MONETARY AND MACROPRUDENTIAL POLICY WITH ENDOGENOUS RISK*

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Abstract. We extend the New Keynesian (NK) model to include endogenous risk. The conditional volatility of the output gap is proportional to the price of risk, giving rise to a "vulnerability channel" of monetary policy: lower interest rates not only shift consumption intertemporally, but also conditional output risk. Policy makers thus face an intertemporal risk-return tradeoff: via the impact on risk-taking, easy monetary policy lowers short-term downside risks to growth, but increases medium-term risks. The model fits estimates of the conditional output gap term structure and can be used to jointly consider monetary and macroprudential policy. The policy prescriptions are very different from those in the standard NK model: central banks' focus purely on inflation and output-gap stabilization can lead to financial and real instability. Macroprudential measures can help reduce the intertemporal risk-return tradeoff of the central bank created by the vulnerability channel.

1. Introduction

Economists have long argued that to make optimal decisions, policymakers should monitor the entire conditional distribution of relevant state variables (Timmermann, 2000). However, to date, the literature on the econometrics of forecast distributions has had little practical impact on monetary policymaking. Crucially, the monetary policy literature has failed to propose structural models that capture the conditional density of relevant state variables in a parsimonious yet policy-relevant fashion.

As a consequence, when central banks do produce forecast distributions, those are usually computed (1) via resampling from linear models; or (2) by combining alternative scenarios

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according to highly judgemental sets of weights. While judgment is very important in policy-making, translating views about alternative scenarios into time-varying forecast distributions is notoriously difficult in practice[1]. Resampling also faces a number of challenges: central banks typically use linearized setups with either constant volatility or models where volatility varies exogenously over time[2]. As we will argue, neither of these approaches is capable of generating empirically relevant conditional densities[3].

In this paper, we present a model with endogenous volatility, which is able to closely match key properties of macroeconomic forecast densities. We label the model NKV, for "New Keynesian Vulnerability," as it extends the three-equation New Keynesian setup by allowing for endogenous movements in risk, and by connecting the resulting vulnerabilities to the evolution of state variables. More specifically, the NKV model extends the textbook NK setup along two dimensions. First, it tightly links the price of risk to the evolution of financial conditions. Second, financial conditions depend on the current and expected levels of the output gap. These relationships pin down the "vulnerability channel," where higher vulnerability is characterized by greater amplification of output gap shocks. Notably, the dependence of financial conditions on endogenous variables also ensures that changes in policies can systematically affect their dynamics. These changes have profound implications for the optimal conduct of monetary policy and macroprudential policy.

Even though our setup retains the standard assumption of Gaussian shocks, it can generate skewed conditional and unconditional forecasts, and it allows for alternative policy path considerations that explicitly take endogenous risk into account[4]. Since the price of risk depends on the level of interest rates, our model captures the "risk-taking channel" of monetary policy[5].

[1]For example, the assessment of Bank of England fan chart performance (Independent Evaluation Office, 2015) found that these tended *not to provide an accurate guide to the eventual distribution of UK GDP growth and inflation outturns*, which in part reflected their tendency to understate the probability of tail events both for GDP growth as well as inflation.

Of course, an additional practical complication is that externally published forecasts can also serve to "manage" private sector expectations, on account of which accuracy may occasionally be sacrificed to "send a clearer message" to market participants.

[2]Importantly, resampled distributions tend not to condition on the state of the economy, with Gonzalez-Astudillo and Vilan (2019) documenting mismatches between the depth, duration and frequency of recessions simulated using the standard bootstrap approach in FRB/US and their empirical counterparts.

[3]Central banks also have access to forecast distributions from surveys. Unfortunately, those cannot be used for counterfactual policy analysis directly. In addition, mapping survey distributions into a model may require setups more complicated than those commonly used in daily policy work (e.g., ones accounting for heterogeneity of beliefs). The end result could still end up unsatisfactory if those models are linearized.

[4]We originally modify the exact linear solution to allow for heteroskedasticity, which implies symmetric one-step-ahead conditional forecasts (though longer-term conditional forecasts and the ergodic distribution are not Gaussian). We subsequently use a second-order perturbation approximation to the non-linear model, under which even the one-step-ahead conditional distribution can be asymmetric.

[5]This could also be termed the "vulnerability channel," as the output gap is affected through the price of risk.

Despite its parsimony, the setup also captures the economics of "leverage cycles": periods of low volatility are associated with high risk-taking, which increases expected future volatility[6]. Significantly, leverage cycles give rise to the "volatility paradox", when risks build up in times of low volatility, as in Brunnermeier and Sannikov (2014).

To empirically validate our model, we match some stylized facts presented in Adrian et al. (2019) and Adrian and Duarte (2018). Loose financial conditions are associated with high expected growth rates and low conditional volatility of the output gap for one- and four-quarters-ahead. The conditional mean and volatility of output gap growth are negatively correlated contemporaneously, giving rise to left-skewed conditional and ergodic distributions. At the same time, loose financial conditions are not associated with higher expected inflation or inflation volatility. Another stylized fact is that loose financial conditions are associated with low volatility of output growth in the near term, but higher volatility in the medium term, as presented in Adrian et al. (2018). That is, the term structure of lower quantiles of output gap growth is upward sloping when the initial price of risk is high, but downward sloping when the initial price of risk is compressed[7]. Importantly, these lower conditional quantiles, called Growth at Risk in Adrian et al. (2018), cross one another over the projection horizon, indicating expected costs of an initially compressed price of risk.

We show how the model can be used to generate paths for the output gap, inflation, and the price of risk under alternative assumptions about monetary and macroprudential policy. Markedly, alternative monetary policy rules not only change the future path of output and inflation, but also the future path of vulnerability. Policymakers can ease monetary policy to lower short-term downside risks to growth via the impact on risk-taking, but at a cost of higher risks in the medium term; in other words, they face an intertemporal risk-return tradeoff.

While our model closely resembles an NK setup, in which the absence of tradeoff inducing shocks implies a "divine coincidence" (Blanchard and Galí, 2007), standard policy prescriptions – that is, attempting to fully stabilize inflation and the output gap – turn out to be problematic. In fact, our model could be considered a stylized and concise illustration of how the Great Moderation (Bernanke, 2012) and the Great Recession are connected: changes in the dynamics of the output gap have a direct impact on the equilibrium law of motion of financial conditions, with "too much" output-gap stability breeding financial condition instability. In other words,

[6] See also Fostel and Geanakoplos (2008), Geanakoplos (2010), Adrian and Shin (2014), and Adrian and Boyarchenko (2015), who have modeled leverage cycles in various forms, and Brunnermeier and Pedersen (2009) and Adrian and Shin (2010) for increasing funding risks.

[7] Although Adrian et al. (2018) actually look at output growth rather than output *gap* growth, similar features also characterize the latter.

by not paying attention to the endogenous component of financial conditions, the central bank risks inadvertently making them unstable.

How should the conduct of monetary policy be adapted? We show two main results. First, a suitable combination of macroprudential and monetary policies can ensure efficiency, potentially even if macroprudential policy is implemented with significant lags. Fundamentally, macroprudential policy can make the financial system inherently more stable, eliminating the risks associated with "overly successful" monetary policy and raising the possibility of full stabilization. Second, when macroprudential tools are not directly available, a Taylor rule augmented for expected financial conditions can increase welfare relative to a standard Taylor rule, effectively reducing volatility by eliminating states of high vulnerability.

The remainder of the paper is organized as follows: Section 2 presents the model and studies its theoretical properties. Section 3 shows the calibration, demonstrating the empirical fit for the whole conditional output gap and inflation distributions. Section 4 discusses optimal monetary policy. Section 5 includes macroprudential considerations. Section 6 employs the model for alternative policy path considerations. Section 7 puts our findings in context by providing a brief overview of the literature. Section 8 concludes.

2. The NKV Model

We incorporate endogenous risk into the NK setup by proposing a parsimonious extension of the three-equation, workhorse model (Woodford, 2003; Galí, 2015). Rather than advocating a "fully nonlinear" approach, our setup ends up being nonlinear along a single dimension and it conveniently nests the textbook NK model as a special case. A key consequence of the non-linearity is that conditional second moments are not constant but instead vary as functions of state variables.

We use a combination of two different approaches to solving the model. Initially, we adapt the linear-homoskedastic solution to account for arbitrary specifications of heteroskedasticity. In that case, the one-step-ahead conditional distributions remain tractably normal, allowing for quick, analytical evaluation of conditional moments[8]. Because the linear solution is certainty-equivalent, another advantage of this approach is that the evolution of the conditional mean will not be affected by the specification of the vulnerability function. This, helpfully, allows us to split the calibration process into two steps, and provides insights into the types of specifications likely to fit the data well. Importantly, however, we subsequently move away from certainty

[8]Importantly, both the k-step ahead conditional distributions, where $k > 1$, and ergodic distributions no longer have to be Gaussian.

equivalence – by using second- and pruned third-order perturbation approximations – to ensure that none of our conclusions crucially hinges on the initial simplifying assumptions[9].

There are many similarities between our setup and the textbook NK model, and the NKV retains many of the appealing features of its standard, linear-homoskedastic counterpart. In addition, the fact that their semi-structural forms are closely related allows us to use standard values for key structural parameters and the coefficients of the welfare loss functions that we use to compare alternative policies. Despite the parsimony, however, we will show that this simple, nonlinear setup is sufficiently rich to capture the key empirical stylized facts of macro-financial linkages, and that some of the policy implications may not carry over.

We now describe the key building blocks of the model. Our starting point is the standard, closed-economy New Keynesian setup (Chapter 3 of Woodford 2003 or Galí 2015), comprising an IS curve, a Phillips curve, and a Taylor rule. For our purposes, the model has two immediate shortcomings.

First, the textbook NK model lacks an explicit role for financial conditions. Since a large literature, surveyed in Section 7, has documented how financial frictions, both on the borrower and lender sides, can be incorporated into the setup, we simply build on extant contributions. More specifically, letting η_t represent financial conditions, with positive (negative) values of η denoting tight (loose) conditions, we represent borrower-side frictions by adding a "financial accelerator" term $-\gamma_\eta \eta_t$ in the IS curve. Since the constant $\gamma_\eta \geq 0$, tighter financial conditions are associated with lower contemporaneous values of the output gap.

To understand the second shortcoming of the textbook NK model, note that its solution can be written as

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{B}\boldsymbol{\epsilon}_t$$

where $\boldsymbol{Y}_t$ denotes a vector of endogenous model variables. Under the standard assumption of normally distributed shocks, with a constant variance-covariance matrix $\boldsymbol{\Sigma}^{\epsilon} \equiv \boldsymbol{E}\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'$ we get

$$\mathcal{P}\left(\boldsymbol{Y}_t|\mathcal{F}_{t-1}\right) = \mathcal{N}\left(\boldsymbol{A}\boldsymbol{Y}_{t-1}, \boldsymbol{B}\Sigma^{\epsilon}\boldsymbol{B}'\right),$$

that is, while the conditional mean $\boldsymbol{A}\boldsymbol{Y}_{t-1}$ is state-dependent, the conditional variance $\boldsymbol{B}\boldsymbol{\Sigma}^{\epsilon}\boldsymbol{B}'$ is constant. This turns out to be important, as the constancy of conditional second moments is

[9] As our results demonstrate, neither of the two solution methods restricts conditional second moments to be constant. Naturally, another advantage of using higher-order perturbation approximations is that the model can be directly solved using standard software such as *dynare*.

strongly rejected by the data, where the conditional mean and volatility of Δy_t^{gap} are negatively correlated (see also Adrian et al. 2019 for further evidence).

Because *any* linear, homoskedastic model will, by construction, feature constant conditional second moments, our NKV extension needs to allow for non-linearities. As alluded to above, introducing endogenous heteroskedastic volatility arguably constitutes a small and relatively tractable deviation from the NK setup, and it is this form of non-linearity that we focus on subsequently. An important question to consider is what shock(s) should be heteroskedastic, however?

We opted to introduce an extra wedge ϵ_t^{ygap} into the IS equation, and to make the variance of that disturbance state dependent. This was, in part, because that is where standard demand shocks would show up. More importantly, we were additionally motivated by the work of Adrian and Duarte (2018), who focus on the role of occasionally binding Value-at-Risk (VaR) constraints of financial intermediaries and who arrive at a similar IS curve specification. Given our focus on macrofinancial interactions, we also restrict attention to fluctuations driven by this wedge, abstracting from productivity and monetary policy shocks.

