The Macroeconomic Implications of Limited Arbitrage

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Abstract

We develop a tractable model to study the macroeconomic impacts of limited arbitrage by linking arbitrage activities with the macroeconomy through collateralization. We show that the interactions between speculative trading and the business cycle can work as a powerful transmission mechanism, where trivial shocks spread, amplify, and trigger simultaneous arbitrage failures and recessions. Collateralization adds extra value to real-sector investments, and ultimately helps boost aggregate production. We solve for the model dynamics analytically and characterize multiple equilibria. Through regime shifts, we account for the non-linear aspects of financial crises as well as the slow and incomplete post-crisis recoveries.

Keywords: limits of arbitrage, financial crises, mispricing, slow recovery, regime shifts, multiple equilibria, collateralization, externality

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1 Introduction

Recent financial crises have featured simultaneous disruptions and slow recoveries in both financial markets and the macroeconomy. In some cases, disturbances from the real sector spread quickly to financial markets and trigger arbitrage failures. For example, by August 2007, what began as some bad news about the souring of the subprime mortgage market had evolved into a full-fledged financial crisis. The plummeting collateral value of illiquid assets forced arbitrageurs to deleverage and unwind their speculative positions. Consequently, the price spreads between similar assets soared, and eventually led to arbitrage crashes.[1] Arbitrage failures of financial intermediaries also gave rise to recessions in the real economy. Acharya and Steffen (2015) find that the European banking crisis can be explained by an arbitrage failure in the "carry trade" of Eurozone banks. Those banks had undertaken long positions in high-yield peripheral sovereign bonds and short positions in low-yield German bunds. They bet on the convergence of those yields. However, the yield spreads continued to diverge, inflicting heavy mark-to-market losses on banks in both legs. In the aftermath of these arbitrage crashes, banks lost on average 70% of their market value, followed by severe contractions in corporate lending and aggregate output.[2] Moreover, not only had recessions lasted longer[3], but noticeably slow recoveries were also evident in many asset markets. Du et al. (2018), for instance, document that deviations from covered interest parity, which skyrocketed during the global financial crisis and the European debt crisis, remained large in magnitude in the post-crisis era as well.

Theoretical literature contains two separate approaches to study the relationship between financial crises and limits of arbitrage: The finance literature focuses on how frictions hinder arbitrage and induce financial market instability; the macroeconomic literature examines how frictions amplify shocks and lead to output contraction with limits to arbitrage in production. The links between arbitrage activities and the macroeconomy, however, are not well understood, especially their role on the transmission of external shocks into simultaneous disruptions in both financial and real sectors.

To fill this gap, we develop a unified framework to examine the interaction between limited arbitrage and the business cycle, and how it affects aggregate production and financial stability. We investigate the impact of arbitrage activities on the macroeconomy, especially on its vulnerability to certain shocks. The model implications help us better understand how arbitrage failures can lead to contractions in the real economy, and how tiny macroeconomic shocks might trigger simultaneous arbitrage crashes and recessions. We also provide a complementary perspective to explain the slow or incomplete post-crisis recoveries in both the real and financial

[1] Mitchell and Pulvino (2012) find that the tumbling collateral value of corporate securities and secured bonds - mortgage-backed securities (MBS), asset-backed securities (ABS), collateralized debt obligations (CDO), etc. - severely impaired primary brokers' financing capability to support arbitrageurs. Consequently, arbitrageurs were unable to maintain similar prices of similar assets.

[2] Acharya et al. (2014) and Acharya and Steffen (2015) find that lending to private sectors contracted substantially in Greece, Ireland, Italy, Portugal, and Spain. In Ireland, Spain, and Portugal, the overall lending volume of newly issued loans fell by 82%, 66%, and 45%, respectively, over the 2008–2013 period. Acharya et al. (2018) present firm-level evidence showing that the lending contraction of banks affected by the crisis depressed investments, job creation, and sales growth of the firms that had significant business relationships to these banks.

[3] Reinhart and Reinhart (2010) and Ball (2014), among others, document highly persistent drops in activity following severe financial crises, with little evidence of an eventual recovery in output back to trend.

sectors of the economy.

To this end, we incorporate convergence trades into a conventional neoclassical growth model via collateralization. Rather than modelling the market collapses through explicit credit cycles, we focus on the breakdown of arbitrage transactions caused by regime shifts. We assume that financially constrained arbitrageurs can collateralize their real-sector investment to support their convergence trades. A shock-triggered regime shift inflicts initial losses in arbitrageurs' speculative positions, forcing them to reduce their capital investment. This leads to a scarcity of collateral for speculation, and widens the price gaps between similar assets. Arbitrageurs thus experience the worst losses when arbitrage opportunities are most profitable. Meanwhile, financial distress also translates into further cuts in capital investment, inducing rapid output contractions. Serving as collateral for speculation adds extra shadow value to the real assets. This in turn increases the marginal return of capital, and ultimately boosts aggregate production. Nevertheless, through regime shifts, such interaction also gives rise to the transmission channel in which tiny shocks can spread, amplify, and eventually instigate simultaneous arbitrage failures and recessions. The economy is thus vulnerable to systemic risks, even with very modest leverage ratios.

In particular, we consider an infinite horizon economy in which household investors from two segmented markets have different asset demands. This leads to price discrepancies between identical assets, and creates arbitrage opportunities for intermediaries with accesses to both markets. While intermediaries can profit from exploiting price differences, their capability to do so is hindered by the collateral requirement. Such a constraint arises naturally because the counterparty cannot compel intermediaries to honor their commitments unless the asset positions are secured. For simplicity, we further assume that, besides trading as arbitrageurs, intermediaries also invest in the real economy. Specifically, they invest capital in production, and pledge it as collateral for their speculative positions. This is consistent with the real-world practice of using corporate securities or structured products originated from real sectors as collateral in the financial market. In this way, the collateral constraints limit intermediaries' trading volume as a function of their capital investment in the production sector.

By deriving a closed-form solution to the model dynamics, we conclude that, in the absence of regime shifts, there is a mutually beneficial relationship between arbitrage activities and aggregate production. Arbitrage profits can be used as extra financing for capital investment. By exploiting the price spreads across markets, intermediaries are essentially obtaining short-term loans from households with zero or even negative interest rates. This effectively lowers the marginal cost of capital and encourages producers to expand their scales. In turn, the amplified capital investment also provides collateral to support more arbitrage trades, allowing for a more liquid market and relatively lower levels of mispricing.

We show, however, that unexpected regime shifts can cause severe disruptions in both financial and real sectors. In particular, under certain parameterizations, there are two distinct steady states, which correspond to different regimes of the economy: The "good" regime features higher trading volume and a narrower price gap between segmented markets, while the "bad" one features less trading and a wider price gap. As more trades support better risk-sharing, there is a Pareto improvement if the economy moves from a bad to a good regime. However, crises can happen when a tiny shock triggers the opposite shift. Consider, for example, the economy is initially in a good regime. A negative shock might inflict only slight losses on intermediaries at first, forcing them to reduce arbitrage positions. As intermediaries cannot internalize the price impact, the collective unwinding widens the price gap further against their initial positions. The resulting losses amplify even more when markets panic and move toward a bad regime. This is because the price gap would widen further to match the bad regime level. Meanwhile, intermediaries are still carrying the large initial positions inherited from the good regime. The product of these two large quantities becomes their realized arbitrage losses. Under financial distress, they have to reduce their capital *en masse*. This not only disrupts aggregate production, but also creates a shortage of collateral for restoring liquidity.

At first glance, it seems that crises like this only occur when the economy, fuelled by pessimistic market sentiment, shifts from a good to a bad regime. However, similar scenarios may also arise even when the economy moves to a good regime. In fact, regardless of market sentiment, as long as the post-shock regime features a lower liquidity than the original one, intermediaries will suffer amplified losses. Especially, if their pre-shock trading volume happens to be large, the subsequent financial distress could pose a serious threat to the overall stability, presenting policy makers with a trade-off between liquidity and fragility.

Moreover, regime shifts can complicate and derail post-crisis recoveries. We show that, in some cases, while aggregate production slowly realizes a full recovery, large mispricing may persist, and the total trading volume only rebounds to a much lower level. This is consistent with the stylized fact that many asset markets after 2008 witnessed a limited recovery in terms of liquidity and the correction of price anomalies. This phenomenon persisted long after aggregate output had slowly rebounded to pre-crisis levels.

To the best of our knowledge, this paper is the first theoretical work to study the interaction between limited arbitrage and the macroeconomy, and evaluate its overall economic impact. We link convergence trades with aggregate production by capturing the collateralization with real assets. Through regime shifts, we investigate the transmission channel that is accountable for a simultaneous arbitrage failure and recession. We provide complementary explanations for slow and incomplete post-crisis recoveries. Moreover, our model is highly tractable, with closed-form solutions to the model dynamics and multiple equilibria.

1.1 Related Literature Review

This paper complements a growing theoretical literature on the limits of arbitrage, especially the strand that stressess arbitrageurs' financial constraints. We contribute to this literature by capturing the practice of collateralization with productive capital, and examining the interaction of limited arbitrage with the real economy. The closest articles to our paper are Gromb and Vayanos (2002, 2018). We pattern our setup of market segmentation on their work. Gromb and Vayanos (2002, 2018) propose equilibrium models in which financially constrained arbitrageurs exploit price discrepancies across segmented markets while providing market liquidity. The financial constraints limit the liquidity supply as a function of arbitrageurs' wealth. One of the major differences between our model and theirs is that we allow a broader range of assets to serve as collateral, as opposed to only the riskless asset. This enables us to capture the practice of securitization, and model the transmission channel that is accountable for the simultaneous arbitrage crashes and recessions. Also, Gromb and Vayanos (2018) derive self-correcting dynamics in which arbitrageurs can fully recover from their initial losses after a negative shock as the asset returns increase. In contrast, we focus on the analytical characterization of multiple equilibria, and the corresponding regime shifts that may lead to incomplete recoveries.

Shleifer and Vishny (1997) were the first to study how financial frictions affect arbitrageurs' capability to eliminate price anomalies. Due to asymmetric information and moral hazard, arbitrageurs bear insolvency risk under margin requirements. Xiong (2001) and Kyle and Xiong (2001) examine the impact of arbitrage capital on asset prices by analyzing the wealth effects of arbitrageurs with log utility in a continuous-time model. And Liu and Longstaff (2004) further study the optimal arbitrage strategy of collateral-constrained arbitrageurs in a partial equilibrium. In Basak and Croitoru (2000, 2006), the mispricing arises endogenously when all agents are portfolio-constrained. Brunnermeier and Pedersen (2009) study the feedback loops of arbitrageurs' funding liquidity and market liquidity, and how they interact through the collateral constraints. Our model differs primarily in the source of arbitrageurs' funding, which comes from arbitrage profits rather than direct borrowing. In Kondor (2009), arbitrageurs face uncertainty over the time point at which the arbitrage opportunity vanishes. Their collective optimal investment strategies may exacerbate mispricing even when they are not collateral constrained. Garleanu and Pedersen (2011) consider collateral constraints in an infinite-horizon setting with multiple assets. They demonstrate that securities with higher margin requirements yield higher expected returns, and are more responsive to changes in the wealth of collateral-constrained agents. In He and Krishnamurthy (2012, 2013), arbitrageurs can raise funds from less sophisticated investors to invest in a risky financial security. This external funding must stay below an exogenous ratio of their own wealth. In Brunnermeier and Sannikov (2014), arbitrageurs are also more efficient holders of productive capital, and can trade a risky claim to that capital. Hugonnier and Prieto (2015) discuss the effects of risky arbitrage on asset pricing and risk sharing. Finally, Kondor and Vayanos (2019) study the interactions among arbitrageurs' wealth, liquidity supply, and assets risk premia, and show that arbitrageurs' capital is the single priced risk factor.

We note that our framework is also related to macroeconomic models that stress the pecuniary externality. In our paper, when intermediaries collectively reduce their arbitrage volume, the price spread widens and moves against their initial positions. The resulting financial distress curbs investment growth in aggregate production and triggers more aggressive unwinding in the financial markets. Some recent work underscores similar externalities, as borrowers do not internalize the impact of their own leverage decisions on systemic risk. Examples include Lorenzoni (2008), Bianchi (2011), Brunnermeier and Sannikov (2014), Chari and Kehoe (2016), and Schmitt-Grohé and Uribe (2016).

This paper also extends the literature on the slow recovery from the Great Recession through regime shifts. Shimer (2012) and Fajgelbaum et al. (2017), among others, study the implications of multiple equilibria and discuss the post-shock recovery with transitions to new steady states.

The remainder of this paper is organized as follows. Section 2 introduces the baseline model, while section 3 characterizes the equilibrium. In section 4, we discuss the steady states, and in section 5, we outline the existence and statics of multiple equilibria. Section 6 extends the multiple equilibria analysis to various impulse responses and recovery patterns following unexpected shocks. Section 7 concludes. The Appendix includes all the proofs.

