs (17), the appreciation rate of housing during booms decreases from ρ_1/η to $\zeta\rho_1/\eta$. In turn, this decreases the price-rent ratio during both busts $p_{0.1}$ and booms $p_{1.1}$, but does not alter the price-rent ratio during recoveries $p_{1.0}$. These results follow from Propositions 2 and 3 with the inequalities: $p_{0a} > p_{0b}$, and $p_{1a} > p_{1b}$. The first result is intuitive. During booms construction reduces the appreciation rate of housing and thereby the expected returns during both busts and booms. Because the total rate of return on housing δ is constant, the price-rent ratio must be lower during booms with construction. Again, recoveries are unaffected because recoveries are assumed to be deterministic.

7. Empirical Implications

In this section empirical implications are identified. This includes predictions from both the minimalist model and its easy extensions. The extensions are described below.

Empirical implications of the model are identified in the Introduction. Rent-price ratios and expected appreciation rates on housing are procyclical. Their procyclical changes lead procyclical changes in prices, which lead procyclical changes in rents, correctly measured, and rental income. Also, housing prices have more procyclical volatility than both price-rent ratios, and rents. These predictions apply to rents and rental income correctly measured with all their cyclical components, as described in the next section. Booms have higher price-rent ratios with either longer, more rapid expansions or shorter, slower contractions. Price-rent ratios during busts are less affected by expansions and contractions. As a result, categories of housing with higher price-rent ratios during booms should also have more procyclical price-rent ratios and thereby housing prices with more procyclical volatility.

Speculation: Different households value differently their home as a consumer durable versus a real asset. For multiple reasons shorter-term occupants value their housing services less than longer-term occupants in comparable housing. In a recent survey owners-occupiers who expect to sell within three years attach relatively less value to their house as a place to live than owners who expect longer tenancy: Zillow (2016). Similarly, landlords should value their rents less than owner-occupants value their implicit rents because tenants typically value their housing services less than owner-occupants. Shorter-term tenants are more poorly matched to their housing than longer-term owner-occupants for several related reasons. Tenants move on average more often than owner-occupants: Zillow (2016). Compared to owner-occupiers, tenants search less intensively for their housing and emphasize less during their search attributes that require physical inspections on site: Zillow (2016). Both behaviors reflect the lower transaction costs of changing leases versus ownership.

Housing has two sources of value for either investors or owner-occupants: rent or implicit rent and expected housing appreciation. By the above argument investors attach less value to rent relative to expected appreciation than owner-occupiers attach to implicit rent relative to appreciation. Other considerations reinforce this result. Rental income is taxed at ordinary rates, whereas implicit rent is untaxed. Also, landlords and tenants have conflicts of interest that reduce the value of tenancy, while owner-occupiers do not. Finally, owners have benefits of occupancy not enjoyed by tenants–notably, greater security of tenure and rights to customize their properties.

The model has identical households. In its equilibria rent-price ratios are counter-cyclical and expected appreciation rates are procyclical. Now add to the model the above heterogeneity. In the resulting equilibrium busts must attract relatively more owner-occupiers, while recoveries and booms attract relatively more investors. This entry by investors during expansions is procyclical speculation. It can have multiple manifestations during recoveries and booms: tenants buying homes, owner-occupiers buying second homes; investors buying rental housing; and speculators flipping builders' purchase and sale agreements for new homes. Also, the exit of tenants, entry of investors, and construction of new housing during booms can decrease the demand for rentals, raise vacancy rates, reduce the quality of tenants, and thereby reduce rental income.

The empirical evidence of procyclical speculation is extensive. Only a few examples are cited here. Tenants with young children and limited savings buy during booms inexpensive starter homes in peripheral suburban subdivisions where builders compete by offering financing with low down payments.[8] During the period, 2000 to 2006, housing prices in-

[8]This phenomenon, commonly called "drive until you qualify," is widely reported in the popular press during housing booms. See, for example, the business blog: https://www. thetruthaboutmortgage.com/should-you-drive-until-you-qualify-for-a-mortgage/. Only credit-constrained households appear to drive until they qualify: Hanson et al (2012).

creased more in zip codes with relatively more rental housing: Nathanson and Zwick (2015). Finally, flipping contracts to purchase homes under construction is, not surprisingly, largely limited to booms when new homes are built.[9]

Procyclical speculation also alters housing cycles. In the model recoveries have the highest expected appreciation rates. Investors in rental housing should then enter the market mostly during recoveries. This raises price-rent ratios and shortens recoveries. Booms begin earlier with construction at a more rapid rate. From the model higher price-rent ratios during both booms and busts amplify the crash in housing values during the next reversal back to bust. In this case, procyclical housing speculation is a not only a necessary consequence of procyclical fundaments but also a contributing factor to the amplitude of housing cycles.[10]

Heterogeneity: All housing and households are identical in the model. In fact, both housing and households are heterogeneous. The discussion below is focused on three types of heterogeneity: degree of customization, intensity of use, and ages of households. In the easy extensions of the model described below, housing or households in each category are ordered by price-rent ratios, speculation and procyclical pricing volatility.