More specifically, letting ε_t^{ygap} be $\mathcal{N}.i.d.\left(0, \sigma_y^2\right)$ with

$$\epsilon_t^{ygap} \equiv V\left(\boldsymbol{X}_t\right) \varepsilon_t^{ygap},$$

we introduce a piecewise-affine, vulnerability function $V(\boldsymbol{X}_t) \equiv \max\left\{\nu - \boldsymbol{\varrho}' \boldsymbol{X}_t, 0\right\}$, where $\boldsymbol{X}_t$ denotes state variables that determine vulnerabilities. This implies that our final IS curve specification is

$$y_t^{gap} = \boldsymbol{E}_t y_{t+1}^{gap} - \frac{1}{\sigma}\left(i_t - \boldsymbol{E}_t \pi_{t+1}\right) - \gamma_\eta \eta_t - V\left(\boldsymbol{X}_t\right) \varepsilon_t^{ygap} \tag{1}$$

where the role of the max operator is to ensure that the affine specification for $V(\cdot)$ doesn't generate negative values of volatility[10, 11]. Here, large values of $V\left(\boldsymbol{X}_t\right)$ mean that even small shock realizations have the propensity to markedly affect model variables, which is why we would refer to the underlying economy as being vulnerable. In contrast, when vulnerability $V\left(\cdot\right)$ is small, or even zero, the economy is well insulated from the impact of ε_t^{ygap} shocks.

[10] Practically, negative volatility would be equivalent to the shock having opposite effects on the output gap under some constellations of the states, which is something we want to explicitly exclude.

[11] Since perturbation methods are incompatible with non-differentiabilities, like the one introduced by the max operator, therefore, when using *dynare*, we use instead $V(\boldsymbol{X}_t) \equiv \sqrt{\left(\nu - \boldsymbol{\varrho}' \boldsymbol{X}_t\right)^2}$. Since for our chosen parameter values $\nu - \boldsymbol{\varrho}' \boldsymbol{X}_t$ is seldom negative, this change doesn't materially impact the properties of the model documented subsequently.

Since the NKV model explicitly accounts for financial conditions η_t, we also need to pin down how these co-move with real activity indicators such as the output gap. Given the lack of a meaningful propagation mechanism, which the underlying three-equation NK model inherits from the real business cycle (RBC) setup (see also Watson 1993, Cogley and Nason 1995), we include two lags of financial conditions in order to allow for persistence as well as financial condition overshoots à la Dornbusch (1976). In addition, financial conditions are assumed to endogenously depend on the contemporaneous and expected levels of the output gap, with current and expected booms associated with looser financial conditions today. Accordingly, we have

$$\eta_t \equiv \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta} \eta_{t-2} - \theta_y y_t^{gap} - \theta_\eta \boldsymbol{E}_t y_{t+1}^{gap}. \tag{2}$$

While we eschew formal derivations here, there are a number of ways in which a specification like Equation (2) could be micro-founded. In the "leverage cycle" literature, for example, when economic conditions tighten, financial intermediaries have to deleverage, reducing balance sheet size and driving up the price of risk. Alternatively, allowing for deviations from rational expectations, as done for instance in Bordalo et al. (2018, 2019), can lead to a similarly rich law of motion for financial conditions[12].

The whole NKV model thus comprises Equations (1)–(2), along with a standard Phillips curve and a Taylor rule[13]:

$$\pi_t = \beta \boldsymbol{E}_t \pi_{t+1} + \kappa y_t^{gap} \tag{3}$$

$$i_t = \phi^\pi \pi_t + \phi^y y_t^{gap}. \tag{4}$$

2.1. **Links between financial conditions and the price of risk.** We now highlight the relationship linking financial conditions η_t and the pricing kernel's conditional volatility, commonly referred to as the "price of risk."In line with the three-equation NK model, we can think of the household block as being entirely standard, with the resulting consumption-based log-SDF $\tilde{m}_t$ given by

$$\tilde{m}_t \equiv \log\left(\tilde{M}_t\right) = \log\left(\beta \frac{u'(C_t)}{u'(C_{t-1})}\right) = \log\beta - \sigma\left(y_t^{gap} - y_{t-1}^{gap}\right)$$

[12]Bordalo et al. (2018) show that the move from rational to diagnostic expectations adds an extra moving average component to the law of motion for spreads, which are a key driver of η. In our setup, equilibrium η ends up being AR(2), though variables such as the output gap or its conditional mean follow. ARMA(2,2) or ARMA(2,1) dynamics, respectively.

[13]Of course, by setting the volatility of ϵ_t^{ygap} to zero, switching-off the financial accelerator ($\gamma_\eta = 0$) and enabling monetary and productivity shocks, we immediately recover the textbook NK model.

where we have exploited the assumption of a CRRA utility function, goods-market clearing $c_t \equiv y_t$ and where we further assumed $y_t^{nat} \equiv 0 \implies y_t^{gap} = y_t$[14]. An important feature of our four-equation specification is that the state variable is $\boldsymbol{X}_t = \{\eta_{t-1}, \eta_{t-2}\}$ and expanding the model by adding in a definition of the log-sdf would enlarge the set of states to $\boldsymbol{X}_t^{\mathbf{1}} = \{\eta_{t-1}, \eta_{t-2}, y_{t-1}^{gap}\}$. Accordingly, the equilibrium solution for the log-sdf will be of the following form

$$m_t = \tilde{m}_t - \log\beta = a_1\eta_{t-1} + a_2\eta_{t-2} + a_3 y_{t-1}^{gap} + b_m V\left(\eta_{t-1}, \eta_{t-2}, y_{t-1}^{gap}\right)\varepsilon_t^{ygap}$$

where the a_i's and b_m can be found by solving the linear, homoskedastic model, and where they are, respectively, the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ characterizing how the stochastic discount factor loads on the state variables and shock ϵ_t^{ygap}[15]. It follows that the conditional mean and variance of m_t are given by

$$E_t m_{t+1} = a_1\eta_t + a_2\eta_{t-1} + a_3 y_t^{gap}$$

and

$$E_t\left(m_{t+1} - E_t m_{t+1}\right)^2 = E_t\left(b_m V\left(\eta_t, \eta_{t-1}, y_t^{gap}\right)\varepsilon_t^{ygap}\right)^2 = \left(b_m V\left(\eta_t, \eta_{t-1}, y_t^{gap}\right)\sigma_y^2\right)^2.$$

Expressed alternatively, after plugging in the definition of $V(\cdot,\cdot,\cdot)$, the conditional volatility of the log pricing kernel m_{t+1} can be expressed as[16]

$$vol\left(m_t|\mathcal{F}_{t-1}\right) = |b_m|\sigma_y\left(\nu - \tilde{\boldsymbol{\varrho}}'\left[\eta_{t-1}, \eta_{t-2}, \eta_{t-3}, \epsilon_{t-1}^{ygap}\right]'\right)^+ \tag{5}$$

where $x^+ \equiv \max\{x, 0\}$[17]. This expression establishes that in our simple NKV model, the pricing kernel's conditional volatility is piecewise-affine in η and the IS curve wedge ϵ^{ygap}[18].

The fact that η_t depends indirectly on interest rates, via the output gap in Equation (2) and the IS curve in Equation (1), is usually referred to as the "risk-taking channel" of monetary policy. The "vulnerability channel," in contrast, is present because lower interest rates directly

[14] While the latter assumption is introduced mainly to simplify the exposition, we do note that the volatility of productivity shocks, which would be expected to move the natural rate of output, is set to zero in the baseline version of our model.

[15] We are exploiting the fact that our model can be rewritten as linear with heteroskedastic shocks. Since any linear model has the certainty-equivalence property, the solution can be found by solving the homoskedastic model and substituting out shocks with $V(\boldsymbol{X}_t)\varepsilon_t^{ygap}$ where ε_t^{ygap} is homoskedastic. In other words, the introduction of heteroskedasticity does not affect the coefficients of the policy function.

[16] Note that, to arrive at this specification, we have substituted out the equilibrium law of motion for y_t^{gap}

[17] We could extend the setup by introducing n other shocks with volatilities σ_i^2 (e.g., productivity and monetary policy shocks). Under the assumption that ϵ_t^{ygap} is the only heteroskedastic shock, the volatility formula generalizes to $vol\left(m_t|\mathcal{F}_{t-1}\right) = \sqrt{b_m^2 V^2\left(\boldsymbol{X}_{t-1}\right) + \sum_{i=1}^n b_{mi}^2\sigma_i^2}$, where b_{mi} characterize how the log-sdf loads on the homoskedastic shocks.

[18] We occasionally refer to η_t as the price of risk or as endogenous output gap volatility, which is only meant to reflect the fact that η effectively pins down the price of risk via Equation (5).

Table 1. New Keynesian Parameter Values

α	β	ϵ	ϕ	ϕ_π	ϕ_y	σ	θ
1/3	0.99	6	1	1.5	0.125	1	2/3

Table 2. Additional, Non-NK Parameter Values

γ_η	λ_η	$\lambda_{\eta\eta}$	σ_y	θ_η	θ_y
0.01	1.97	-1.01	0.17	0.31	0.08

Table 3. Fit to Targeted Moments

	$corr(\Delta y_t^{gap}, \Delta y_{t-1}^{gap})$	$corr(E_t \Delta y_{t+1}^{gap}, E_{t-1} \Delta y_t^{gap})$	$corr(\Delta y_t^{gap}, \eta_t)$	$corr(E_t \Delta y_{t+1}^{gap}, \eta_t)$
Data	**0.33**	**0.82**	**-0.43**	**-0.82**
VAR	0.30	0.77	-0.45	-0.81
NKV	0.42	0.75	-0.44	-0.36

impact the price of risk and $V(\boldsymbol{X})$, that is, the conditional volatility of output. It follows that when making monetary policy decisions, the policymaker has to consider not only the output-inflation tradeoff, but also an intertemporal risk-return tradeoff introduced by the "vulnerability channel."

While easier monetary policy leads to lower volatility, thus allowing short-term risk-taking, a key question for the next sections is whether such lower short-run volatility is associated with larger medium-term risk. Theories of leverage cycles predict precisely that: low volatility boosts risk-taking and hence activity in the short term, but leads to the buildup of medium-term risks. This intuition is formalized in Adrian and Boyarchenko (2015), where leverage cycles are associated with the endogenous buildup of systemic risk.

3. Empirical Evidence

We now discuss the parametrization and empirical properties of the model.

3.1. **Data.** For the stylized facts reported below, we use the log-difference between real GDP and the Congressional Budget Office's estimate of potential as a measure of the output gap. In addition, we use annual core personal consumption expenditures (PCE) inflation and the National Financial Conditions Index (NFCI) compiled by the Federal Reserve Bank of Chicago. That index aggregates 105 financial market, money market, credit supply, and shadow bank indicators to compute a single index using the filtering methodology of Stock and Watson (1998).

3.2. **Model calibration.** To impose discipline on our exercise and ensure that our specification ends up nesting the three-equation New Keynesian workhorse model, we restrict parameters common to both to equal the values proposed in Chapter 3 of the Galí (2015) textbook. These are reproduced in Table 1[19].

The remaining parameter values are provided in Table 2. They have been selected to match the first-order auto-correlations of Δy_t^{gap} and $E_t \Delta y_t^{gap}$ (columns 1 and 2) and their correlations with η_t (columns 3 and 4, respectively). In addition, the coefficients of the vulnerability adjustment $V(\boldsymbol{X})$ were chosen to match the same negative conditional mean-volatility relationship observed in the data. As shown in Table 3, the overall fit of the NKV is comparable to a model with Equations (1) and (2) replaced by an unrestricted, first-order VAR in $\boldsymbol{X}_t$[20].

We now turn to the five stylized facts on the empirical output gap and inflation distributions, and we document how close the NKV model comes to matching them.

3.3. **Stylized Fact 1: Financial variables predict the tail of the output gap distribution.** Adrian et al. (2019) show that financial conditions explain shifts in the conditional output growth distribution. A similar pattern can be seen in Panel (a) of Figure 1, where we show the 5th conditional quantile, the conditional median, and the 95th conditional quantile of the output gap growth distribution. In line with Adrian et al. (2019), we consistently estimate all the conditional moments using quantile regressions. These feature the variable of interest on the left hand side, and lags of inflation, the change in the output gap, and financial conditions on the right hand side. The figure also reports the p-value associated with the level of financial conditions, which indicates that these are significant at the 1 percent level.

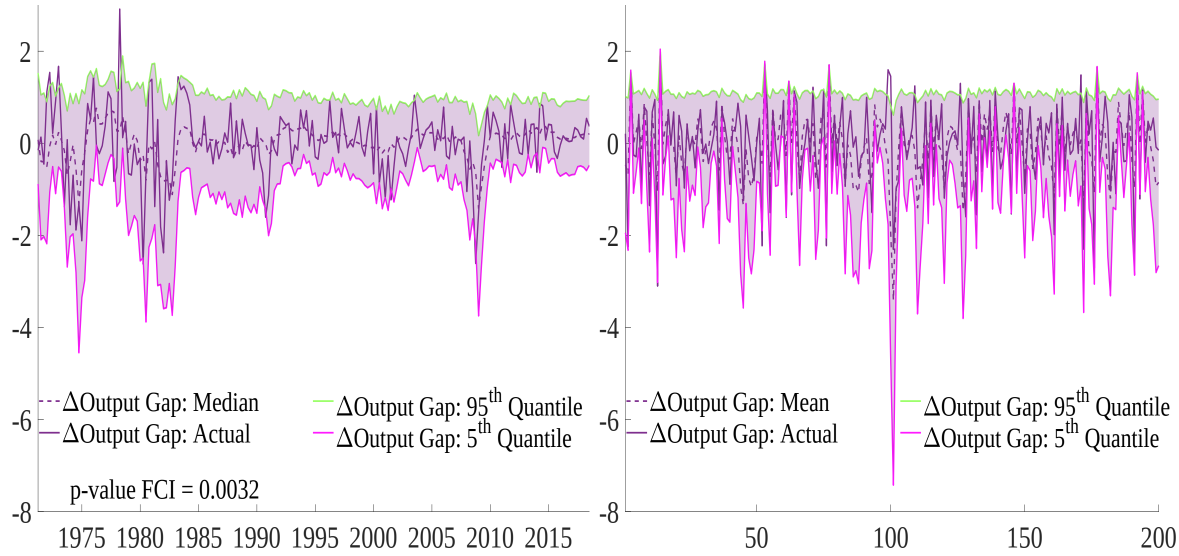
Figure 1 reveals that the output gap growth distribution is highly skewed: while upside risk to output gap growth is more or less constant, downside risk varies sharply over time. Importantly, the conditional median and the conditional 5th quantile are strongly correlated, and both are largely explained by the Financial Conditions Index (FCI): when financial conditions are easy, growth is high and volatility is low, resulting in low downside risk. When financial conditions deteriorate, the conditional median shifts down, and the conditional volatility increases, thus leading to a sharp fall in the 5th quantile, corresponding to an increase in downside risk.