2 Baseline Model

2.1 Representative Households

We consider two segmented markets, A and B, in an infinite horizon economy with one perishable good. Each market is populated by an identical continuum of representative households (hereinafter HH), which constitute a measure of l. In each period, individual HH from market i receive exogenous natural endowments e_t^i.

$$e_t^i = b + u_{t-1}^i \theta_t, \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

where b is a constant, and $u_{t-1}^i \theta_t$ is the endowment shock. In particular, $\{\theta_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables that follows a symmetric distribution around zero on a bounded support $\mathcal{S} = \left[-\bar{\theta}, \bar{\theta}\right]$, where $\bar{\theta} > 0$. In brief, we denote θ_t as the shock unit, and u_{t-1}^i as the shock intensity. The latter is always revealed one period earlier, so HH know their hedging demand in advance.

We assume that the shock intensity in the two markets is constant, and identical in magnitude, but opposite in direction:

$$u_t^{\mathrm{A}} = -u_t^{\mathrm{B}} = u > 0.$$

Without further intermediation, the consumption paths in the two markets are perfectly negatively correlated. Thus, HH from different markets have opposite hedging demands.

HH's expected utility is given by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \log \left(C_t^i \right) \right], \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

where C_t^{i} is HH's time t consumption in market i, and $0 < \beta < 1$. To ensure positive consumptions for HH, we assume that $b - u\bar{\theta} > 0$.

2.2 Financial Assets

Within each market, there exists an identical financial asset that is long-lived, in zero net supply, and pays out a dividend equal to θ_t in t. Because this dividend exactly mimics the shock unit in each period, the asset can serve as a perfect hedging instrument for HH.

Due to HH's opposite hedging demands, the asset prices, i.e., P_t^i, will differ across markets without further intermediation. As HH in market A always experience a positive amount of shock units, i.e., $u = u_t^{\mathrm{A}} > 0$, they are eager to sell the asset in order to neutralize their endowment shocks. To the extent that market A has negative asset demands, while market B has positive ones, prices in market A tend to be lower than those in market B. We define the price difference between the two markets as:

$$\phi_t := P_t^{\mathrm{B}} - P_t^{\mathrm{A}}.$$

2.3 Intermediaries

Outside the two markets, there also exists a continuum of competitive, risk-averse, and infinitely lived intermediaries (hereinafter IM). Their total measure is normalized to be one. Unlike HH, IM can trade financial assets simultaneously in both markets. Thus, the price gaps create arbitrage opportunities for them. They can obtain immediate profits by entering long positions in the low-price market, and by going short in the high-price market. In this process, they also provide market liquidity to HH in both markets. For simplicity, we assume further that IM will incur prohibitive costs if they fail to take balanced positions across markets. We denote x_t as IM's position in market A, thus:

$$x_t := x_t^{\mathrm{A}} = -x_t^{\mathrm{B}}.$$

Following Gromb and Vayanos (2002, 2018), we use x_t as a measure of market liquidity.

Similarly to Brunnermeier and Sannikov (2014), we assume that IM are also more efficient holders of productive capital. They are uniquely able to convert consumption goods into durable capital on a one-to-one basis, and vice versa. They run the production sector in the economy by providing capital input, and by hiring HH from both markets as labor. The output function follows a Cobb-Douglas form:

$$Y(K_t, L_t) = aK_t^{\alpha} L_t^{\gamma},$$

where a is the total productivity factor, α and $\gamma = 1 - \alpha$ are the output elasticity of capital K_t and labor L_t, respectively. In addition, capital depreciates at a rate of $\delta \in (0, 1)$. IM compensate HH with a competitive wage for their labor. Figure 1 illustrates the basic setup in the economy.

IM's expected utility is given by:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \rho^t \log \left(C_t^{\mathrm{IM}} \right) \right],$$

where C_t^{IM} is IM's time t consumption, and $0 < \rho < 1$.

2.4 Collateral Constraints

IM face collateral requirements when they arbitrage across markets—they must pledge enough collateral such that they have no incentive to escape from their liability in the next period. This constraint arises from the limited-liability enforcement: IM won't get excluded from future trading even if they renege on their obligations. When IM take arbitrage positions, their portfolios have negative value in the following period, whereas HH's positions have positive value. This is because financial assets here are long-lived; unlike one-period assets, whose value collapses to

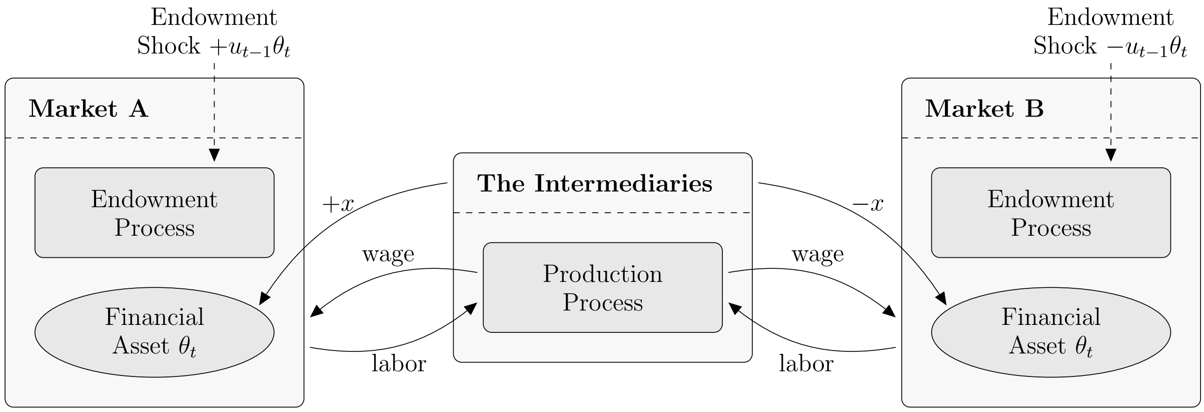

Figure 1: The structure of the baseline model.

zero in the future, they remain valid and may bear value in all following periods. Therefore, if IM refuse to honor their previous trading contracts, HH's positions will become worthless.

In particular, IM can pledge their capital investment; but only the residue part from depreciation–$1-\delta$–counts as effective collateral. This is consistent with limited liability literature (see e.g., Kübler and Schmedders (2003) and Chien and Lustig (2010)): in case of default, HH can confiscate IM's depreciated capital, but not their capital rent—their "labor" income. Also, δ can be interpreted as the capital haircut, reflecting the liquidity discount relative to cash.

IM's collateral constraint is thus given by:

$$(1-\delta)K_t - x_t\phi_{t+1} \geq 0,$$

where $-x_t\phi_{t+1} = \sum_{i\in\{\mathrm{A,B}\}} x_t^i P_{t+1}^i$ is the liquidation value of their previous positions, or HH's losses in case of default.

Our financial constraint differs with Gromb and Vayanos (2002, 2018) mainly in two aspects. First, we expand the collateral set to include capital, which is illiquid to HH and links real sectors. We can thus capture the collateralization practice in reality and examine its impact on price discovery and aggregate economy. Second, in Gromb and Vayanos (2002, 2018), agents have finite investment horizons for any given risky asset pair. This makes dividend payments a dominant part of IM's liability. So their margin constraint focuses on zero cross-netting of dividend payments—IM must cover for the maximal possible dividend payments market-by-market. In our model, both agents and assets have infinite horizon; individual dividend is small in magnitude relative to IM's liability. Our collateral constraint is thus to prevent IM from making HH's positions invalid by dishonouring previous contracts. It allows for partial cross-netting—IM don't have to collateralize for dividend payments that eventually offset each other.

2.5 Utility Optimization and Equilibrium

Since HH draw no utility from leisure, the labor input is constant, $L_t = L = 2l$. Given initial K_{-1} and x^i_{-1}, IM's optimization problem can be written as:

$$\max_{C_t^{\mathrm{IM}}, x_t^i, K_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \rho^t \log\left(C_t^{\mathrm{IM}}\right) \right], \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

subject to the budget constraint:

$$C_t^{\mathrm{IM}} + K_t = \underbrace{\sum_{i \in \{\mathrm{A},\mathrm{B}\}} x_{t-1}^i P_t^i}_{\text{obligations}} - \underbrace{\sum_{i \in \{\mathrm{A},\mathrm{B}\}} x_t^i P_t^i}_{\text{arbitrage gain}} + F\left(K_{t-1}\right) + (1-\delta) K_{t-1}$$

$$= \underbrace{-x_{t-1}\phi_t}_{\text{obligations}} + \underbrace{x_t \phi_t}_{\text{arbitrage gain}} + F(K_{t-1}) + (1-\delta)K_{t-1}, \tag{1}$$

and the collateral constraint:

$$(1-\delta)K_t - x_t \phi_{t+1} \geq 0, \tag{2}$$

where $F\left(K_{t-1}\right) = a(1-\gamma)K_{t-1}^{\alpha} L^{\gamma}$ is IM's capital rent.

Note that HH cannot invest in physical capital. They earn their labor income from the production process. Similarly to Gromb and Vayanos (2002, 2018)'s setting, they are subject only to budget constraints. Moreover, given initial positions y^i_{-1}, in each period t, they choose their consumption C_t^i and asset position y_t^i to solve the following problem:

$$\max_{C_t^i, y_t^i} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \log\left(C_t^i\right) \right], \quad i \in \{\mathrm{A}, \mathrm{B}\},$$

subject to

$$C_t^i = \underbrace{\left(y_{t-1}^i \left(P_t^i + \theta_t\right) - y_t^i P_t^i\right)}_{\text{income from trading financial assets}} + \underbrace{a\gamma K_{t-1}^{\alpha} L^{\gamma-1}}_{\text{labor income}} + \underbrace{\left(b + u_{t-1}^i \theta_t\right)}_{\text{endowment}}$$
$$= P_t^i \left(y_{t-1}^i - y_t^i\right) + a\gamma K_{t-1}^{\alpha} L^{\gamma-1} + b + \left(u_{t-1}^i + y_{t-1}^i\right)\theta_t. \tag{3}$$

Ideally, HH prefer to take a position equal to $y_t^i = -u_t^i$ in period t, so that they are fully protected from endowment shocks in $t+1$.

Our notion of equilibrium is standard. It is a collection of prices, consumption plans, capital investment, and asset positions, such that 1) each agent maximizes her utility given the prices and subject to her constraints, and 2) both markets for the risky asset clear.

To keep the model parsimonious, we do not model explicitly the risk-free storage technology. However, we show that when there is a safe asset accessible to all agents in the economy, all the following model implications still hold quantitatively.[4]

[4] We prove that in an otherwise identical economy, where there is a safe asset with relatively low return and

2.6 Interpretation

Our assumptions fit settings in which assets with similar payoffs are traded at different prices in markets with certain degrees of segmentation. These include Siamese twin stocks, covered interest arbitrage across currencies, corporate bonds, credit default swaps (CDS), and sovereign bonds.[5] Our setting, for example, can model covered interest arbitrage by interpreting the two assets as a currency forward and its synthetic counterpart. HH in market A represent hedgers in the forward market who have little expertise or access to synthetic forwards, as illustrated in Borio et al. (2016). HH in market B resemble investors in the synthetic forward market who are unable to trade forwards due to restricted mandates. Likewise, the price gaps between corporate bonds and the corresponding CDS, as documented by Duffie (2010), can also be partially explained by market segmentation. Individual investors have access to corporate bonds, but not to CDS. Moreover, sovereign bonds with similar payoff structures can be traded at significantly different yields. In this context, HH capture investors who are mandated to hold bonds with certain coupon rates and times to maturity, or with a "home bias" such as insurance companies, pension funds, and domestic banks.

IM in our model correspond to arbitrageurs and market makers in financial markets who also invest or intermediate in real economic activities. Examples are hedge funds, investment banks, dealer-broker banks with proprietary trading desks, and influential regional commercial banks who engage in the "carry trade" of sovereign debt[6].

We use IM's capital to model those illiquid, real-sector assets or projects intermediated by IM, especially those IM have significant advantage over other investors in managing. Thus, it does not represent all the real investments in the economy, but only those relying on IM's financial conditions and expertise for intermediation. Investment banks, for example, may be better at reducing risks by having a more diversified investment pool; they can afford more R&D efforts in deciphering payoff distributions of otherwise opaque, complicated assets and in profiting from regulatory limits; they can better securitize, finance, and collateralize illiquid investments; etc. Given IM's growing influence in real sectors, their intermediation capacity is relevant to the aggregate economic growth.

Other investors, including HH in this model, can also invest in productive assets. Yet these investments are not captured by IM's capital: their outputs can be reflected by—perhaps a time-varying version of—HH's endowment b in our model. Also, assuming HH's indirect investment in production through labor is only for symbolic reflection—L_t may as well represent other production factors: lands, funding via equity or debt, machines, market access, etc. We use this form to stress that HH and IM's production inputs are not perfect substitute.

