Systemic Portfolio Diversification *

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Abstract

We study the implications of fire-sale externalities on balance sheet composition. Banks choose their asset holdings accounting for the liquidation costs incurred when they sell assets to manage their leverage. Our analysis highlights the fundamental trade-off between diversification at the bank and at the systemic level. While sacrificing diversification benefits to reduce portfolio commonality may increase the bank's idiosyncratic probability of liquidation, it also lowers the endogenous probability of a costly widespread sell-off. We show that higher heterogeneity in banks' leverage is socially beneficial because it gives banks stronger incentives in achieving systemic diversification. The socially optimal level of systemic diversification can be attained by taxing banks for creating interlinked balance sheets with high concentration on illiquid assets.

Key words: systemic diversification, leverage, fire-sale externalities, liquidity risk, illiquidity concentration.

JEL Classification: G01, G21, G38

1 Introduction

The classical paradigm in financial investment prescribes asset diversification as a means to minimize risk. Standard pre-crisis policies argued for the unlimited benefits of diversification, with little emphasis on balancing those against the downside risks of contagion. However, the global 2007-2009 financial crisis highlighted potential vulnerabilities resulting from balance sheet interconnectedness: in a crisis, investors exposed to the same shocked asset may be forced to simultaneously liquidate their positions in this asset. The liquidation of an asset carried out simultaneously by many financial institutions exacerbates losses for all investors involved in the sell-off.

Prior literature has analyzed banks deleveraging, and studied the feedback between tightening liquidity and falling asset prices during financial crises (e.g., Brunnermeier and Pedersen (2008),

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Khandani and Lo (2011), Manconi et al. (2012)). Fire-sale spillovers due to asset commonality among institutions have been recognized as a major source of systemic risk (e.g., Allen and Carletti (2013), Billio et al. (2012)). Concerns about fire-sale externalities led, for example, to the initiation of asset purchase programs under TARP by the U.S. Treasury and to the emergency cash bailout of Bear Stearns by J.P.Morgan Chase and the New York Federal Reserve in March, 2008.[1] All this indicates the importance of balancing asset diversification (optimal in isolation) with diversification of fire-sale risk *across financial institutions*.

A financial firm may mitigate the idiosyncratic risk of each asset by holding a diversified portfolio. This reduces the portfolio's variance, and therefore the firm's individual probability of asset liquidation. At the system level, instead, "systemic" diversification, i.e., the reduction of portfolio overlaps across different institutions, lowers the likelihood of concurrent asset liquidation and, therefore, of costly widespread sell-offs.

A decline in the price of a security may force financial investors to liquidate their holdings. For instance, the findings of Adrian and Shin (2008, 2014) indicate that banks sell risky assets during periods of market downturns to bring their Value-at-Risk in line with their available equity. Mutual funds may be forced to liquidate assets to repay investors who redeemed their shares in response to a price drop (Capponi et al. (2019)). Hedge funds may need to liquidate their positions to meet margin calls from their prime brokers (Khandani and Lo (2011)). Insurance companies must liquidate assets to cover for policyholders' losses after a natural disaster (Girardi et al. (2018)). While the exact constraints that prompt asset liquidation at fire-sale prices depend on the specific institutional structure, the underlying mechanism is similar.

We consider a financial system consisting of leveraged institutions (henceforth, called *banks*) subject to a leverage constraint: if after an initial market shock the asset value of a bank falls and its resulting leverage exceeds a given threshold, then the bank is required to liquidate assets to reduce its leverage ratio. Asset liquidation is costly, and imposes a downward pressure on prices proportional to the quantity that is being liquidated. A bank is then exposed to cross-agent externalities if its portfolio significantly overlaps with the portfolios of other banks facing similar constraints. Each bank chooses its asset holdings ex-ante, i.e., before the market shock is realized, accounting for potential vulnerability to fire-sale spillovers.

The proposed model highlights the mechanism through which systemic risk affects the banks' portfolio choice, and quantifies the externalities imposed by the banks on the system. We argue that systemic risk from fire-sale spillovers should play an important role on the banks' balance sheet decisions: a portfolio that is optimal for an agent in isolation may be far from optimal if cross-agent externalities are accounted for. Our analysis shows that even though banks reduce portfolio commonality to mitigate the risk of fire-sale spillovers, they do not reduce it enough relative to the social optimum. This result is a consequence of the fact that each bank only accounts for the costs

[1] The provision of liquidity by the Federal Reserve was taken to avoid a potential resale of nearly U.S. $210 billion of Bear Stearns' assets. The Chairman of the Fed, Ben Bernanke, defended the bail-in by stating that Bear Stearns' bankruptcy would have affected the economy, causing a "chaotic unwinding" of investments across the U.S. markets and a further devaluation of other securities across the banking system.

that other banks impose on it, but disregards the externalities imposed on the rest of the system through its own liquidation actions.

Our model allows assessing the impact of welfare enhancing policies. Any regulatory intervention based on the banks' current balance sheet allocations is subject to the Lucas critique, as banks may adapt to the new regulatory environment in unexpected and, potentially, socially damaging ways. Hence, policies should account for the banks' optimal response, i.e., their equilibrium asset allocations. Monetary policy tools such as asset purchase programs run by government or regulatory bodies may have the unintended consequence of incentivizing banks to hold excessively correlated financial exposures. For example, Acharya et al. (2010) argue that providing unconditional liquidity support to banks decreases their incentives to hold a liquid portfolio. We show how a tax on the banks' balance sheet interconnectedness may align the private optimum with the socially optimal asset allocation. The externality that a bank imposes on the system is increasing in the size of its balance sheet, the liquidity of its asset holdings, its leverage ratio, and the illiquidity-weighted portfolio overlap with other banks. This externality, and the corresponding Pigovian tax, is related to the systemicness of a bank, as defined in Greenwood et al. (2015): the tax is a weighted average of an adjusted version of the bank's systemicness over different asset price shocks.[2]

Our model predicts that a higher heterogeneity in the financial system reduces the expected aggregate liquidation costs. Even if each bank were to ignore fire-sale spillovers, it would still select asset holdings based on its current leverage because the latter determines the incentives of holding liquid assets. Therefore, banks with different leverages hold different portfolios. When banks account for fire-sale spillovers, the diversity in their portfolios becomes even higher because each bank runs away from the externalities imposed by others. From a policy perspective, these findings suggest that mergers of banks may have unintended consequences, and increase the fragility of the system. Consolidation would reduce the level of heterogeneity in the system and, as a result, the possibility of diversifying fire-sale risk across banks. While a single large bank may optimally choose its asset allocation, it may not be able to diversify its fire-sale risk. By contrast, in a system of two heterogeneous banks, each bank can adjust its portfolio to lower the likelihood of joint asset liquidation.

A growing financial literature analyzes the aggregate vulnerability of the banking system to fire-sale risk (e.g Greenwood et al. (2015),Capponi and Larsson (2015), and Duarte and Eisenbach (2018)). Unlike these studies, we do not take banks' portfolios as exogenously given. We frame the decision making problem of banks' portfolio selection as a game in which each bank maximizes the expected return of its own portfolio. Each bank's allocation decision affects the likelihood and magnitude of forced asset liquidations and the bank's contribution to systemic risk. We show that this game may be casted as a potential game, and therefore a Nash equilibrium can always be guaranteed to exist under mild conditions on the distribution of the initial asset price shocks.

[2]In the U.S., banks file the form FR Y-9C every quarter with the Federal Reserve, a report that collects their consolidated balance sheet data. This information allows a regulator to monitor common exposures in the financial system, infer –after accounting for size and leverage ratio– each bank's contribution to systemic risk, and impose a tax that makes the bank internalize such a contribution.

There are multiple economic forces that affect the Nash equilibrium of the game. First, portfolio diversification reduces each bank's likelihood of forced liquidation; second, highly leveraged banks have a stronger incentive to hold liquid assets as they are more vulnerable to fire sales; third, banks seek to reduce portfolio commonality to limit fire-sale spillover costs. Because of the intricate patterns of interactions between these three forces, the game may admit multiple equilibria.[3] We show that a financial system that is sufficiently homogeneous in both the banks' characteristics (size and leverage) and the assets' liquidity levels admits a unique equilibrium. In such a setting, the more leveraged bank adjusts its position towards the more liquid asset.

Literature Review

Existing literature has analyzed the implications on asset pricing and financial stability resulting from banks' leverage management. Adrian and Shin (2010) provide empirical evidence that banks react to asset price changes by actively managing their balance sheets. Greenwood et al. (2015) introduce a model to explain the propagation of shocks in a system of leverage-targeting banks with common asset holdings. They focus on the first-order effects of fire-sale losses caused by spillovers, and measure the contribution of each bank to the fragility of the systemCapponi and Larsson (2015) generalize their analysis and introduce the systemicness matrix to show that higher-order effects of fire-sale externalities can be substantial during periods of financial distress. Duarte and Eisenbach (2018) empirically study the historical vulnerability to fire-sale spillovers of American banks. "Illiquidity concentration", i.e., the concentration of illiquid assets among large and levered banks, is shown to have increased significantly up to early 2007. This measure demonstrates the importance of balance sheet linkages in the propagation of market shocks, and corroborates our claim that large banks should account for the portfolio composition of other systemically important banks.

Our work is related to earlier studies on counterparty risk networks. Acemoglu et al. (2015*a*) analyze the resilience to shocks of different network architectures. They conclude that a completely interconnected system, i.e., in which all institutions completely diversify their counterparty credit risk, may increase the fragility of the system if a large shock hits the network. A similar behavior is observed in the network of portfolio holdings, where two institutions share a link if their portfolios overlap: in an interconnected network multiple agents hold similar portfolios and, after a large market shock, they may all be forced to simultaneously sell assets, exacerbating the costs for all agents participating in the sell-off. While Acemoglu et al. (2015*a*) analyze an ex-post scenario where shocks have already hit the balance sheets of banks in the network, we consider an ex-ante scenario where the shock is yet to occur. Our network of portfolio holdings is endogenously determined by

[3]Consider a system with two assets and two banks, and assume that the more leveraged bank is significantly smaller. In one equilibrium, the more leveraged bank holds a larger position in the liquid asset than the bigger bank, because its incentive to hold the liquid asset is stronger. In another equilibrium, the lowly leveraged bank, which dominates in the system because of its size, adjusts its portfolio towards the liquid asset. The smaller bank, which now has a stronger incentive to run away from the externality imposed by the bigger bank than to hold the liquid asset, increases its holdings of the illiquid asset.

the equilibrium choice of banks.

Farboodi (2017) and Acemoglu et al. (2015*b*) consider endogenous intermediation and highlight the inefficiencies arising from overexposure to counterparty risk by banks which make risky investment. Our work shows that, even in the absence of direct credit linkages, banks are exposed to excessive systemic risk because in equilibrium they hold portfolios that are too similar.

Our study is connected to to that of Wagner (2011), who studies the trade-off between diversity and diversification in financial exposures. While in our model, it is balance sheet heterogeneity that incentives banks to reduce portfolio overlaps, in the model by Wagner (2011) even if agents are identical their portfolios are not. The reason is that he considers a continuum of agents, each of infinitesimal size, and therefore each agent is not subject to any liquidation cost if it sells assets while all others do not. Furthermore, because in our model agents are large, it is possible to quantify the externalities that each bank imposes on the rest of the system and analyze policy implications.[4]

The tax on systemic risk we propose brings analogies with that considered in the study by Acharya et al. (2017). They construct an aggregate indicator for the occurrence of a systemic crisis, which is exogenously specified in terms of total assets and capital of the banks in the system. Each bank is charged a tax in the amount equal to the share of expected aggregate loss it generates during the crisis scenario. Different from their top-down approach, we infer the tax directly from the banks' balance sheet information: the tax amount is equal to the endogenous cost that each bank imposes on the rest of the system due to simultaneous asset liquidation, rather than being determined via an exogenously defined systemic event.

The rest of the paper is organized as follows. We introduce the model primitives and the assumptions in Section 2. We describe the game theoretical model of strategic banks' holdings in Section 3. We solve for the Nash equilibrium and discuss its properties in Section 4. We study the social planner problem and discuss policy implications in Section 5. We discuss operational challenges in Section 6. Section 7 concludes the paper. Technical proofs are deferred to the Appendix.

2 Model Setup

We consider a two-period economy consisting of K assets and N banks. Let d_i be the initial debt of bank i, and e_i its initial equity. Denote by $w_i := d_i + e_i$ the total initial asset value of bank i. The leverage ratio of bank i is

$$\lambda_i := \frac{d_i}{e_i},$$

which may be equivalently rewritten as $d_i = \frac{\lambda_i}{1+\lambda_i} w_i$. Each bank wishes to maintain its leverage below the target threshold $\lambda_{M,i}$, i.e., $\lambda_i \leq \lambda_{M,i}$ for every i.[5]

[4]Other studies have considered different settings to investigate the dangers and social costs of asset diversification, e.g., Shaffer (1994) and Ibragimov et al. (2011).

[5]Empirical findings of Adrian and Shin (2008, 2014) and Greenlaw et al. (2008) confirm that banks manage their leverage based on internal value at risk models.

At date 1, each bank chooses its asset allocation. Denote by $\pi_{i,k}$ the weight of asset k in bank i's portfolio. Portfolio weights are positive and satisfy the relation $\sum_{k=1}^{K} \pi_{i,k} = 1$ for $i = 1, \cdots, N$.

At date 2, each asset k is subject to a return shock Z_k. Hence, the return of bank i's portfolio is $R_i = \pi_i^T Z$, where $\pi_i := (\pi_{i,k})_{1 \leq k \leq K}$ is the vector of bank i's weights and $Z := (Z_k)_{1 \leq k \leq K}$ is the vector of shocks, which are assumed to be identically distributed. Ex-post, the leverage ratio of bank i is

$$\lambda_i^{post} = \frac{d_i}{w_i(1 + R_i) - d_i}.$$

Assumption 2.1. *After a shock, banks whose leverage exceeds the threshold liquidate the minimum amount of assets needed to restore the leverage threshold.*

If $\lambda_i^{post} \geq \lambda_{M,i}$, then bank i sells assets and uses the cash proceeds from the sale x_i to repay its debt. To attain the leverage threshold, x_i needs to satisfy the relation $\frac{d_i - x_i}{w_i(1+R_i)-d_i} = \lambda_{M,i}$. The focus of the paper is on large liquidity shocks, and it is well known that raising equity during distressed market conditions is prohibitively costly. Despite selling assets in a depressed market environment is also difficult, empirical evidence provided by Adrian and Shin (2008) –see the scatter plot in Figure 6 therein– indicates that firms manage leverage primarily through adjustments in the size of debt (e.g. through asset disposals), leaving equity unchanged, rather than through direct changes in equity.[6]

We remark that Greenwood et al. (2015) assume that banks target their leverage, i.e., that they immediately sell assets to return to their initial leverage ratio. In our model, it is only the assets required by the bank to meet its leverage threshold that are liquidated at discounted prices. A bank that intends to restore its initial leverage ratio may do so on a longer time scale and incur lower execution costs.