Homes differ in their degree of customization. Larger, more expensive homes are on average more customized for their initial buyers than smaller, less expensive homes. The most customized are true custom homes: large, luxury homes built for buyers by contractors. Semi-custom, speculative homes are finished with customized upgrades selected by buyers during construction. Least customized are entry-level, production homes in subdivisions. Second and subsequent buyers of more customized homes have an incentive to search more intensively than buyers of less customized, pre-owned homes. This more costly, optimal search requires that buyers of more customized homes be compensated in equilibrium with higher implicit rents and thereby higher rent-price ratios than buyers of less customized, pre-owned homes. By the above argument, larger, more expensive, more customized, pre-owned homes are predicted to have less procyclical pricing volatility than smaller, less expensive, less customized homes.

This prediction is consistent with the limited empirical evidence. Homes with more valuable physical attributes are more likely to have both higher rent-price ratios and owner-occupiers: Halket et all (2016). During the housing bust, 2007-2012, mansions in metropolitan Phoenix decreased in price relatively less than smaller homes: Liu et al (2016). During the housing boom, bust, and subsequent boom between November 2001 and September 2018, more expensive homes had less procyclical pricing volatility than less expensive homes in multiple metropolitan areas of the United States: Piazzesi and Schneider (2016) and S&P Dow-Jones (2018).

Primary and secondary homes have a similar distinction. Primary homes are used on average more intensively by their owner-occupants than secondary homes. Thereby,

[9] An example is Fu and Qian (2014).

[10] The latter result is the focus of Gao et al (2018).

primary homes produce more housing services than comparable secondary homes. The resulting higher rent-price ratios of primary homes are predicted to produce more procyclical speculation among secondary homes. This should raise the procyclical pricing volatility of secondary homes relative to primary homes. This prediction is also consistent with the limited evidence. During the years, 1997-2017, procyclical pricing volatility was greater in housing markets with higher percentages of vacation homes: Zillow (2018).

Homes owned and occupied by older versus younger households also have a similar distinction. Older households move less frequently on average than younger households.[11] A major motive for not moving is family and friends living nearby.[12] These differences should be reflected in higher implicit rents for older households relative to younger households in comparable homes. In neighborhoods with fewer homes for sale, buyers with downward sloping demand curves also have higher implicit rents.[13] By the above argument, older homes in older neighborhoods with older households or housing in age-restricted communities should then have higher rent-price ratios and thereby less procyclical pricing volatility than housing occupied by young households. Examples of the latter include new tract homes built during booms on the outer edges of expanding cities for first-time buyers with limited cash for down payments.[14] Other explanations reinforce this result. For example, older households have less leverage and thereby lower user-costs of capital: Diaz and Luengo-Prado (2012). With less leverage older households in older homes within older neighborhoods have lower likelihoods of countercyclical foreclosures that depress prices of both foreclosed and neighboring housing: Landvoight (2015) and Campbell et al (2011).

8. Sticky Rents

Rents reported in empirical studies have very little procyclical volatility, less so for smaller multifamily properties with fewer units: Genesove (2003). The latter result for smaller properties suggests that the implicit rents of owner-occupied homes have even less procyclical volatility. These sticky or rigid rents are puzzling. In what equilibrium can sticky rents support procyclical, forward-looking housing prices? Also, how can price-rent ratios lead real housing prices but not housing rents, as reported in Gallin (2008)? To be fair, the existing evidence is less than conclusive. For example, indicies of prices and matched rents are cointegrated: Baltagi, J. and J. Li (2015) and the references therein. Also, measured rents and prices must come from the same set of rental properties: Begley et al (2018). Finally, measured rents must include all components of rental income.

[11] The propensity to move peaks in the early 20s, decreases to about age 50-55, and remains roughly constant thereafter: Green and Lee (2016).

[12] In a survey of 2250 Americans aged 60 or more, 66% of the respondents selected "family and friends nearby" as a reason for not moving: AARP (2012).

[13] Long-run price elasticites of demand for housing have been estimated at -0.45 for Phoenix and -0.64 for Pittsburgh: Hanushek and Quigley (1980).

[14] See the next subsection below.