Opaque payoffs and illiquidity of real assets often make it hard for non-specialist investors to evaluate their fair prices; these assets may not even be tradable among non-institutional investors. So we model IM as the marginal investors and first-best users of capital; and the

HH have exponential utility (to separate wealth effects on asset prices), all the lemma, corollary and propositions 3-8 still hold, except price gaps need minor adjustments. The proof is available upon request.

[5]See section II.B.3 in Gromb and Vayanos (2018) for more examples and illustrations.

[6]See Acharya and Steffen (2015).

capital price in our setting equals unity—their marginal cost. Admittedly, this simple setting limits us in studying the capital fire-sales in our crisis analysis. Since our main focus is to reveal the macroeconomic impacts of IM's speculative trading on aggregate output, investments and employment, the pure quantity of their capital investment is sufficient to serve this purpose.[7]

Specifically, the capital in our settings fits a wide range of investments/financing in the real sectors, such as bank or firm loans, corporate bonds, certificates of deposit, covered bonds, and securitized products like MBS, ABS, CDOs, and collateralized loan obligations (CLOs). As illustrated in Gorton and Metrick (2012) and Mitchell and Pulvino (2012), it is a common practice for hedge funds or prime brokers to pledge corporate securities or structured products as collateral in order to obtain financing or fulfill margin requirements. In light of the increasing use of non-marketable real assets as collateral in the financial markets, this setup captures the ongoing trend of collateralization in recent years.[8]

3 Equilibrium Characterization

In this section, we explore the model implications of asset prices, market liquidity supply, collateral value, and equilibrium dynamics. We begin with IM's Euler equations:

$$1 = C_t^{\mathrm{IM}} \mathbb{E}_t \frac{\rho}{C_{t+1}^{\mathrm{IM}}} \left(F'\left(K_t\right) + 1 - \delta \right) + \bar{\lambda}_t (1 - \delta), \tag{4}$$

$$\phi_t = C_t^{\mathrm{IM}} \mathbb{E}_t \frac{\rho \phi_{t+1}}{C_{t+1}^{\mathrm{IM}}} + \bar{\lambda}_t \phi_{t+1}, \tag{5}$$

where $\bar{\lambda}_t = \lambda_t C_t^{\mathrm{IM}} \geq 0$ is the shadow collateral price, and λ_t is the Lagrange multiplier for the collateral constraint (2).

Equations (4) and (5) are IM's first-order conditions with respect to capital investment K_t and trading volume x_t. With binding collateral constraints, IM's capital investment bears extra collateral value $\bar{\lambda}_t(1-\delta)$. We can interpret δ as the haircut rate of corporate securities used as collateral. Equation (4) indicates that, because of the shadow collateral value, IM's capital now has a higher marginal return than what would result from pure production. Likewise, the marginal benefit of taking an additional position is the immediate arbitrage gain, measured by ϕ_t. This must be counterbalanced by the next period obligation and the collateral cost $\bar{\lambda}_t \phi_{t+1}$.

Similarly, because HH are unconstrained, their first-order conditions lead to the following standard pricing relationship:

$$\frac{P_t^i}{C_t^i} = \beta \mathbb{E}_t \left[\frac{P_{t+1}^i + \theta_{t+1}}{C_{t+1}^i} \right], \quad i \in \{\mathrm{A}, \mathrm{B}\}. \tag{6}$$

Obviously, one trivial equilibrium is $x_t = u_t,\ \phi_t = 0,\ \forall t$. That is, IM provide full liquidity

[7]Zhang (2019) studies the interactions between capital fire-sales and that of arbitrage portfolios by incorporating endogenous capital price dynamics.

[8]Tamura and Tabakis (2013) document that the use of credit claims as collateral, i.e., non-marketable bank loans, has increased significantly since they were included in the Eurosystem's single list of eligible collateral in 2007.

and eliminate any price gaps at all times. IM realize no arbitrage profits, and their collateral constraints always remain slack. However, as we clarify later, for the settings in which alternative equilibria are possible, such a trivial equilibrium is not as robust as others. This is because, when competitive IM earn a profit of zero, they may not have an incentive to provide full liquidity. Especially in the real-world markets where transaction costs are proportional to the trading volume, such equilibrium is not feasible. In contrast, the equilibrium with binding collateral constraints leaves IM no other options except sticking to the equilibrium positions. We thus regard the trivial equilibrium as a degenerate case, and exclude it from our subsequent discussion.

3.1 Equilibrium Prices

Lemma 1. *Define $p_t^A(\theta_t)$ and $p_t^B(\theta_t)$ as the equilibrium prices in markets A and B, as functions of θ_t. It follows that:*

$$p_t^A(\varepsilon) = -p_t^B(-\varepsilon),$$

where $\varepsilon \in \left[-\bar{\theta}, \bar{\theta}\right]$.

Proposition 1. *In equilibrium, the asset prices are given by:*

$$P_t^A = -\frac{C_t^A}{C_t^A + C_t^B}\phi_t = -\left(\frac{1}{2} + \frac{(u - x_{t-1}/l)\,\theta_t}{2w\,(K_{t-1})}\right)\phi_t, \tag{7}$$

$$P_t^B = \frac{C_t^B}{C_t^A + C_t^B}\phi_t = \left(\frac{1}{2} - \frac{(u - x_{t-1}/l)\,\theta_t}{2w\,(K_{t-1})}\right)\phi_t, \tag{8}$$

and the price difference is:

$$\phi_t = \frac{2w\,(K_{t-1})}{M_t + (x_t - x_{t-1})\,/l}, \tag{9}$$

where:

$$w\,(K_{t-1}) := a\gamma K_{t-1}^{\alpha} L^{\gamma-1} + b, \quad M_t := \left(\mathbb{E}_t\left[\sum_{j=1}^{\infty}\beta^j \frac{\theta_{t+j}}{C_{t+j}^B}\right]\right)^{-1}.$$

Moreover, ϕ_t is independent of the realization of the shock units θ_t, $\forall t$.

Although prices in both A and B are decreasing functions of θ_t, the price difference as a whole is independent of the specific realization of the shock units, θ_t. From IM's perspective, because they take a net zero position in financial assets, their consumptions and first-order conditions are not affected by the realization of θ_t. We find further that the independence of ϕ_t on θ_t also holds for the cases where HH have CRRA utility, and where θ_t follows a two-point distribution.

3.2 Market Liquidity

Next, we discuss the market liquidity in the equilibrium.

Proposition 2. *Given shock intensity u and the distribution of $\{\theta_t\}$, there exists a unique threshold value $\bar{\rho} > 0$ for the IM's discount factor, such that:*

1. *If $\rho > \bar{\rho}$, then IM's positions are $x_t = ul$, and the price spread is $\phi_t = 0$ for all t.*
2. *Else, if $0 < \rho \leq \bar{\rho}$:*
   - *When IM's collateral constraints are binding, their asset positions x_t follow:*

$$x_t \in (0, ul) \quad \textit{and} \quad \phi_{t+1} = \frac{(1-\delta)K_t}{x_t}. \tag{10}$$

   - *When IM's collateral constraints are slack, then:*

$$x_t \leq 0, \quad \textit{and} \quad \frac{\phi_{t+1}}{\phi_t} = F'(K_t) + 1 - \delta. \tag{11}$$

Proposition 2 shows that IM's varying degrees of patience lead to different forms of equilibrium. In particular, if IM are sufficiently patient, markets expect they will accumulate enough collateral to eliminate all price gaps over the long run. Such expectation feeds back into the current equilibrium, resulting in a zero price spread from the beginning, even when their initial capital investment levels are low. Consequently, IM are always able to provide full liquidity and remove any price difference that arises from market segmentation. In this case, the economy resembles the one in the neoclassical growth model, with frictionless financial markets.

On the other hand, if IM are not as patient, i.e., $\rho < \bar{\rho}$, then two possible scenarios may arise. First, when IM's collateral constraints are binding, they provide market liquidity to HH by taking arbitrage positions. In other words, they satisfy HH's asset demands by entering long positions in the low-price market A, and short positions in the high-price market B. Due to their binding collateral requirement, IM can only provide HH with partial liquidity. Later, in the steady-state discussion, we illustrate that, as long as $\rho < \bar{\rho}$, the economy will eventually end up in this binding equilibrium, regardless of its initial states.

Second, when IM are not collateral-constrained, they compete with HH for market liquidity, instead of providing liquidity. In other words, they take opposite positions relative to the convergence trading. This special case is only possible in the first few periods, when IM are endowed with huge initial wealth. In order to smooth their consumptions, IM tend to save to transfer their large initial resources into future periods. IM thus treat financial assets as saving devices parallel to their capital investment, instead of as arbitrage instruments. Because IM's positions are opposite those of the convergence trades, they expect positive cash flows, instead of obligated payments, from the next period liquidation. Thus, IM are not subject to collateral constraints, and their capital input has no extra collateral value. Both serve as simple savings instruments. Thus, IM's asset investment and capital input should have the same marginal return in equilibrium, as indicated by Equation (11).[9]

[9] In an otherwise identical economy with risk-free assets, this no longer hold. We can prove that when IM's initial wealth exceeds certain threshold, i.e., $W_0 > \bar{W}$, IM will close out all the price gap at the beginning and for all future periods by providing full liquidity, i.e., $x_t = ul$, for all t. Thus, the equilibrium dynamic reduces to the degenerate case resembling the frictionless neoclassic growth model, given the risk-free asset's return is lower than that of the steady-state capital.

This result differs from that of Gromb and Vayanos (2018), who conclude that price spreads decrease with arbitrageurs' wealth until they are fully closed out. Here, due to the motive of consumption smoothing, however, IM's excessive wealth tends to induce greater scarcity in the liquidity supply, serving to further widen the price gaps. Such a disparity arises mainly from the different settings of IM's saving instruments' marginal returns. In Gromb and Vayanos (2018), arbitrageurs can resort to a risk-free asset with a constant interest rate, while in our model IM face decreasing marginal returns from their capital investment.

3.3 Model Dynamics

Below, we analytically derive the model dynamics of IM's consumption, wealth, and capital investment, when they are collateral-constrained.

Proposition 3. *With initial wealth $W_0 > 0$, when IM's collateral constraints are binding in t, their consumption and capital investments evolve according to:*

$$C_t^{IM} = (1-\alpha\rho)W_t, \quad K_t = \alpha\rho\mu_t W_t, \tag{12}$$

where the leverage for capital investment is:

$$\mu_t := \frac{\phi_{t+1}}{\phi_{t+1} - (1-\delta)\phi_t} > 1.$$

IM's wealth dynamics are given by:

$$W_{t+1} = F(K_t) + (1-\delta)K_t - x_t\phi_{t+1} = F(K_t) = F(\alpha\rho\mu_t W_t). \tag{13}$$

When IM's collateral constraints are binding, their remaining capital $(1-\delta)K_t$ exactly offsets their liquidation payments $-x_t\phi_{t+1}$. Thus, IM are left with capital rent $F(K_t)$ as wealth, which will be allocated later to consumption and savings.

Equation (12) reveals the myopic feature of IM arised from log utility. Independent of their income levels, they allocate fixed portions of their wealth to consumption and savings. In particular, they save $S_t = \alpha\rho W_t$. The savings ratio increases with the capital share and with IM's degree of patience to delay consumption. However, compared to their savings, IM's capital investment levels are much higher. The multiplying effect comes from their arbitrage income, $x_t\phi_t$. In fact, IM reinvest all their immediate arbitrage gains in the production sector. Thus,

$$K_t = \alpha\rho W_t + x_t\phi_t = \alpha\rho W_t + \frac{(1-\delta)K_t\phi_t}{\phi_{t+1}} = \alpha\rho\mu_t W_t.$$

The multiplier μ_t also measures IM's arbitrage profitability. The more profits IM obtain from speculations, the more they can leverage up their capital investment.

Proposition 3 demonstrates the mutually beneficial relationship between limited arbitrage and capital investment in the production sector. From a producer's perspective, the arbitrage gain $x_t\phi_t$ can be viewed as a one-period loan borrowed from HH. Likewise, the corresponding obligation $x_t\phi_{t+1}$ in the next period can be seen as the repayment that will be due. Because IM bring the prices closer over time through their trading, i.e., $\phi_{t+1} \leq \phi_t$, the effective interest rate of such a loan can be negative. Favourable external financing significantly reduces the marginal cost of capital investment, thus encouraging producers to augment their production scale. Meanwhile, the expanded production also means higher wage/employment in both markets, raising the overall welfare of the economy. In this sense, we can conclude that arbitrage activities help boost the real economy by providing favourable financing to producers. A higher level of capital investment also generates more collateral for arbitrageurs through securitization. This allows them to provide more market liquidity, and to better correct asset mispricing.

3.4 Collateral Prices in the Time Series and Cross-Section

Given the model dynamics, we obtain the following implications about the shadow collateral price.