The mechanism that induces financial firms to liquidate assets in distressed market conditions depends on their institutional structure. While excessive leverage may trigger forced sales in the banking industry, different constraints play a similar role in other type of financial institutions. For instance, investors in open-end mutual funds may withdraw their deposits in response to negative returns. If the number of redeeming investors is too high, and the cash buffer held by the fund is not sufficient to meet redemption requests, the fund is forced to sell assets (see Capponi et al. (2019)). Even though we present the model for leveraged banking institutions, its implications hold for any financial firm subject to a constraint that gets violated when asset values significantly decline, and which requires asset liquidation to be satisfied again.

Assumption 2.2. *Banks liquidate assets proportionally to their initial allocation.*

As in Greenwood et al. (2015) and Duarte and Eisenbach (2018), we assume that if bank i needs to raise a total amount of cash x, then it liquidates $\pi_{i,k}x$ for each asset k. This assumption may be interpreted as a stationarity condition on the composition of the banks' portfolio: in a hypothetical

[6]If banks use a combination of equity issuance and asset liquidation to reduce leverage, the size of fire-sale externalities would be lower but our qualitative conclusions would remain unaltered.

multi-period model, the proportional liquidation strategy would yield a terminal portfolio that is close to the initial portfolio, and therefore still resilient to subsequent market shocks. We remark here that there is no agreement in the empirical literature on the liquidation strategy adopted by financial firms when they liquidate assets. On the one hand, selling liquid assets first reduces the cost of fire sales. On the other hand, holding liquid assets carries an option value because of the prospectus that markets may become more illiquid in the future (see Ang et al. (2014)). Because of the lack of conclusive evidence on what force dominates in this trade-off, we consider the case of proportional liquidation.[7]

Asset liquidation is costly. If a large return shock causes the bank's leverage to exceed the threshold, the bank needs to readjust its positions and revert its leverage ratio to the threshold. If assets are liquidated on a very short notice, then the bank may need to sell them at discounted prices relative to their fundamental values, i.e., a fire sale would occur. The initial price of each asset k is normalized to one dollar. If the aggregate amount of asset k that banks liquidate is q_k, the execution price per share of the asset is

$$p_k^{post} := 1 + Z_k - \gamma_k q_k,$$

where $\gamma_k > 0$ is the price impact parameter of asset k. The limiting case $\gamma_k \downarrow 0$ corresponds to the case of a perfectly liquid asset.

Our model abstracts from the underlying source of market illiquidity, and captures the knocked down effect of sales on prices in reduced form through the parameter γ. The price impact function can be viewed as a representation of outside investors with limited capital and other investment opportunities. This form of price impact function captures the mechanics of typical theoretical models of fire sales, as explained next. Suppose outside investors with a fixed dollar amount of outside wealth were to step in and provide liquidity to the banking sector during a fire sale. These investors would then face a trade-off between the returns from investing in new projects and the gains from purchasing assets sold by banks at fire-sale prices; see also Shleifer and Vishny (2011) for a related discussion.

The total amount of shares of asset k that banks liquidate is

$$\sum_{j=1}^{N} \pi_{j,k} x_j \chi_{\{\lambda_j^{post} > \lambda_{M,j}\}},$$

where $\chi_{\{\lambda_j^{post} > \lambda_{M,j}\}}$ is the indicator function of the event $\{\lambda_j^{post} > \lambda_{M,j}\}$. Hence, the liquidation

[7] Greenwood et al. (2015) and Duarte and Eisenbach (2018) also consider –as an alternative to the proportional liquidation strategy– a pecking order of liquidation where banks first sell off their most liquid assets. They show that in a calibrated model of fire-sale spillovers this strategy reduces the magnitude of fire-sale losses. In the same context of leverage targeting, Capponi and Larsson (2015) show analytically that fire-sale externalities are smaller if banks first sell liquid and then illiquid assets. In a two-period game-theoretical model, forcing banks to follow a pecking order strategy may lead to counterintuitive results: all banks would simultaneously first sell the most liquid asset, making its liquidation costly. Some banks may therefore prefer not to hold any share of the most liquid asset only as an artificial consequence of the pecking order constraint.

cost per share of asset k is $\gamma_k \sum_{j=1}^N \pi_{j,k} x_j \chi_{\{\lambda_j^{post} > \lambda_{M,j}\}}$. Let $Diag[\gamma]$ be the diagonal matrix with entries γ_k on the diagonal. Then the total liquidation cost incurred by bank i at time 2 is

$$cost_i(\pi_i, \pi_{-i}, Z) := \underbrace{x_i \chi_{\{\lambda_i^{post} > \lambda_{M,i}\}} \pi_i^T}_{\text{assets liquidated by bank } i} \quad \text{Diag}[\gamma] \quad \underbrace{\sum_{j=1}^N \pi_j x_j \chi_{\{\lambda_j^{post} > \lambda_{M,j}\}}}_{\text{total quantities traded}},$$

where $\pi_{-i} := (\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_N)$.

At date 1, bank i chooses the portfolio weights that maximize the expected return of its portfolio at date 2. We outline the technical assumptions A.1–A.4 in the Appendix. We refer to them whenever they are required in our results.

3 Equilibrium Asset Allocations

The banks' portfolio allocations are described by a game theoretical model, in which the N banks are the players. The space of strategies is the set $X := \{x \in [0,1]^K : \sum_{k=1}^K x_k = 1\}$ of admissible portfolio weights. Each bank maximizes an objective function given by its expected portfolio return, i.e.,

$$\text{PR}_i(\pi_i, \pi_{-i}) := E[\pi_i^T Z - \text{cost}_i(\pi_i, \pi_{-i}, Z)].$$

A Nash equilibrium is a set of banks' asset allocation decisions $\pi^* := (\pi_1^*, \cdots, \pi_N^*)$ such that no bank has any incentive to unilaterally deviate from it, i.e., $\text{PR}_i(\pi_i^*, \pi_{-i}^*) \geq \text{PR}_i(\tilde{\pi}_i, \pi_{-i}^*)$, for any i and strategy $\tilde{\pi}_i$ of bank i.

Because asset returns are identically distributed, it holds that $E[\pi_i^T Z] = E[Z_1]$. Hence, for each bank maximizing its expected portfolio return is equivalent to minimizing its expected total liquidation cost $\text{EC}_i(\pi_i, \pi_{-i}) := E[\text{cost}_i(\pi_i, \pi_{-i}, Z)]$. Bank i's liquidation cost function EC_i may be rewritten as $S_i(\pi_i) + \sum_{j \neq i} M_{i,j}(\pi_i, \pi_j)$, where

$$S_i(\pi_i) := E\left[\pi_i^T \text{Diag}[\gamma] \pi_i x_i^2 \chi_{\{\lambda_i^{post} > \lambda_{M,i}\}}\right],$$
$$M_{i,j}(\pi_i, \pi_j) := E\left[\pi_i^T \text{Diag}[\gamma] \pi_j x_i x_j \chi_{\{\lambda_i^{post} > \lambda_{M,i}\} \cap \{\lambda_j^{post} > \lambda_{M,j}\}}\right].$$

The term S_i is the idiosyncratic component of the expected liquidation cost incurred by bank i. Such a cost is due to the price dislocation caused by bank i's asset sales, and would be incurred even in the absence of other banks in the system. The term $M_{i,j}$ is the systemic component of the liquidation cost, i.e., the additional cost incurred by bank i due to the presence of bank j in the system. This term captures the externality that bank j imposes on bank i due to their overlapping portfolios.

Fix the asset holdings π_{-i} of all other banks except i. Bank i's optimization problem is equiv-

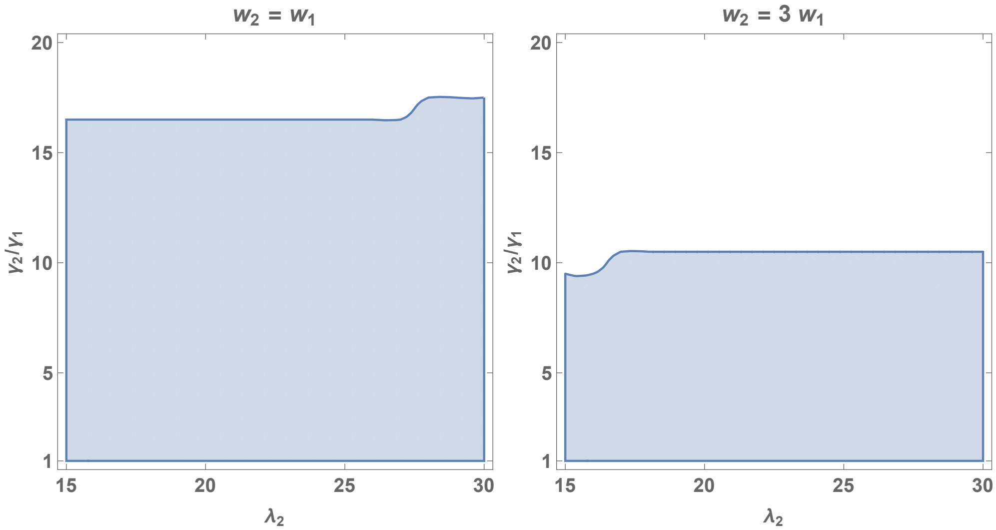

Figure 1: The region of parameters in which the function $P(\pi)$ is convex for $N = 2$, $K = 2$. The leverage thresholds $\lambda_{M,1}$, $\lambda_{M,2}$ are set to 30 and λ_1 is set to 15.

alent to choosing the portfolio weight vector π_i that minimizes

$$P(\pi) := \sum_{m=1}^{N} \left(S_m(\pi_m) + \sum_{j<m} M_{m,j}(\pi_m, \pi_j) \right).$$

Because each bank minimizes the same objective function, the problem can be formulated in terms of a *potential game*, where $P(\pi)$ is the potential function.

Proposition 3.1. *The game specified by (i)-(iii) is a potential game. Moreover, if Z is a continuous random variable with values in $\mathbb{R}^K$, the game admits a Nash equilibrium.*

Even though a Nash equilibrium exists, its uniqueness cannot be always guaranteed. This is due to the complex trade-offs faced by each bank in the system. On the one hand, diversification reduces the likelihood of asset liquidation. On the other hand, concentration on liquid assets and avoidance of portfolio overlapping with other banks reduces the realized liquidation costs. The multiple economic forces that drive allocation decisions imply that the bank's optimization problem is in general non-convex. Hence, the game may admit multiple equilibria. If the system is sufficiently homogeneous, then the incentives to diversify and hold a liquid portfolio are sufficiently aligned, and the game admits a unique equilibrium (see Theorem 3.2). In Figure 1 we illustrate that the equilibrium is unique if the economy is not too heterogeneous.

Proposition 3.2. *Let $N = 2$, $K = 2$. Under Assumption A.1, for any $\lambda, \lambda_M, w, \gamma > 0$ there exist $\lambda_* < \lambda < \lambda^*$, $\lambda_{M,*} < \lambda < \lambda_M^*$, $w_* < w < w^*$, $\gamma_* < \gamma < \gamma^*$ such that if $\lambda_i \in (\lambda_*, \lambda^*)$, $\lambda_{M,i} \in (\lambda_{M,*}, \lambda_M^*)$, $w_i \in (w_*, w^*)$ for $i = 1, 2$ and $\gamma_k \in (\gamma_*.\gamma^*)$ for $k = 1, 2$, then there exists a unique Nash equilibrium.*

4 The Nash Equilibrium of Banks' Portfolio Holdings

We start by introducing the distance to liquidation, a convenient reparameterization of the leverage ratio that will be convenient for the analysis conducted in this section.

Definition 4.1. *The distance to liquidation of bank i is $\ell_i := \frac{\lambda_{M,i}-\lambda_i}{(1+\lambda_i)\lambda_{M,i}}$, for $i = 1, \cdots, N$.*

Distance to liquidation can be viewed as a rescaling of the leverage ratio in the units of portfolio returns. A highly leveraged bank that breaches its leverage threshold even for a moderate decrease of its asset value has a low distance to liquidation. In contrast, a bank with a low leverage ratio has a large distance to liquidation. More precisely, the distance to liquidation quantifies the minimal portfolio return that the bank can absorb without needing to raise cash through asset liquidation. For instance, if a bank has distance to liquidation equal to 5%, then it would be forced to liquidate assets if its portfolio return drops below -5%.

The optimal portfolio allocation of a bank in isolation, i.e., if $N = 1$, provides a benchmark for analyzing the impact of the system on the equilibrium allocation. If either all assets are equally liquid or all banks have the same distance to liquidation, then the presence of other banks in the system does not impact the portfolio holdings of an individual bank. However, in the case of a heterogeneous financial system, each bank seeks to run away from the systemic externalities by reducing its portfolio overlap with other banks.

4.1 Single Bank Benchmark

In the absence of systemic externalities, the bank's portfolio allocation decision is driven by two main forces: the likelihood of breaching the leverage threshold and the total execution costs from the bank's liquidation strategy. If all assets are equally liquid, the individual minimization of these two criteria yields the same outcome: complete diversification is optimal (see Proposition 4.2 for the formal statement). First, the portfolio's variance –hence the probability of violating the leverage threshold– is minimized when the portfolio is fully diversified. Second, the marginal cost of asset liquidation is increasing in the quantity that is sold, therefore liquidating smaller positions in multiple assets results in a lower cost than liquidating a large position in a single asset.

Proposition 4.2. *Let $N = 1$ and $\gamma_k = \gamma$ for each k. Under Assumption A.1, the optimal allocation is $\pi^S_{1,k} = \frac{1}{K}$ for all k.*

Next, we analyze an economy consisting of assets with different levels of liquidity.Observe that the probability of violating the leverage constraint does not depend on the liquidity of the assets, but only on the distribution of assets' returns. On the one hand, a portfolio with equal asset weights minimizes the likelihood of forced liquidation because returns are identically distributed. On the other hand, to reduce the total costs in the event of a forced sale the bank should allocate a larger portion of its wealth to the most liquid asset. More precisely, in a market consisting of two assets with liquidity parameters $\gamma_1 < \gamma_2$, the optimal liquidation policy is attained when the marginal cost from the sale of each asset is identical. In other terms, if a bank has to liquidate assets, it is optimal to sell a proportion x of the first asset and $1-x$ of the second asset, where x satisfies the indifference condition $1+\gamma_1 x = 1+\gamma_2(1-x)$, i.e., $x = \frac{\gamma_2}{\gamma_1+\gamma_2}$. Because of the trade-off between these two forces, the optimal portfolio weight $\pi^S_{1,1}$ lies in the interval $(\frac{1}{2}, \frac{\gamma_2}{\gamma_1+\gamma_2})$.