The answers here have two components. First, measured monthly rents miss the procyclical components of rental income realized by landlords. Second, competitive, profit-maximizing landlords optimally smooth their monthly rents relative to the cyclical fundamentals that drive housing demand and thereby housing prices.

Measured Rents: Rents used in empirical studies come from surveys. Respondents report monthly rents, which are contractual for tenants with leases.[15] Monthly rents miss two major components of contractual rent not collected by landlords: initial free rent offered to new tenants and rent not paid by defaulting tenants.[16] Landlords' rental income is also realized net of turnover costs: legal costs of evictions, damages not covered by security deposits, losses from vacancies, and costs of releasing vacated units. All these credits and costs are countercyclical. Free and uncollected rent have more cyclical volatility than collected rent, which has more cyclical volatility than gross potential rent: NAA (2015). Gross potential rent, measured per rental unit, is the closest match for contractual rent and thereby measured rent.

Smoothed Rents: Measurement errors cannot explain all the rigidity of reported rents. Landlords have an incentive to smooth their monthly rents relative to housing prices. To see this, separate a landlord's tenants into two types: those who must move and all others. Tenants must move for multiple reasons, including loss of local employment. Landlords then maximize their expected net operating income by selecting contractual rents that maximize their rental income from their continuing tenants with jobs. This optimal rent is increasing in the wages of tenants in comparable rental properties. Thereby, nominal monthly rents are rigid or sticky if nominal wages are also sticky. Sticky wages were once conventional wisdom among macroeconomists: Blanchard and Fischer (1989), page 19. Recent empirical evidence of sticky wages is mixed: Barattieri et al (2014), Elsby et al (2016), and the citations therein.

Landlords also respond to their rising vacancy rates during downturns by offering initial discounts to new tenants. Deeper initial discounts are commonly advertised during deeper downturns. These "move-in specials" frequently include free or reduced rent for the first month, reduced security deposits, and no application fees.[17] As such, the same discounts are not offered to current or renewing tenants. This legal price-discrimination allows the

[15] In Genesove (2003) rent is obtain from answers by owners and operators of rental properties to the question: "What is the monthly rent" (in the year of the interview)? Estimation of implicit rent is described in Section 1 of Katz (2017).

[16] For example, tenants are not asked to report initial free rent, commonly called "move-in specials." See, for example, U.S. Department of Housing and Urban Development, Principles for Conducting Area Rent Surveys, Attachments 1 and 2: https://www.huduser.gov/principalsforpha-conductedarearentsurveys.pdf

[17] Landords' initial discounts during downturns are described in realtor.com/advice/rent/what-you-need-to-know-about-move-in-specials/ and realestate.usnews.com/real-estate/articles/7-concessions-to-ask-about-while-you-search-for-your-next-apartment.

landlord to offer lower rents to new tenants who have more price-elastic demand for rental units in the landlord's property than existing tenants with sunk moving costs. Thereby, landlords optimally smooth their contractual rents for existing tenants and incur countercyclical leasing costs with new tenants. This equilibrium has additional properties in practice that require careful analysis–a topic for future research.

A similar argument can help to explain why smaller properties with fewer rental units have more rigid rents: Genesove (2003). Tenants are heterogenous. An increase in contractual rent that induces one employed tenant to stay may induce another to leave. Because tenants talk to each other, dissimilar rents for similar apartments can antagonize tenants and thereby create potential legal liability for landlords. With similar rents for similar apartments and limited information about their tenants's reservation rents, landlords rationally expect that raising rents will induce more tenants to leave. In this case, landlords with higher costs of managing turnover, measured per apartment, should raise rents less rapidly during expansions than landlords with lower costs. Smaller properties with fewer rental units have fewer economies of scale and thereby higher turnover costs per apartment than larger properties: NAA (2015). Also, smaller, older multifamily buildings are frequently owned and managed by individuals who are poorly diversified relative to investors with partial interests in larger, newer properties. In a competitive rental market, owners of smaller properties should then reduce their turnover costs and idiosyncratic risks relative to larger properties by smoothing their rents for renewing tenants more than managers of larger properties.

9. Conclusion

In model the housing market switches at random times between two states: cold markets with contracting demand and hot markets with expanding demand. The aggregate demand for perishable housing services is isoelastic. Housing services are priced in a spot rental market. Housing prices are expected present values of future housing rents and appreciation. All investors have a common, constant, discount rate. All are equally informed about future rents. Construction of new homes is instantaneous once started. Unit construction costs grow proportionally with aggregate construction.