Proposition 4. *Suppose IM's collateral constraints are binding. The dynamics of shadow collateral price $\bar{\lambda}_t$ would then be given by:*

$$\bar{\lambda}_t = \frac{\phi_t}{\phi_{t+1}} - \frac{\rho W_t}{F(\alpha \rho \mu_t W_t)}. \tag{14}$$

Holding all other characteristics constant:

- *For a given economy, $\bar{\lambda}_t$ increases with the gross arbitrage return ϕ_t/ϕ_{t+1}, and decreases with IM's wealth W_t.*
- *$\bar{\lambda}_t$ increases with the total productivity factor a and capital output share α. It decreases with depreciation rate δ.*

If we use IM's wealth W_t as an indicator of their funding condition or of the overall tightness of the credit standards of the financial intermediary sectors, then Proposition 4 predicts IM's shadow collateral cost will increase when funding liquidity is low. Garleanu and Pedersen (2011) provide empirical support for this notion. They document that the required return on a low-margin asset (e.g., a CDS) is significantly lower than that on a high-margin security with the same cash flows (e.g., proxied by the corresponding corporate bond), when the financial sector's funding condition is tighter. Likewise, if we interpret the gross arbitrage profitability ϕ_t/ϕ_{t+1} as a measure of relative mispricing, and proxy for the shadow collateral price by the interest spread between uncollateralized and collateralized loans, then Proposition 4 is also consistent with Garleanu and Pedersen (2011). They document that the interest rate spread between LIBOR and the general collateral repo rate widens with the deviation from covered interest parity (CIP).

Similarly, the cross-sectional part of Proposition 4 indicates that IM's shadow collateral cost increases with the return of the asset used as collateral. A natural empirical prediction is that the interest rate spread between uncollateralized and collateralized loans is wider in those economies where assets used as collateral have, on average, higher expected risk-adjusted returns.

3.5 Deterministic Equilibrium

Given that IM hold net zero financial assets, and are thus immune from endowment shock units, their optimization problem is a deterministic one. We can prove that there exists at least one deterministic equilibrium.

Proposition 5. *Given IM's initial wealth $W_0 > 0$, at least one equilibrium exists in which price differences ϕ_t, IM's capital investment K_t and market liquidity x_t are deterministic. In equilibrium, prices are non-negative in the market with negative shock intensity, i.e., market B, and non-positive in the market with positive shock intensity, i.e., market A.*

4 Steady States

When shock intensity is constant, i.e., $u_t = u$, there exit steady states in which the price difference, market liquidity, and capital investment remain the same across time. There are also multiple steady states under certain conditions.

Proposition 6. *Given shock intensity u and the distribution of $\{\theta_t\}$, there exists a unique threshold value $\bar{\rho} > 0$ for the IM's discount factor, such that:*

1. *If $\rho > \bar{\rho}$, there is a unique steady state with slack collateral constraints for IM. In particular,*
   - *IM's capital investment and consumption converge to $K_s^* = F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right)$ and $C_s^* = F\left(K_s^*\right) - \delta K_s^* = \frac{\delta(1-\alpha\rho)+(1-\rho)(1-\delta)}{\alpha\rho} K_s^*$.*
   - *IM provide full liquidity and eliminate any price gaps in the steady state, i.e., $x_s^* = ul$ and $\phi_s^* = 0$.*
   - *The shadow collateral price of the capital is zero, i.e., $\bar{\lambda}_s^* = 0$ and the leverage multiplier equals to one, i.e., $\mu_s^* = 1$.*
2. *Else, if $0 < \rho \leq \bar{\rho}$, there is a steady state(s) with binding collateral constraints. Moreover, in these steady state(s),*
   - *IM's capital investment, consumption, shadow collateral price, and leverage multiplier converge to:*

$$
\begin{aligned}
&K_b^* = F'^{-1}\left(\frac{\delta}{\rho}\right) > F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right), \\
&C_b^* = \frac{\delta(1-\alpha\rho)}{\alpha\rho} K_b^* > \frac{\delta(1-\alpha\rho)+(1-\rho)(1-\delta)}{\alpha\rho} F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right), \\
&\bar{\lambda}_b^* = 1-\rho, \quad \mu_b^* = 1/\delta > 1.
\end{aligned}
$$

   - *IM provide only partial liquidity, and thus a positive price difference persists: $0 < x_b^* < ul$, $\phi_b^* > 0$. In particular, given the steady-state position size x_b^*, the price* *spread converges to:*

$$\phi_b^* = \frac{2\beta w\left(K_b^*\right)}{1-\beta}\mathbb{E}\left[\frac{\theta}{w\left(K_b^*\right) - \left(u - x_b^*/l\right)\theta}\right] > 0. \tag{15}$$

- *IM's total transaction volume from arbitrage amounts to* $x_b^* \phi_b^* = (1-\delta)K_b^*$.

3. *Discontinuity exists in IM's steady-state capital investment, consumptions, shadow collateral price, and leverage ratio at the threshold value* $\bar{\rho}$. *We can always find a* $\epsilon_\rho > 0$, *such that*

$$K^*_{\bar{\rho}+\epsilon_\rho} < K^*_{\bar{\rho}-\epsilon_\rho}, \quad C^*_{\bar{\rho}+\epsilon_\rho} < C^*_{\bar{\rho}-\epsilon_\rho}, \quad \bar{\lambda}^*_{\bar{\rho}+\epsilon_\rho} < \bar{\lambda}^*_{\bar{\rho}-\epsilon_\rho}, \quad \mu^*_{\bar{\rho}+\epsilon_\rho} < \mu^*_{\bar{\rho}-\epsilon_\rho}.$$

Intuitively, if IM are patient (i.e. $\rho > \bar{\rho}$), they will tend to save sufficient capital as collateral in order to eliminate all the potential price differences. As a result, there are no unexploited arbitrage opportunities in the long-run steady state. Accordingly, there is zero shadow collateral value attached to the capital investment, and IM gain no leverage for their production. In this case, IM's steady-state capital investment, consumption, and wealth resemble those in the neoclassical growth model.

In contrast, impatient IM would not save enough capital to absorb all arbitrage opportunities. As they are collateral-constrained, they can only provide partial liquidity. Therefore, one would observe persistent price differences across segmented markets. Those unexploited arbitrage opportunities, however, allow capital to have extra collateral value, i.e., $\bar{\lambda}_b^* = 1 - \rho > 0$, which reduces the marginal cost of capital and leads to higher capital investments than in the neoclassical growth model. Empirical support abounds: Chaney et al. (2012) show that over the 1993-2007 period, a \$1 increase in collateral value leads the representative US public corporation to raise its investment by \$0.06; Cahn et al. (2017) report that extension of the Eurosystem's universe of eligible collateral leads to an 8-9% increase in bank credit to small and medium enterprises in France; Bekkum et al. (2018) show that after ECB allowed lower-rated RMBS to be eligible collateral, there is an expansion in mortgage credit in terms of both lower interest rates and greater loan volumes; and Barthélemy et al. (2017) find one standard deviation increase in the volume of illiquid collateral pledged with the Eurosystem corresponded to a 1.1% increase in real-sector loans.

An alternative interpretation is that the presence and exploitation of arbitrage opportunities allow IM to obtain external financing from HH to leverage up their capital, i.e., $\mu_b^* > 1$. IM's arbitrage gain in the steady state is essentially a nominal zero-interest one-period loan. Because the price spreads are constant, IM earn the same amount of arbitrage gains each period as the amount of their liability. In other words, IM receive new loans each period, which exactly offset their repayment of the previous debt. Thus, their trading is indeed arbitrage in nature, because they can renew and roll over their short-term loans infinitely in the steady state. In particular, the scale of this periodical external financing equals a fixed ratio of IM's capital, $x_b^* \phi_b^* = (1-\delta)K_b^*$.

Due to the extra funding from arbitrage, IM's welfare improves as their steady-state capital investment and consumption, i.e., $K_b^* = F'^{-1}(\delta/\rho)$ and $C_b^* = (1-\alpha\rho)F(K_b^*)$, are higher than their counterparts in the frictionless neoclassical growth model. Interestingly, because of the varying access to external financing, there exists discontinuity in the steady states around the cutoff value $\bar{\rho}$. For $\rho_1 < \bar{\rho} < \rho_2$, the corresponding capital investment follows the converse order: $K_1^* > K_2^*$ as $\bar{\lambda}_1^* > \bar{\lambda}_2^* = 0$.

5 Multiple Equilibria

In this section, we further explore steady states implications for cases with multiple collateral-constrained equilibria.

Proposition 7. *If $\rho \in (0, \bar{\rho})$, there exist two distinct collateral-constrained steady states, denoted as $SS_1 := (x_1^*, \phi_1^*, K_1^*)$ and $SS_2 := (x_2^*, \phi_2^*, K_2^*)$. They share the same level of capital investment, $K_1^* = K_2^* = K_b^* = F'^{-1}(\delta/\rho)$, but differ as follows:*

- *One steady state features higher market liquidity and a smaller price gap, while the other features a lower trading size and a wider price spread, i.e., $0 < x_1^* < x_2^* < ul$ and $\phi_1^* > \phi_2^* > 0$.*
- *HH's welfare is higher with higher market liquidity in SS_2 than in SS_1. IM are indifferent between the two.*
- *K_b^*, x_1^*, and ϕ_2^* increase with ρ, while x_2^* and ϕ_1^* decrease with ρ.*

From Proposition 6, we know that IM's liability (the product of x^* and ϕ^*) equals their total collateral $(1-\delta)K_b^* = (1-\delta)F'^{-1}(\delta/\rho)$—a constant that is independent of any financial market characteristics. Also, the price spread ϕ^* decreases with the trading volume x^* in the steady state, since HH are risk averse. These two factors make it possible to have multiple solutions to market liquidity x^* and price gap ϕ^*: For economies with impatient IM (i.e., $\rho \in (0, \bar{\rho}]$), there exist exactly two robust collateral-constrained steady states, with $x_1^* \leq x_2^*$ and $\phi_1^* \geq \phi_2^*$. As the right-hand graph in Figure 2 shows, a special case occurs when $\rho = \bar{\rho}$, when the two steady states happen to be identical.

The two steady states indicate distinctive welfare implications for IM and HH. Clearly, IM are indifferent between the two regimes. This is because their utility and consumption are determined solely by the constant capital investment level K_b^*, which is independent of trading volume x^* or price gap ϕ^*. In contrast, the welfare levels for HH at the two steady states can differ significantly. As illustrated by the numerical example in the left-hand graph in Figure 2, given fixed capital level K_b^*, HH receive the same labor income in both states. However, there is less market liquidity in the steady state SS_1, which exposes HH to more unhedged endowment shocks. Thus, holding all model characteristics constant, there is a Pareto improvement when the economy transitions from SS_1 to SS_2. In the following, we refer to SS_1 as the bad steady state/regime, and to SS_2 as the good one. We also define the regime shift as the economy switches from one steady state to the other.

In a nutshell, this proposition suggests that the same amount of collateralizable, real-sector assets can support different liquidity supply levels in financial markets. In reality, varying market microstructure (e.g., transaction costs or rebates for providing liquidity) and collateral policies (such as rehypothecation limits, eligibility scope, circulation velocity, etc.) can give rise to distinct liquidity outcomes, even with the same quantities of illiquid collateral. In this light, the bad regime corresponds to those less liquid, heavily regulated markets with rigid collateral re-use limit, low circulation speed, or limited acceptance of non-marketable assets as eligible collateral; the good regime captures a more liquid and less regulated trading environment featuring the opposite. It is likely that the liquidity supply moves from one equilibrium level to another in response to changes in collateral policies, trading platforms, general market regulations, and other micro- or macroeconomic factors.

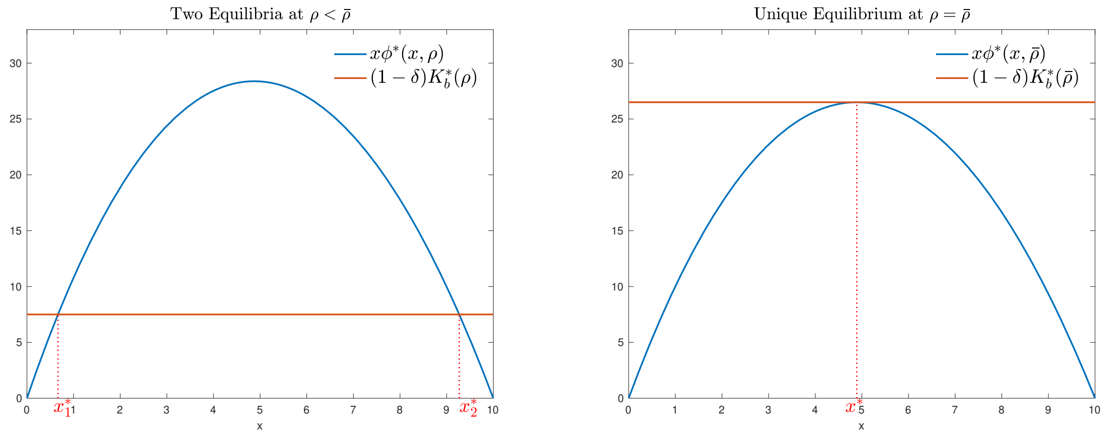

Figure 2: $a = 4$, $b = 60$, $u = 10$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal line is $(1-\delta)K_b^*(\rho)$, where K_b^* is the steady state capital. The left chart is when $\rho = 0.5$, and the right one is when $\rho = \bar{\rho} = 0.94$. The blue lines are the possible products of equilibrium x and ϕ given that the steady state capital is $K_b^*(\rho)$. The interaction point(s) are the steady state position size(s) x_1^* and x_2^* (x^*).