Proposition 4.3. *Let $N=1$, $K=2$, and $0<\gamma_1<\gamma_2$. Assume that $S_1(\pi_1)$ is convex. Under Assumption A.1, it holds that $\pi^S_{1,1} \in (\frac{1}{2}, \frac{\gamma_2}{\gamma_1+\gamma_2})$, where $(\pi^S_{1,1}, 1-\pi^S_{1,1})$ minimizes the function S_1 on X. Furthermore, $\pi^S_{1,1}(\ell)$ is a decreasing function of the distance to liquidation ℓ.*

Proposition 4.3 also states that banks with different distances to liquidation weigh these two forces differently, and therefore their optimal portfolio choice is different. A highly leveraged bank, i.e., with low distance to liquidation, is more likely to breach the leverage threshold regardless of its portfolio allocation, and should therefore aim at reducing the costs of its asset liquidation strategy. Vice versa, a bank with a low leverage ratio is less concerned about its realized liquidation costs, and constructs a portfolio that is more diversified but less liquid.

4.2 Homogeneous Economy

Consider an economy that is homogeneous either across assets –all assets are equally liquid– or across banks –all banks have the same distance to liquidation–. Then, banks hold identical portfolios in equilibrium. Furthermore, as formalized in the next Proposition, this portfolio is the same as in the single bank benchmark where each bank does not account for the liquidation actions of other banks.

Proposition 4.4. *Under Assumptions A.1 and A.2:*

(1) If $\gamma_k = \gamma$ for each k, then $\pi^*_{i,k} = \frac{1}{K}$ is the unique Nash equilibrium.

(2) Let π^S be the vector of optimal weights determined in Proposition 4.3 when the distance to liquidation is ℓ. If $\ell_i = \ell$ for each i, then $\pi^*_i = \pi^S$ is the unique Nash equilibrium.

While the presence of other banks holding the same portfolio π^* results in higher expected liquidation costs, it does not alter the portfolio holdings of each individual bank. To see this, consider two banks with equal distance to liquidation in an economy with two assets. The optimal portfolio π^* that each bank holds if it were the only institution in the economy is such that the expected marginal liquidation costs for each asset are identical. If this were not the case, the bank would invest more in the asset with lower marginal cost to reduce the total expected costs. However, in an economy with two banks, each bank also accounts for the externalities imposed by the other bank. These externalities can be decomposed across assets: a higher portfolio weight in an asset implies a larger externality resulting from the liquidation of that asset. If both banks hold the same portfolio π^*, the additional expected liquidation cost that each bank bears due to the presence of the other bank is the same for each asset. In particular, the expected marginal liquidation costs for each asset, after accounting for the extra cost imposed by the other bank, are also identical.

4.3 Heterogeneous Economy

In this section, we consider an economy in which there is heterogeneity both with respect to assets' illiquidity and banks' leverage. Then the systemic externalities arising from joint liquidation affect the banks' optimal portfolio allocations. As argued in Section 4.1, a bank whose leverage is closer to the threshold values liquid assets more than a bank with higher distance to liquidation. In an economy consisting of two assets with different liquidity, the bank with higher distance to liquidation holds a more diversified portfolio and allocates a higher proportion of wealth to the more illiquid asset relative to the more leveraged. Hence, the presence of the bank with lower leverage contributes to increase the costs of holding the illiquid asset for the highly leveraged bank, which in turn readjusts its portfolio to increase its position in the liquid asset even further. Analogously, the bank with lower leverage shifts its portfolio towards the less liquid asset. Both banks adjust their positions further and the prevailing Nash equilibrium is the outcome of this process. In each step of this iterative procedure, each bank runs away from the externalities imposed by the other bank in the economy.

Theorem 4.5 states that banks reduce portfolio overlapping, and therefore cross-bank externalities, when they account for the presence of other banks in the economy.

Theorem 4.5. *Let $N = 2$, $K = 2$. Assume $\gamma_1 < \gamma_2$ and $\ell_1 < \ell_2$. Under Assumptions A.1, A.2, and A.3, $|\pi^*_{1,1} - \pi^*_{2,1}| > |\pi^S_{1,1} - \pi^S_{2,1}|$, where $\pi^S_{i,1}$ is bank i's optimal allocation in asset 1 under the single bank benchmark.*

Taken together, Proposition 4.4 and Theorem 4.5 show that it is the heterogeneity in the financial system that gives banks incentives to reduce their common exposures. We provide a graphical illustration of this phenomenon in Figure 2. When assets have the same liquidity, i.e., $\gamma_1 = \gamma_2$, all banks hold the same perfectly diversified portfolio. If the assets have different liquidities, then each bank's holdings would not be equally split across assets, even in the single bank benchmark. Cross-bank externalities increase diversity in banks' asset holdings, because banks seek to reduce their portfolio commonality. A similar mechanism arises if we consider heterogeneity in banks' initial leverages: assuming assets have different liquidity levels, banks hold identical portfolios if they are equally levered and thus have the same distance to liquidation. Vice versa, if banks have initially different leverage ratios, they reduce their common exposures significantly compared to the optimal allocations in their corresponding single bank benchmark (see the right panel in Figure 2).

Even though, prior to the global 2007-2009 financial crisis, banks may not have fully accounted for systemic externalities created by portfolio commonality, empirical evidence suggests that they have accounted for the risk of fire-sale spillovers in more recent years. Duarte and Eisenbach (2018) define a measure of portfolio overlap on illiquid assets by large leveraged banks, called "illiquidity concentration". This measure has increased steadily until 2007 and started to drop in 2013. Such a decrease in illiquidity concentration means either a higher awareness of portfolio contagion or an effectiveness of proposed regulatory measures.

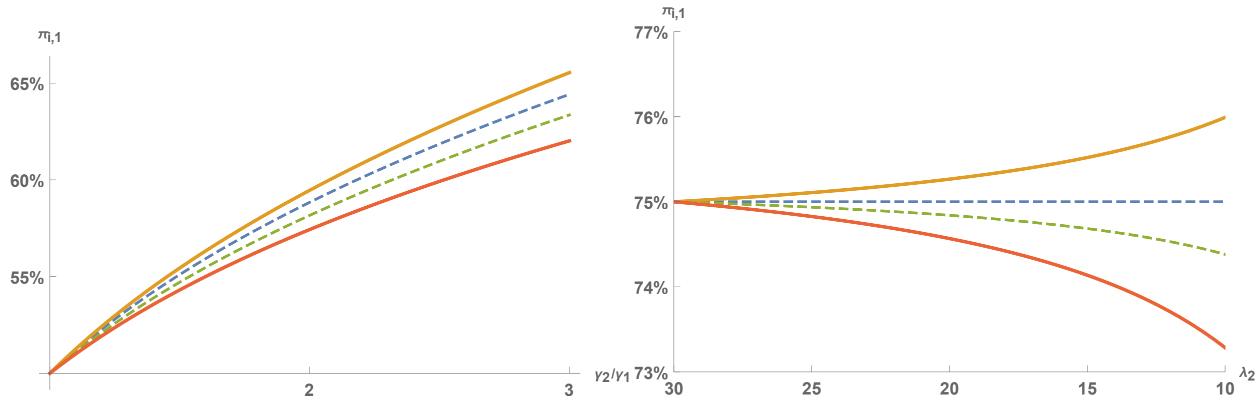

Figure 2: Banks' portfolio allocations in asset 1 in an economy consisting of two assets and two banks. We increase heterogeneity across assets (left panel) and across banks (right panel). Solid lines represent the allocations in the two-bank economy (orange for bank 1, red for bank 2), dashed lines the allocations of each bank in the single-bank benchmark (blue for bank 1, green for bank 2). We fix $\lambda_{M,1} = \lambda_{M,2} = \lambda_1 = 30$. In the left panel, we choose $\lambda_2 = 5$. In the right panel, we choose $\gamma_2/\gamma_1 = 6$.

The probability of joint asset liquidation is determined endogenously, as it depends on the portfolio holdings of each bank in the system. Consider, for example, an economy consisting of two banks and two asset, in which bank 1 only holds asset 1 and bank 2 only holds asset 2. Joint asset liquidation occurs only when both assets are simultaneously hit by large price shocks. If only one asset is hit by a shock, then only one bank will be forced to liquidate assets. By contrast, if two equally leveraged banks hold the same diversified portfolio, then either both banks would be forced to liquidate assets or neither of them would. This leads to a fundamental trade-off between asset diversification on the individual firm level and systemic portfolio diversification. Define $A_{\text{liq}}(\pi_1, \pi_2)$ to be the event that at least one bank liquidates assets when bank 1 and bank 2 hold, respectively, portfolio π_1 and portfolio π_2. Define $A_{\text{sim}}(\pi_1, \pi_2)$ as the event that both banks liquidate assets.

Proposition 4.6. *Let $N = 2$, $K = 2$. Assume $\gamma_1 < \gamma_2$ and $\ell_1 < \ell_2$. Under Assumptions A.1 and A.3, $P(A_{liq}(\pi_1^*, \pi_2^*)) > P(A_{liq}(\pi_1^S, \pi_2^S))$ and $P(A_{sim}(\pi_1^*, \pi_2^*)) < P(A_{sim}(\pi_1^S, \pi_2^S))$.*

If neither bank accounts for the externalities imposed by the other, the probability of simultaneous liquidation increases because of the larger portfolio overlap. As banks reduce their portfolio commonality, the bank with higher leverage further increases its exposure to the more liquid asset, resulting in a less balanced portfolio. Therefore, it has a higher probability of liquidating after a shock. In contrast, the likelihood that the bank with lower leverage liquidates is smaller, because such a bank has reduced its position in the more liquid asset and therefore holds a more balanced, even though less liquid, portfolio. The net effect on the economy is described in Proposition 4.6: on the one hand, the probability that *some* bank sells assets to reduce its leverage increases if banks account for systemic externalities. On the other hand, the probability of a systemic asset sell-off, in which *both* banks liquidate assets, is lower. Hence, a new concept of systemic diversification arises: in equilibrium the system diversifies the likelihood of asset liquidation across banks, reducing the

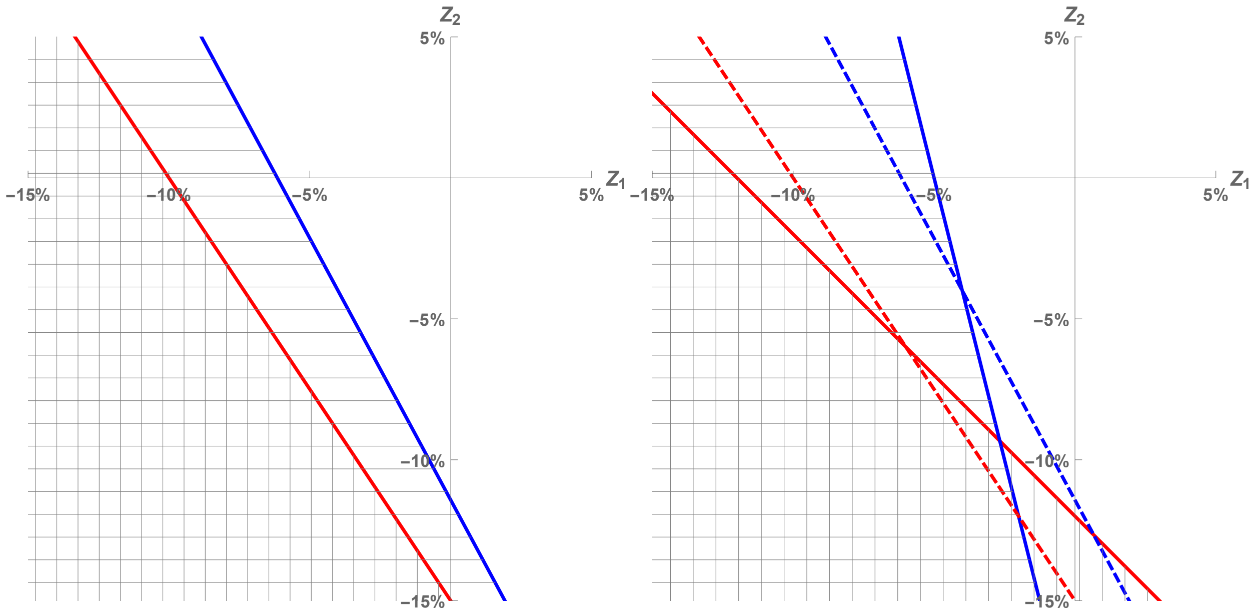

Figure 3: The horizontal lines identify the asset returns for which bank 1 liquidates assets, and the vertical lines the asset returns for which bank 2 liquidates assets. Left panel: solid colored lines represent the portfolio composition of each bank in the single bank benchmark. Right panel: solid colored lines represent the portfolio composition of each bank in equilibrium, and dashed colored lines represent the portfolio composition in the single bank benchmark. An asset price shock is less like to fall in the region where both banks liquidate in the right panel.

probability of a widespread fire-sale event. Figure 3 illustrates the event of joint liquidation, both for the case that they account for systemic externalities and they do not. The probability that the realized price shock falls in the area where both banks simultaneously liquidate is lower if banks do account for these externalities.

4.4 A System with Multiple Equilibria

In general, the uniqueness of a Nash equilibrium cannot be guaranteed even in an economy consisting of only two banks and two assets. This is because the cost function is not necessarily convex in the portfolio weights. To understand why this is the case, consider two banks of significantly different size: a highly leveraged small bank and a lowly leveraged large bank. Because the externalities imposed by the small bank on the large bank are small relative to the size of the latter bank, the large bank's portfolio allocation is close to the one prevailing in an economy where it is the only bank. The small bank, instead faces the following trade off. First, the small bank would like to reduce the likelihood of exceeding the leverage threshold, and hence it aims at holding a fully diversified portfolio. Second, the small bank, being highly leveraged, has a stronger incentive to hold the liquid asset because this lowers the cost of asset liquidation. Third, to minimize its portfolio overlap with the large bank, the small bank may significantly increase its position in either asset. Hence, there exists a region of the parameter space in which two equilibria are possible: the small

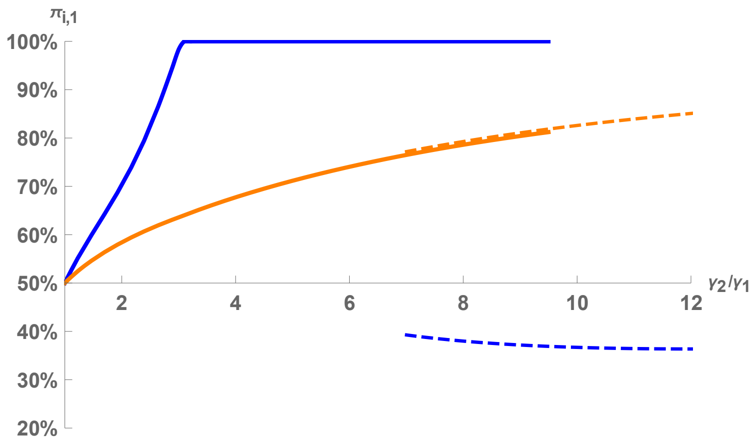

Figure 4: Banks' portfolio allocations in asset 1 for a market of two banks and two assets as a function of the assets' heterogeneity. Solid lines represent the allocations in one equilibrium (orange for bank 1, blue for bank 2), dashed lines the allocations of each bank in another equilibrium. We fix $\lambda_{M,1} = \lambda_{M,2} = 30$, $\lambda_1 = 10$, $\lambda_2 = 25$, $w_1 = 50w_2$. There exist two prevailing equilibria in the γ_2/γ_1 interval [7, 9.5].

bank may run away from the externalities imposed by the large bank by either holding a much higher position in the liquid asset compared to the one it would hold in the absence of the large bank, or shifting its portfolio towards the illiquid asset. We illustrate these two potential outcomes in Figure 4. If the difference in liquidity of the two assets increases, the large bank also holds a significant position in the liquid asset thus further increasing portfolio overlap. If the expected liquidation costs from the presence of the large bank are prohibitively high, it is preferable for the small bank to load more on the illiquid asset and therefore reduce its portfolio overlap with the large bank. In the example of Figure 4, for values of γ_2/γ_1 that belong to the interval [7, 9.5], two Nash equilibria exist.