The model's endogenous price-rent ratios are procyclical: lower during contractions and higher during expansions. Forward-looking prices change abruptly during instantaneous transitions between contractions and expansions, while short-term rents change only over time between transitions. During the latter periods prices and rents change together, continuously through time, increasing during expansions and decreasing during contractions. If the price-rent ratio changes, it does so continuously in response to new information unrelated to the previous reversal. Thereby, procyclical changes in price-rent ratios lead procyclical changes in prices, which lead procyclical changes in and rents. Also, the procyclical volatility of prices exceeds the procyclical volatility of rental income.

Procyclical price-rent ratios require countercyclical rent-price ratios and thereby procyclical expected appreciation rates on housing. The latter result holds in the model because investors must always expect a total return on housing equal to their constant discount rate. That total return is calculated as the rent-price ratio plus the expected appreciation rate. Countercyclical rent-price ratios combined with procyclical appreciation rates have two important implications. The value to owner-occupants of housing as a consumer durable is countercyclical, while its remaining value as a speculative real asset must be procyclical. Also, procyclical speculation is necessary for equilibrium in the housing market. The magnitude of speculation depends partly on the parameters that determine the expected appreciation rate of housing.

Table 1: Notation

Functions and variables:

h	Housing stock or aggregate supply.
i	State of housing market.
	Stagnant cold market: cold market, $i = 0$, or hot market, $i = 1$.
	Contracting cold market: bust, $i = 0.1$, recovery, $i = 1.0$, or boom, $i = 1.1$.
P_i	Pricing function for housing in market i.
p_i	Price-rent ratio in market i.
q	Aggregate construction.
R	Rental function for housing services in (1).
r_i	Rent-price ratio in market i.
r_{ai}	Rent-price ratios defined above Proposition 2: $i = 0, 1$.
r_{bi}	Rent-price ratios defined above Proposition 3: $i = 0, 1$.
x	Exogenous component of aggregate demand for housing services.
$\bar{x}$	Historic high of exogenous demand.

Parameters:

α_i	Transition probability per unit of time from state i to state j.
a_0	Transformed transition probability (7).
γ_i	Construction cost of one house.
δ	Discount rate per unit of time.
ζ	Composite parameter defined in (17).
$-\eta$	Elasticity of aggregate demand for housing services with respect to rent.
θ	Elasticity of unit construction costs wrt aggregate construction.
λ	Composite parameter defined in (8).
ρ_i	Growth rate of exogenous component x of aggregate demand.
ϕ_a	Composite parameter defined above Proposition 2.
ϕ_b	Composite parameter defined above Proposition 3.
ψ	Composite parameter defined above Proposition 2.

Table 2: Comparative statics

Stagnant Cold Markets: $\rho_0 = 0$ **Increase in parameter value**	**Price-rent ratios during** stagnation	expansion
Expected duration of stagnation $1/\alpha_0$	$-$	$-$
Expected duration of expansion $1/\alpha_1$	$+$	$+$
Growth rate during expansion ρ_1	$+$	$+$

Contracting Cold Markets: $\rho_0 < 0$ **Increase in parameter value**	**Price-rent ratios during** bust	boom
Expected duration of contraction $1/\alpha_0$	?	$-$
Expected duration of expansion $1/\alpha_1$	$+$	$+$
Growth rate during contraction ρ_0	$+$	$+$
Growth rate during expansion ρ_1	?	$+$
Elasticity of unit construction cost θ	$-$	$-$

Note: Cold markets have two types: stagnation with unchanging demand for housing services and contractions with decreasing demand. Hot markets are expansions with increasing demand. With contracting cold markets, expanding hot markets have two phases. During recoveries the excess supply of housing from the previous contraction is absorbed. During subsequent booms the increasing demand for housing services is supplied by new construction. The above comparative statics for the price-rent ratio p_i, are reversed for both the rent-price ratio, $r_i = 1/p_i$ and the expected appreciation rate of housing, $m_i = \delta - r_i$. The index i is the state of the market: cold, $i = 0$, and hot, $i = 1$.

Parameters: α_0 constant expected transition rate from cold to hot market, α_1 constant expected transition rate from hot to cold market, ρ_0 constant growth rate during cold markets of fundamental factor driving the demand for housing services, ρ_1 constant growth rate during hot markets of fundamental factor, and θ elasticity of unit construction costs with respect to aggregate construction.