Next, we show the comparative statics of multiple equilibria with respect to the shock intensity.[10]

Proposition 8. *Holding all other characteristics constant, the cut-off value $\bar{\rho}$ of IM's discount factor increases with shock intensity u. For two otherwise identical economies with differing shock intensities, i.e., $u_1 < u_2$, if IM's discount factor ρ is below either cut-off value, then, in the steady states:*

- *The capital investments are identical: $K_b^*[u_1] = K_b^*[u_2]$.*
- *The market liquidity in the bad regime decreases with u, i.e., $x_1^*[u_1] > x_1^*[u_2]$, while the price spread increases, i.e., $\phi_1^*[u_1] < \phi_1^*[u_2]$.*
- *The market liquidity in the good regime increases with u, i.e., $x_2^*[u_1] < x_2^*[u_2]$, while the price spread decreases, i.e., $\phi_2^*[u_1] > \phi_2^*[u_2]$.*

[10] We also show other comparative statics in our Internet Appendix.

Holding all else equal, an economy with a higher shock intensity allows more patient IM to leverage up aggregate production through arbitrage. Also, as Figure 3 shows, as long as IM remain collateral-constrained, the shock intensity level does not affect their steady-state capital investment. However, the gap between good and bad regimes in terms of market liquidity and the price spread expands with increasing shock intensity. Thus, the potential economic impact of a regime shift rises concurrently with shock intensity.

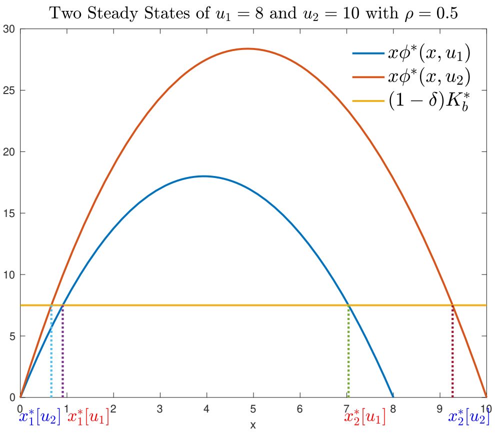

Figure 3: $a = 8$, $b = 40$, $\alpha = \gamma = 0.5$, $\delta = 0.4$, $\beta = 0.9$, $\bar{\theta} = 2$ and θ follows a two-point distribution. The horizontal line is $(1-\delta)K_b^*$, where K_b^* is the binding steady-state capital. The $x\phi^*$ lines are the possible product of equilibrium x and ϕ given K_b^* and the corresponding shock intensity. Each interaction point corresponds to a binding steady-state position size x^*.

6 Crisis and Recovery

This section explores the model implications for shock reactions, financial crises, post-crisis recoveries, and the policy trade-off. First, we examine the self-recovery case as the benchmark shock reactions. Then, through regime shifts, we show that the interaction between limited arbitrage and the business cycle can work as a powerful transmission channel in which a tiny shock can lead to simultaneous arbitrage crashes and recessions. Furthermore, we illustrate that such a crisis does not only happen when markets panic, or the economy switches to a bad regime. Rather, crises can be unavoidable even when the economy shifts to a good post-shock regime. In particular, we only consider shock responses relative to the steady state of the economy with multiple equilibria, as in Proposition 7. Finally, we discuss the policy indications about regime trade-off between welfare and risks.

6.1 Self-Recovery without Regime Shift

To study the benchmark shock reactions, we consider the following thought experiment. Suppose at t IM encounter an unexpected negative shock in their wealth, and must unwind part of their arbitrage positions. This kind of disturbance in the real world may correspond to a temporary productivity shock, an abrupt hit to IM's balance sheet, or a sudden drop in the price of the asset used as collateral in their margin account, etc. Assume further there is no regime shift after the initial shock. Below we examine the short- and long-run responses of the economy.

Corollary 1.1. *Suppose there is a sudden negative shock in IM's wealth at t, with no concurrent regime shift in the economy.*

- *The immediate effect is that capital investment and market liquidity fall below pre-shock levels, while the price gap widens:*

$$K_t < K^*, \quad x_t < x^*, \quad \phi_t > \phi^*.$$

- *Following this immediate reaction, capital investments, market liquidity and price spreads gradually regain their pre-shock steady-state levels:*

$$K_t < K_{t+1} < \cdots < K^*, \quad x_t < x_{t+1} < \cdots < x^*, \quad \phi_t > \phi_{t+1} > \cdots > \phi^*.$$

Corollary 1.1 indicates that, when there is no regime shift, the shock reactions are consistent with the self-correcting behavior in Gromb and Vayanos (2018). Following an adverse shock, the economy experiences a quick and full recovery. Clearly, the resulting wider price gap and lower capital from the initial shock help drive up the arbitrage profitability and the marginal return of production. These two favourable factors allow IM to quickly regain their wealth and eventually stabilize the economy.

However, this kind of "ideal" recovery pattern only occurs in our model as a special case. After a negative shock, market sentiment about future economic prospects can turn pessimistic. This may shift the economy toward a bad regime. As a result, the recovery paths would not necessarily exhibit such self-correcting behavior. Nevertheless, we use this ideal case as the benchmark for recovery in our later discussion.

6.2 Crisis and Incomplete Recovery: Regime Shift to Bad Equilibrium

As a comparison, we now examine the case in which the economy, starting at a good steady state, switches to a bad regime following the same initial shock as in section 6.1.

Corollary 1.2. *Suppose the economy is at a good regime $SS_g : \left(K^*,\ x_g^*,\ \phi_g^*\right)$ and, following the same shock in IM's wealth as in section 6.1, moves toward the bad regime $SS_b : (K^*,\ x_b^*,\ \phi_b^*)$. If equilibrium exists, relative to the benchmark case:*

- *The immediate reaction would be that IM's initial loss would get amplified, both capital investment and market liquidity would drop more significantly, and the price spread would widen:*

$$W_t < W^{bm} < W^*, \quad K_t < K_t^{bm} < K^*, \quad x_t < x_t^{bm} < x_g^*, \quad \phi_t > \phi_t^{bm} > \phi_g^*,$$

  where $\left(W_t^{bm},\ K_t^{bm},\ x_t^{bm},\ \phi_t^{bm}\right)$ are the corresponding benchmark levels.

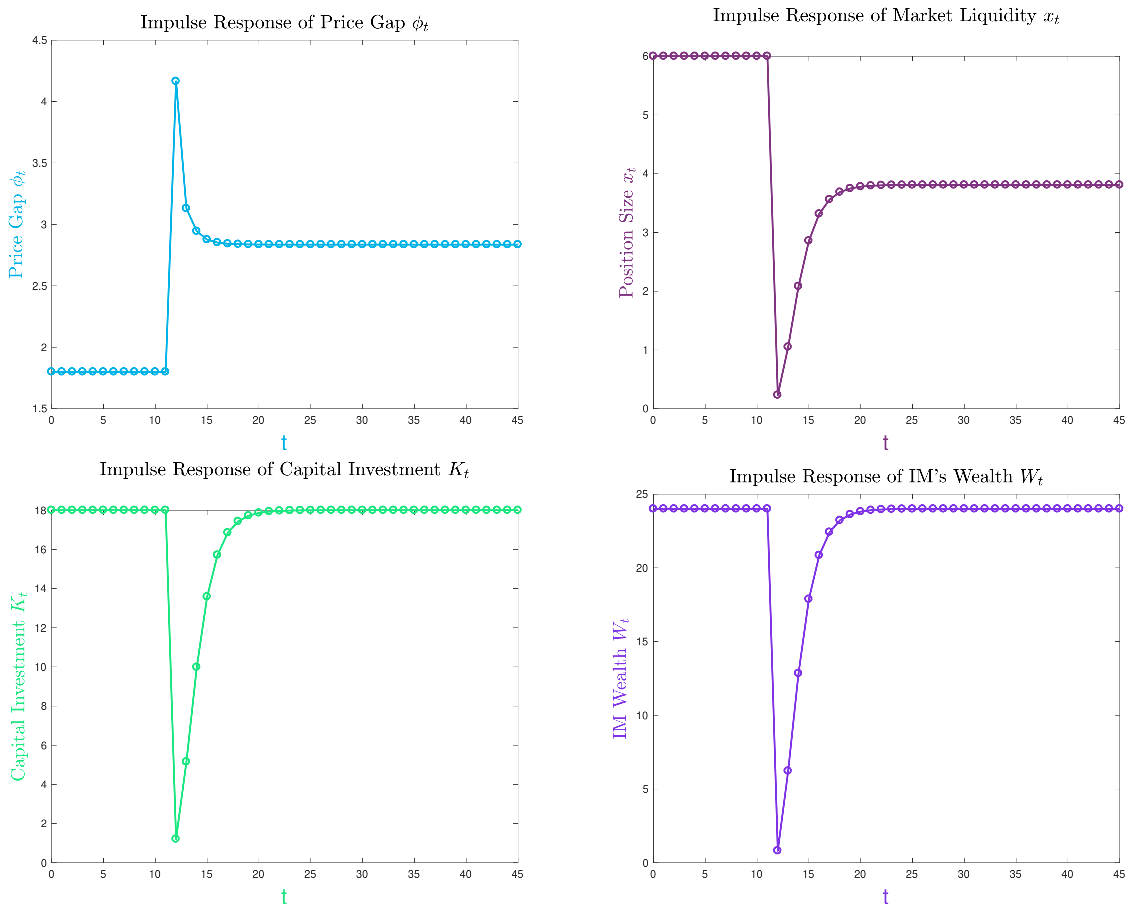

Figure 4: Impulse response from a good to a bad regime after a sudden loss in IM's wealth. Parameter set: $a = 8$, $b = 60$, $u = 10$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$, initial loss $\Delta W = 9$. θ_t follows a two-point distribution.

- *In the long run, IM's capital and wealth would recover gradually to pre-shock levels. However, the price spread and market liquidity would only converge to the bad steady-state levels:*

$$K_t < K_{t+1} < \cdots < K^*, \quad x_t < x_{t+1} < \cdots < x_b^* < x_g^*, \quad \phi_t > \phi_{t+1} > \ldots \phi_b^* > \phi_g^*.$$

Figure 4 illustrates the equilibrium paths when the negative shock triggers a regime shift. Relative to the benchmark case, the economy reacts more dramatically. With the regime shift, IM's financial losses are amplified, because the price gap immediately jumps further to match the higher level expected in the bad regime. Meanwhile, when the shock occurs, IM are still carrying the large previous positions inherited from the good regime, i.e., x_g^*. The product of these two large quantities consitutes IM's realized arbitrage losses.

Under such financial distress, IM have to cut down more of their capital *en masse* and therefore lack enough collateral to restore liquidity supply. The price gap thus widens further with IM's collective unwinding, exacerbating their arbitrage losses. This is consistent with scenarios during Global Financial Crisis (GFC): amid worsening financial conditions and doubt over potential defaults of the collateral assets, IM were forced to reduce their holdings of illiquid corporate securities and other structured instruments, ultimately making a large part of them "toxic"—no longer acceptable as collateral. This created a market-wide scarcity of eligible collateral and sharply reduced the investment demands of these assets[11]. As a result, financially stressed arbitrageurs found it even harder to fulfil their margin requirements, causing asset mispricing to spike persistently; meanwhile real investments also plummeted.

In particular, if the price gap widens to such a degree that IM's inflated liability exceeds their total production income, then equilibrium no longer exists. This matches the extreme case, where financial distress renders the whole intermediary industry insolvent, followed by a major recession and a market meltdown. Without external aids to IM, the economy cannot recover by itself.

In the long run, surviving IM gradually regain their capital investment, and increase their liquidity supply. However, because of the amplified financial losses, it generally takes longer for IM to recover. This is consonant with the stylized fact of slow recoveries in the real sectors after the GFC. Reinhart and Rogoff (2009), Reinhart and Reinhart (2010), and IMF (2009), among others, have documented the slow recoveries in various measures of economic activity in many economies since the 2007-2008 crisis, such as aggregate output, investment, capital stocks, and employment. GDP growth in the U.S. and the Euro area, for example, did not rebound to pre-crisis levels for several years after the financial crisis.