[19]We also retain all structural parameter relationships, i.e.,

$$\omega \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}, \qquad \lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}\omega, \qquad \kappa \equiv \lambda\left(\sigma + \frac{(\phi+\alpha)}{1-\alpha}\right).$$

[20]The model also matches signs of auto-correlations of y_t^{gap} and its conditional mean and their cross-correlations with η_t.

Figure 1. Financial Variables Predict the Tail of the Output Gap Distribution



(a) Data **(b)** Simulation

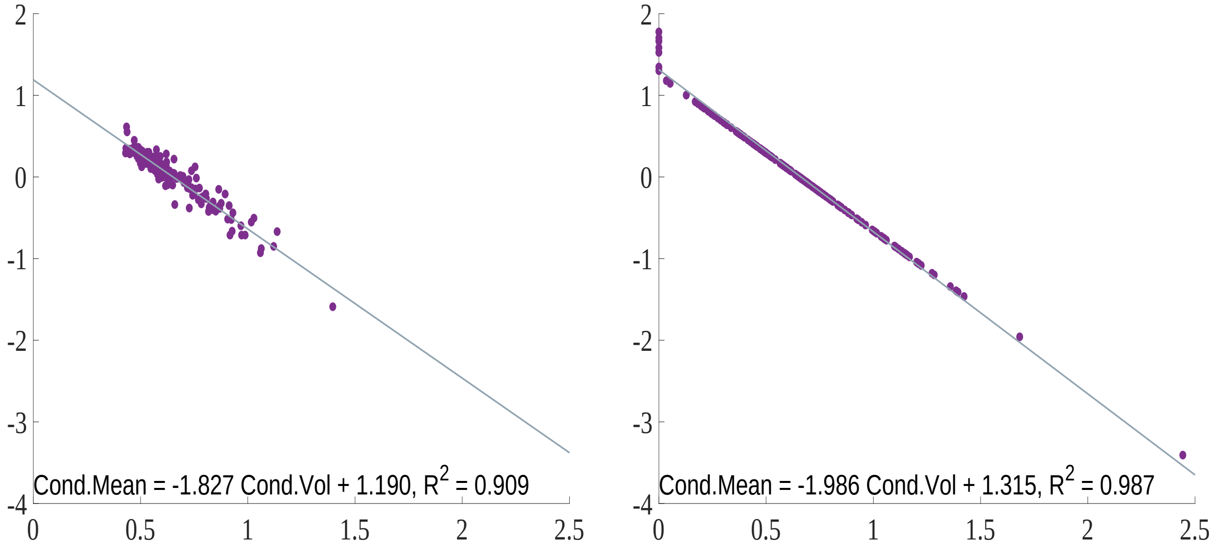
Note: The 5th conditional quantile, the conditional median, and the 95th conditional quantile of the output gap growth distribution. The conditional moments are estimated using quantile regressions featuring Δy^{gap}_{t+1} on the left hand side, and its lag, inflation, and financial conditions on the right hand side. Panel (a) shows the data while Panel (b) shows data simulated from the NKV model.

A simulated path from the NKV model matches these features of the data, as can be seen in Panel (b) of Figure 1[21].

3.4. **Stylized Fact 2: Conditional output gap growth median and volatility correlate negatively.** The fact that the upper quantile of output gap growth is largely constant, while the lower quantile varies strongly with financial conditions, is due to the negative correlation between the conditional median of output gap growth and its conditional volatility, as discussed in Adrian et al. (2019). Importantly, using both semi-parametric and nonparametric estimators, those authors show that movements in higher moments such as the conditional skewness and kurtosis are quantitatively small. Hence the conditional output distribution is well described by conditional first and second moments that vary systematically with the state variables, giving rise to the negative correlation shown in Figure 2. The figure shows that the simulations of the NKV model reproduce the negative correlation between the conditional median and the conditional volatility of the output gap. The predictive powers of the underlying univariate regressions are close as well.

[21]While we could have backed out a shock sequence to exactly recover the observed realizations of Δy^{gap}_t, or its conditional mean, Figure 1 shows the result of drawing a random sequence of shocks, with the corresponding conditional moments evaluated analytically.

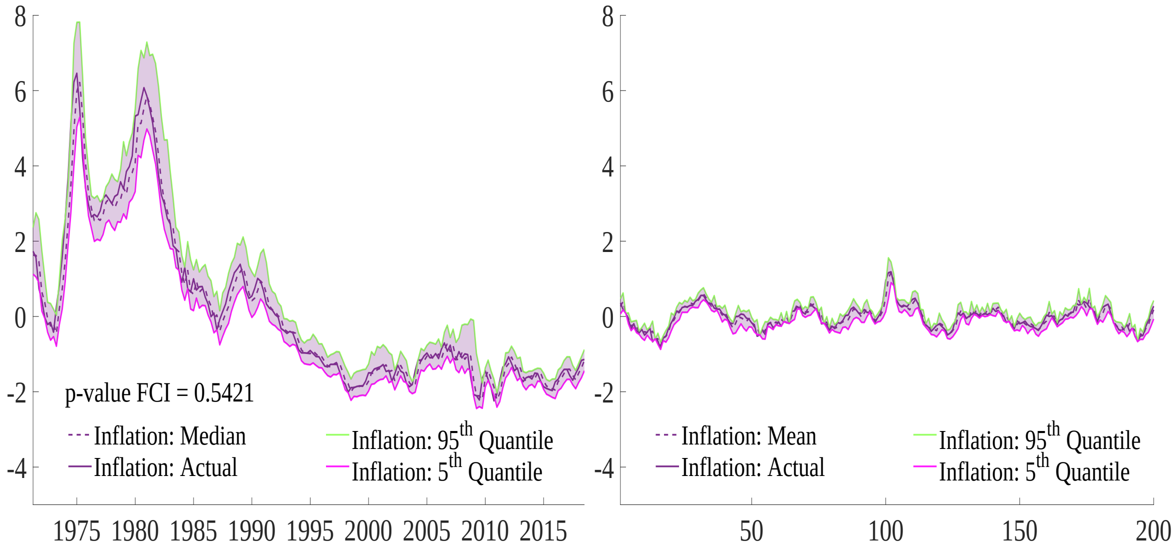
Figure 2. Conditional Output Gap Growth Median and Volatility Correlate Negatively



(a) Data **(b)** Simulation

Note: Panel (a) shows estimates of the conditional median and conditional volatility while Panel (b) shows the conditional median and volatility simulated from the NKV model.

Figure 3. Financial Variables Do Not Predict Tails of Inflation



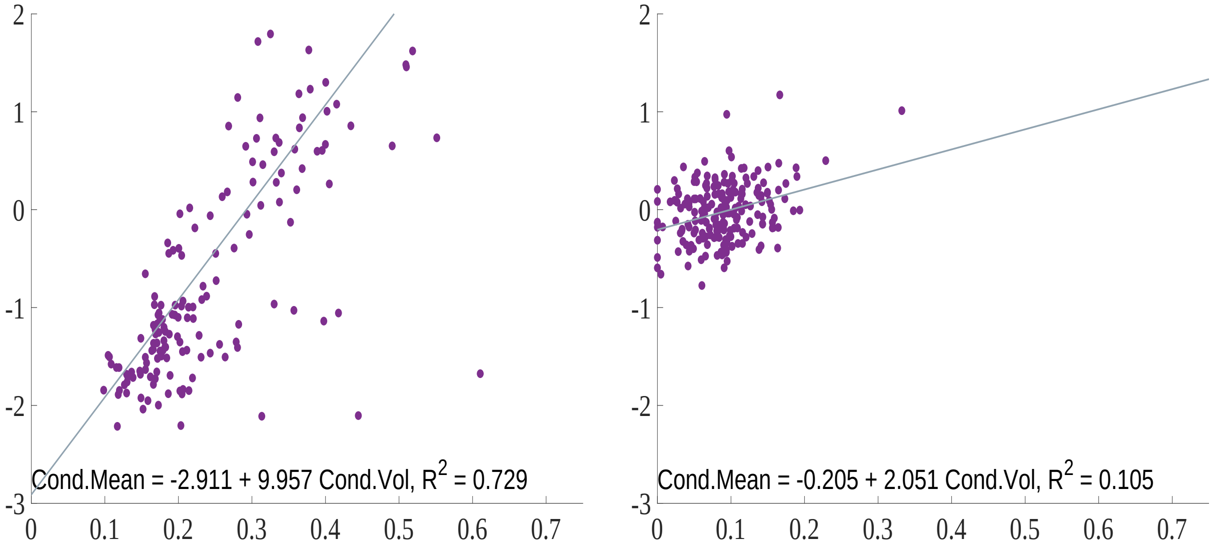
(a) Data **(b)** Simulation

Note: The 5th conditional quantile, the conditional median, and the 95th conditional quantile of the inflation distribution. The series are estimated using quantile regressions with one-quarter-ahead inflation on the left hand side, and current output gap, inflation, and financial conditions on the right hand side. Panel (a) shows the data while Panel (b) shows data simulated from the NKV model.

Because inflation is zero in the model's deterministic steady state we have demeaned the data to make the two panels more directly comparable.

3.5. **Stylized Fact 3: Financial variables do not predict tails of inflation.** While financial conditions are "highly significant" in forecasting the shape of the conditional output gap distribution, they do not forecast the tails of the inflation distribution in a statistically significant manner (with the FCI coefficient lacking significance at the 50 percent level). In fact, conditional heteroskedasticity of inflation is well described by the level of past inflation itself, with the co-movement pattern in Figure 3 very different from the one in Figure 1. The NKV model captures these stylized facts qualitatively, which can be inferred from Figure 4, showing that it replicates the positive slope of the relationship between inflation's conditional median and volatility[22].

Figure 4. Inflation Conditional Median and Conditional Volatility Correlate Positively.



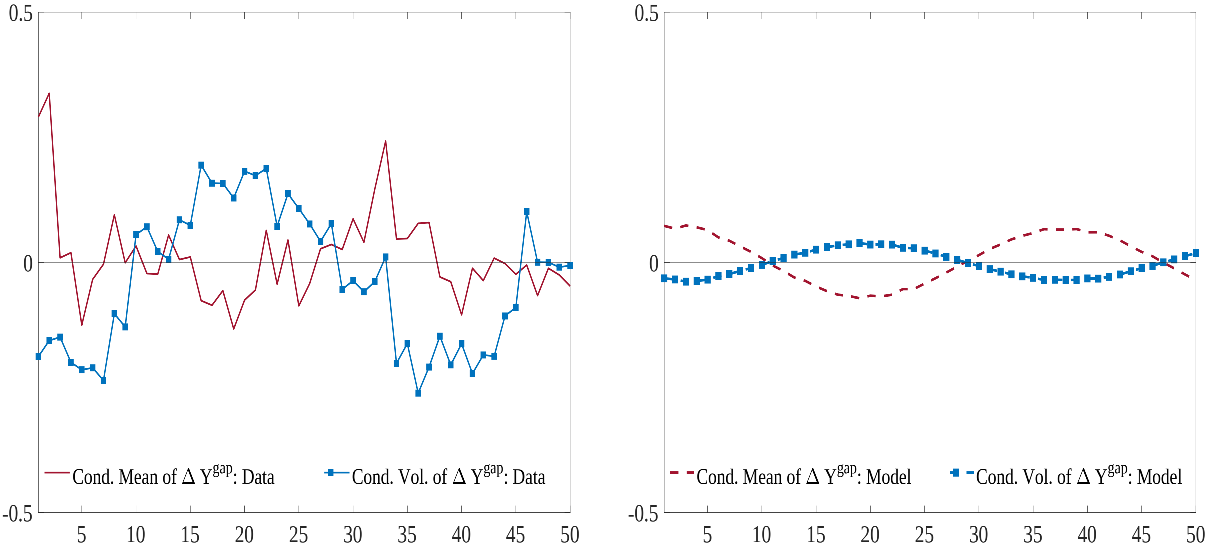
(a) Data **(b)** Simulation

Note: Panel (a) shows estimates of the conditional median and conditional volatility while Panel (b) shows the conditional median and volatility simulated from the NKV model.