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Appendix

Proof of Proposition 1: Restrict the rent-price ratios, r_0 and r_1, to constants. Also, construct the composite constant:

$$g \equiv \alpha_1 + \delta - r_1 - \frac{\alpha_0 \alpha_1}{\alpha_0 + \delta - r_0}.$$

With $\rho_0 = 0$, the differential equation in (3) for cold markets, $i = 0$, simplifies to the ratio:

$$\frac{P_0(x)}{P_1(x)} = \frac{\alpha_0}{\alpha_0 + \delta - r_0}. \tag{A.1}$$

Insert this result into the corresponding differential equation for hot markets, $i = 1$, with $\rho_1 > 0$. This yields for hot markets the differential equation:

$$\frac{P_1'}{P_1} = \frac{g}{\rho_1 x}. \tag{A.2}$$

With (A.2) housing prices grow during hot markets at the rate: $P_1' \dot{x} / P_1 = g$.

Given a constant rent-price ratio r_1, housing prices and rents must change during hot markets at the same rate from (1):

$$\frac{P_1'}{P_1} = \frac{R'}{R} = \frac{1}{\eta x}. \tag{A.3}$$

With (A.3) housing prices grow during hot markets at the rate ρ_1 / η. Because housing can have only one price at each time, the two appreciation rates from (A.2) and (A.3) must be equal:

$$g = \frac{\rho_1}{\eta}. \tag{A.4}$$

Together, (A.2) and (A.4) require the housing price:

$$P_1(x) = p_1 x^{1/\eta}.$$

With $\eta > 0$, as previously assumed, this price satisfies the initial condition (4).

Finally, rents (1) remain unchanged during instantaneous transitions between hot and cold markets. This produces the first equality in (A.5):

$$\frac{r_1}{r_0} = \frac{P_0(x)}{P_1(x)} = \frac{\alpha_0}{\alpha_0 + \delta - r_0} < 1. \tag{A.5}$$

The second equality in (A.5) is (A.1). The inequality holds since $r_0 < \delta$. Together, (A.4) and (A.5) generate two linear equations in the two price-rent ratios:

$$(\alpha_0 + \delta) p_0 - \alpha_0 p_1 = 1 = -\alpha_1 p_0 + (\alpha_1 + \delta - \rho_1/\eta) p_1. \tag{A.6}$$

The solution to these equations produces the rent-price ratios, r_0 and r_1 in (5).

Comparative Statics below Proposition 1: From (5) and (A.1), calculate the ratio of price-rent ratios:

$$\frac{p_1}{p_0} \;=\; \frac{\alpha_0+\alpha_1+\delta}{\alpha_0+\alpha_1+\delta-\rho_1/\eta} \;>\; 1.$$

From this ratio and (A.2), it follows that

$$\frac{\partial p_i}{\partial \alpha_0} > 0, \qquad \frac{\partial p_i}{\partial \alpha_1} < 0, \qquad \frac{\partial p_i}{\partial \rho_1} > 0,$$

for $i = 0, 1$, and

$$\frac{\partial}{\partial \alpha_0}\left(\frac{p_1}{p_0}\right) < 0, \quad \frac{\partial}{\partial \alpha_1}\left(\frac{p_1}{p_0}\right) < 0, \quad \frac{\partial}{\partial \rho_1}\left(\frac{p_1}{p_0}\right) > 0.$$

Discontinuous Demand: With the discontinuous demand for housing services specified in the last paragraph of Section 4, prices and rents are constant between market reversals: $\rho_0 = 0 = \rho_1$. In this case, the valuation equations (3) simplify to the relationships:

$$\frac{P_j}{P_i} = \frac{\alpha_i + \delta - r_i}{\alpha_i},$$

for $i, j = 0, 1$ with $i \neq j$. For $i = 0$ this matches (A.1); for $i = 1$ it replaces (A.2). Also, housing has before and after transitions from market i to market j the relative price:

$$\frac{P_j}{P_i} = \frac{R_j}{R_i}\frac{p_j}{p_i} = \beta_i^{1/\eta}\frac{p_j}{p_i}.$$

for $i, j = 0, 1$ with $i \neq j$.. This is analogous to (A.5). Together, these four equations generate two linear equations in the two price-rent ratios:

$$(\alpha_0 + \delta)\, p_0 - \alpha_0\beta_0^{1/\eta} p_1 \;=\; 1 \;=\; (\alpha_1 + \delta)\, p_1 - \alpha_1\beta_1^{1/\eta} p_0.$$

These equations are analogous to (A.6). Construct the new constant:

$$\xi \equiv \frac{\alpha_0 + \alpha_1\beta_1^{1/\eta} + \delta}{\alpha_1 + \alpha_0\beta_0^{1/\eta} + \delta} < 1.$$

The inequality follows from the previous assumptions: $0 < \alpha_0 \leq \alpha_1$ and $\beta_0 > 1 \geq \beta_1$. With this new notation the two linear equations have the unique solution:

$$p_0 = \frac{1}{\alpha_0 - \alpha_0\beta_0^{1/\eta}\xi + \delta}, \qquad p_1 = \xi p_0. \tag{A.7}$$

The solution satisfies $p_0 > p_1$.