4.5 A Multi-Bank Multi-Asset Economy

Most of the results discussed in the paper focus on an economy consisting of two banks and two assets. Precise mathematical statements are difficult to make in an economy consisting of several banks holding multiple assets. In this section, we provide numerical evidence that the qualitative implications of the model remain valid in such an extended economy.

Consider, first, an economy of two banks which hold three types of assets, each identified by a different level of liquidity, as exemplified in Figure 5. If both banks have the same distance to liquidation, they hold the same portfolio. If, instead, bank 1 has a lower distance to liquidation than bank 2, then bank 1 has a stronger incentive to hold the most liquid asset. If asset 1 is significantly more liquid than the other two assets, bank 1 forgoes the benefits of diversification: it loads more on its most liquid asset, and decreases its holdings in either of the other two assets. As bank 1 increases its position in asset 1, this asset becomes more costly to hold for the other bank, due to the potential fire-sale externalities imposed by bank 1 on bank 2. Hence, bank 2 reduces its exposure to asset 1 and concurrently increases its positions in the less liquid assets.

The presence of other banks in the system alters each institution's portfolio composition. In fact, banks reduce their portfolio overlap in *each* asset. The bank that is more likely to exceed the leverage threshold increases its holdings of the most liquid asset and reduces its holdings of the least liquid asset. How much the bank alters its holdings of the medium liquid asset crucially depends on the liquidity of this asset relative to the most and least liquid assets. Suppose that in the three-asset economy, the two most liquid assets have close (or equal) levels of liquidity. Then the bank with lower distance to liquidation, i.e., bank 1, increases its position in both of the two most liquid assets. Hence, bank 1 diversifies across these two assets, and increases the overall liquidity of its portfolio. Conversely, bank 2 only increases its exposure to the most illiquid asset. If instead, the two most illiquid assets have similar illiquidity levels, bank 1 only increases its position in the most liquid asset. In fact, there exists an illiquidity level γ^* of the medium liquid asset such that each bank maintains the same exposure to it as in the single-bank benchmark. To summarize, in an economy consisting of multiple assets, banks behave as in a two-asset economy: bank 1 increases its holdings of all assets whose illiquidity parameter is lower than γ^*, and reduces its holdings in the remaining assets. Bank 2 instead increases its holdings of all assets with illiquidity parameter higher than γ^*.

Next, consider an economy of three banks which hold two types of assets, as shown in Figure 6. The bank with lowest distance to liquidation, i.e., bank 1, has the strongest incentive to hold the more liquid asset. Bank 1 increases its position in the liquid asset relative to both the single-bank benchmark and the two-bank economy. All other banks maintain a lower position in the more liquid asset than bank 1, even in the single-bank benchmark. Hence, by holding an even higher position in the more liquid asset, bank 1 reduces the portfolio overlap with all other banks. Because the externalities are stronger in the presence of many other banks, bank 1's holdings of the more liquid asset are higher even relative to the case of a two-bank economy. Conversely, the bank with highest distance to liquidation, i.e., bank 3, holds higher quantities of the less liquid asset than if it were in a two-bank economy.

There are two opposing forces which drive the portfolio allocation of bank 2, i.e., the bank with intermediate distance to liquidation, when it accounts for the externalities imposed by other banks in the system. To reduce portfolio overlap with bank 1, bank 2 should increase its position in the less liquid asset. Instead, to reduce overlapping with bank 3's portfolio, bank 2 should increase its position in the more liquid asset. The aggregate impact of these externalities on bank 2's portfolio depends both on balance sheet characteristics of the other two banks, and on the relative liquidity of the assets.

5 First-Best Allocations

An ample literature has discussed the systemic risk implications of fire sales (see, for example, Shleifer and Vishny (2011) and Schwarcz (2008)). Persistent price-drops may lead investors to

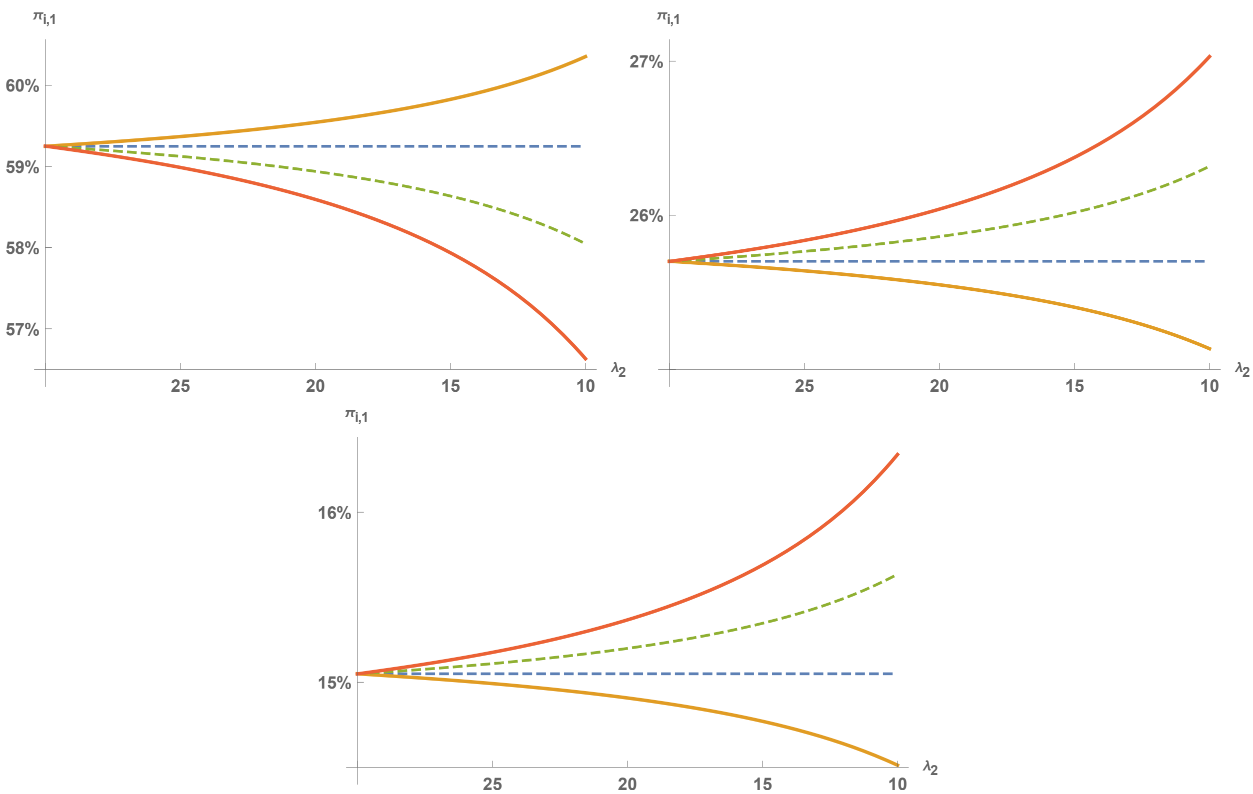

Figure 5: Banks' portfolio allocations in a model economy consisting of three assets and two banks. As λ_2 increases, there is a higher heterogeneity across banks. Solid lines represent the equilibrium allocations in the two-bank economy (orange for bank 1, red for bank 2), dashed lines the allocations of each bank in the single-bank benchmark (blue for bank 1, green for bank 2). We fix $\lambda_{M,1} = \lambda_{M,2} = \lambda_1 = 30$. The relative assets' illiquidities are $\gamma_2/\gamma_1 = 5$ and $\gamma_3/\gamma_1 = 10$.

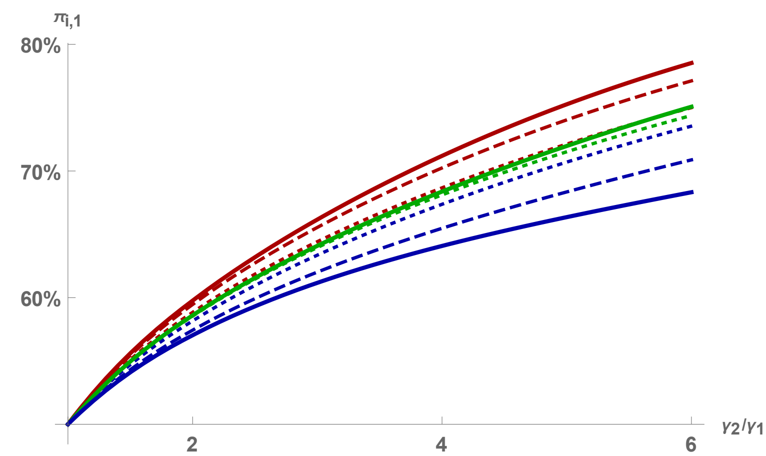

Figure 6: Banks' portfolio allocations in asset 1 for an economy of three banks and two assets as a function of the relative asset illiquidity. Solid lines represent the equilibrium allocations (red for bank 1, green for bank 2 and blue for bank 3), dotted lines the allocations of each bank in the single-bank benchmark. Dashed lines (red for bank 1 and blue for bank 3) represent the equilibrium allocations in an economy with only bank 1 and bank 3. We fix $\lambda_{M,1} = \lambda_{M,2} = 30$, $\lambda_1 = 30$, $\lambda_2 = 10$, $\lambda_3 = 5$.

lose confidence and withdraw funds from institutions, undermining financial intermediation and weakening the wider economy. The objective of the benevolent social planner is to maximize the expected aggregate banks' portfolio returns. Because assets have identically distributed returns, this is equivalent to minimizing the expected liquidation costs $TC(\pi) = \sum_{i=1}^{N} EC_i(\pi)$. The following result shows that, in a heterogeneous economy, the equilibrium banks' allocations are not socially optimal. Even if banks account for the presence of other banks and reduce portfolio overlapping, their holdings still exhibit excessive asset commonality relative to the social optimum.

Theorem 5.1. *Let $K = 2$ and $N = 2$. Under the assumptions of Theorem 4.5 and Assumption A.4, $|\pi_1^{SP} - \pi_2^{SP}| > |\pi_1^* - \pi_2^*|$, where π_i^{SP} is the bank i's asset 1 allocation that maximizes the expected aggregate banks' portfolio returns.*

Theorem 5.1 states that banks choose to hold excessively overlapping exposures compared to the social optimum. This is because each bank does not internalize the externalities it imposes on all other banks, but only accounts for the externalities imposed by other banks on itself when it makes its allocation decision. By contrast, in a homogeneous economy the social optimum is aligned with the banks' portfolio holdings obtained in equilibrium. In other words, if banks have the same distance to liquidation, the equilibrium prescribed by Proposition 4.4 is socially optimal. In this equilibrium all banks hold the same portfolio, i.e., there is complete portfolio overlap. However, this does not imply that a homogeneous economy is socially preferable to a heterogeneous one. As the following Proposition shows, the opposite result holds, i.e., aggregate liquidation costs are maximized in a homogeneous economy.

Proposition 5.2. *Assume $K = 2$ and $N = 2$. Let w be the aggregate asset value and d the*

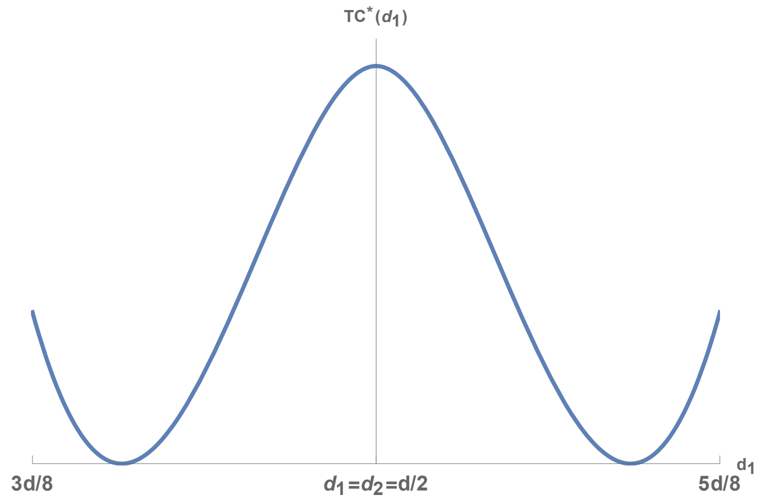

Figure 7: Total expected liquidation costs for different levels of leverage heterogeneity. If $d_1 = d_2$, then the system is homogeneous.

aggregate debt in the system. Assume that the banks' individual asset values are $w_1 = w_2 = \frac{w}{2}$*, and that the debt levels* d_1 *and* d_2 *are such that* $\lambda_1 := \frac{d_1}{w_1 - d_1} \leq \lambda_M$*,* $\lambda_2 := \frac{d_2}{w_2 - d_2} \leq \lambda_M$ *where* $d_1 + d_2 = d$ *and* $\lambda_M := \lambda_{M,1} = \lambda_{M,2}$*. Define* $TC^*(d_1)$ *to be the total expected liquidation costs in equilibrium when the debt of bank 1 is* d_1 *(and therefore the debt of bank 2 is* $d - d_1$*). Then* $d_1 = \frac{d}{2}$ *is a local maximum of* $TC^*(d_1)$*.*

Theorem 4.5 shows that it is the heterogeneity in the economy that incentives banks to reduced portfolio overlap. A lower portfolio commonality in turn reduces the likelihood and severity of liquidity crises. Therefore, aggregate costs are higher in a homogeneous economy where all banks hold the same portfolio, as stated in Proposition 5.2. See also Figure 7 for a plot of the total expected liquidation costs as a function of the leverage heterogeneity in the economy.