3.6. **Stylized Fact 4: The volatility paradox.** An important feature of the data – and the NKV model – is the volatility paradox, as seen in Figure 5. When η is low, indicating loose financial conditions, volatility is low in the short term. But this effect eventually reverts, with the elasticity of the conditional volatility of the output gap indicating that medium-term volatility increases somewhat with respect to an initial easing of financial conditions.

[22]Since our model wasn't designed to account for the 1970s oil price shock or its aftermath, it is unsurprising that it fails to generate inflation of a corresponding magnitude, and hence understates the slope implied by the conditional median-conditional volatility univariate regression.

Figure 5. The Volatility Paradox



(a) Data **(b)** Simulation

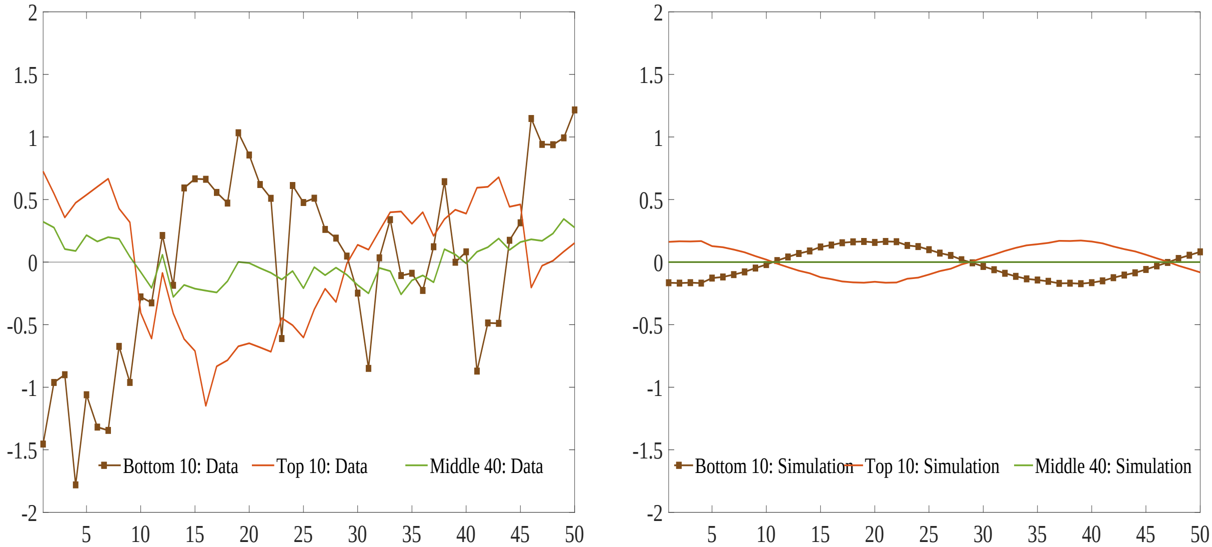
Note: Elasticity of the conditional output gap median and volatility with respect to changes in η. Panel (a) shows estimates of the elasticity, while Panel (b) shows estimates based on data simulated from the NKV model.

"Volatility paradox" (Brunnermeier and Sannikov, 2014), refers to the observation that forward-looking risk builds during good times, when contemporaneous risk is low and growth is high.

3.7. **Stylized Fact 5: Term structures of growth-at-risk cross.** Adrian et al. (2018) study the term structure of downside risk to GDP growth. Growth-at-risk is measured by the (lower) 5th percentile of future GDP growth conditional on financial conditions. The shape of the estimated growth-at-risk term structure is consistent with endogenous risk-taking and the volatility paradox. Periods of easy financial conditions are characterized in the short run by low downside risks and high expected growth, but there is a reversal in the medium term as easy financial conditions forecast a large increase in downside risks. Conversely, in times of recessions or crises, when financial conditions are particularly tight, downside risks in the near term are high, but diminish in the medium term (and display cyclical patterns in the long run).

Figure 6 shows that the NKV model replicates the crossing of the lower quantiles of the output gap distribution when conditioning on a compressed (easy) price of risk in comparison to an average or elevated price of risk.

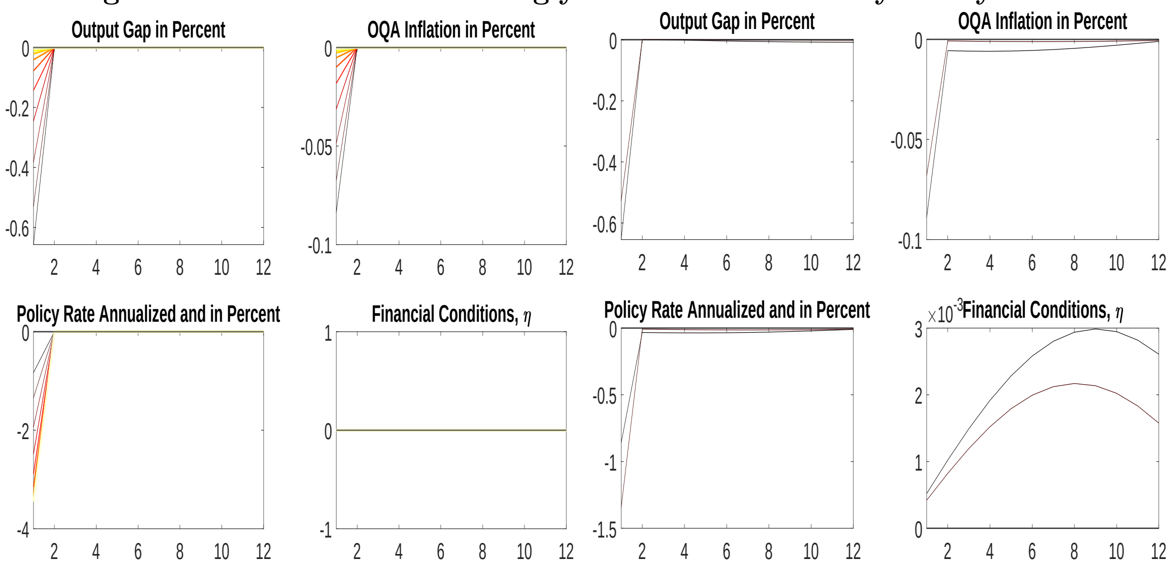
Figure 6. Term Structures of Growth-at-Risk Cross



(a) Data **(b)** Simulation

Note: The figure shows term structures of output-gap-at-risk, the 5th quantile of the Δy^{gap} distribution. The three lines condition on easy, average, and tight financial conditions (Top 10, Middle 40, Bottom 10, respectively). Panel (a) shows the empirical term structures, while Panel (b) shows the simulated term structures from the NKV model.

Figure 7. IRFs under Increasingly "Activist" Monetary Policy Rules.



(a) Standard NK Model **(b)** NKV Model

Note: The progressively brighter lines correspond respectively to greater weights on deviations of inflation and the output gap in the Taylor rule. These increase by a factor of approximately 1,000, with each brighter line corresponding to a doubling of Taylor rule coefficients (from their initially assumed baseline values).

Missing lines in the RHS panels correspond to instability on account of violations of the Blanchard Kahn conditions.

4. Optimal Monetary Policy

Having documented how well our model replicates the macrofinancial stylized facts, we now turn to its implications for monetary policymakers. Given that our setup nests the three-equation NK model, it is perhaps most natural to consider whether the standard NK policy prescriptions carry over. To that effect we note that, by construction, our model is one in which the "divine coincidence" holds: the only shock is isomorphic to a demand shock and directly affects only the dynamic IS curve[23]. As such, it would seem natural to expect that optimal policy under discretion would entail full stabilization of both inflation and the output gap. We also know from the standard NK model that while a Taylor rule does not fully stabilize the economy, it can approximate that outcome arbitrarily well (Galí, 2015, p.114): as the weights on inflation or the output gap increase, the demand shock would have less and less of an impact (as illustrated in Panel (a) of Figure 7)[24]. Since the case of a standard Taylor rule forms our benchmark, we ask whether an increasingly "activist" monetary policy rule would also deliver full stabilization in our proposed NKV setup.

There are good reasons to expect such a result to hold. First, if monetary policy was able to achieve full stabilization, then both the level and the expectation of the output gap would equal their respective steady state of zero. Accordingly, in such circumstances, the process for financial conditions η_t would approximately reduce to

$$\eta_t = \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta} \eta_{t-2}.$$

This shows that under full output gap stabilization, η_t would only depend on its own lags. It follows that if we initialized the system in its steady state, then financial conditions would stay in that steady state forever. As a consequence, vulnerability would also be constant, because

$$V_t = \max\left\{0, a_1 + a_2\eta_{t-1} + a_3 y_{t-1}^{gap} + a_4\eta_{t-2}\right\} \Longrightarrow \lim_{y_t^{gap}\to 0, \eta_t \to 0} V_t = a_1^+.$$

With constant vulnerability, our model would become linear and homoskedastic, and in that case, we know that an aggressive Taylor rule can deliver full output gap stabilization. As such, it would seem that even though the NKV model is nonlinear (and hence the problem of optimal policy under discretion is no longer tractably linear-quadratic), a sufficiently aggressive

[23] Expressed alternatively, there are no tradeoff-inducing wedges showing up in the Phillips curve.

[24] As these coefficients increase, the output gap and inflation respond less and less to the same initial shock. In the limit, they wouldn't respond at all – which corresponds to optimal policy and full stabilization. Notably, while monetary policy is reacting by more and more, the rate cuts don't diverge to minus infinity: effectively, inflationary expectations are affected by less and less as the Taylor rule coefficients increase, so the (absolute) size of the monetary policy interventions necessary to ensure stability can be shown not to increase without bound.

monetary policy rule should be able to achieve full stabilization. In other words, no "leaning against the wind" and no macro-prudential policy would be required here, with traditional monetary policy effective at eliminating inefficient fluctuations.

While intuitively compelling, Panel (b) of Figure 7 demonstrates that the argument fails to apply to the NKV model. Taylor rule coefficients cannot be increased without bound: once they get too large, the model becomes explosive, which accounts for the missing impulse responses in Figure 7 (b).

The underlying story has a theme familiar from Minsky (1992): too much stability is capable of breeding instability. And, in fact, this is precisely what happens in our simple NKV model. As we show below, the fact that monetary policy is fixated on inflation and output gap volatility implies that when the corresponding Taylor rule weights are increased, financial conditions will become unstable. Since financial conditions directly affect the real economy via the financial accelerator and through the vulnerability channel, our model highlights the possibility that a period of low volatility, such as the Great Moderation, may be more likely to be followed by undesirable outcomes through increased sensitivity to shocks (Bernanke, 2012).

To illustrate what exactly is happening, consider again the process for financial conditions

$$\eta_t = \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta} \eta_{t-2} - \theta_y y_t^{gap} - \theta_\eta E_t y_{t+1}^{gap}.$$

This specification comprises backward-looking autoregressive components along with forward-looking endogenous variables, namely the contemporaneous and expected levels of the output gap (i.e. y_t^{gap} and $E_t y_{t+1}^{gap}$ respectively). In equilibrium, this semi-structural specification, combined with all the other market clearing and optimality conditions, gives rise to a "solved" specification for η_t of the following form[25]

$$\eta_t = \gamma_\eta \eta_{t-1} + \gamma_{\eta\eta} \eta_{t-2} + \gamma_\varepsilon \varepsilon_t^y. \tag{6}$$

Crucially, in our baseline model, the coefficients γ_η and $\gamma_{\eta\eta}$ will be different from the λ_η and $\lambda_{\eta\eta}$ in the semi-structural form. This is because the equilibrium law of motion (Equation 6) effectively accounts for the equilibrium laws of motion for y_t^{gap} and $E_t y_{t+1}^{gap}$ (which are themselves functions of the state variables η_{t-1},η_{t-2} and ε_t^y). It is also the case that our equilibrium γ_η and $\gamma_{\eta\eta}$ corresponding to the baseline specification imply a stable AR(2) process. In other words, agents' expectations of output gap and expected output gap volatility, along with the lag structure built into our process for financial conditions, imply a stable process for η_t.

[25] We're abstracting from heteroskedastic volatility here as it is not central to the argument.