Moreover, even after IM's wealth fully recovers, there is only a limited recovery in the financial markets, which results in persistent and large post-crisis mispricing. HH thus end up trapped in the bad regime, with less protection against their endowment shocks. This is consistent with the observation that many asset markets have witnessed slow and limited recoveries long after aggregate output bounced back to pre-crisis levels in 2007. Prior to the GFC, for instance, only small deviations from CIP could be found in the data.[12] Since the GFC, there is a plethora of evidence of significant deviations from CIP. Du et al. (2018) document large deviations for ten currency pairs, even of the same order of magnitude as the interest rate differential, have persisted in tranquil markets after the crisis. Bai and Collin-Dufresne (2019) also find similar patterns in the CDS-bond basis for investment-grade (IG) and high-yield (HY) bonds. They show that the average basis for IG firms, which generally hovered around -17 basis points (bps) before the crisis, fell to -243 bps during the crisis, and the average basis for HY firms dropped from 12 bps to -560 bps. The bases for both IG and HY firms remained negative even after the crisis, i.e., -101 bps and -237 bps, respectively. In addition, Boyarchenko et al. (2018) find a prolonged widening of spreads in the CDX-CDS (index CDS - the single-name CDS) basis trades between segments of the CDS market after the GFC.

The regime shift also provides a complementary perspective to account for the post-crisis liquidity drop in many asset markets. According to a recent BIS report by Aldasoro and Ehlers (2018), for example, the global CDS market has shrunk significantly since GFC, when compared to its peak at end 2007. Outstanding notional amounts of CDS contracts fell markedly, from

[11]Here for simplicity, we do not capture the price drops during the fire-sale of collateral. As can be proved, incorporating endogenous capital price dynamic will only serve to strengthen our results.

[12]See Akram et al. (2008) and Levich (2017) for more details.

$61.2 trillion at the end of 2007, to $9.4 trillion just ten years later. A similar pattern can be seen for the gross market value of outstanding positions.

6.3 Crisis and Incomplete Recovery: Regime Shift to Good Equilibrium

As we noted before, the previous discussions may imply that crises and incomplete recoveries only occur if markets are pessimistic and are headed toward a bad regime. However, next, we demonstrate that similar crises can occur even when the economy moves to a good regime. Consider the following thought experiment. Suppose that, at t, shock intensity suddenly plunges (for example, from u_2 to u_1 as illustrated in Figure 3). Following this shock, the economy shifts to a good regime under the new shock intensity. This type of shock may correspond to an abrupt drop in the asset demands or underlying volatility, a sudden change in the regulatory environment, or an unexpected increase in the market integration.[13]

Corollary 1.3. *Suppose the economy is in a good regime, $SS : \left(K^*,\ x_g^*,\ \phi_g^*\right)$. There is then a sudden decrease in shock intensity in both markets, $u \to u_1$ at t. The economy subsequently moves toward a good regime under the new shock intensity u_1, i.e., $SS_{n,g} : \left(K^*,\ x_{n,g}^*,\ \phi_{n,g}^*\right)$. If equilibrium exists,*

- *The immediate reaction would be that the price spread would jump, and IM would suffer losses from previous arbitrage positions. Capital investment and market liquidity would slump.*

$$\phi_t > \phi_{n,g}^* > \phi_g^*, \quad W_t < W^*, \quad K_t < K^*, \quad x_t < x_{n,g}^* < x_g^*, \quad \phi_t > \phi_{n,g}^* > \phi_g^*,$$

- *In the long run, IM's capital and wealth would gradually recover to pre-shock levels. However, the price spread and market liquidity would converge to the new good steady-state levels:*

$$K_t < K_{t+1} < \cdots < K^*, \quad x_t < x_{t+1} < \cdots < x_{n,g}^* < x_g^*, \quad \phi_t > \phi_{t+1} > \ldots \phi_{n,g}^* > \phi_g^*.$$

As shown in Figure 5, even if the economy moves to a good regime, it may still encounter a simultaneous arbitrage crash and recession in the real sector, similar to the case discussed in section 6.2. As Proposition 8 shows, the new good steady state features a higher price spread and lower market liquidity than the pre-shock regime. Thus, even without pessimistic market sentiment, the price gap widens immediately to accommodate the higher anticipated long-run level. Because IM must liquidate the relatively large positions that were inherited from the previous regime, the surge in the price gap tends to inflict unexpected arbitrage losses on them. Again, they have to reduce their capital stocks and investments, leading to collateral shortage and output contractions.

[13] We also discuss and document other shock responses, under different pre-shock regimes and with other types of shocks, in our Internet Appendix.

Similarly, in the long run, there is only a limited recovery in the correction or mispricing, as well as in the market liquidity supply. In this regard, HH remain worse off afterward, although the sudden shock lowers their initial consumption risk.

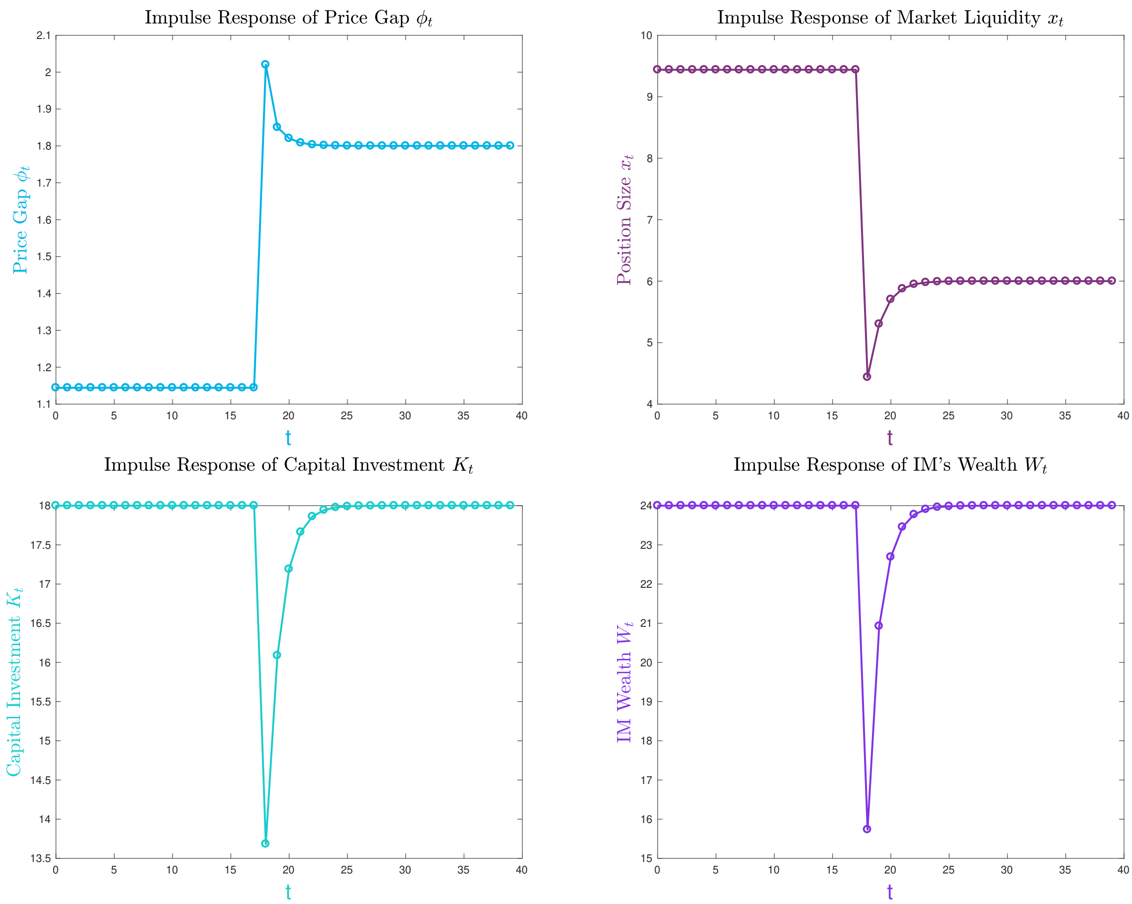

Figure 5: Impulse responses from a previous healthy steady state to a new healthy steady state after the shock intensity drops from $u_1 = 12$ to $u_2 = 10$. Parameter set: $a = 8$, $b = 60$, $\alpha = 0.5$, $\beta = 0.8$, $\gamma = 0.5$, $\delta = 0.4$, $\bar{\theta} = 2$, $\rho = 0.6$, $L = 2$. θ_t follows a two-point distribution.

Therefore, we show that, independent of market sentiment, as long as the post-shock regime features a larger price gap or lower market liquidity than the previous regime, IM end up suffering from amplified losses in their arbitrage positions. Especially if their pre-shock trading size happens to be significant, the resulting financial distress could pose a serious threat to market stability and the overall economy. Also, because of the potential regime shift, different sectors of the economy may have varying post-crisis recovery paths.

6.4 Regime Trade-off

In this section, we compare the welfare of different regimes in response to unexpected regime shifts.

Proposition 9. *Consider two economies initially in the bad regime SS_1 and the good regime SS_2 as described in Proposition 7. For any given sudden shock at t and the post-shock regime, the* *economy in the bad pre-shock regime is (weakly) better off than the economy in the good regime after the shock; and the post-shock liquidity and capital follow:* $x_{t+j}^{SS_1} \geq x_{t+j}^{SS_2}$, $K_{t+j}^{SS_1} \geq K_{t+j}^{SS_2}$, $\forall j = \{0, 1, 2, \dots\}$.

Proposition 9 implies that although the good regime features more liquidity and higher welfare for HH, it is also more vulnerable: an unexpected regime shift is more likely to trigger crises and causes more negative impacts on liquidity supply and the real economy. Thus, when considering relevant regulation changes, policy makers may face a trade-off between maintaining stability and facilitating market liquidity. A new collateral policy aiming to improve price discoveries or liquidity supply in financial markets, for example, may work at the cost of raising systemic risks and fuelling contagion to the real economy.

7 Conclusion

In this paper, we develop a tractable model to study the interplay between limited arbitrage and the aggregate economy through collateralization. We primarily address two issues: 1) how such interactions affect asset mispricing, market liquidity, investment, welfare, and aggregate output, and 2) how they contribute to the creation of financial crises, and shape the post-crisis economic recoveries.

To address the first question, we derive an analytical solution to the model dynamics, and find that securitization increases the amount of collateral, which supports a higher supply of market liquidity. In turn, through collateralization, limited arbitrage leads to extra shadow value for capital investment, because it serves as collalteral. This effectively raises the marginal return of production, and boosts overall investment, aggregate output, and welfare. We also provide empirical predictions on the shadow collateral value that is embedded in the assets.

To address the second issue, we analytically solve the two equilibria in the model, which correspond to different possible economic regimes. We show that, in the wake of unexpected shocks, the interaction between speculation and aggregate production gives rise to a powerful transmission channel, where tiny disturbances can spread, amplify, and ultimately trigger simultaneous arbitrage crashes and economic recessions. The possibility of shifting to a different post-crisis regime may also derail a subsequent recovery, which sheds light on the slow and incomplete recoveries observed in many asset markets.

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Appendices

A Proof of Lemma 1

Proof. We prove the lemma through backward induction. Suppose at $s = t + T$,

$$p^{\mathrm{A}}_{t+T}(\varepsilon) = -p^{\mathrm{B}}_{t+T}(-\varepsilon).$$

We define $c^i_s(\theta_s)$ as the equilibrium consumption of HH in i at time s as a function of θ_s, for $i \in \{\mathrm{A}, \mathrm{B}\}$. Since $\{\theta_s\}$ follows a symmetric distribution around zero, then it must hold that:

$$\frac{p^{\mathrm{A}}_{t+T}(\varepsilon)}{c^{\mathrm{A}}_{t+T}(\varepsilon)} = -\frac{p^{\mathrm{B}}_{t+T}(-\varepsilon)}{c^{\mathrm{B}}_{t+T}(-\varepsilon)}.$$

Thus,

$$\beta^T \mathbb{E}\left[\frac{P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right] = -\beta^T \mathbb{E}\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right].$$

as $c^{\mathrm{A}}_{t+T}(\varepsilon) = c^{\mathrm{B}}_{t+T}(-\varepsilon)$, which follows from households' budget constraints.