From a policy making perspective, Proposition 5.2 implies that bank mergers may have undesired effects on financial stability. A homogeneous economy behaves as a single large bank, and can thus be interpreted as the outcome of bank consolidation. A single bank cannot diversify its liquidation risk, as it is either affected by the liquidation event or not. If instead the economy is heterogeneous, banks manage their assets to account for systemic externalities, i.e., adjust their portfolios to reduce the likelihood of simultaneous sell-offs. A consolidated banking system decreases the available options for diversifying fire-sale risk across banks. Hence, the total quantity that the economy is required to liquidate cannot be as optimally controlled as in an economy of multiple smaller banks.

Next, we discuss how the imposition of a tax on the interconnectedness of the banking system may align the private banks' incentives with the social optimum. The next Proposition provides an explicit formula for such a tax, through which each bank fully internalizes the externalities imposed on the rest of the economy.

Proposition 5.3. *Under Assumptions A.2 and A.4, if each bank* i *is charged a tax in the amount* *equal to $T_i(\pi) := \sum_{j \neq i} M_{i,j}(\pi)$, then the private equilibrium allocation is equal to the social planner's optimum.*

The tax amount $T_i(\pi)$ is equal to the sum of externalities $M_{i,j}(\pi)$, $j \neq i$, that bank i imposes on every other bank in the economy. This externality is increasing in the size of the bank's balance sheet, bank's leverage ratio and the concentration of the bank's holdings on illiquid assets. This tax changes the ex-ante banks' incentives, aligning their equilibrium asset allocations with the social optimum. In practice, a tax on portfolio overlapping may be combined with the initiation of an asset purchase program in the event of a liquidity crisis. The tax would not only incentivize banks to reduce their common exposures, and hence the likelihood of asset liquidation spirals, but would also fund such a relief program to mitigate fire-sale losses during a crisis.

The tax $T_i(\pi)$ is related to the systemicness of bank i, as defined in Greenwood et al. (2015): the amount a bank should be charged equals the component of its expected systemicness that is not borne by the bank itself. In other words, such a tax amount can be seen as the weighted average of banks' contributions to the aggregate vulnerability of the rest of the system over a number of stress tests with different initial market shocks. Cont and Schaanning (2017) describe stress tests in line with our model, as we assume that banks –rather than being leverage targeting like in Greenwood et al. (2015)– only sell assets if their leverage threshold is breached.

In the United States, the Financial Stability Oversight Council (FSOC) uses total consolidated assets, gross notional credit default swaps, derivative liabilities, total debt outstanding, leverage ratio, and short-term debt ratio as factors for designating systemically important financial institutions (SIFIs). An institution is designed as SIFI if these factors exceed certain thresholds.[8] Our study highlights another dimension to consider in the designation of SIFI institutions, in addition to too-big-to-fail measures of default costs. A highly central node in the network of asset holdings should be taxed more because it would cause higher disruption in the provision of services to the real economy during fire-sale events (e.g. interruption of project financing, and termination of productive investments due to suspension of loans).

6 Operational Challenges and Model Extensions

We discuss limitations of the current model and outline potential extensions as well as related operational challenges. Because our focus is on fire-sale spillovers, we ignore the possibility of a bank's default. If bank i's portfolio return falls below $-\frac{1}{1+\lambda_i}$, its equity becomes negative. A slight extension of the model would cap the amount a bank can liquidate to the total amount of assets held by the bank. Because the cost function is well-defined for all values of asset shocks Z, we assume the cost function to be uncapped.

In our model, the leverage ratio is updated only after the initial market shock, but not marked to market following price changes due to asset liquidation. While we assume that the initial shock on

[8] The BIS has developed a methodology to identify systemically important financial institutions (SIFIs) based on asset size, interconnectedness, and the availability of substitutes for the services they provide.

asset prices is permanent, the knocked-down effect on prices is only temporary. Marking to market would make leverage procyclical, because the initial asset liquidation –and not further fundamental market changes– would cause new rounds of deleveraging. We also remark that commercial banks in the US and universal banks in Europe do not mark the value of their assets to market.[9]

In our study, we assume that there is only one round of deleveraging. Because asset liquidation is costly, the revenue loss due to fire sales would result in the bank violating again the leverage threshold, and thus trigger a new round of deleveraging. Hence, there would be infinite rounds of deleveraging. Realistically, banks are likely to target a leverage ratio that is strictly smaller than their threshold, and the resulting safety buffer protects the banks from subsequent rounds of deleveraging. To preserve tractability and highlight the main economic forces, we assume that banks target $\lambda_{M,i}$ and perform only one round of asset liquidation. A similar assumption has also been made by Greenwood et al. (2015), who consider only the first round of deleveraging in their leverage targeting model.

Our model assumes that banks have full knowledge on the portfolio composition of other banks. This assumption is standard in existing literature on leverage targeting banks, in which the computation of systemic risk measures relies on publicly available data. In the U.S., financial institutions file form FR Y-9C with the Federal Reserve every quarter. These forms provide consolidated information on each bank's exposures and are available through the Board's Freedom of Information Office. Duarte and Eisenbach (2018) build their empirical model on FR Y-9C balance sheet data. The studies on the vulnerability of the European banking system by Greenwood et al. (2015), and Cont and Schaanning (2017) rely on publicly available data released by the European Banking Authority.

7 Conclusions

Existing literature on fire sales has analyzed the mechanism through which hard balance sheet constraints and portfolio commonality exacerbate fire-sale externalities in the presence of distressed financial institutions. Our paper fills an important gap in the literature because it views banks as strategic as opposed to mechanical: Banks adjust their balance sheets to be more resilient to fire-sale spillovers. As such, our model does not simply provide a tool to study the propagation of financial contagion through the network of asset holdings due to overlapping portfolios. Rather, it sheds light on how fire-sale risk affects banks' ex-ante asset holding decisions. Furthermore, our model can be used to assess the welfare implications of government intervention as banks adapt to the new regulatory environment.

A natural extension of the model includes assets with heterogeneous returns. Investments in more illiquid assets would be compensated with higher risk premia. In such an extension, banks

[9] Regulators generally try to lean against mark-to-market valuation. One mechanism to enforce this is by requiring buffers in good times that can be run down in bad times. In other words, officials encourage banks to have lower leverage in good times than in bad times. This is the purpose of the Basel III conservation buffer and the countercyclical buffer.

would be profit seeking and risk-averse. Furthermore, our model may be extended to investigate the dynamics of systemic diversification. In a model economy consisting of multiple periods, each subject to the occurrence of asset price shocks, banks would face a trade off between liquidating liquid assets and reduce execution costs now versus holding a more liquid portfolio when severe shocks occur in the future. In such an economy, decoupling portfolio holdings and liquidation strategy would allow studying the time evolution of each bank's optimal portfolio and dynamic asset pricing implications in the presence of fire-sale externalities.

A Technical Assumptions

Assumption A.1. *For each $k = 1, \ldots, K$, Z_k has continuous probability density function, increasing on $[-\infty, 0]$, and the random vector Z is spherically symmetric.*

Assumption A.1 implies that all assets have the same distribution of returns. Such an assumption allows isolating the effect of fire sales from that of mean-variance optimization of returns. Furthermore, spherical symmetry guarantees that full diversification yields the lowest likelihood of liquidation. Because each bank's cost function only depends on the truncated distribution of its portfolio return, Assumption A.1 could be relaxed to also include probability density functions with asymmetric tails. This assumption is satisfied if Z is a centered Gaussian random vector, and the examples provided in the paper will be based on Gaussian returns.

Assumption A.2. *The potential function $P(\pi)$ is strictly convex on X^N.*

Assuming that the potential function is strictly convex ensures uniqueness of the equilibrium asset allocations. For $N = 2$, this assumption is implied by assumptions on the primitives of the model (see Theorem 3.2).

Assumption A.3. *For each bank i, $\ell_i \leq \bar{\ell}$ for a sufficiently small $\bar{\ell}$.*

Under Basel III, the required leverage constraint is $\lambda_M = 33$. This means that a leverage ratio of 20 implies a distance to liquidation equal to 1.9%. Even a leverage ratio of 10 implies a distance to liquidation of just 6.3%.

Assumption A.4. *The social planner's objective function $TC(\pi)$ is strictly convex on X^N.*

The assumption guarantees that the social planner admits a unique local minimum.

B Proofs

Lemma B.1. *If bank i's portfolio return is R_i, the amount x_i that bank i is required to raise is*

$$\lambda_{M,i} w_i (R_i + \ell_i)^-.$$

Proof. Bank i liquidates if $\frac{d_i}{w_i(1+R_i)-d_i} \geq \lambda_{M,i}$. After substituting $d_i = \frac{\lambda_i}{1+\lambda_i} w_i$, it can be seen that the inequality is equivalent to $R_i + \ell_i \leq 0$. Solving for x in the equation $\frac{d_i - x}{w_i(1+R_i)-d_i} = \lambda_{M,i}$, yields the quantity $x = \lambda_{M,i} w_i (R_i + \ell_i)^-$ that bank i is required to trade to comply with regulations. $\square$

Proof of Proposition 3.1

Recall that the game is a potential game with potential function $P : X^N \to \mathbb{R}$ if $\forall i \in \{1, \cdots, N\}$, $\forall \pi_{-i} \in X^{N-1}$, $\forall \pi_i', \pi_i'' \in X$,

$$P(\pi_i', \pi_{-i}) - P(\pi_i'', \pi_{-i}) = \mathrm{EC}_i(\pi_i', \pi_{-i}) - \mathrm{EC}_i(\pi_i'', \pi_{-i}).$$

It can be immediately verified that $P(\pi)$ satisfies this condition.

If Z is a continuous random variable, then $P(\pi)$ is a continuous function over the compact set X^N. Hence, there exists $\pi^* \in X^N$ that minimizes $P(\pi)$. It can be verified that π^* is a Nash equilibrium. $\square$

Proof of Proposition 3.2

Recall the definition of distance to liquidation ℓ_i in Definition 4.1. First, we assume $\ell_1 = \ell_2$, $\lambda_{M,1} = \lambda_{M,2}$, $w_1 = w_2$ and $\gamma_1 = \gamma_2$, and prove that the potential function $P(\pi)$ is strongly convex. Notice that with $\lambda_{M,1} = \lambda_{M,2}$, $\ell_1 = \ell_2$ is equivalent to $\lambda_1 = \lambda_2$.

With a slight abuse of notation, we denote $\pi_{i,1}$ simply by π_i, and hence $\pi_{i,2} = 1 - \pi_i$. We will show that the Hessian matrix H of $\frac{1}{\lambda_M^2 w^2 \gamma} P(\pi)$ is positive definite. For the first part of the proof we will consider any $N > 1$.

Define $A_i := \{\pi_i Z_1 + (1-\pi_i) Z_2 + \ell \leq 0\}$ and $A_{i,j} := \{\pi_i Z_1 + (1-\pi_i) Z_2 + \ell \leq 0, \pi_j Z_1 + (1 - \pi_j) Z_2 + \ell \leq 0\}$ for any $1 \leq i, j \leq N$. A simple calculation shows that $H_{i,i} = \frac{1}{\lambda_M^2 w^2 \gamma} \frac{\partial^2}{\partial \pi_i^2} P(\pi)$ is

$$\begin{aligned}
E\Big[& 2(Z_1 - Z_2)^2 (\pi_i^2 + (1-\pi_i)^2) 1_{A_i} + 8(\pi_i Z_1 + (1-\pi_i) Z_2 + \ell_i)(Z_1 - Z_2)(2\pi_i - 1) 1_{A_i} \\
& + 4(\pi_i Z_1 + (1-\pi_i) Z_2 + \ell_i)^2 1_{A_i} + \sum_{j \neq i} \Big(2(\pi_j Z_1 + (1-\pi_j) Z_2 + \ell_j)(Z_1 - Z_2)(2\pi_j - 1) 1_{A_{i,j}} \\
& + (\pi_i \pi_j + (1-\pi_i)(1-\pi_j)) \frac{\partial}{\partial \pi_i} \left[(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j) Z_2 + \ell_j) 1_{A_{i,j}} \right] \Big) \Big].
\end{aligned}$$

The off-diagonal element $H_{i,j} = \frac{1}{\lambda_M^2 w^2 \gamma} \frac{\partial^2}{\partial \pi_i \partial \pi_j} P(\pi)$ of the Hessian matrix H is

$$\begin{aligned}
E\Big[& (Z_1 - Z_2)^2 (\pi_i \pi_j + (1-\pi_i)(1-\pi_j)) 1_{A_{i,j}} + (\pi_j Z_1 + (1-\pi_j) Z_2 + \ell_j)(Z_1 - Z_2)(2\pi_i - 1) 1_{A_{i,j}} \\
& + (\pi_i Z_1 + (1-\pi_i) Z_2 + \ell_i)(Z_1 - Z_2)(2\pi_j - 1) 1_{A_{i,j}} \\
& + 2(\pi_i Z_1 + (1-\pi_i) Z_2 + \ell_i)(\pi_j Z_1 + (1-\pi_j) Z_2 + \ell_j) 1_{A_{i,j}} \Big].
\end{aligned}$$

Next, we construct a positive semidefinite matrix M_1 with the same off-diagonal elements as H. Define the random vectors $v^{(1)} := (\pi_i (Z_1 - Z_2) 1_{A_i})_{1 \leq i \leq N}$, $v^{(2)} := ((1-\pi_i)(Z_1 - Z_2) 1_{A_i})_{1 \leq i \leq N}$, $v^{(3)} :=$

$((\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)1_{A_i})_{1\le i\le N}$, $v^{(4)} := ((Z_1 - Z_2)(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)1_{A_i})_{1\le i\le N}$ and $v^{(5)} := ((2\pi_i - 1)1_{A_i})_{1\le i\le N}$. The random matrix $M_2 := v^{(1)}{v^{(1)}}^T + v^{(2)}{v^{(2)}}^T + 2v^{(3)}{v^{(3)}}^T + v^{(4)}{v^{(5)}}^T + v^{(5)}{v^{(4)}}^T$ is such that $E[M_2]$ has the same off-diagonal elements of H. Recall that for any couple of linearly independent vectors x, y, the matrix $xy^T + yx^T$ has exactly one negative eigenvalue, while the matrix xx^T has only one non-zero eigenvalue, which is positive. Now, we want to find a positive definite diagonal matrix D_1 such that $D_1 + M_2$ is almost surely positive semidefinite and the elements of D_1 are as small as possible. An immediate application of Woodbury matrix identity and the matrix determinant lemma shows that for any symmetric invertible matrix A, we have $\det(A + xy^T + yx^T) = ((1 + x^T A^{-1} y)^2 - (x^T A^{-1} x)(y^T A^{-1} y))\det(A)$. Therefore, $D_1 + v^{(4)}{v^{(5)}}^T + v^{(5)}{v^{(4)}}^T$ is positive semidefinite if and only if $a := (1 + {v^{(4)}}^T D_1^{-1} v^{(5)})^2 - ({v^{(4)}}^T D_1^{-1} v^{(4)})({v^{(5)}}^T D_1^{-1} v^{(5)}) \ge 0$. Define $b := \min_{\pi\in X^N} a$. It can be verified that the matrix D_1 with entries $d^{(1)}_{i,i} = \frac{N}{4}(2\ell + Z_1 + Z_2)^2 1_{A_i}$ is such that $b = 0$. Therefore, $D_1 + M_2$ is a positive semidefinite matrix.