We can now consider what happens when monetary policy becomes increasingly aggressive in targeting inflation and the output gap. As argued above, if both y_t^{gap} and $E_t y_{t+1}^{gap}$ almost surely converge to zero (in a probabilistic sense), then the coefficients of their equilibrium laws of motion, that is,

$$y_t^{gap} = a_1\eta_{t-1} + a_2\eta_{t-2} + a_3\varepsilon_t^y$$
$$E_t y_{t+1}^{gap} = b_1\eta_{t-1} + b_2\eta_{t-2} + b_3\varepsilon_t^y$$

also have to converge to zero (that is, we would have $\forall_{i\in\{1,2,3\}} \lim_{y_t^{gap}\to^{as}0} a_i = \lim_{y_t^{gap}\to^{as}0} b_i = 0$). It follows that in this particular situation, because the impact of the endogenous components vanishes, the coefficients γ_η and $\gamma_{\eta\eta}$ actually converge to their semi-structural counterparts:

$$\lim_{y_t^{gap}\to^{as}0} \gamma_\eta = \lambda_\eta \qquad \text{and} \qquad \lim_{y_t^{gap}\to^{as}0} \gamma_{\eta\eta} = \lambda_{\eta\eta}.$$

As a consequence, if

$$\eta_t = \lambda_\eta\eta_{t-1} + \lambda_{\eta\eta}\eta_{t-2}$$

happens to be an unstable process, then eliminating output gap volatility, somewhat paradoxically, pushes the equilibrium specification from $\eta_t = \gamma_\eta\eta_{t-1} + \gamma_{\eta\eta}\eta_{t-2} + \gamma_\varepsilon\varepsilon_t^y$, which was stable, toward $\eta_t = \lambda_\eta\eta_{t-1} + \lambda_{\eta\eta}\eta_{t-2} + \tilde{\gamma}_\varepsilon\varepsilon_t^y$ which is not!

This is precisely what happens in the NKV model, in which the AR coefficients in the specification for $\eta_t = 1.97\eta_{t-1} - 1.01\eta_{t-2}$ are unstable, but the reduced form process corresponding to a standard Taylor rule ends up with different, stable coefficients. This is also exactly why monetary policy that ends up being too successful in stabilizing the output gap runs the risk of destabilizing financial conditions, and, ultimately, the entire economy[26].

Our model points to the possibility, absent from the standard NK model, that having a central bank focused solely on eliminating inflation and output gap volatility may be suboptimal. By not paying attention to the endogenous nature of financial conditions, the central bank risks inadvertently making them unstable. While this does not automatically have to hold in our setup, and indeed, the macroprudential section highlights when full stabilization may be possible, we believe this eventuality is important enough to highlight and consider more seriously. Clearly, it is also possible to have monetary policy explicitly depend on expected financial conditions, which, as we shall show and explain in Section 6, can improve upon the outcome associated with a standard Taylor rule.

[26]Of course, in the model both financial conditions and the output gap would end up simultaneously unstable, which manifests itself as a violation of Blanchard-Kahn conditions.

5. Macroprudential Policy

The use of cyclical macroprudential tools can mitigate downside risks to GDP. The NKV framework is well suited to analyzing monetary and cyclical macroprudential policies simultaneously, as it is tractable yet empirically relevant, with its focus on endogenous output risk.

We expand the NKV model to study the joint determination of monetary and macroprudential policy with a hypothetical policy instrument that impacts the level of financial conditions η: tighter macroprudential policy is assumed to increase the price of risk and, via the financial accelerator effect, it also impacts output growth. More specifically, we assume that a state contingent macroprudential tool μ_t is capable of affecting contemporaneous financial conditions, that is, that

$$\eta_t = \mu_t + \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta}\eta_{t-2} - \theta_y y_t^{gap} - \theta_\eta E_t y_{t+1}^{gap}.$$

We now illustrate the possibility that a combination of macroprudential policy and monetary policy achieves full stabilization. To that effect, we posit that macroprudential policy satisfies,

$$\mu_t = \nu_\eta \eta_{t-1} + \nu_{\eta\eta}\eta_{t-2}$$

which immediately implies that the semi-structural specification for financial conditions would be

$$\eta_t = (\lambda_\eta + \nu_\eta)\,\eta_{t-1} + (\lambda_{\eta\eta} + \nu_{\eta\eta})\,\eta_{t-2} - \theta_y y_t^{gap} - \theta_\eta E_t y_{t+1}^{gap}.$$

If the policy coefficients ν_η and $\nu_{\eta\eta}$ were set such that the process

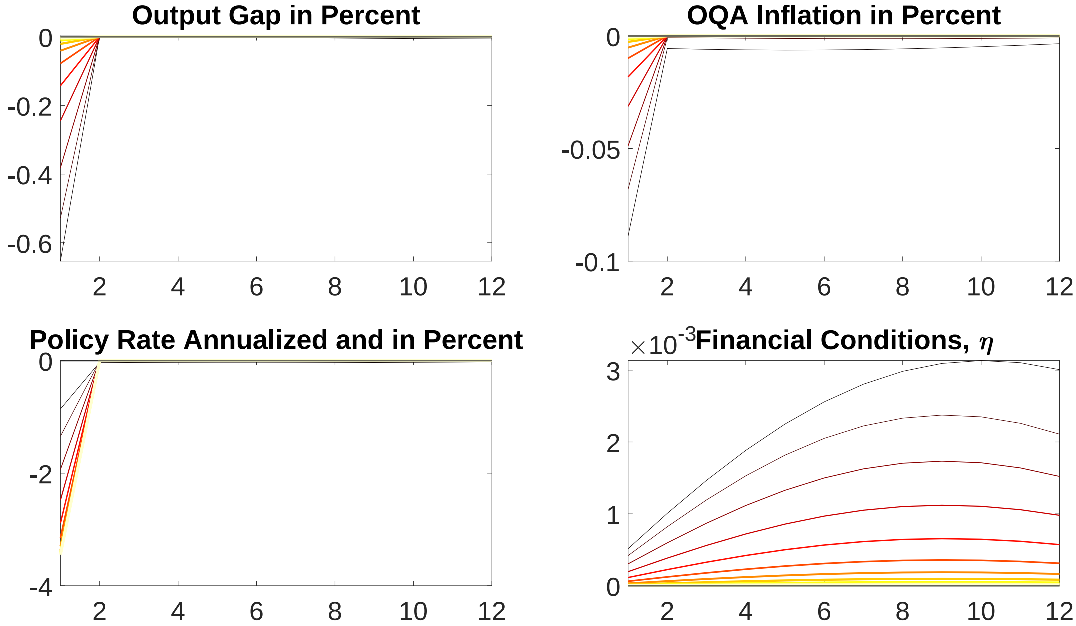
$$\eta_t = (\lambda_\eta + \nu_\eta)\,\eta_{t-1} + (\lambda_{\eta\eta} + \nu_{\eta\eta})\,\eta_{t-2}$$

was stable, then the risk of explosive dynamics associated with "overly successful" monetary policy would be taken off the table. That eventuality is precisely what we illustrate in Figure 8, which shows that, in an NKV model with an appropriately altered specification for η, increasingly aggressive monetary policy can achieve outcomes arbitrarily close to full stabilization. This, naturally, would be a desirable outcome.

Importantly, even if macroprudential policy only affected financial conditions with a lag, for example, in the following fashion:

$$\eta_t = \mu_{t-1} + \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta}\eta_{t-2} - \theta_y y_t^{gap} - \theta_\eta E_t y_{t+1}^{gap}$$

Figure 8. IRFs for Increasingly "Activist" Monetary Policy Rules under a Stabilizing Macroprudential Policy Affecting η.



Note: The progressively brighter lines correspond to greater weights on deviations of inflation and the output gap in the Taylor rule. These increase by a factor of approximately 1,000, with each brighter line corresponding to a doubling of Taylor rule coefficients (from their initially assumed baseline values).

then a specification in which

$$\mu_t = \nu_\eta \eta_t + \nu_{\eta\eta}\eta_{t-1}$$

would still make it possible for monetary policy to fully stabilize the economy. More generally, any macroprudential rule, for which

$$\mu_t + \lambda_\eta \eta_{t-1} + \lambda_{\eta\eta}\eta_{t-2}$$

is a stable linear process would allow this to hold. And of course, the stability properties of an $AR(k)$ process depend on their corresponding $k-$th order characteristic polynomials. So in principle, systematically affecting financial conditions at any lag could create conditions under which the strict separation of monetary and macroprudential policies leads to efficient outcomes.

How to translate these fairly abstract results into practical policy recommendations? What would happen if there were constraints on how often macroprudential tools could be adjusted? What would happen if, say, macroprudential tools directly affected inflation and the output gap? Would an uncoordinated policy approach still be possible? Clearly, our setup is too stylized to provide answers to such questions, which we believe would be worth studying in a model with a fully micro-founded specification for financial conditions.

What our results do show, though, is that real-world macroprudential policy would need to ensure that financial conditions remain stable even during periods such as the Great Moderation, when the temptation may be to increase risk exposures and hope for stability to persist. If this prerequisite is not satisfied, then the buildup in vulnerabilities, proxied by our $V(\cdot)$ function, could mean that a small shock is all it takes to start off an intrinsically unstable spiral of events (of which only policies much richer than those accounted for in our model could be capable of stabilizing).

6. Alternative Policy Paths with Endogenous Risk

We now highlight the benefits of analyzing alternative policy paths using setups such as the NKV model, which accounts for endogenous conditional risk, and thus more fully captures the challenging tradeoffs facing policymakers. This section also aims to highlight the potential benefits of monetary policy accounting for financial conditions directly, when suitable macroprudential tools may not be available.

To that effect, we compare responses under a standard Taylor (1993) rule (solid line in Figure 9) to responses under an alternative rule that makes interest rates additionally depend on the expected price of risk (dashed line in Figure 9):

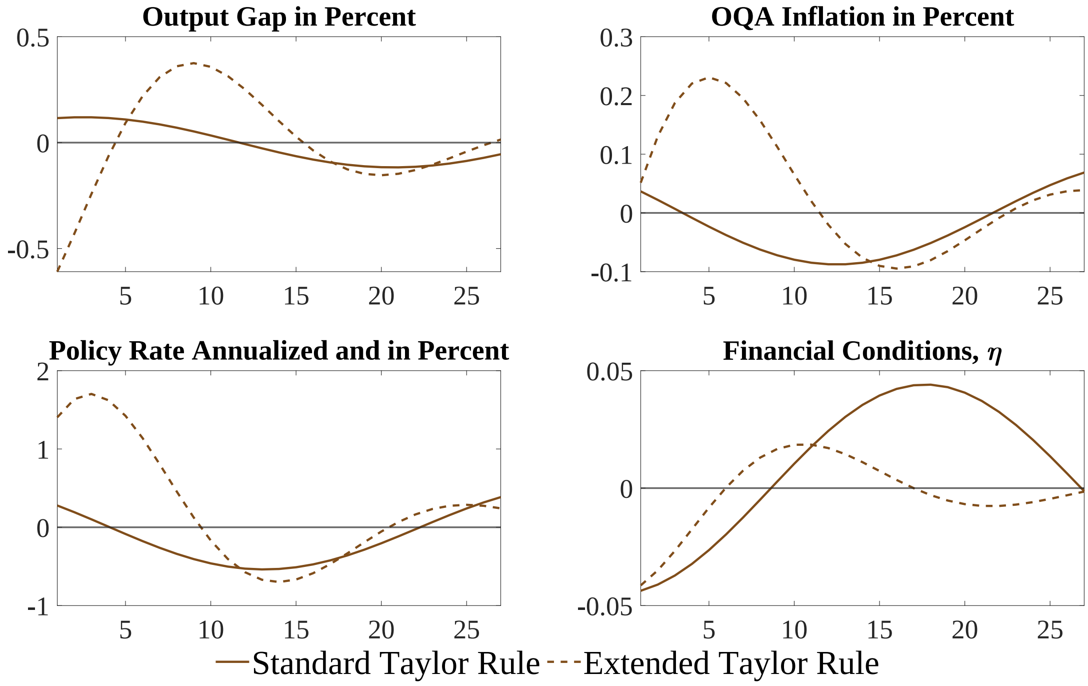
$$i_t = \phi^\pi \pi_t + \phi^y y_t^{gap} - \phi^\eta E_t \eta_{t+1} \tag{7}$$

where ϕ^η is set equal to 0.1.

For the impulse responses depicted in Figure 9, we initialize the model by setting initial conditions to be one standard deviation below their steady state, that is, $\eta_0 = \eta_{-1} = -\sigma^\eta$. In equilibrium, under a standard Taylor rule, loose financial conditions are associated with higher levels of inflation and a positive output gap (top two panels) leading the central bank to tighten rates by just over 25bps (bottom left panel). This results in falls in inflation and the output gap, and a gradual tightening of financial conditions. Crucially, under the standard Taylor rule, financial conditions "overshoot." This leads to elevated vulnerability from the ninth quarter.

Under the extended Taylor rule of Equation (7), policymakers additionally account for fluctuations in financial conditions. This causes them to tighten by an extra 115bps, which is associated with an immediate output gap contraction. Somewhat surprisingly, the larger interest rate hike is associated with higher equilibrium inflation, suggesting that an additional target for monetary policy may weaken the central bank's inflation-fighting credentials (as it means relatively less weight on deviations of inflation from the target). One benefit of the

Figure 9. Alternative Paths with Endogenous Risk



policy, however, is that financial conditions tighten faster and overshoot by less, meaning that the economy is not as vulnerable to shocks as under a standard rule.

While the initial recession observed when using the extended Taylor rule is relatively short-lived – with higher levels of the output gap from the sixth quarter until almost the end of the simulation – the comparison in Figure 9 may not make it immediately clear whether the medium-term benefit exceeds the short-term costs. In particular, inflation and the output gap appear to be more volatile under the extended rule. What the path conceals, however, is that the extended Taylor rule effectively eliminates periods of very tight (and very loose) financial conditions (whereas the standard rule does not), meaning that policymakers implementing it would seldom find themselves facing the situation considered here.