At $s = T + t - 1$, from the first-order condition of households, we have

$$P^{\mathrm{A}}_{t+T-1} = \beta C^{\mathrm{A}}_{t+T-1}\mathbb{E}\left[\frac{\theta_{t+T} + P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right], \quad P^{\mathrm{B}}_{t+T-1} = \beta C^{\mathrm{B}}_{t+T-1}\mathbb{E}\left[\frac{\theta_{t+T} + P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right].$$

Substituting C^{A}_{t+T-1} and C^{B}_{t+T-1} with the households' budget constraints at $t+T-1$, it follows that:

$$p^{\mathrm{A}}_{t+T-1}(\varepsilon) = -p^{\mathrm{B}}_{t+T-1}(-\varepsilon), \quad c^{\mathrm{A}}_{t+T-1}(\varepsilon) = c^{\mathrm{B}}_{t+T-1}(-\varepsilon).$$

Likewise, we can derive:

$$p^{\mathrm{A}}_{t}(\varepsilon) = -p^{\mathrm{B}}_{t}(-\varepsilon), \quad c^{\mathrm{A}}_{t}(\varepsilon) = c^{\mathrm{B}}_{t}(-\varepsilon).$$

On the other hand, we can rewrite $p^i_t(\varepsilon)$ as:

$$\begin{aligned} p^{\mathrm{A}}_t(\varepsilon) &= c^{\mathrm{A}}_t(\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C^{\mathrm{A}}_{t+j}}\right] + \beta^T\mathbb{E}\left[\frac{P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right]\right) \\ &= -p^{\mathrm{B}}_t(-\varepsilon) = -c^{\mathrm{B}}_t(-\varepsilon)\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C^{\mathrm{B}}_{t+j}}\right] + \beta^T\mathbb{E}\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right]\right). \end{aligned}$$

When $T \to \infty$, according to the TVC in markets A and B,

$$\lim_{T\to\infty} -\beta^T\frac{P^{\mathrm{A}}_T}{C^{\mathrm{A}}_T}y^{\mathrm{A}}_T = 0, \quad \lim_{T\to\infty} \beta^T\frac{P^{\mathrm{B}}_T}{C^{\mathrm{B}}_T}y^{\mathrm{B}}_T = 0.$$

If the steady state prices $\lim_{T\to\infty} P^i_{t+T} \neq 0$, then it must hold $y^i_{t+T} \neq 0$ in equilibrium. Otherwise, some IM can make an arbitrage profit by increasing liquidity provisions. Therefore, in this case,

$$\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{P^{\mathrm{A}}_{t+T}}{C^{\mathrm{A}}_{t+T}}\right] = -\lim_{T\to\infty}\beta^T\mathbb{E}\left[\frac{P^{\mathrm{B}}_{t+T}}{C^{\mathrm{B}}_{t+T}}\right] = 0. \tag{16}$$

Else if $\lim_{T\to\infty} P^i_{t+T} = 0$, Equation (16) obviously holds as well.

Therefore, we have

$$\begin{aligned} p^{\mathrm{A}}_t(\varepsilon) &= c^{\mathrm{A}}_t(\varepsilon)\lim_{T\to\infty}\left(\sum_{j=1}^{T}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{c^{\mathrm{A}}_{t+j}}\right] + \beta^T\mathbb{E}\left[\frac{p^{\mathrm{A}}_{t+T}}{c^{\mathrm{A}}_{t+T}}\right]\right) = c^{\mathrm{A}}_t(\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C^{\mathrm{A}}_{t+j}}\right]\right) \\ &= -c^{\mathrm{B}}_t(-\varepsilon)\left(\sum_{j=1}^{\infty}\beta^j\mathbb{E}\left[\frac{\theta_{t+j}}{C^{\mathrm{B}}_{t+j}}\right]\right) = -p^{\mathrm{B}}_t(-\varepsilon). \end{aligned}$$

□

B Proof of Proposition 1

Proof. From Lemma 1, we know that $p_t^{\mathrm{A}}(\varepsilon) = -p_t^{\mathrm{B}}(-\varepsilon)$. Thus,

$$c_t^{\mathrm{A}}(\varepsilon) = -p_t^{\mathrm{A}}(\varepsilon)\left(y_t^{\mathrm{A}} - y_{t-1}^{\mathrm{A}}\right) + w\left(K_{t-1}\right) + \left(u + y_{t-1}^{\mathrm{A}}\right)\varepsilon = p_t^{\mathrm{A}}(\varepsilon)\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\varepsilon$$

where $w(K) = a\gamma K^{\alpha}L^{\gamma} + b$ is a function of K. Similarly,

$$c_t^{\mathrm{B}}(\varepsilon) = -p_t^{\mathrm{B}}(\varepsilon)\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\varepsilon.$$

Hence, it is obvious that

$$c_t^{\mathrm{A}}(\varepsilon) \;=\; c_t^{\mathrm{B}}(-\varepsilon), \quad \mathbb{E}\left[\frac{\theta_{t+s} + P_{t+s}^{\mathrm{A}}}{C_{t+s}^{\mathrm{A}}}\right] \;=\; -\mathbb{E}\left[\frac{\theta_{t+s} + P_{t+s}^{\mathrm{B}}}{C_{t+s}^{\mathrm{B}}}\right], \qquad \forall s \in \{1, 2, \dots\}.$$

From households' first-order conditions, it follows that

$$\phi_t \;\equiv\; P_t^{\mathrm{B}} - P_t^{\mathrm{A}} \;=\; \beta C_t^{\mathrm{B}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] - \beta C_t^{\mathrm{A}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \;=\; \beta\left(C_t^{\mathrm{A}} + C_t^{\mathrm{B}}\right)\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right].$$

Thus,

$$P_t^{\mathrm{B}} = \beta C_t^{\mathrm{B}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] = \frac{C_t^{\mathrm{B}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}}\phi_t.$$

Likewise,

$$P_t^{\mathrm{A}} = -\frac{C_t^{\mathrm{A}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}}\phi_t.$$

Substituting C_t^i with HH's budget constraints in i, $i \in \{\mathrm{A}, \mathrm{B}\}$, one can obtain the following after rearranging:

$$P_t^{\mathrm{B}} \;=\; \frac{w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\theta_t}{2w\left(K_{t-1}\right)}\phi_t, \quad P_t^{\mathrm{A}} \;=\; -\frac{w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\theta_t}{2w\left(K_{t-1}\right)}\phi_t.$$

If we continue to decompose ϕ_t,

$$\begin{aligned}
\phi_t &= \beta C_t^{\mathrm{B}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] - \beta C_t^{\mathrm{A}}\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \\
&= \beta\left[-P_t^{\mathrm{B}}\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) - \left(u - x_{t-1}/l\right)\theta_t\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right] \\
&\quad - \beta\left[P_t^{\mathrm{A}}\left(x_t - x_{t-1}\right)/l + w\left(K_{t-1}\right) + \left(u - x_{t-1}/l\right)\theta_t\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{A}}}{C_{t+1}^{\mathrm{A}}}\right] \\
&= \beta\left[-\phi_t\left(x_t - x_{t-1}\right)/l + 2w\left(K_{t-1}\right)\right]\mathbb{E}\left[\frac{\theta_{t+1} + P_{t+1}^{\mathrm{B}}}{C_{t+1}^{\mathrm{B}}}\right].
\end{aligned}$$

After rearranging and repeatedly substituting with HH's first-order condition, we can obtain:

$$\phi_t \equiv P_t^{\mathrm{B}} - P_t^{\mathrm{A}} = \frac{2w\left(K_{t-1}\right)}{M_t + \left(x_t - x_{t-1}\right)/l}.$$

where

$$1/M_t := \sum_{j=1}^{\infty} \beta^j \mathbb{E}_t \left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right],$$

which is independent of the realization of θ_t.

Because IM take a net zero position in the financial markets, when u_t is constant, their optimization problems are deterministic. Accordingly, x_t, x_{t-1}, K_{t-1} are all independent of the realization of θ_t in equilibrium. Hence the price difference ϕ_t doesn't depend on any particular θ_t realization.

□

C Proof of Proposition 2

Proof. First, for $\rho > \bar{\rho}$, we prove that any $x_t \neq ul$ or $\phi_t \neq 0$ will not hold in equilibrium. Second, for $\rho \leq \bar{\rho}$, we prove that, when the collateral constraint is slack for IM, it must hold that $x_t \leq 0$ and $\phi_t > 0$. Third, for $\rho \leq \bar{\rho}$, we prove that, when the collateral constraint is binding for IM, $x_t \in (0, ul)$. The pricing for the price gap follows naturally from the collateral constraint.

As proven in Proposition 6, when $\rho > \bar{\rho}$, the steady state price gap and market liquidity are $\phi^* = 0$ and $x^* = ul$. Suppose IM only reaches the steady state position $x^* = ul$ in period t and before t they choose $x_s \neq ul$ in equilibrium, $\forall s < t$. Without any loss of generality, suppose that, in $t-1$, IM choose $x_{t-1} \neq ul$, and from the pricing formula in equilibrium $\phi_{t-1} \neq 0$. We know that, in period t, $\phi_t = 0$. Thus, in $t-1$, IM are not collateral-constrained. Given $\phi_{t-1} \neq 0$, a certain IM can make an arbitrage profit by taking $x'_{t-1} = x_{t-1} + \Delta x$, such that $\phi_{t-1}\Delta x > 0$, without assuming any increased obligation in t. Therefore, $x_{t-1} \neq ul$ cannot hold in equilibrium, and $x_{t-1} = ul$ and $\phi_{t-1} = 0$ must hold. Similarly, this applies to all $s < t$.

As proven in Proposition 6, when $\rho \leq \bar{\rho}$, the steady state price gap and market liquidity are $\phi^* > 0$ and $0 < x^* < ul$. Suppose, in this case, when the collateral constraint is slack for IM, $x_t > 0$ holds in equilibrium. Then, from the budget constraints and the first-order condition of IM, we have:

$$\begin{aligned} c_t^{\mathrm{IM}} &= F\left(K_{t-1}\right) + (1-\delta)K_{t-1} - \left(x_{t-1} - x_t\right)\phi_t - K_t, \\ c_{t+1}^{\mathrm{IM}} &= F\left(K_t\right) + (1-\delta)K_t - \left(x_t - x_{t+1}\right)\phi_{t+1} - K_{t+1}, \end{aligned}$$

$$F'\left(K_t\right) + 1 - \delta = \frac{\phi_{t+1}}{\phi_t}, \tag{17}$$

and

$$-x_t\phi_{t+1} + (1-\delta)K_t > 0.$$

Next, assume certain IM choose to take $x'_t = x_t - \epsilon$ and $K'_t = K_t - \epsilon\phi_t$ in t, where $\epsilon\phi_t > 0$

and $-x_t'\phi_{t+1}+(1-\delta)K_t'>0$. From the budget constraint, we can find that, in this way, the consumption in t stays the same as in the others, i.e., $c_t'=c_t^{\mathrm{IM}}$. However, in $t+1$ if this IM choose the same K_{t+1} and x_{t+1}, then we will have:

$$\begin{aligned}
c_{t+1}' &= F\left(K_t'\right)+(1-\delta)K_t'-\left(x_t'-x_{t+1}\right)\phi_{t+1}-K_{t+1}\\
&= F\left(K_t-\epsilon\phi_t\right)+(1-\delta)\left(K_t-\epsilon\phi_t\right)-\left(x_t-\epsilon-x_{t+1}\right)\phi_{t+1}-K_{t+1},\\
c_{t+1}'-c_{t+1}^{\mathrm{IM}} &= F\left(K_t-\epsilon\phi_t\right)-F\left(K_t\right)+(1-\delta)\left(-\epsilon\phi_t\right)+\epsilon\phi_{t+1}\\
&= F\left(K_t-\epsilon\phi_t\right)-F\left(K_t\right)+F'\left(K_t\right)\epsilon\phi_t>0.
\end{aligned}$$

The last equation follows from Equation (17), and the inequality holds because $F(\cdot)$ is a concave function of K_t. Thus, these IM can increase utility by deviating from x_t. Therefore, if $\rho\le\bar{\rho}$, $x_t>0$ will not hold in equilibrium under the slack collateral constraints. The relationship between price spreads between t and $t+1$ follows from Equation (17).

We also prove that, when the collateral constraints are binding for IM in t, their positions in equilibrium must satisfy $x_t\in(0,ul)$. Specifically, we can show this by invalidating the $x_t\le 0$ and $x_t\ge ul$ cases.

Moreover, suppose in equilibrium that $x_t\le 0$ exists when $\rho\le\bar{\rho}$, and IM's collateral constraints are binding in t, i.e., $x_t\phi_{t+1}=(1-\delta)K_t$. Thus, $\phi_{t+1}<0$. From IM's first-order condition with respect to x_t, it must also hold that $\phi_t<0$. Otherwise, certain IM could make an arbitrage profit by taking $x_t'>0$. On HH's side, this means $P_{t+1}^{\mathrm{B}}<0$ and $P_t^{\mathrm{B}}<0$, as $P_t^{\mathrm{B}}=C_t^{\mathrm{B}}\phi_t/\left(C_t^{\mathrm{A}}+C_t^{\mathrm{B}}\right)$.