We now show that

$$E\left[\frac{\partial}{\partial \pi_i}\left[(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)1_{A_{i,j}}\right]\right] > 0.$$

Assume that $\pi_i + \varepsilon < \pi_j$, for some small $\varepsilon > 0$, and define the event $A^{\varepsilon}_{i,j} := \{(\pi_i + \varepsilon)Z_1 + (1 - \pi_i - \varepsilon)Z_2 + \ell \le 0, \pi_j Z_1 + (1-\pi_j)Z_2 + \ell \le 0\}$. We have that $A_{i,j} \subset A^{\varepsilon}_{i,j}$ and that $Z_1 < Z_2$ on the event $A^{\varepsilon}_{i,j} \setminus A_{i,j}$. Therefore, $(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell) > 0$ on $A^{\varepsilon}_{i,j} \setminus A_{i,j}$. Hence, for $\pi_i < \pi_j$, we have shown that the derivative is positive. Similarly, if $\pi_i > \pi_j$, $A^{\varepsilon}_{i,j} \subset A_{i,j}$ and $(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell) < 0$ on $A_{i,j} \setminus A^{\varepsilon}_{i,j}$. It follows that also in this case the derivative is positive.

Hence, the diagonal matrix D_2 with elements

$$d^{(2)}_{i,i} = E\left[\sum_{j\ne i}(\pi_i\pi_j + (1-\pi_i)(1-\pi_j))\frac{\partial}{\partial \pi_i}\left[(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)1_{A_{i,j}}\right]\right]$$

is positive definite.

Next, we show that the diagonal matrix $M_3 := H - E[M_2 + D_1] - D_2$ is positive semidefinite. It follows then that H is positive definite. The i-th element on the diagonal of M_3 is

$$\begin{aligned}
E\Bigg[&(Z_1 - Z_2)^2(\pi_i^2 + (1-\pi_i)^2)1_{A_i} + 6(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A_i} \\
&+ 2(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)^2 1_{A_i} - \frac{N(Z_1 + Z_2 + 2\ell)^2}{4}1_{A_i} \\
&+ \sum_{j\ne i} 2(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1)1_{A_{i,j}}\Bigg].
\end{aligned}$$

Assume now $N = 2$. First, we prove that $E\left[\left((Z_1 - Z_2)^2(\pi_i^2 + (1-\pi_i)^2) - 2(\frac{Z_1}{2} + \frac{Z_2}{2} + \ell)^2)\right)1_{A_i}\right] > 0$. Define $d_{ax+by}(z_1, z_2) := \frac{|az_1 + bz_2|}{(a^2+b^2)^{1/2}}$ the distance of the point (z_1, z_2) to the line $ax + by = 0$. The inequality can then be rephrased as $E\left[\left(2d^2_{y-x}(Z_1,Z_2)(\pi_i^2+(1-\pi_i)^2)-d^2_{x/2+y/2+\ell}(Z_1,Z_2)\right)1_{A_i}\right]>0$. Since $\pi_i^2+(1-\pi_i)^2\geq\frac{1}{2}$, it is enough to prove that $E\left[\left(d^2_{y-x}(Z_1,Z_2)-d^2_{x/2+y/2+\ell}(Z_1,Z_2)\right)1_{A_i}\right]>0$. Define $A_i^{(1)}=A_i\cap\{\pi_iZ_1-(1-\pi_i)Z_2+(2\pi_i-1)\ell<0\}$ and $A_i^{(2)}=A_i\cap\{\pi_iZ_1-(1-\pi_i)Z_2+(2\pi_i-1)\ell\geq 0\}$. Notice that the line $\pi_ix-(1-\pi_i)y+(2\pi_i-1)\ell=0$ is the reflection of the line $\pi_ix+(1-\pi_i)y+\ell=0$ over $y=-\ell$. We will show that $E\left[\left(d^2_{y-x}(Z_1,Z_2)-d^2_{x/2+y/2+\ell}(Z_1,Z_2)\right)1_{A_i^{(1)}}\right]>0$, the case for $A_i^{(2)}$ is analogous. In the set $A_i^{(1)}$, consider the points $(x_1,y_1)=(z_1,z_2)$ and $(x_2,y_2)=(z_1,-2\ell-z_2)$ with $z_2>-\ell$, which are symmetric with respect to the line $y=-\ell$. They are such that $d_{y-x}(x_1,y_1)>d_{x/2+y/2+\ell}(x_1,y_1)$, $d_{y-x}(x_1,y_1)=d_{x/2+y/2+\ell}(x_2,y_2)$ and $d_{y-x}(x_2,y_2)=d_{x/2+y/2+\ell}(x_1,y_1)$. In particular,

$$(d^2_{y-x}(x_1,y_1)-d^2_{x/2+y/2+\ell}(x_1,y_1))+(d^2_{y-x}(x_2,y_2)-d^2_{x/2+y/2+\ell}(x_2,y_2))=0.$$

Notice also that $\|(x_1,y_1)\|<\|(x_2,y_2)\|$. Since the distribution of Z is rotationally invariant and Z_1 has an increasing probability density function on $(-\infty,0]$, we have $\varphi(x_1,y_1)>\varphi(x_2,y_2)$, where $\varphi(\cdot,\cdot)$ is the probability density function of (Z_1,Z_2). Therefore,

$$\begin{aligned}E\left[\left(d^2_{y-x}(Z_1,Z_2)-d^2_{x/2+y/2+\ell}(Z_1,Z_2)\right)1_{A_i^{(1)}}\right]&=\int_{A_i^{(1)}}(d^2_{y-x}(z_1,z_2)-d^2_{x/2+y/2+\ell}(z_1,z_2))\varphi(z_1,z_2)dz_1dz_2\\&=\int_{A_i^{(1)}\cap\{z_2>-\ell\}}(d^2_{y-x}(z_1,z_2)-d^2_{x/2+y/2+\ell}(z_1,z_2))(\varphi(z_1,z_2)-\varphi(z_1,-2\ell-z_2))dz_1dz_2.\end{aligned}$$

Both terms in the product of the last integrand are positive. It follows that the expectation on $A_i^{(1)}$ is positive. The same arguments hold for the expectation on $A_i^{(2)}$, with $(x_1,y_1)=(z_1,z_2)$ and $(x_2,y_2)=(-2\ell-z_1,z_2)$ where $z_1>-\ell$.

Next, we prove that

$$E\left[(\pi_iZ_1+(1-\pi_i)Z_2+\ell)(Z_1-Z_2)(2\pi_i-1)1_{A_i}+(\pi_jZ_1+(1-\pi_j)Z_2+\ell)(Z_1-Z_2)(2\pi_j-1)1_{A_{i,j}}\right]>0.$$

Assume that $\pi_i>1/2$, $\pi_j<1/2$ and $|\pi_i-1/2|<|\pi_j-1/2|$ (analogous arguments apply for all other combinations). Notice that $(\pi_iz_1+(1-\pi_i)z_2+\ell)(z_1-z_2)(2\pi_i-1)>0$ for any $(z_1,z_2)\in A_i\setminus A_{i,j}$. Hence, it is enough to show $E[((\pi_iZ_1+(1-\pi_i)Z_2+\ell)(Z_1-Z_2)(2\pi_i-1)+(\pi_jZ_1+(1-\pi_j)Z_2+\ell)(Z_1-Z_2)(2\pi_j-1))1_{A_{i,j}}]\geq 0$. Define $A_{i,j}^{(1)}=A_{i,j}\cap\{(1-\pi_j)Z_1+\pi_jZ_2+\ell<0\}$ and $A_{i,j}^{(2)}=A_{i,j}\cap\{(1-\pi_j)Z_1+\pi_jZ_2+\ell>0\}$. On $A_{i,j}^{(2)}$ we have $d_{\pi_ix+(1-\pi_i)y+\ell}<d_{\pi_jx+(1-\pi_j)y+\ell}$.

Therefore, for any $(z_1, z_2) \in A^{(2)}_{i,j}$, since $(1+x^2)^{1/2}x$ is increasing on $[0,1]$, we get

$$
\begin{aligned}
&(\pi_i z_1 + (1-\pi_i)z_2 + \ell)(2\pi_i - 1) + (\pi_j z_1 + (1-\pi_j)z_2 + \ell)(2\pi_j - 1) = \\
&\quad -\frac{1}{\sqrt{2}} d_{\pi_i x + (1-\pi_i)y+\ell}(z_1, z_2)(1 + (2\pi_i - 1)^2)^{1/2}(2\pi_i - 1) \\
&\quad + \frac{1}{\sqrt{2}} d_{\pi_j x + (1-\pi_j)y+\ell}(z_1, z_2)(1 + (2\pi_j - 1)^2)^{1/2}|2\pi_j - 1| > \\
&\quad \frac{1}{\sqrt{2}}(-d_{\pi_i x + (1-\pi_i)y+\ell}(z_1, z_2) + d_{\pi_j x + (1-\pi_j)y+\ell}(z_1, z_2))(1 + (2\pi_i - 1)^2)^{1/2}(2\pi_i - 1) > 0.
\end{aligned}
$$

Since $z_1 - z_2 > 0$ on $A^{(2)}_{i,j}$, we get $E[((\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1) + (\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1))1_{A^{(2)}_{i,j}}] > 0$. Next, we show that both $E[(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A^{(1)}_{i,j}}]$ and $E[(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1))1_{A^{(1)}_{i,j}}]$ are positive. Consider the point $(z_1, z_2) \in A^{(1)}_{i,j}$ with $z_2 > z_1$, then also $(z_2, z_1) \in A^{(1)}_{i,j}$. Since $d_{\pi_i x + (1-\pi_i)y+\ell}(z_1, z_2) > d_{\pi_i x + (1-\pi_i)y+\ell}(z_2, z_1)$, we get $(\pi_i z_1 + (1-\pi_i)z_2 + \ell)(z_1 - z_2)(2\pi_i - 1) + (\pi_i z_2 + (1-\pi_i)z_1 + \ell)(z_2 - z_1)(2\pi_i - 1) > 0$. It follows that $E[(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A^{(1)}_{i,j}}] > 0$. Similarly, it can be shown that $E[(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1)1_{A^{(1)}_{i,j}}] > 0$.

The remaining terms in the diagonal elements of M_3 are positive. Hence, H is positive definite for all $(\pi_1, \pi_2) \in [0,1]^2$, and therefore $P(\pi)$ is strongly convex.

Because of the smoothness of the potential function with respect to the parameters, there exist $\ell_* < \ell < \ell^*$ (equivalently, $\lambda_* < \lambda < \lambda^*$), $w_* < w < w^*$, $\gamma_* < \gamma < \gamma^*$ such that if $\ell_i \in (\ell_*, \ell^*)$ (equivalently, if $\lambda_i \in (\lambda_*, \lambda^*)$), $w_i \in (w_*, w^*)$ for each i and $\gamma_k \in (\gamma_*.\gamma^*)$ for each k, then $P(\pi)$ is strictly convex on X^N. The uniqueness of the Nash equilibrium then follows from Theorem 2.3 in Lã et al. (2016). □

Proof of Proposition 4.2

Recall the definition of distance to liquidation ℓ in Definition 4.1. By Lemma B.1, the bank's expected liquidation costs are given by $\lambda_M^2 w^2 \gamma \|\pi\|_2^2 E[(\pi^T Z + \ell)^2 1_{\{\pi^T Z + \ell \le 0\}}]$. Since Z is spherically symmetric, $\frac{1}{\|\pi\|_2}\pi^T Z$ has the same distribution as Z_1. Hence, $\|\pi\|_2^2 E[(\pi^T Z + \ell)^2 1_{\{\pi^T Z + \ell \le 0\}}] = \|\pi\|_2^2 E[(\|\pi\|_2 Z_1 + \ell)^2 1_{\{\|\pi\|_2 Z_1 + \ell \le 0\}}]$. It follows that the expected liquidation costs are minimized when the bank minimizes $\|\pi\|_2$. The minimum of $\|\pi\|_2$ is attained at $\pi_k = \frac{1}{K}$ for each k. □

Proof of Proposition 4.3

Recall the definition of distance to liquidation ℓ in Definition 4.1. Define $f(x, \ell) := E[(xZ_1 + (1-x)Z_2 + \ell)^2 1_{\{xZ_1 + (1-x)Z_2 + \ell \le 0\}}]$ and $g(x) := x^2\gamma_1 + (1-x)^2\gamma_2$. The minimizer of $f(\cdot, \ell)$ is $\frac{1}{2}$ and the minimizer of $g(\cdot)$ is $x_g := \frac{\gamma_2}{\gamma_1 + \gamma_2} > \frac{1}{2}$. We write $f_x(x, \ell)$ for $\frac{\partial}{\partial x} f(x, \ell)$. Since $\frac{d}{d\pi_{1,1}} S_1 = f_x(\pi_{1,1})g(\pi_{1,1}) + f(\pi_{1,1})g_x(\pi_{1,1})$, $f \ge 0$, $g > 0$, $f_x(x) < 0$ on $[0, \frac{1}{2})$ and $f_x(x) > 0$ on $(\frac{1}{2}, 1]$, and $g_x(x) < 0$ on $[0, x_g)$ and $g_x(x) > 0$ on $(x_g, 1]$, we get that $\frac{d}{d\pi_{1,1}} S_1 < 0$ on $[0, \frac{1}{2}]$ and $\frac{d}{d\pi_{1,1}} S_1 > 0$ on $[x_g, 1]$. Hence, $\pi^S_{1,1} \in (\frac{1}{2}, \frac{\gamma_2}{\gamma_1 + \gamma_2})$.