In fact, Figure 10, which compares the ergodic distributions of η, shows that extreme realizations are much less likely under the extended Taylor rule. This also translates into less output gap volatility, as shown in Panel (a) of Figure 11[27]. In particular, under the extended Taylor rule, outcomes closer to the mean are more likely, precisely because the risk-conscious approach is more effective at eliminating states of high output gap volatility. Risk-averse agents would prefer less output gap volatility; the evidence in Figure 11 suggests an additional reason they might prefer the extended rule over the standard one. Of course, if a volatile output gap

[27]Note, however, that the "fatness" of Δy^{gap} left tails is hardly affected - as made clear by Panel (b) of Figure 11.

Figure 10. Ergodic Distribution of Financial Conditions

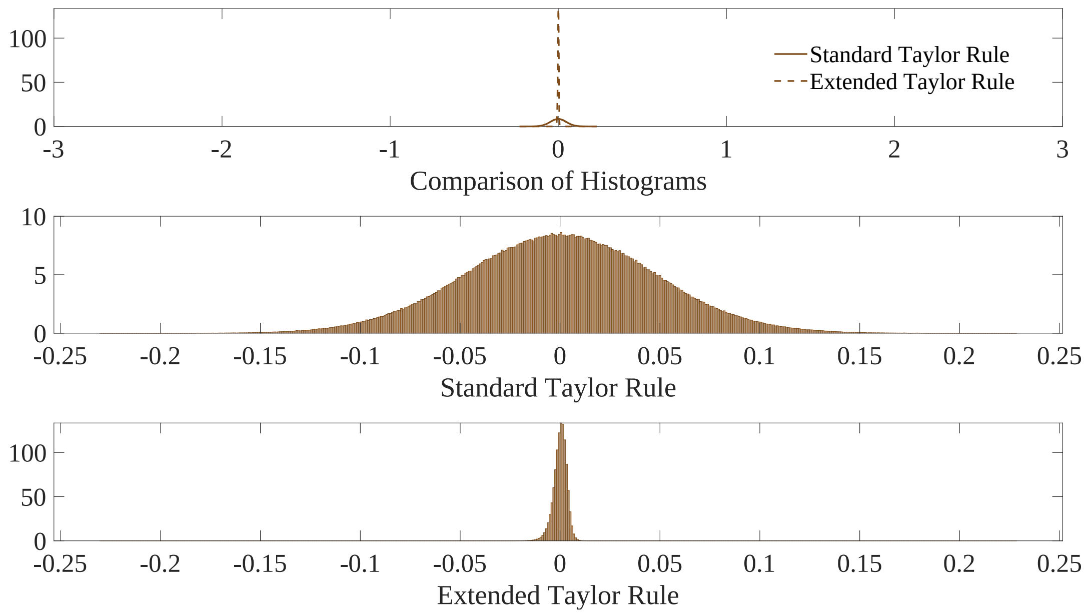
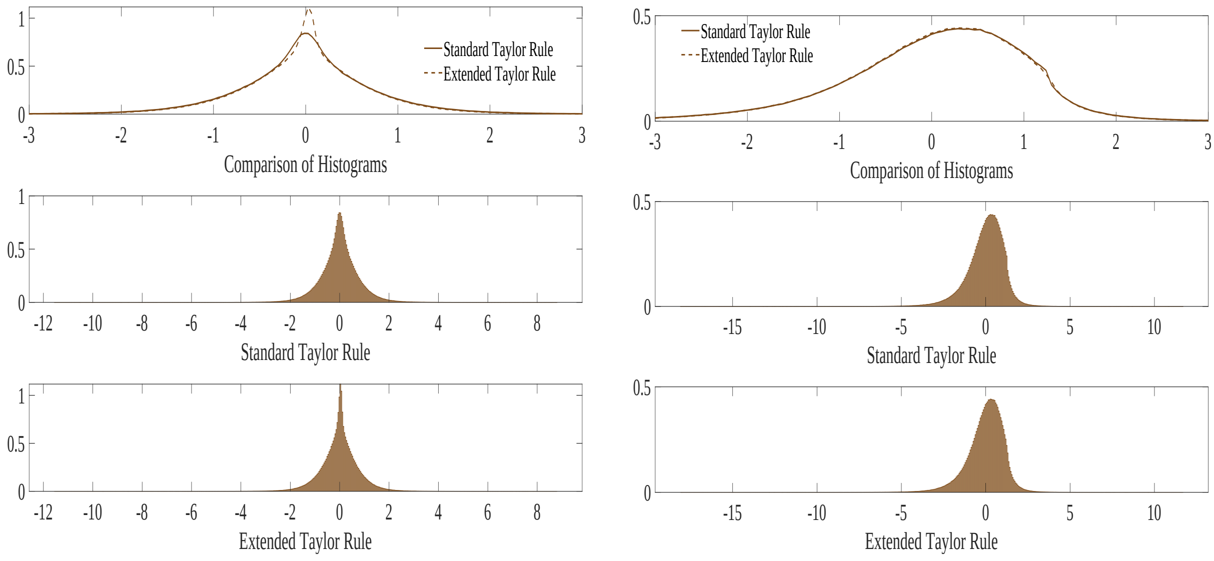


Figure 11. Ergodic Output Gap Distribution

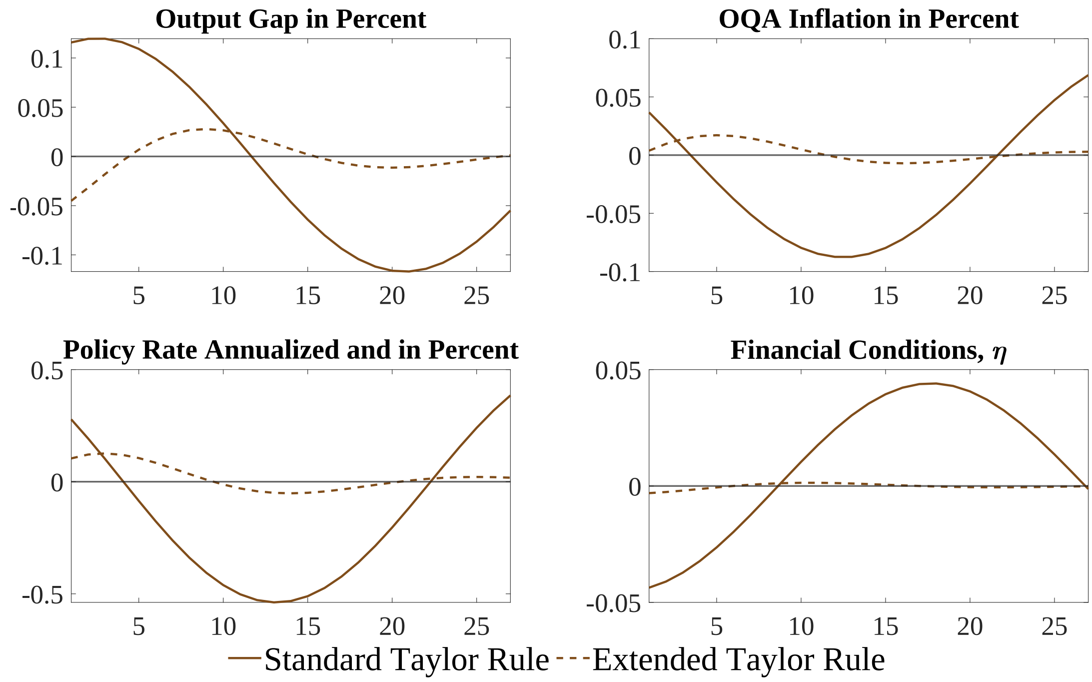


(a) Levels **(b)** Changes

was associated with additional inefficiencies, as is the case in the standard NK model, that would only provide *more* reasons to seriously consider the extended Taylor rule of Equation (7).

These different observations are summarized in Figure 12, which mirrors Figure 9 except for one crucial exception. Rather than initializing both experiments using the volatility of η in the economy under the standard Taylor rule (as done in Figure 9), this chart uses the "economy-specific" eta volatilities (that is, it accounts for the fact that the extended Taylor rule considerably lowers the volatility of η). What the figure makes clear is that when these adjusted responses are compared, the extended Taylor rule delivers markedly lower volatility for all variables of interest.

Figure 12. Alternative Policy Paths Adjusted for Changes in Volatility



Of course, in an actual monetary policy setting, any decision would require policymakers' judgment, given lack of precision in measuring the output gap in real time. Additionally, one might want to use a more realistic medium-sized dynamic stochastic general equilibrium (DSGE) model that fits the data along various additional dimensions. Moreover, any outcome would also reflect the ability of policymakers to communicate objectives clearly and credibly commit to implementing them (which was implicitly assumed in the exercises considered here).

Our key takeaway is that policy decisions, whether consciously or not, do affect the price of risk and so can have a marked impact on the dynamics of inflation and the output gap. We argue that in such an environment, monetary policy should aim to curb vulnerability and the excessive volatility of the output gap associated with it.

7. Related Literature

Our paper is related to research that positions the financial sector at the heart of macroeconomic fluctuations and the transmission mechanism. There is an active research agenda directly focused on incorporating credit conditions into New Keynesian policy models. Woodford (2010), for example, augments a Keynesian IS-LM model with financial intermediary frictions, based on Curdia and Woodford (2010). In that setting, the additional friction gives rise to an extra state variable that can be mapped into credit spreads, and optimal policy is shown to explicitly depend on credit supply conditions. Relatedly, Woodford (2012) characterizes optimal monetary policy in a setting with financial crises, and finds that inflation-targeting rules should be modified to explicitly consider the possibility of such crises occurring. Gertler and Kiyotaki (2015) add a banking sector featuring liquidity mismatches, and focus on the implications of bank runs, while Adrian and Duarte (2018) analyze optimal policy in a setting in which financial intermediaries are subject to VaR constraints.

While some of our modeling choices are informed by the analysis of Adrian and Duarte (2018), our aim here is not to propose a novel model of macrofinancial linkages, nor to take a stand on the specific economic mechanism(s) driving the empirical regularities we highlight. Instead we focus on two simple modifications of the standard NK model: (1) the inclusion of financial conditions, assumed to have an endogenous, forward-looking component; and (2) the recognition that the economy may become more vulnerable to shocks following periods of loose financial conditions. We show that the interplay of these two features allows our model to generate more realistic conditional and unconditional forecasts, and that it also adds an extra dimension to the policy maker's problem.

To make these points, we build a semi-structural model, featuring just four equations, that directly extends the macroeconomic framework of Woodford (2003) and Galí (2015). Specifically, variations in financial conditions are allowed to affect the conditional mean and volatility of the output gap through financial accelerator and vulnerability channels.

A number of mechanisms have been put forward to explain observed macrofinancial linkages. When lenders face asymmetric information so that there is an external finance premium for borrowers, easier monetary policy and financial conditions can improve the net worth of borrowers, and, through a financial accelerator effect, increase credit for households and businesses (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997; Bernanke et al., 1999). Our setup incorporates a financial accelerator term, and related models have been used to address policy issues. In practice though, macrofinancial models, which are widely used in central banks today, rely predominantly on first-order solution methods, and approximate financial frictions up to first order[28]. We have argued that, by construction, such linear setups are unable to match an important number of stylized macro-financial facts. In particular, our approach featuring

[28]A substantial effort is underway to develop methods to solve such models globally, or up to high order, as they are generically nonlinear (see, for example, Andreasen et al. 2018 or Dou et al. 2018).

endogenous risk, modeled as a function of endogenous financial conditions, can match important stylized facts of the conditional distribution of output while remaining analytically and numerically tractable. Hence our approach can be readily implemented in policy applications, contrary to much of the existing literature on macro-financial linkages.

Importantly, output fluctuations can be magnified when financial intermediaries respond endogenously to looser financial conditions, and these changes can occur through a number of different channels. Easier policy can increase net worth and relax capital constraints of banks, which may affect the supply of credit in a procyclical way (Bernanke and Blinder, 1988; Gertler and Kiyotaki, 2010). Low interest rates can lead to compressed risk premia because investors "reach for yield" on account of fixed nominal rate targets tied to their liabilities (Rajan, 2005). To achieve those targets, they may increase leverage and funding risks (Brunnermeier and Pedersen, 2009; Adrian and Shin, 2010, 2014). These risks can also manifest themselves as a deterioration in asset quality (Altunbas et al., 2010; Jimenez et al., 2012; Dell'Ariccia et al., 2017). Accordingly, low rates and a low price of risk can boost current growth while simultaneously making the economy more vulnerable to future shocks and future financial instability. The observation that periods of low volatility and endogenous risk-taking contribute to a buildup of imbalances and future negative growth is the "volatility paradox" (Brunnermeier and Sannikov, 2014) discussed earlier, and our model's ability to account for it forms one of the key litmus tests considered.