Modification of (3) and (4): With the modifications in Section 5, the differential equations (3) are replaced by

$$\begin{aligned} 0 &= \rho_0 x P_0' - (\alpha_0{+}\delta{-}r_0)P_0 + \alpha_0 P_{1.0}, \\ 0 &= \rho_1 x P_{1.0}' - (\delta{-}r_{1.0})P_{1.0}, \qquad P_{1.0}(\bar{x}) = P_{1.1}(\bar{x}), \\ 0 &= \rho_1 x P_{1.1}' - (\alpha_1{+}\delta{-}r_{1.1})P_{1.1} + \alpha_1 P_0. \end{aligned} \tag{A.8}$$

Also, the initial conditions (4) become

$$P_i(0) = 0, \tag{A.9}$$

for $i = 0, 1.0, 1.1$.

Proof of Proposition 2: Focus first on booms: $i = 1.1$. By the argument in the text above Proposition 2, housing prices during booms $P_{1.1}$ can be calculated from (3) for hot markets, $i = 1$, with the following substitutions: α_0 is replaced by a_0 from (7), and the constant rent $\bar{x}^{1/\eta}$ is replaced by $\lambda\bar{x}^{1/\eta}$ with the parameter, λ from (8). Both a_0 and λ are calculated below.

The parameter, a_0 from (7), is calculated first. During a bust with the remaining duration or time t_1, exogenous demand decreases from its current value, $x \le \bar{x}$, to its trough: $\underline{x} = x\exp(\rho_0 t_1)$. During the initial phase of the subsequent recovery with the duration t_2, exogenous demand then returns to its previous value: $x = \underline{x}\exp(\rho_1 t_2)$. Together, these two durations must satisfy the constraint: $0 = \rho_0 t_1 + \rho_1 t_2$. As a result, the subsequent partial recovery must have the random duration or length: $t_2 = -(\rho_0/\rho_1)t_1$. The time t_1 has a negative exponential distribution with the mean $1/\alpha_0$. Hence, the remaining bust and partial recovery must have a negative exponential duration with the mean: $1/a_0 = (1-\rho_0/\rho_1)/\alpha_0$. The latter equality is (7). This equality also applies to the time of transition from boom to bust. Therefore, the entire bust and complete recovery must have at the time of transition a negative exponential duration with the same mean.

By the above argument a longer bust must also have a longer expected time to recovery. A bust with the above current value, $x \le \bar{x}$, has the duration to date t_0 satisfying $x = \bar{x}\exp(\rho_0 t_0)$. By the above argument, the remaining recovery after the above partial recovery has the corresponding duration t_3 satisfying $\bar{x} = x\exp(\rho_1 t_3)$. These two times must satisfy the same constraint: $0 = \rho_0 t_0 + \rho_1 t_3$. Therefore, the remaining time to recovery has at time t_0 the value: $t_3 = -\rho_0/\rho_1 t_0$. This generates at time t_0 the expected time to recovery: $1/a_0 - (\rho_0/\rho_1)t_0$.

The parameter, λ in (8), is identified next. During a bust starting at time 0, rent from (1) has the relative value: $(x/\bar{x})^{1/\eta} = \exp(\rho_0 t/\eta)$ for all times $0 \le t \le t_0 = \ln(\underline{x}/\bar{x})/\rho_0$. Here, t_0 is the time from the start of the bust to its trough $\underline{x}$. Conditional on the duration of the bust t_0, rent during a bust has at the beginning of a bust the present value:

$$V_0(t_0, \bar{x}) = \bar{x}^{1/\eta}\int_0^{t_o} e^{-\delta t} e^{(\rho_0/\eta)t} dt = \frac{\bar{x}^{1/\eta}}{\delta - \rho_0/\eta}\left[1 - e^{-(\delta-\rho_0/\eta)t_0}\right].$$

Again, the time to the trough t_0 has a negative exponential distribution with the mean $1/\alpha_0$. Hence, the above present value has the expectation:

$$E[V_0(t_0, \bar{x})] = \frac{\bar{x}^{1/\eta}}{\alpha_0 + \delta - \rho_0/\eta}. \tag{A.10}$$