Next, we show that $\frac{f_x(x,\ell)}{f(x,\ell)}$ is an increasing function of ℓ for $x > \frac{1}{2}$. Since Z is rotationally invariant, $f(x) = E[(n(x)Z_1 + \ell)^2 1_{\{n(x)Z_1+\ell\le 0\}}]$, where $n(x) = (x^2 + (1-x)^2)^{1/2}$. An explicit calculation shows that $f_{x\ell}(\pi_{1,1})f(\pi_{1,1}) - f_x(\pi_{1,1})f_\ell(\pi_{1,1})$ is equal to

$$-\frac{2(2\pi_{1,1}-1)}{n(\pi_{1,1})}(2E[Z_1^2 1_A]P(A)\ell n(\pi_{1,1}) + \ell^2 E[Z_1 1_A]P(A) + E[Z_1 1_A]E[Z_1^2 1_A]n(\pi_{1,1})^2),$$

where $A = \{n(\pi_{1,1})Z_1 + \ell \le 0\}$. Because the distribution of Z_1 is increasing on $(-\infty, 0]$, we get $\frac{\ell}{n(\pi_{1,1})}P(A) + E[Z_1 1_A] < 0$ and $E[Z_1^2 1_A] + \frac{\ell}{n(\pi_{1,1})}E[Z_1 1_A] < 0$. It follows that $f_{x\ell}(\pi_{1,1})f(\pi_{1,1}) - f_x(\pi_{1,1})f_\ell(\pi_{1,1}) > 0$ for $\pi_{1,1} > \frac{1}{2}$. Therefore, if $\ell_2 > \ell_1$ we have $\frac{f_x(\pi^S_{1,1}(\ell_1),\ell_2)}{f(\pi^S_{1,1}(\ell_1),\ell_2)} > \frac{f_x(\pi^S_{1,1}(\ell_1),\ell_1)}{f(\pi^S_{1,1}(\ell_1),\ell_1)} = \frac{g_x(\pi^S_{1,1}(\ell_1))}{g(\pi^S_{1,1}(\ell_1))}$. In other terms, $\frac{d}{d\pi_{1,1}}S_1(\pi^S(\ell_1),\ell_2) > 0$. Since $\frac{d}{d\pi_{1,1}}S_1(\pi_{1,1},\ell)$ is increasing in $\pi_{1,1}$, we get that $\pi^S_{1,1}(\ell_2) < \pi^S_{1,1}(\ell_1)$. Because ℓ is a decreasing function of λ, we obtain the thesis. $\square$

Proof of Proposition 4.4

Rewrite the allocation vector $\pi_i \in X$ of bank i as $\left(\pi_{i,1}, \cdots, \pi_{i,K-1}, 1 - \sum_{k=1}^{K-1}\pi_{i,k}\right)$. Using Lemma B.1, we get that the derivative $\frac{\partial}{\partial \pi_{i,h}}P(\pi)$ is

$$E\Bigg[2w_i^2(\pi_i^T Z + \ell_i)\pi_i^T \mathrm{Diag}(\gamma)\pi_i(Z_h - Z_K)1_{A_i} + 2w_i^2(\pi_i^T Z + \ell_i)^2(\pi_{i,h}\gamma_h - (1 - \sum_{k=1}^{K-1}\pi_{i,k})\gamma_K)1_{A_i} +$$
$$\sum_{j\neq i} w_i w_j(\pi_j^T Z + \ell_j)\pi_i^T \mathrm{Diag}(\gamma)\pi_j(Z_h - Z_K)1_{A_{i,j}} +$$
$$\sum_{j\neq i} w_i w_j(\pi_i^T Z + \ell_i)(\pi_j^T Z + \ell_j)(\pi_{j,h}\gamma_h - (1 - \sum_{k=1}^{K-1}\pi_{j,k})\gamma_K)1_{A_{i,j}}\Bigg],$$

where, without loss of generality, we have replaced $\lambda_{M,i}w_i$ with w_i. To prove (1), assume that $\gamma_k = \gamma$ and $\pi_{i,k} = \frac{1}{K}$ for all i and k. Because Z is spherically symmetric, $E[(\sum_{k=1}^K Z_k + K\ell_i)Z_h 1_{\{\sum_{k=1}^K Z_k + K\ell_i \le 0\}}] = E[(\sum_{k=1}^K Z_k + K\ell_i)Z_K 1_{\{\sum_{k=1}^K Z_k + K\ell_i \le 0\}}]$ for each h. Hence, the first term in the derivative is 0. The second term is 0, because one of its coefficients is 0. The same arguments yield that the third and fourth term are also 0. Since the potential function $P(\pi)$ is convex, $\pi_{i,k} = \frac{1}{K}$ for all i and k is the unique Nash equilibrium.

To prove (2), first notice that π^* is a critical point for $S_i(\pi_i)$. It follows that π^* solves the equations $E[(\pi^{*T}Z + \ell)\pi^{*T}\mathrm{Diag}(\gamma)\pi_i^*(Z_h - Z_K)1_{A_i} + (\pi^{*T}Z + \ell)^2(\pi_h^*\gamma_h - (1 - \sum_{k=1}^{K-1}\pi_k^*)\gamma_K)1_{A_i}] = 0$ for every h. If $\ell_i = \ell_j = \ell$ and $\pi_i = \pi_j = \pi^*$, then $A_{i,j} = A_i$. Hence, the sum of the first two terms in the derivative of $P(\pi)$ and the sum of the last two terms are both 0. From the convexity of $P(\pi)$ it follows that $\pi_i = \pi^*$ for each i is the unique Nash equilibrium. $\square$

Lemma B.2. *Assume $K = 2$. Under Assumptions A.1 and A.3, $\frac{\partial^2 P}{\partial \pi_{i,1}\pi_{j,1}}(\pi) > 0$ for $1 \le i \neq j \le N$ and $\pi \in X^N$.*

Proof. A simple calculation shows that $\frac{1}{\lambda_{M,i}\lambda_{M,j}w_i w_j}\frac{\partial^2 P}{\partial \pi_{i,1}\pi_{j,1}}(\pi) = E[(\gamma_1(\ell_i + 2\pi_{i,1}Z_1 + (1-2\pi_{i,1})Z_2)(\ell_j + 2\pi_{j,1}Z_1 + (1-2\pi_{j,1})Z_2) + \gamma_2(\ell_i + (2\pi_{i,1}-1)Z_1 + 2(1-\pi_{i,1})Z_2)(\ell_j + (2\pi_{j,1}-1)Z_1 + 2(1-\pi_{j,1})Z_2))1_{A_i\cap A_j}]$, where $A_i := \{\pi_{1,1}Z_1 + (1-\pi_{i,1})Z_2 + \ell_i \leq 0\}$ and $A_j := \{\pi_{j,1}Z_1 + (1-\pi_{j,1})Z_2 + \ell_j \leq 0\}$. Assume $\ell_i = \ell_j = 0$. From the spherical symmetry of the distribution of Z, it follows that the distribution is uniform along every circle. Next, we consider the change of variable $(Z_1, Z_2) = (\rho\cos(t), \rho\sin(t))$. In terms of the new variables (ρ, t), if $\pi_{i,1} > \pi_{j,1}$, the integration region $A_i \cap A_j$ translates into the range $(0, +\infty) \times [t_l, t_r] := (0, +\infty) \times \left[\arctan\left(-\frac{\pi_{j,1}}{1-\pi_{j,1}}\right) + \pi, \arctan\left(-\frac{\pi_{i,1}}{1-\pi_{i,1}}\right) + 2\pi\right]$. For a fixed ρ, the integral over t reduces to

$$\begin{aligned}&\gamma_1(2(t_r - t_l) + 2(-(\pi_{i,1}+\pi_{j,1}) + 4\pi_{i,1}\pi_{j,1})(2t_r + \cos(2t_r) - 2t_l - \cos(2t_l)) \\ &+ (2(\pi_{i,1}+\pi_{j,1}) - 1)(\sin(2t_r) - \sin(2t_l))) + \gamma_2(\cdots),\end{aligned}$$

up to a positive coefficient. The expression that multiplies γ_1 is strictly positive. The same calculations and arguments hold for the term that multiplies γ_2. Hence, the integral over ρ is also strictly positive. By continuity, the expectation is strictly positive for $\ell_1, \ell_2 \leq \bar{\ell}$, for a sufficiently small $\bar{\ell}$. $\square$

Proof of Theorem 4.5

With a slight abuse of notation, we denote $\pi_{i,1}$ simply by π_i, and hence $\pi_{i,2} = 1 - \pi_i$. The Nash equilibrium (π_1^*, π_2^*) solves the system of equations

$$\begin{aligned}\frac{\partial}{\partial \pi_i}P(\pi_i, \pi_j) = E\Big[&2w_i^2(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)(\gamma_1\pi_i^2 + \gamma_2(1-\pi_i)^2)(Z_1 - Z_2)1_{A_i} \\ &+ 2w_i^2(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)^2((\gamma_1+\gamma_2)\pi_i - \gamma_2)1_{A_i} \\ &+ w_i w_j(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)(\gamma_1\pi_i\pi_j + \gamma_2(1-\pi_i)(1-\pi_j))(Z_1 - Z_2)1_{A_{i,j}} \\ &+ w_i w_j(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)((\gamma_1+\gamma_2)\pi_j - \gamma_2)1_{A_{i,j}}\Big] = 0 \\ &\text{for } i = 1, j = 2 \text{ and } i = 2, j = 1,\end{aligned} \tag{B.1}$$

where $A_i := \{\pi_1 Z_1 + (1-\pi_i)Z_2 + \ell_i \leq 0\}$, $A_j := \{\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j \leq 0\}$, and $A_{i,j} := A_i \cap A_j$, and where, without loss of generality, we have replaced $\lambda_{M,i}w_i$ with w_i. If $\ell_2 = \ell_1$, then there exists $\pi^{\ell_1} \in (\frac{1}{2}, \frac{\gamma_1+\gamma_2}{\gamma_2})$ such that $\pi_1 = \pi_2 = \pi^{\ell_1}$ is the Nash equilibrium. In particular, since $\frac{\partial}{\partial \pi_i}S_i(\pi_i) = 0$ for $\pi_i = \pi^{\ell_1}$, both the sum of the first two terms, i.e., $\frac{\partial}{\partial \pi_i}S_i(\pi_i)$, and the sum of the last two terms, i.e., $\frac{\partial}{\partial \pi_i}M_{i,j}(\pi_i, \pi_j)$, are zero. Consider equation (B.1) where $i = 1, j = 2$. For $\ell_1 < \ell_2$ and $\pi_1 = \pi_2 = \pi^{\ell_1}$, we will show that the sum of the last two terms is negative, i.e., $\frac{\partial}{\partial \pi_i}M_{i,j}(\pi_i, \pi_j) < 0$.

This is equivalent to proving that

$$\frac{E\Big[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_2)(Z_1-Z_2)1_{A_{1,2}}\Big]}{E\Big[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_1)(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_2)1_{A_{1,2}}\Big]}<$$

$$-\frac{((\gamma_1+\gamma_2)\pi^{\ell_1}-\gamma_2)}{(\gamma_1{\pi^{\ell_1}}^2+\gamma_2(1-\pi^{\ell_1})^2)}=\frac{E\Big[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_1)(Z_1-Z_2)1_{A_1}\Big]}{E\Big[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_1)^2 1_{A_1}\Big]},$$

where the last equality follows from $\frac{\partial}{\partial\pi_1}S_1(\pi_1)=0$ for $\pi_1=\pi^{\ell_1}$. Define $h(\ell_2):=E[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_2)(Z_1-Z_2)1_{A_{1,2}}]$ and $k(\ell_2):=E[(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_1)(\pi^{\ell_1}Z_1+(1-\pi^{\ell_1})Z_2+\ell_2)1_{A_{1,2}}]$. The inequality we want to prove can thus be rewritten as $\frac{h(\ell_2)}{k(\ell_2)}<\frac{h(\ell_1)}{k(\ell_1)}$. Notice that $h(\ell_2)=\frac{1}{2}\frac{\partial}{\partial\pi}E[(\pi Z_1+(1-\pi)Z_2+\ell_2)^2 1_{A_{1,2}}]|_{\pi=\pi^{\ell_1}}=\frac{1}{2}\frac{\partial}{\partial\pi}E[(n(\pi)Z_1+\ell_2)^2 1_{A_{1,2}}]|_{\pi=\pi^{\ell_1}}=E[(n(\pi^{\ell_1})Z_1+\ell_2)\frac{2\pi^{\ell_1}-1}{n(\pi^{\ell_1})}Z_1 1_{A_{1,2}}]$, where $n(\pi)=(\pi^2+(1-\pi)^2)^{1/2}$. Also, $k(\ell_2)=E[(n(\pi^{\ell_1})Z_1+\ell_1)(n(\pi^{\ell_1})Z_1+\ell_2)1_{A_{1,2}}]$. Explicit computations show that $h'(\ell_2)k(\ell_2)-h(\ell_2)k'(\ell_2)=(2\pi^{\ell_1}-1)\ell_1(E[Z_1 1_{A_{1,2}}]^2-E[Z_1^2 1_{A_{1,2}}]P(A_{1,2}))$. It follows from the Cauchy-Schwartz inequality and $\pi^{\ell_1}>\frac{1}{2}$ that this expression is negative. Hence, the fraction $\frac{h(\ell)}{k(\ell)}$ is decreasing in ℓ, which proves the inequality. In particular, it follows that $\frac{\partial}{\partial\pi_1}P(\pi_1,\pi_2)<0$ for $(\pi_1,\pi_2)=(\pi^{\ell_1},\pi^{\ell_1})$. From the convexity of the potential function $P(\pi_1,\pi_2)$, we get that $\frac{\partial}{\partial\pi_1}P(\pi_1,\pi_2)=0$ for $(\pi_1,\pi_2)=(\pi_1^{(0)},\pi^{\ell_1})$, where $\pi_1^{(0)}>\pi^{\ell_1}$. Similarly, $\frac{\partial}{\partial\pi_2}P(\pi_1,\pi_2)=0$ for $(\pi_1,\pi_2)=(\pi^{\ell_2},\pi_2^{(0)})$, where $\pi_2^{(0)}<\pi^{\ell_2}$. In particular, $\pi_2^{(0)}<\pi^{\ell_2}<\pi^{\ell_1}<\pi_1^{(0)}$, where the second inequality follows from Proposition 4.3.

Next, we show that the optimal weight π_1 of bank 1 is a decreasing function of π_2. For a fixed π_2, let $\bar\pi_1(\pi_2)$ be the optimal response by bank 1, i.e., $\frac{\partial}{\partial\pi_1}P(\bar\pi_1(\pi_2),\pi_2)=0$. Differentiating both the left and right hand side with respect to π_2 yields $\frac{\partial^2}{\partial\pi_1\partial\pi_2}P(\bar\pi_1(\pi_2),\pi_2)+\frac{\partial^2}{\partial\pi_1^2}P(\bar\pi_1(\pi_2),\pi_2)\times\frac{\partial}{\partial\pi_2}\bar\pi_1=0$. From the convexity of P and Lemma B.2, it follows that $\bar\pi_1(\pi_2)$ is decreasing, and therefore $\bar\pi_1^{-1}(\pi_1)$ is decreasing. Analogously, $\bar\pi_2(\pi_1)$ is a decreasing function.