Our set-up is also consistent with leverage cycles, which have been shown to arise as more optimistic investors who are leveraged compress risk premia during a boom, but are then forced to sell when prices start to fall. This leaves assets in the hands of more pessimistic buyers who value them less (Geanakoplos, 2010). Diagnostic expectations of investors can give rise to extrapolative forecasting behavior and lead to the neglect of tail risk, and can generate predictable credit spreads dynamics (Bordalo et al., 2018). In addition, extrapolative beliefs in the stock market can amplify technology shocks, giving rise to booms and busts in stock prices and the real economy (Adam and Merkel, 2019). Deviations from rational expectations potentially play a more powerful role during times of low interest rates. One interpretation of our price-of-risk dynamics is that they reflect leverage cycles, another is that they reflect endogenous belief formation.

Our paper is also related to those studying how monetary and macroprudential policy could reduce risks to financial stability. In particular, we revisit the separation principle, forcefully put forward in Svensson (2017). He states that monetary policy should focus on price stability and real activity, while macroprudential policies should be directed to reduce vulnerabilities consistent with an acceptable level of financial stability risk[29]. In that framework (Svensson, 2017) the probability of a financial crisis depends on credit growth and the costs of a crisis are very high, many multiples of the benefits. This leads him to conclude that monetary policy should not be used for these purposes (see also Ajello et al., 2016)[30]. However, Adrian and Liang (2018) show that the net benefits are very sensitive to assumptions made about the costs of a crisis. In contrast, in our model a powerful risk-taking channel of monetary policy is present. While effective macroprudential policy can restore the separation principle, in practice, it is more reasonable that monetary and macroprudential policy decisions should be made jointly.

The semi-structural model in this paper allows us to evaluate monetary policy with endogenous risk and the effects of also using macroprudential policy, and to vary its assumed effectiveness. Relatedly, in extensions of Svensson's framework featuring more persistent output effects, Gourio et al. (2018) document that the benefits associated with monetary policy rules that respond systematically to excess credit can outweigh the costs. In a similar vein, Filardo and Rungcharoenkitkul (2016) incorporate a financial cycle in which booms and busts are recurring, rather than one-off events. They show that monetary policy can constrain the accumulation of imbalances and hence lessen the duration and costs associated with crises. Mirroring our findings, they also find a role for monetary policy to monitor and potentially curb the buildup of financial imbalances, rather than acting ex-post, once the crisis probability becomes unacceptably high[31]. Relative to those contributions, our approach is much more parsimonious and tractable, and our model can be directly used in policy applications. Furthermore, we conjecture that our approach can easily be included in larger models, such as those typically used in central banks for policy purposes.

A number of papers model the interaction of monetary policy and macroprudential policies. In line with our results, several find that there are benefits from coordination, with a significant

[29]According to the argument, macroprudential policies are best suited to address financial vulnerabilities in part because the effects of monetary policy are broad and it cannot directly address high leverage and funding risks of financial intermediaries.

[30]Svensson (2017) estimates the costs and benefits of using monetary policy to prevent a financial crisis in a model where the costs are related to higher unemployment, while the benefits are associated with a lower probability of future crises on account of reduced household borrowing.

[31]Caballero and Simsek (2019) provide another rationale for using monetary policy to lean against the wind. In their model, monetary policy affects the discount rate (not the risk premia on risky assets) of heterogenous investors (optimists and pessimists), but can act like a leverage limit (especially valuable when the policy rate is near the zero lower bound). Thus it reduces asset prices in booms, which will soften the asset price bust when the economy moves into a recession. Their model allows for monetary policy and macroprudential policy to be both complements and substitutes, depending on the tightness of macroprudential policy.

role for macroprudential policies to reduce the intensity of systemic stresses[32]. Korinek and Simsek (2016) also highlight that a rise in rates to reduce borrowing may be less effective than a limit on debt growth. This is because the rate increase could prompt a recession or could exacerbate borrowing driven by consumption smoothing. Kiley and Sim (2017) point out that a rise in rates to offset a rise in credit-to-GDP would reduce welfare if it followed a positive technology shock. Our paper complements this literature in presenting an empirically tractable, parsimonious NK model that is augmented with vulnerability, thus giving rise to the NKV model.

8. Conclusion

We present a parsimonious semi-structural model with endogenous volatility to capture important empirical properties of the distribution of the output gap and inflation forecasts. We incorporate a financial accelerator and allow for endogenous risk with a new financial vulnerability channel (consistent with macrofinancial linkages) that has been documented widely in the literature. In particular, we match a strong contemporaneous negative correlation between the conditional mean and volatility of output gap growth, and a term structure for a lower quantile output gap growth. The lower quantile of forecast GDP growth conditioned on financial conditions declines following periods when financial conditions are loose. Moreover, we match a crossing of the term structures.

We use the model to evaluate monetary policy and macroprudential policy, and find very different policy prescriptions from those in the standard New Keynesian model. In the NKV model, monetary policy that would stabilize the output gap and inflation in a standard NK model runs the risk of instability when it ignores financial conditions. A monetary policy rule augmented with expected financial conditions can increase welfare. The introduction of a cyclical macroprudential policy implemented as an offset to financial conditions, together with standard monetary policy, can deliver full stabilization of the output gap, inflation, and financial conditions.

The NKV model presented in this paper is analytically tractable, yet generates rich dynamics for the entire output gap distribution, including for the term structure of the distribution. We thus conjecture that the methods proposed in this paper can be applied in many situations due to their empirical relevance.

[32] In terms of trade-offs, Martinez-Miera and Repullo (2019) show that tighter monetary policy raises costs for both safe and risky entrepreneurs, while raising capital requirements can lead banks to shift from risky to safe entrepreneurs.

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Appendix A: The Analytics of the Conditional Mean-Volatility Trade-off Slope

We start by asking the following question: what is the lowest approximation order for which a DSGE model can generate a non-trivial relationship between the conditional mean and conditional variance of its variable? To fix attention, we consider a simple model with two variables, y^{gap} and π, which we'll jointly denote as $y \equiv (y^{gap}, \pi)$, approximated around some point $y^{ss} = (y^{gap,ss}, \pi^{ss})$ and driven by a vector of $\mathcal{N}$.i.d. shocks ϵ_t. In what follows we shall analyze the conditional distribution $\mathcal{P}(y_{t+1}|\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(\epsilon_t)$ is the filtration generated by ϵ_t.

A.1. **The linear model.** In this case, the first-order approximation to the policy function equals

$$y_{t+1} = y^{ss} + A(y_t - y^{ss}) + B\epsilon_{t+1}.$$

It is immediately clear that only the mean of the conditional distribution can vary over time. Specifically

$$\mathcal{P}(y_{t+1}|\mathcal{F}_t) = \mathcal{N}(y^{ss} + A(y_t - y^{ss}), B\Sigma^{\epsilon})$$

that is, the variance of the conditional distribution (that is, the conditional variance of y_{t+1}) equals $B\Sigma^{\epsilon}$ and so is independent of the state y_t[33].

Remark 1. *To fix attention Σ^{ϵ} denotes the standard deviation of the exogenous disturbance. Letting $p!! = 1 \times 3 \times \ldots \times (p-1)$ we then have*

$$\mathbf{E}\epsilon_{t+1}^{p} = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \Sigma^{\epsilon,p}p!! & \text{if } p \text{ is even.} \end{cases}$$

A.2. **Second-order approximation.** In this case, the policy function is

$$y_{t+1} = y^{ss} + \frac{1}{2}g_{\sigma\sigma} + A(y_t - y^{ss}) + B\epsilon_{t+1} + C(y_t - y^{ss})^2 + D(y_t - y^{ss})\varepsilon_{t+1} + E\epsilon_{t+1}^2.$$

Clearly, the *conditional* distribution will no longer be normal, because of the final term (that is, $E\epsilon_{t+1}^2$ which is $\chi^2(1)$). The resulting conditional moments are

$$\mu^{2nd} = y^{ss} + \frac{1}{2}g_{\sigma\sigma} + A(y_t - y^{ss}) + C(y_t - y^{ss})^2 + E\Sigma^{\epsilon,2}$$

and

$$\begin{aligned}
\left(\sigma^{2nd}\right)^2 &= \mathbf{E}\left(y_{t+1} - \left(y^{ss} + \frac{1}{2}g_{\sigma\sigma} + A(y_t - y^{ss}) + C(y_t - y^{ss})^2 + E\Sigma^{\epsilon,2}\right)\right)^2 \\
&= \mathbf{E}\left((B + D(y_t - y^{ss}))\epsilon_{t+1} + E\left(\epsilon_{t+1}^2 - \Sigma^{\epsilon,2}\right)\right)^2 \\
&= (B + D(y_t - y^{ss}))^2\Sigma^{\epsilon,2} + E^2\left(\Sigma^{\epsilon,4}4!! - \Sigma^{\epsilon,4}\right) \\
&= (B + D(y_t - y^{ss}))^2\Sigma^{\epsilon,2} + E^2\Sigma^{\epsilon,4}(3-1).
\end{aligned}$$

So here it becomes crucial whether D is zero or not. With $D = 0$, the conditional variance of y_t is constant and equal to some function of shock moments (up to order 4), that is

$$D = 0 \Longrightarrow \left(\sigma^{2nd}\right)^2 = B^2\Sigma^{\epsilon,2} + 2E^2\Sigma^{\epsilon,4}$$

and so we would have no chance to witness changes in simulated conditional variances.

Via a similar arithmetic as above

$$skew = \mathbf{E}\left((B + D(y_t - y^{ss}))\varepsilon_{t+1} + E\left(\varepsilon_{t+1}^2 - \Sigma^{\varepsilon,2}\right)\right)^3$$

so the skew will be a mixture of normal and χ^2-distributed variables. One can show that all higher moments will be time/state invariant unless $D \neq 0$.

[33]This suggests why the conditional mean is likely to be more volatile than the conditional volatility: changes in the conditional mean are a first-order phenomenon, whereas changes in the conditional volatility are not.

Appendix B: Analytical Derivations of Correlation Coefficients

Under the specification assumed in Equations (1) – (4), η_{t-1} and η_{t-2} are the only two state variables in the model. Assuming that a unique equilibrium exists, this implies that the reduced form for η_t and the output gap y_t^{gap} will be given by

$$\begin{aligned} \eta_t &= F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon_t^{ygap} \\ y_t^{gap} &= P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon_t^{ygap} \end{aligned}$$

where the coefficients $[F_1, F_2, F_3]$ and $[P_1, P_2, P_3]$ are complicated, non-linear functions of the underlying structural parameters.

We can now characterize the laws of motion satisfied by $y_t^{gap}, E_t y_{t+1}^{gap}, dy_t^{gap}$ and $E_t dy_{t+1}^{gap}$ as a function of the Fs and Ps. This is done in the following sequence of Lemma's.

Lemma 2. *In the model considered, the level of the output gap is an ARMA(2,2) process given by*

$$y_t^{gap} = F_2 y_{t-1}^{gap} + F_3 y_{t-2}^{gap} + P_1\epsilon_t^{ygap} + (F_1P_2 - F_2P_1)\,\epsilon_{t-1}^{ygap} + (F_1P_3 - F_3P_1)\,\epsilon_{t-2}^{ygap}$$

Proof. We know that

$$\begin{aligned} y_{t-1}^{gap} - P_2\eta_{t-2} - P_3\eta_{t-3} - P_1\epsilon_{t-1}^{ygap} &= 0 \\ y_{t-2}^{gap} - P_2\eta_{t-3} - P_3\eta_{t-4} - P_1\epsilon_{t-2}^{ygap} &= 0 \end{aligned}$$

and so the second equation can be equivalently rewritten as

$$\begin{aligned} y_t^{gap} = \kappa_1\left(y_{t-1}^{gap} - P_2\eta_{t-2} - P_3\eta_{t-3} - P_1\epsilon_{t-1}^{ygap}\right) \\ + \kappa_2\left(y_{t-2}^{gap} - P_2\eta_{t-3} - P_3\eta_{t-4} - P_1\epsilon_{t-2}^{ygap}\right) + P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon_t^{ygap} \end{aligned}$$

where κ_1 and κ_2 are arbitrary constants. This can be rearranged as

$$\begin{aligned} y_t^{gap} =\; & \kappa_1 y_{t-1}^{gap} + \kappa_2 y_{t-2}^{gap} + P_1\epsilon_t^{ygap} - \kappa_1P_1\epsilon_{t-1}^{ygap} - \kappa_2P_1\epsilon_{t-2}^{ygap} \\ & + P_2\left(\eta_{t-1} - \kappa_1\eta_{t-2} - \kappa_2\eta_{t-3}\right) + P_3\left(\eta_{t-2} - \kappa_1\eta_{t-3} - \kappa_2\eta_{t-4}\right). \end{aligned}$$

By setting

$$\kappa_1 = F_2 \qquad \text{and} \qquad \kappa_2 = F_3$$

and exploiting

$$\forall i \in \{1,2\} : \eta_{t-i} - F_2\eta_{t-i-1} - F_3\eta_{t-i-2} = F_1\epsilon_{t-i}^{ygap}$$

this simplifies to

$$y_t^{gap} = F_2 y_{t-1}^{gap} + F_3 y_{t-2}^{gap} + P_1\epsilon_t^{ygap} + (P_2F_1 - F_2P_1)\,\epsilon_{t-1}^{ygap} + (P_3F_1 - F_3P_1)\,\epsilon_{t-2}^{ygap}$$

which completes the proof. □

Remark 3. *Note that we have so far assumed that $\epsilon_t^{ygap} \sim N(0,1)$, but we could equally introduce $\tilde{\epsilon}_t^{ygap} = P_1\epsilon_t^{ygap} \sim N\left(0, P_1^2\right)$ and express the output gap as*

$$y_t^{gap} = F_2 y_{t-1}^{gap} + F_3 y_{t-2}^{gap} + \tilde{\epsilon}_t^{ygap} + \frac{(P_2F_1 - F_2P_1)}{P_1}\tilde{\epsilon}_{t-1}^{ygap} + \frac{(P_3F_1 - F_3P_1)}{P_1}\tilde{\epsilon}_{t-2}^{ygap}$$

that is, as a standard ARMA(2,2) process in which the noise has some non-unitary variance (P_1^2).