During recoveries rent has the relative value: $(x/\bar{x})^{1/\eta} = \exp(\rho_0 t_0/\eta)\exp[\rho_1(t - t_0)/\eta]$ for all times $t_0 \leq t \leq t_0 + t_1$. Here, t_1 is the time from the trough to the end of the recovery: $t_1 = -(\rho_0/\rho_1)t_0$ from above. Conditional on the time t_0, rent during recoveries has at the beginning of a bust the corresponding present value:

$$\begin{aligned} V_{1.0}(t_0, \bar{x}) &= \bar{x}^{1/\eta} e^{-\delta t_0} e^{\rho_0 t_0/\eta} \int_{t_0}^{t_0+t_1} e^{-\delta(t-t_0)} e^{\rho_1(t-t_0)/\eta} dt \\ &= \frac{\bar{x}^{1/\eta}}{\delta - \rho_1/\eta} \left[e^{-(\delta-\rho_0/\eta)t_0} - e^{-\delta(1-\rho_0/\rho_1)t_0} \right]. \end{aligned}$$

This present value has the expectation:

$$E[V_{1.0}(t_0, \bar{x})] = \frac{\alpha_0 \bar{x}^{1/\eta}}{\delta - \rho_1/\eta} \left[\frac{1}{\alpha_0 + \delta - \rho_0/\eta} - \frac{1}{\alpha_0 + \delta(1 - \rho_0/\rho_1)} \right]. \tag{A.11}$$

Finally, stagnant cold markets with the same duration $t_0 + t_1$ have the corresponding present value:

$$V_0(t_0, \bar{x}) = \bar{x}^{1/\eta} \int_0^{t_0+t_1} e^{-\delta t} dt = \frac{\bar{x}^{1/\eta}}{\delta} \left[1 - e^{-\delta(t_0+t_1)} \right].$$

Again, the total time $t_0 + t_1$ has a negative exponential distribution with the mean $1/a_0$. Therefore, this present value has the expectation:

$$E[V_{0.1}(t_0), \bar{x}] = \frac{\bar{x}^{1/\eta}}{a_0 + \delta}. \tag{A.12}$$

Add (A.10) and (A.11); divide the sum by (A.12); and simplify the result. This yields the ratio, λ in (8).

Given (7) and (8), the pricing of housing during booms continues as follows. Construct the constant:

$$g_a = \alpha_1 + \delta - r_{1a} - \frac{a_0 \alpha_1}{a_0 + \delta - r_{0a}}.$$

This matches g above (A.1) with three substitutions: a_0 for α_0 and r_{ia} for r_i with $i = 0, 1$. With the substitution, g_a for g, the argument leading to (A.4) yields

$$g_a = \frac{\rho_1}{\eta}. \tag{A.13}$$

Also, (A.5) again applies with the stagnant cold market replaced by its synthetic substitute. This substitution includes the altered rents from (8); the boom ends with the rent, $R_{1a}(\bar{x}) = \bar{x}^{1/\eta}$, and the subsequent synthetic cold market starts with the rent, $R_{0a}(\bar{x}) = \lambda\bar{x}^{1/\eta}$. It yields the analogue to (A.5):

$$\frac{r_{1a}}{r_{0a}} = \frac{R_{1a}(\bar{x})}{R_{0a}(\bar{x})}\frac{P_{0.1}(\bar{x})}{P_{1.1}(\bar{x})} = \frac{1}{\lambda}\frac{a_0}{a_0+\delta-r_{0a}}. \tag{A.14}$$

The two equalities in (A.14) follow from the construction of the synthetic cold market. Regarding the first equality, the price at the end of the boom $P_{1.1}(\bar{x})$ is the same in the two economies: one with the boom followed by bust and the second with the boom followed by the synthetic cold market. This equality for $P_{1.1}(\bar{x})$, combined with (3) for hot markets, $i = 1$, followed by the synthetic cold market, generates the same result for the price at the start of the bust $P_{0.1}(\bar{x})$. The latter price is the same as the price at the start of the synthetic cold market. Regarding the second equality, the ratio of rents is identified above (A.14). The ratio of prices satisfies (A.1) with the indicated adjustments for the synthetic cold market.

Together, (A.13) and (A.14) generate two linear equations in the two price-rent ratios:

$$(a_0+\delta)p_0 - \frac{a_0}{\lambda}p_1 = 1 = -\lambda\alpha_1 p_0 + \left(\alpha_1+\delta-\frac{\rho_1}{\eta}\right)p_1.$$

This is analogous to (A.6). The solution to these equations produces the constant rent-price ratios, r_{0a} and r_{1a} below (8). The rent-price ratio during booms, $r_{1.1}$ in (12), then follows from the equal price-rent ratios during booms in the two economies.