We have already proved that $\bar\pi_1(\pi^{\ell_1})=\pi_1^{(0)}$ and $\bar\pi_2(\pi^{\ell_2})=\pi_2^{(0)}$, where $\pi_2^{(0)}<\pi^{\ell_2}<\pi^{\ell_1}<\pi_1^{(0)}$. Notice that $\bar\pi_1^{-1}(\pi_1^{(0)})=\pi^{\ell_1}>\pi_2^{(0)}=\bar\pi_2(\pi^{\ell_2})>\bar\pi_2(\pi_1^{(0)})$. Define $\pi_{1,r}:=\bar\pi_1(0)$. It follows that $0=\bar\pi_1^{-1}(\pi_{1,r})\le\bar\pi_2(\pi_{1,r})$. By continuity, there exists $\pi_1^*\in(\pi_1^{(0)},\pi_{1,r}]$ such that $\bar\pi_1^{-1}(\pi_1^*)=\bar\pi_2(\pi_1^*)$. The point $(\pi_1^*,\bar\pi_2(\pi_1^*))$ is by definition the Nash equilibrium. Because $\pi_1^*>\pi^{\ell_1}$ and $\bar\pi_2(\pi_1^*)<\bar\pi_2(\pi^{\ell_2})=\pi_2^{(0)}<\pi^{\ell_2}$, we get the thesis. □

Proof of Proposition 4.6

For $i=1,2$, define $A_i^*:=\{\pi_{i,1}^*Z_1+\pi_{i,2}^*Z_2+\ell_i\le 0\}$ and $A_i^S:=\{\pi_{i,1}^S Z_1+\pi_{i,2}^S Z_2+\ell_i\le 0\}$. Hence, $A_{\text{liq}}^*:=A_{\text{liq}}(\pi_1^*,\pi_2^*)=A_1^*\cup A_2^*$ and $A_{\text{liq}}^S:=A_{\text{liq}}(\pi_1^S,\pi_2^S)=A_1^S\cup A_2^S$. Similarly, $A_{\text{sim}}^*:=A_{\text{sim}}(\pi_1^*,\pi_2^*)=A_1^*\cap A_2^*$ and $A_{\text{sim}}^S:=A_{\text{sim}}(\pi_1^S,\pi_2^S)=A_1^S\cap A_2^S$. From Proposition 4.3 and Theorem 4.5 it follows that $\pi_{2,1}^*<\pi_{2,1}^S<\pi_{1,1}^S<\pi_{1,1}^*$. Because ℓ_1 and ℓ_2 are sufficiently close, these allocations belong to the interval $[\frac{1}{2},\frac{\gamma_2}{\gamma_1+\gamma_2}]$. First, compare the events $B_1:=A_2^S\cap(A_2^*)^c$ and $B_2:=A_2^*\cap(A_2^S)^c$, where A^c denotes the complement of A. It follows from assumption A.1 and the inequality $\frac{1}{2}\le\pi_{2,1}^*<\pi_{2,1}^S$

that $P(B_1) \geq P(B_2)$. Since $A^*_{\text{sim}} \subset (A^S_{\text{sim}} \setminus B_1) \cup B_2$, we get that $P(A^*_{\text{sim}}) < P(A^S_{\text{sim}})$.

Next, compare the events $C_1 := A^*_1 \cap (A^S_1)^c$ and $C_2 := A^S_1 \cap (A^*_1)^c$. From Assumption A.1 and the fact that $\frac{1}{2} \leq \pi^S_{1,1} < \pi^*_{1,1}$, it follows that $P(C_1) \geq P(C_2)$. Since $A^S_{\text{liq}} \subset (A^*_{\text{liq}} \setminus C_1) \cup C_2$, we get that $P(A^*_{\text{liq}}) > P(A^S_{\text{liq}})$, which concludes the proof. $\square$

Proof of Theorem 5.1

Notice that $TC(\pi) = 2P(\pi) - S(\pi)$. From the proof of Theorem 4.5 we know that $\frac{\partial}{\partial \pi_1}(2P - S)(\pi^{(0)}_1, \pi^{\ell_1}) < 0$. From the convexity of $TC(\cdot)$, it follows that there exists a unique $\pi^{(1)}_1$ such that $\frac{\partial}{\partial \pi_1} TC(\pi^{(1)}_1, \pi^{\ell_1}) = 0$, where $\pi^{(1)}_1 > \pi^{(0)}_1$. Similarly, $\frac{\partial}{\partial \pi_2} TC(\pi^{\ell_2}, \pi^{(1)}_2) = 0$ for $\pi^{(1)}_2 < \pi^{(0)}_2$. Hence, $\pi^{(1)}_2 < \pi^{(0)}_2 < \pi^{\ell_2} < \pi^{\ell_1} < \pi^{(0)}_1 < \pi^{(1)}_1$.

Given π_2, let $\bar{\pi}^{SP}_1(\pi_2)$ be the minimizer of $TC(\cdot, \pi_2)$, i.e., $\frac{\partial}{\partial \pi_1} TC(\bar{\pi}^{SP}_1(\pi_2), \pi_2) = 0$. As in the proof of Theorem 4.5, differentiating the left and right hand side with respect to π_2 yields $\frac{\partial^2}{\partial \pi_1 \partial \pi_2} TC(\bar{\pi}^{SP}_1(\pi_2), \pi_2) + \frac{\partial^2}{\partial \pi_1^2} TC(\bar{\pi}^{SP}_1(\pi_2), \pi_2) \times \frac{\partial}{\partial \pi_2} \bar{\pi}^{SP}_1 = 0$. It follows that

$$\frac{\partial}{\partial \pi_2} \bar{\pi}^{SP}_1 = -\frac{\frac{\partial^2}{\partial \pi_1 \partial \pi_2} TC(\bar{\pi}^{SP}_1(\pi_2), \pi_2)}{\frac{\partial^2}{\partial \pi_1^2} TC(\bar{\pi}^{SP}_1(\pi_2), \pi_2)} = -\frac{\frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\bar{\pi}^{SP}_1(\pi_2), \pi_2)}{\frac{\partial^2}{\partial \pi_1^2}(P - 1/2S)(\bar{\pi}^{SP}_1(\pi_2), \pi_2)} < \frac{\partial}{\partial \pi_2} \bar{\pi}_1 < 0.$$

Therefore, $0 > \frac{\partial}{\partial \pi_1} (\bar{\pi}^{SP}_1)^{-1} > \frac{\partial}{\partial \pi_1} \bar{\pi}^{-1}_1$. Since $\bar{\pi}^{SP}_2(\pi^{\ell_2}) = \pi^{(1)}_2 < \pi^{(0)}_2 = \bar{\pi}_2(\pi^{\ell_2})$ and $\frac{\partial}{\partial \pi_2} \bar{\pi}^{SP}_1 < \frac{\partial}{\partial \pi_2} \bar{\pi}_1$, we get $\bar{\pi}^{SP}_2 < \bar{\pi}_2$ on $[\pi^{\ell_2}, 1]$. Analogously, $(\bar{\pi}^{SP}_1)^{-1} > \bar{\pi}^{-1}_1$ on $[\pi^{\ell_1}, \pi^{SP}_{1,r}]$, where $\pi^{SP}_{1,r} := \bar{\pi}^{SP}_1(0)$. By continuity, there exists $\pi^{*,SP}_1 \in (\pi^{(1)}_1, \pi^{SP}_{1,r}]$ such that $(\bar{\pi}^{SP}_1)^{-1}(\pi^{*,SP}_1) = \bar{\pi}^{SP}_2(\pi^{*,SP}_1)$ and $\pi^{*,SP}_1 > \pi^*_1$. This completes the proof of the theorem. $\square$

Proof of Proposition 5.3

It is enough to observe that $EC_i(\pi) + T_i(\pi) - TC(\pi)$ does not depend on π_i. Hence, for any π_{-i} the allocation π_i that minimizes $EC_i(\pi_i, \pi_{-i}) + T_i(\pi_i, \pi_{-i})$ also minimizes $TC(\pi_i, \pi_{-i})$. $\square$

Proof of Proposition 5.2

By symmetry, $TC^*(d/2 - \varepsilon) = TC^*(d/2 + \varepsilon)$. It follows that $\frac{d}{2}$ is a critical point for $TC^*(\cdot)$.

Notice that if ℓ (resp. ℓ_1) is the distance to liquidation for the bank with asset value w and debt d (resp. $w_1 := \frac{w}{2}$ and d_1), then a bank with asset value $w_2 := \frac{w}{2}$ and debt $d_2 := d - d_1$ has distance to liquidation $\ell_2 := 2\ell - \ell_1$. Let $\pi^*_1(\ell_1)$ and $\pi^*_2(\ell_1)$ be the allocations in equilibrium for bank 1 and bank 2 in a system with two banks of equal size $\frac{w}{2}$ and distance of liquidation ℓ_1 and ℓ_2. Next, we prove that $\frac{\partial}{\partial \ell_1} \pi^*_1|_{\ell_1 = \ell} = -\frac{\partial}{\partial \ell_1} \pi^*_2|_{\ell_1 = \ell}$. Because $\pi^*_1(\ell_1)$ and $\pi^*_2(\ell_1)$ solve the system of equations $\frac{\partial}{\partial \pi_1} P(\pi_1, \pi_2) = 0$, $\frac{\partial}{\partial \pi_2} P(\pi_1, \pi_2) = 0$, differentiating with respect to ℓ_1 yields

$$\begin{aligned} \frac{\partial^2}{\partial \pi_1 \partial \ell_1} P(\pi^*_1, \pi^*_2) + \frac{\partial^2}{\partial \pi_1^2} P(\pi^*_1, \pi^*_2) \frac{\partial}{\partial \ell_1} \pi^*_1 + \frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\pi^*_1, \pi^*_2) \frac{\partial}{\partial \ell_1} \pi^*_2 = 0, \\ \frac{\partial^2}{\partial \pi_2 \partial \ell_1} P(\pi^*_1, \pi^*_2) + \frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\pi^*_1, \pi^*_2) \frac{\partial}{\partial \ell_1} \pi^*_1 + \frac{\partial^2}{\partial \pi_2^2} P(\pi^*_1, \pi^*_2) \frac{\partial}{\partial \ell_1} \pi^*_2 = 0. \end{aligned} \tag{B.2}$$

This is a linear system in $\frac{\partial}{\partial \ell_1}\pi_1^*$ and $\frac{\partial}{\partial \ell_1}\pi_2^*$. After evaluating its solution at $\ell_1 = \ell$, and noticing that $\pi_1^*(\ell) = \pi_2^*(\ell)$, we get that $\frac{\partial}{\partial \ell_1}\pi_1^*|_{\ell_1=\ell} = -\frac{\partial}{\partial \ell_1}\pi_2^*|_{\ell_1=\ell}$.

Consider now TC^* as a function of (ℓ_1, π_1, π_2). Recall that $\frac{\partial}{\partial \pi_1}TC^*|_{\ell_1=\ell,\pi_1=\pi_1^*(\ell),\pi_2=\pi_2^*(\ell)} = 0$, $\frac{\partial}{\partial \pi_2}TC^*|_{\ell_1=\ell,\pi_1=\pi_1^*(\ell),\pi_2=\pi_2^*(\ell)} = 0$. Furthermore, it can be easily verified that $\frac{\partial^2}{\partial \ell_1^2}TC^*|_{\ell_1=\ell,\pi_1=\pi_1^*(\ell),\pi_2=\pi_2^*(\ell)} = 0$. It follows that, after evaluating at $\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)$,

$$
\begin{aligned}
\frac{d^2}{d\ell_1^2}TC^* &= 2\frac{\partial^2}{\partial \ell_1 \partial \pi_1}TC^* \frac{\partial}{\partial \ell_1}\pi_1^* + 2\frac{\partial^2}{\partial \ell_1 \partial \pi_2}TC^* \frac{\partial}{\partial \ell_1}\pi_2^* \\
&\quad + \frac{\partial^2}{\partial \pi_1^2}TC^* \left(\frac{\partial}{\partial \ell_1}\pi_1^*\right)^2 + \frac{\partial^2}{\partial \pi_2^2}TC^* \left(\frac{\partial}{\partial \ell_1}\pi_2^*\right)^2 + 2\frac{\partial^2}{\partial \pi_1 \partial \pi_2}TC^* \frac{\partial}{\partial \ell_1}\pi_1^* \frac{\partial}{\partial \ell_1}\pi_2^* \\
&= 2\left(\frac{\partial^2}{\partial \ell_1 \partial \pi_1}TC^* - \frac{\partial^2}{\partial \ell_1 \partial \pi_2}TC^*\right)\frac{\partial}{\partial \ell_1}\pi_1^* \\
&\quad + \left(\frac{\partial^2}{\partial \pi_1^2}TC^* + \frac{\partial^2}{\partial \pi_2^2}TC^* - 2\frac{\partial^2}{\partial \pi_1 \partial \pi_2}TC^*\right)\left(\frac{\partial}{\partial \ell_1}\pi_1^*\right)^2.
\end{aligned}
$$

From Theorem 4.5 we get that $\frac{\partial}{\partial \ell_1}\pi_1^* < 0$. Hence, we need to prove that

$$
2\left(\frac{\partial^2}{\partial \ell_1 \partial \pi_1}TC^* - \frac{\partial^2}{\partial \ell_1 \partial \pi_2}TC^*\right) + \left(\frac{\partial^2}{\partial \pi_1^2}TC^* + \frac{\partial^2}{\partial \pi_2^2}TC^* - 2\frac{\partial^2}{\partial \pi_1 \partial \pi_2}TC^*\right)\left(\frac{\partial}{\partial \ell_1}\pi_1^*\right) > 0.
$$

Rewrite TC^* as $P + M = 2P - S$, where P is the potential function, S the sum of idiosyncratic terms in the potential function and M the mixed term. Since $\frac{\partial}{\partial \ell_1}\pi_1^*$ solves equations B.2, the expression simplifies as

$$
2\left(\frac{\partial^2}{\partial \ell_1 \partial \pi_1}M - \frac{\partial^2}{\partial \ell_1 \partial \pi_2}M\right) - \left(\frac{\partial^2}{\partial \pi_1^2}S + \frac{\partial^2}{\partial \pi_2^2}S\right)\left(\frac{\partial}{\partial \ell_1}\pi_1^*\right).
$$

It is enough now to show that both terms are positive. $\frac{\partial^2}{\partial \pi_1^2}S > 0$ and $\frac{\partial^2}{\partial \pi_2^2}S > 0$ because the Nash equilibrium minimizes the idiosyncratic terms of the potential function. For $\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)$ we have

$$
\frac{\partial^2}{\partial \ell_1 \partial \pi_1}M - \frac{\partial^2}{\partial \ell_1 \partial \pi_2}M = -2w_1^2((\pi_1^*)^2\gamma_1 + (1-\pi_1^*)^2\gamma_2)E[(Z_1 - Z_2)1_{\{\ell+\pi_1^* Z_1 + (1-\pi_1^*)Z_2 \le 0\}}].
$$

Hence, we are left to show that the expectation is negative. This follows immediately from the fact that $\pi_1^* \geq \frac{1}{2}$. □

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