Lemma 4. *In the model considered, the change in the output gap is an ARMA(2,3) process given by*

$$\begin{aligned} dy_t^{gap} = F_2 dy_{t-1}^{gap} + F_3 dy_{t-2}^{gap} + P_1\epsilon_t^{ygap} + (F_1P_2 - (F_2+1)\,P_1)\,\epsilon_{t-1}^{ygap} \\ + (F_1(P_3 - P_2) - (F_3 - F_2)\,P_1)\,\epsilon_{t-2}^{ygap} - (F_1P_3 - F_3P_1)\,\epsilon_{t-3}^{ygap} \end{aligned}$$

Proof. Letting

$$y_t^{gap} = A_1 y_{t-1}^{gap} + A_2 y_{t-2}^{gap} + A_3\epsilon_t^{ygap} + A_4\epsilon_{t-1}^{ygap} + A_5\epsilon_{t-2}^{ygap}$$

we immediately obtain

$$\begin{aligned}
dy^{gap}_{t+1} = y^{gap}_{t+1} - y^{gap}_{t} &= \left(A^1 y^{gap}_t + A^2 y^{gap}_{t-1} + A^3 \epsilon^{ygap}_t + A^4 \epsilon^{ygap}_{t-1} + A^5 \epsilon^{ygap}_{t-2}\right) \\
&\quad - \left(A^1 y^{gap}_{t-1} + A^2 y^{gap}_{t-2} + A^3 \epsilon^{ygap}_{t-1} + A^4 \epsilon^{ygap}_{t-2} + A^5 \epsilon^{ygap}_{t-3}\right) \\
&= A^1 dy^{gap}_t + A^2 dy^{gap}_{t-1} + A^3 \epsilon^{ygap}_t + \left(A^4 - A^3\right) \epsilon^{ygap}_{t-1} + \left(A^5 - A^4\right) \epsilon^{ygap}_{t-2} - A^5 \epsilon^{ygap}_{t-3}.
\end{aligned}$$

Plugging in $A_1 = F_2$, $A_2 = F_3$, $A_3 = P_1$, $A_4 = (P_2F_1 - F_2P_1)$, $A_5 = (P_3F_1 - F_3P_1)$ from the previous Lemma and rearranging terms then immediately establishes the result. □

Lemma 5. *In the model considered above, the conditional mean of the output gap is an ARMA(2,1) process satisfying*

$$E_t y^{gap}_{t+1} = F_2 E_{t-1} y^{gap}_t + F_3 E_{t-2} y^{gap}_{t-1} + P_2 F_1 \epsilon^{ygap}_t + P_3 F_1 \epsilon^{ygap}_{t-1}$$

Proof. We know that $y^{gap}_t = P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon^{ygap}_t$ and so

$$\begin{aligned}
E_t y^{gap}_{t+1} = \left(P_2\eta_t + P_3\eta_{t-1}\right) &= P_2\left(F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon^{ygap}_t\right) + P_3\eta_{t-1} \\
&= \left(P_2F_2 + P_3\right)\eta_{t-1} + P_2F_3\eta_{t-2} + P_2F_1\epsilon^{ygap}_t
\end{aligned}$$

which is an AR(2) in η_t. Accordingly, applying the first Lemma and rearranging terms, we know that $E_t y^{gap}_{t+1}$ will be an ARMA(2,2) process with the following coefficients

$$\begin{aligned}
E_t y^{gap}_{t+1} &= F_2 E_{t-1} y^{gap}_t + F_3 E_{t-2} y^{gap}_{t-1} + P_2F_1\epsilon^{ygap}_t \\
&\quad + \left(\left(P_2F_2 + P_3\right)F_1 - F_2P_2F_1\right)\epsilon^{ygap}_{t-1} + \left(P_2F_3F_1 - F_3P_2F_1\right)\epsilon^{ygap}_{t-2}
\end{aligned}$$

which after simplifying yields the ARMA(2,1) process above. □

Lemma 6. *In the model considered above, the conditional mean of the change in the output gap is an ARMA(2,2) process satisfying*

$$\begin{aligned}
E_t y^{gap}_{t+1} - y^{gap}_t &= F_2 E_{t-1} dy^{gap}_t + F_3 E_{t-2} dy^{gap}_{t-1} + \left(P_1 - F_1\right)\epsilon^{ygap}_t \\
&\quad + \left(F_1P_2 - F_2P_1 - F_1\right)\epsilon^{ygap}_{t-1} + \left(F_1P_3 - F_3P_1\right)\epsilon^{ygap}_{t-2}
\end{aligned}$$

Proof. We can combine the two previous results, namely

$$\begin{aligned}
y^{gap}_t &= A^1 y^{gap}_{t-1} + A^2 y^{gap}_{t-2} + A^3\epsilon^{ygap}_t + A^4\epsilon^{ygap}_{t-1} + A^5\epsilon^{ygap}_{t-2} \\
E_t y^{gap}_{t+1} &= B^1 E_{t-1} y^{gap}_t + B^2 E_{t-2} y^{gap}_{t-1} + B^3\epsilon^{ygap}_t + B^4\epsilon^{ygap}_{t-1} + B^5\epsilon^{ygap}_{t-2}
\end{aligned}$$

to find, after noting that $A^1 = B^1 = F^2$ and $A^2 = B^2 = F^3$, that

$$\begin{aligned}
E_t y^{gap}_{t+1} - y^{gap}_t &= F_2\left(E_{t-1} y^{gap}_t - y^{gap}_{t-1}\right) + F_3\left(E_{t-2} y^{gap}_{t-1} - y^{gap}_{t-2}\right) \\
&\quad + \left(B^3 - A^3\right)\epsilon^{ygap}_t + \left(B^4 - A^4\right)\epsilon^{ygap}_{t-1} + \left(B^5 - A^5\right)\epsilon^{ygap}_{t-2}
\end{aligned}$$

which, after plugging in for the remaining A^i and B^i from the previous Lemmas, completes the proof. □

Having characterized the laws of motion for $y^{gap}_t, E_t y^{gap}_{t+1}, dy^{gap}_t$ and $E_t dy^{gap}_{t+1}$ it will also be helpful to establish how these depend on η, as that will allow us to quickly compute their respective correlations with η_t and autocorrelations.

Lemma 7. *If*

$$\begin{aligned}
\eta_t &= F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon^{ygap}_t \\
y^{gap}_t &= P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon^{ygap}_t
\end{aligned}$$

then

$$\begin{aligned}
E_t y^{gap}_{t+1} &= \left(P_2F_2 + P_3\right)\eta_{t-1} + P_2F_3\eta_{t-2} + P_2F_1\epsilon^{ygap}_t \\
dy^{gap}_t &= P_2\eta_{t-1} + \left(P_3 - P_2\right)\eta_{t-2} - P_3\eta_{t-3} + P_1\epsilon^{ygap}_t - P_1\epsilon^{ygap}_{t-1} \\
E_t dy^{gap}_{t+1} &= \left(P_2F_2 + \left(P_3 - P_2\right)\right)\eta_{t-1} + \left(P_2F_3 - P_3\right)\eta_{t-2} + \left(P_2F_1 - P_1\right)\epsilon^{ygap}_t
\end{aligned}$$

Proof. Straight from the respective definitions, we have

$$\begin{aligned} E_t y_{t+1}^{gap} &= E_t\left(P_2\eta_t + P_3\eta_{t-1} + P_1\epsilon_t^{ygap}\right) \\ &= P_2\left(F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon_t^{ygap}\right) + P_3\eta_{t-1} \\ &= \left(P_2F_2 + P_3\right)\eta_{t-1} + P_2F_3\eta_{t-2} + P_2F_1\epsilon_t^{ygap} \end{aligned}$$

and

$$\begin{aligned} dy_t^{gap} &= y_t^{gap} - y_{t-1}^{gap} \\ &= P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon_t^{ygap} - \left(P_2\eta_{t-2} + P_3\eta_{t-3} + P_1\epsilon_t^{ygap}\right) \\ &= P_2\eta_{t-1} + \left(P_3 - P_2\right)\eta_{t-2} - P_3\eta_{t-3} + P_1\epsilon_t^{ygap} - P_1\epsilon_{t-1}^{ygap} \\ &= P_2\eta_{t-1} + \left(P_3 - P_2\right)\eta_{t-2} - P_3\eta_{t-3} + P_1\epsilon_t^{ygap} - P_1\epsilon_{t-1}^{ygap}. \end{aligned}$$

Using the result above we can then write

$$\begin{aligned} E_t dy_{t+1}^{gap} &= E_t\left(P_2\eta_t + \left(P_3 - P_2\right)\eta_{t-1} - P_3\eta_{t-2} + P_1\epsilon_{t+1}^{ygap} - P_1\epsilon_{t-1}^{ygap}\right) \\ &= P_2\left(F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon_t^{ygap}\right) + \left(P_3 - P_2\right)\eta_{t-1} - P_3\eta_{t-2} - P_1\epsilon_t^{ygap} \\ &= \left(P_2F_2 + P_3 - P_2\right)\eta_{t-1} + \left(P_2F_3 - P_3\right)\eta_{t-2} + \left(P_2F_1 - P_1\right)\epsilon_t^{ygap} \end{aligned}$$

which completes the proof. □

Remark 8. *It then immediately follows that*

$$\begin{aligned} cov\left(\eta_t, y_t^{gap}\right) &= E_t\eta_t\left(P_2\eta_{t-1} + P_3\eta_{t-2} + P_1\epsilon_t^{ygap}\right) \\ &= P_2\gamma(1) + P_3\gamma(2) + P_1F_1 \\ cov\left(\eta_t, E_t y_{t+1}^{gap}\right) &= E_t\eta_t\left(\left(P_2F_2 + P_3\right)\eta_{t-1} + P_2F_3\eta_{t-2} + P_2F_1\epsilon_t^{ygap}\right) \\ &= \left(P_2F_2 + P_3\right)\gamma(1) + P_2F_3\gamma(2) + P_2F_1^2 \\ cov\left(\eta_t, dy_t^{gap}\right) &= E_t\eta_t\left(P_2\eta_{t-1} + \left(P_3 - P_2\right)\eta_{t-2} - P_3\eta_{t-3} + P_1\epsilon_t^{ygap} - P_1\epsilon_t^{ygap}\right) \\ &= P_2\gamma(1) + \left(P_3 - P_2\right)\gamma(2) - P_3\gamma(3) + P_1F_1 - P_1F_2F_1 \\ cov\left(\eta_t, E_t dy_{t+1}^{gap}\right) &= E_t\eta_t\left(P_2F_2 + P_3 - P_2\right)\eta_{t-1} + \left(P_2F_3 - P_3\right)\eta_{t-2} + \left(P_2F_1 - P_1\right)\epsilon_t^{ygap} \\ &= \left(P_2F_2 + P_3 - P_2\right)\gamma(1) + \left(P_2F_3 - P_3\right)\gamma(2) + \left(P_2F_1 - P_1\right)F_1 \end{aligned}$$

Where $\gamma(i)$ is the i-th order autocovariance of η_t.

Remark 9. *Of course, since*

$$\eta_t = F_2\eta_{t-1} + F_3\eta_{t-2} + F_1\epsilon_t^{ygap}$$

therefore the autocovariances $\gamma(1)$, $\gamma(2)$ and $\gamma(3)$ are straightforward to compute. Furthermore, we can also solve for the first three autocorrelation coefficients $\tau(i), i \in \{1,3\}$ directly from

$$\tau(i) \equiv corr\left(\eta_t, \eta_{t-i}\right) = \frac{cov\left(\eta_t, \eta_{t-i}\right)}{\sqrt{var\left(\eta_t\right) var\left(\eta_{t-i}\right)}} = \frac{\gamma(i)}{\gamma(0)}$$

with

$$\tau(1) = \frac{F_2}{1 - F_3} \qquad \tau(2) = F_3 - \frac{F_2^2}{F_3 - 1} \qquad \tau(3) = \frac{-F_2^3 + F_2\left(F_3 - 2\right)F_3}{F_3 - 1}.$$

The companion Mathematica files contain all the underlying derivations.