Consider next housing during recoveries. By the argument in the text above the proposition, the valuation equation during recoveries is (3) with $i = 1.0$ and $\alpha_{1.0} = 0$:

$$0 = \rho_1 x P'_{1.0} - (\delta - r_{1.0})P_{1.0}.$$

Because recoveries become booms when exogenous demand first reaches its historical maximum, $x = \bar{x}$, the pricing function during recoveries $P_{1.0}$ has the upper boundary condition: $P_{1.0}(\bar{x}) = P_{1.1}(\bar{x})$. With this boundary condition and the constant rent-price ratio $r_{1.0}$, the above differential equation has the unique solution:

$$P_{1.0}(x) = P_{1.1}(\bar{x})\left(\frac{x}{\bar{x}}\right)^{(\delta-r_{1.0})/\rho_1}. \tag{A.15}$$

With the constant rent-price ratio $r_{1.0}$, the appreciation rate of housing in (A.15) must equal at all times during recoveries the growth rate of rents from (1): $\delta - r_{1.0} = \rho_1/\eta$. This determines the rent-price ratio during recoveries, $r_{1.0}$ in (12).

Finally, focus on housing during busts. To simplify the notation, construct the composite constants: $A \equiv \alpha_0 + \delta$ and $B \equiv 1 + \alpha_0/(\delta - \rho_1/\eta)$. For busts the valuation equation is (3) with $i = 0.1$ and $j = 1.0$:

$$0 = \rho_0 x P'_{0.1} - (\alpha_0 + \delta - r_{0.1})P_{0.1} + \alpha_0 P_{1.0} = \rho_0 x P'_{0.1} - (A - Br_{0.1})P_{0.1}. \tag{A.16}$$

The second equality in (A.16) exploits both the constant rents during transitions from busts to recoveries, $P_{1.0}/P_{0.1} = r_{0.1}/r_{1.0}$, and the constant rent-price ratio during recoveries, $r_{1.0} = \delta - \rho_1/\eta$ in (12). With (A.16) and the notation above (A.16), housing has during busts the appreciation rate:

$$\frac{P'_{0.1}}{P_{0.1}} = \frac{A - Br_{0.1}}{\rho_0 x}. \tag{A.17}$$

The rent-price ratio during busts, $r_{0.1} = R/P_{0.1}$, is assumed to be everywhere continuously differentiable in its argument x. With respect to x, the price, $P_{0.1} = R/r_{0.1}$, then grows at the rate:

$$\frac{P'_{0.1}}{P_{0.1}} = \frac{R'}{R} - \frac{r'_{0.1}}{r_{0.1}} = \frac{1}{\eta x} - \frac{r'_{0.1}}{r_{0.1}}. \tag{A.18}$$

The second equality in (A.18) follows from the rental function, R in (1).

Temporarily simplify the subsequent notation: $r \equiv r_{0.1}$. The two appreciation rates, (A.17) and (A.18) must be equal at all times during busts:

$$\frac{A - Br}{\rho_0 x} = \frac{1}{\eta x} - \frac{r'}{r}.$$

This is analogous to (A.4). Rearrange terms using the new constant: $C \equiv A - \rho_0/\eta = \alpha_0 + \delta - \rho_0/\eta > 0$. This generates the Bernoulli equation:

$$\rho_0 x r' = (Br - C)r.$$

Under a standard transformation, this equation is linear: Zwillinger (1992), p. 194. Its general solution has under the reverse transformation the form:

$$\frac{1}{r} = \frac{B}{C} + \left(K - \frac{B}{C}\right)\left(\frac{x}{\bar{x}}\right)^{-A/\rho_0},$$

with the undetermined constant K.

With (7) and (8) housing has the same price at the beginning of a bust and its synthetic cold market. As a result, the bust and synthetic stagnation have an initial ratio of rent-price ratios equal to their ratio of rents:

$$\frac{r_{0.1}(\bar{x})}{r_{0a}} = \frac{\bar{x}^{1/\eta}}{\lambda \bar{x}^{1/\eta}} = \frac{1}{\lambda}.$$

With thisboundary condition, $r_{0.1}(\bar{x}) = r_{0a}/\lambda$, the above general solution has the value:

$$\frac{1}{r_{0.1}} = \frac{B}{C} + \left(\frac{\lambda}{r_{0a}} - \frac{B}{C}\right)\left(\frac{x}{\bar{x}}\right)^{-A/\rho_0}.$$

This is (11).

Properties below Proposition 2: The constants, ϕ_a and ψ above (11), are decreasing in $-\rho_0$. Both rent-price ratios, r_{0a} and r_{1a} in (11), are increasing in $-\rho_0$.

Proof of Proposition 3: The new growth rate of relative exogenous demand, x in (13), alters only the price-rent ratios for the synthetic economy, booms paired with the synthetic stagnant cold market. Thereby, (14) and (15) replace (9) and (10). All other calculations in the proof of Proposition 2 are unchanged.