Since $x_t<0$, it follows:

$$\begin{aligned}
\mathbb{E}\left[\frac{\theta_{t+1}}{C_{t+1}^{\mathrm{B}}}\right] &= \mathbb{E}\left[\frac{\theta_{t+1}}{w\left(K_t\right)+P_{t+1}^{\mathrm{B}}\left(x_t-x_{t+1}\right)/l-\left(u-x_t/l\right)\theta_{t+1}}\right]\\
&= \mathbb{E}\left[\frac{\theta_{t+1}}{\left(w\left(K_t\right)-\left(u-x_t/l\right)\theta_{t+1}\right)\left(1+\frac{\phi_{t+1}\left(x_t-x_{t+1}\right)/l}{2w\left(K_t\right)}\right)}\right]>0.
\end{aligned}$$

And the last equation comes from:

$$P_{t+1}^{\mathrm{B}}=\frac{w\left(K_t\right)-\left(u-x_t/l\right)\theta_{t+1}}{2w\left(K_t\right)}\phi_{t+1}.$$

The inequality holds, because, when $\theta_{t+1}=\epsilon>0$, $c_{t+1}^{\mathrm{B}}(\epsilon)<c_{t+1}^{\mathrm{B}}(-\epsilon)$ when $x_t\le 0$.

Assume that, from $t+1$ onwards, IM's position sequence $\{x_{t+s}\}$ in equilibrium stays below ul, i.e., $x_{t+s}<ul$, until some period $t+T$, for $s\in\{1,2,\ldots,T-1\}$. Thus,

$$\mathbb{E}\left[\frac{\theta_{t+s}}{C_{t+s}^{\mathrm{B}}}\right]>0,\qquad\forall s\le T.$$

If $T=\infty$, then, from the pricing formula,

$$\frac{P_t^{\mathrm{B}}}{C_t^{\mathrm{B}}}=\lim_{T\to\infty}\beta\mathbb{E}_t\left[\frac{\theta_{t+1}}{C_{t+1}^{\mathrm{B}}}\right]+\ldots+\beta^T\mathbb{E}_t\left[\frac{\theta_{t+T}}{C_{t+1}^{\mathrm{B}}}\right]+\beta^T\mathbb{E}_t\left[\frac{P_{t+T}^{\mathrm{B}}}{C_{t+T}^{\mathrm{B}}}\right],$$

we can conclude, that if $P_t^{\mathrm{B}}<0$,

$$\lim_{T\to\infty}\beta^T\mathbb{E}_t\left[\frac{P_{t+T}^{\mathrm{B}}}{C_{t+T}^{\mathrm{B}}}\right]<0$$

must hold. However, this violates HH's TVC.

Else, if $T<\infty$, i.e., $x_{t+T}\geq ul$, suppose $P_{t+T}^{\mathrm{B}}\leq 0$ holds. In this case, if $\phi_{t+T+1}\leq 0$, then at $t+T$ IM's collateral constraints are not binding, $x_{t+T}\phi_{t+T+1}\leq 0$. From previous proof, however, we know that if they are slack, it must have $x_{t+T}<0$. Thus, $P_{t+T}^{\mathrm{B}}\leq 0$ cannot hold. Now, suppose $\phi_{t+T+1}>0$. Given $\phi_{t+T}<0$ and $\phi_{t+T+1}>0$, certain competitive IM can earn arbitrage profits by changing their positions from $x_{t+T}\geq ul$ to $x'_{t+T}<0$.

Therefore, when $x_t\leq 0$, $\phi_t\leq 0$, and $\phi_{t+1}\leq 0$ cannot possibly hold. If $x_t\leq 0$, we must have $\phi_t>0$ and $\phi_{t+1}>0$. However, this contradicts with the binding condition, i.e., $x_t\phi_{t+1}=(1-\delta)K_t>0$. Thus, $x_t\leq 0$ cannot hold in equilibrium when the collateral constraints are binding. In the same fashion, when $x_t\geq ul$, we can derive $\phi_{t+1}\leq 0$. This again fails to satisfy IM's binding conditions. Consequently, when IM are collateral-constrained in equilibrium, $x_t\in(0,ul)$. By definition then, $\phi_{t+1}=(1-\delta)K_t/x_t$.

□

D Proof of Proposition 3

Proof. We begin proving the proposition by assuming that some IM only live for $T=2$ periods. We suppose further that all IM's collateral constraints are binding in $t=1$, and their initial wealth is W_1. The two-period living IM's optimization problems therefore become:

$$\max_{C_1^{\mathrm{IM}},C_2^{\mathrm{IM}},x_1,K_1}\log\left(C_1^{\mathrm{IM}}\right)+\rho\log\left(C_2^{\mathrm{IM}}\right),$$

subject to

$$\begin{aligned}
&(i) && C_1^{\mathrm{IM}}+K_1=W_1+x_1\phi_1,\\
&(ii) && C_2^{\mathrm{IM}}=F\left(K_1\right)+(1-\delta)K_1-x_1\phi_2,\\
&(iii) && (1-\delta)K_1\geq x_1\phi_2.
\end{aligned}$$

Because they have binding collateral constraints, $(1-\delta)K_1=x_1\phi_2$.

Applying first-order conditions, we have:

$$-\frac{1}{C_1^{\mathrm{IM}}}+\frac{\rho\left(F'\left(K_1\right)+1-\delta\right)}{C_2^{\mathrm{IM}}}+\lambda_1(1-\delta)=0,\quad \frac{\phi_1}{C_1^{\mathrm{IM}}}-\frac{\phi_2}{C_2^{\mathrm{IM}}}-\lambda_1\phi_2=0,$$

where $\lambda_1 > 0$ is the Lagrange multiplier for the collateral constraint at $t = 1$.

Solving all the above equations, we obtain:

$$C_1^{\text{IM}} = \frac{W_1}{1+\alpha\rho}, \quad K_1 = \frac{\alpha\rho W_1}{(1+\alpha\rho)S_1}, \text{ where } S_1 = 1 - \frac{(1-\delta)\phi_1}{\phi_2} = \frac{1}{\mu_1}.$$

Similarly, for some IM that live for periods of $T \in \{3, \ldots\}$, and where the collateral constraints are binding for $t \in \{1, 2, \ldots, T-1\}$, we obtain:

$$C_1^{\text{IM}} = \frac{W_1}{1+\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}}, \quad K_1 = \frac{\left(\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}\right)\mu_1 W_1}{1+\alpha\rho+\alpha^2\rho^2+\ldots+\alpha^{T-1}\rho^{T-1}}.$$

When $T \to \infty$,

$$C_1^{\text{IM}} = \lim_{T\to\infty} \frac{W_1}{1+\alpha\rho+\alpha^2\rho^2+\cdots+\alpha^{T-1}\rho^{T-1}} = (1-\alpha\rho)W_1, \quad K_1 = \alpha\rho\mu_1 W_1.$$

Likewise, starting with period t, and when

$$W_t = F\left(K_{t-1}\right) + (1-\delta)K_{t-1} - x_{t-1}\phi_t = F\left(K_{t-1}\right)$$

due to binding collateral constraints, obviously, we can extend the above to period t, for all $t = 1, 2, \ldots$.

$$C_t^{\text{IM}} = (1-\alpha\rho)W_t, \quad K_t = \alpha\rho\mu_t W_t, \quad W_{t+1} = F\left(K_t\right).$$

Moreover, we can easily confirm that, with binding collateral constraints, the steady-state level of IM's consumption, capital investment, and market liquidity, as shown in Proposition 5, are also consistent with IM's TVC.

□

E Proof of Proposition 4

Proof. Equation (14) follows from the first-order conditions in Equations (4) and (5), given $C_t^{\text{IM}} = (1-\alpha\rho)W_t$ and $K_t = \alpha\rho\mu_t W_t$. Taking partial derivatives of $\bar{\lambda}_t$, we obtain the following:

$$\frac{\partial\bar{\lambda}_t}{\partial a} = \frac{\partial\bar{\lambda}_t}{\partial F(\alpha\rho\mu_t W_t)}\frac{\partial F(\alpha\rho\mu_t W_t)}{\partial a} = \frac{\rho W_t}{aF(\alpha\rho\mu_t W_t)} > 0.$$

$$\frac{\partial\bar{\lambda}_t}{\partial \alpha} = \frac{\partial\bar{\lambda}_t}{\partial F(\alpha\rho\mu_t W_t)}\frac{\partial F(\alpha\rho\mu_t W_t)}{\partial \alpha} = \frac{\rho W_t(1+\log(\alpha\rho\mu_t W_t))}{F(\alpha\rho\mu_t W_t)} > 0.$$

$$\frac{\partial\bar{\lambda}_t}{\partial \delta} = \frac{\partial\bar{\lambda}_t}{\partial F(\alpha\rho\mu_t W_t)}\frac{\partial F(\alpha\rho\mu_t W_t)}{\partial \mu_t}\frac{\partial \mu_t}{\partial \delta} = \frac{-\alpha\rho\mu_t W_t\phi_t}{F(\alpha\rho\mu_t W_t)\phi_{t+1}} < 0. \quad \frac{\partial\bar{\lambda}_t}{\partial W_t} = -\frac{\rho(1-\alpha)}{F(\alpha\rho\mu_t W_t)} < 0.$$

$$\frac{\partial \bar{\lambda}_t}{\partial(\phi_t/\phi_{t+1})} = \frac{d\bar{\lambda}_t}{d(\phi_t/\phi_{t+1})} + \frac{\partial \bar{\lambda}_t}{\partial F(\alpha\rho\mu_t W_t)} \frac{\partial F(\alpha\rho\mu_t W_t)}{\partial \mu_t} \frac{\partial \mu_t}{\partial(\phi_t/\phi_{t+1})} = 1 + \frac{(1-\delta)\alpha\rho\mu_t W_t}{F(\alpha\rho\mu_t W_t)} > 0.$$

□

F Proof of Proposition 5

Proof. From Proposition 2, we know that, for certain given technology and endowment shocks, if $\rho > \bar{\rho}$, the economy will resemble one in a neoclassical model with frictionless financial markets, i.e., $x_t = ul$. Obviously, in this instance, equilibrium exists.

Otherwise, if $\rho \leq \bar{\rho}$, when IM's collateral constraints are binding, we can solve the equilibrium backwards through Equations (9), (10), (12), (13), and (15). When the collateral constraints are slack for the initial few periods, we can also solve for the equilibrium through Equations (11), (4), and (9), as well as both types of agents' budget constraints. Meanwhile, IM hold opposite positions in two markets and $x_t^i = -y_t^i l$ ensures the markets clear for financial assets.

With the assumption of constant shock intensity, i.e., $u_t = u$, and with IM having net zero positions in financial assets, IM are not exposed to any idiosyncratic shocks from θ_t. Hence, IM's optimization problems are deterministic ones. In equilibrium, the quantities $\left(\phi_t,\ x_t,\ y_t^i,\ K_t\right)$ are also deterministic.

As implied by Proposition 2, in equilibrium, IM's position in the markets with positive shock intensity will not exceed total asset demands, i.e., $x_t \leq ul$. Therefore, $\phi_t \geq 0$, and from Equation (7) and (8), $P_t^{\mathrm{A}} \leq 0$ and $P_t^{\mathrm{B}} \geq 0$.

□

G Proof of Proposition 6

Proof. To facilitate later proving, we first derive the steady-state price spreads independent of whether collateral constraints are slack.

$$\begin{aligned}\phi_t^* \; &:= P_t^{\mathrm{B}*} - P_t^{\mathrm{A}*} \;=\; \beta\left(C_t^{\mathrm{B}*} + C_t^{\mathrm{A}*}\right)\mathbb{E}\left[\frac{P_{t+1}^{\mathrm{B}*} + \theta_{t+1}}{C_{t+1}^{\mathrm{B}*}}\right] \;=\; 2\beta w\left(K^*\right)\left(\frac{\phi_{t+1}}{C_{t+1}^{\mathrm{A}*} + C_{t+1}^{\mathrm{B}*}} + \mathbb{E}\left[\frac{\theta_{t+1}}{C_{t+1}^{\mathrm{B}*}}\right]\right) \\ &= 2\beta w\left(K^*\right)\left(\frac{\phi_{t+1}^*}{2w\left(K^*\right)} + \mathbb{E}\left[\frac{\theta_{t+1}}{w\left(K^*\right) - \left(u - x^*/l\right)\theta_{t+1}}\right]\right),\end{aligned}$$

where the first equation is by definition, the second is derived directly from the first-order condition of HH, the third is from Proposition 1 and Lemma 1, the fourth follows from $P_t^{\mathrm{B}*} = \frac{\phi_t C_t^{\mathrm{B}}}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}}$, and the final equation is straight derived by HH's steady state budget constraints. Thus, rearranging the above, and equating $\phi_t^* = \phi_{t+1}^*$ as the steady state property, we obtain:

$$\phi^* = \frac{2\beta w\left(K^*\right)}{1-\beta}\mathbb{E}\left[\frac{\theta}{w\left(K^*\right) - \left(u - x^*/l\right)\theta}\right].$$

It is straightforward that, when $x^* = ul$, $\phi^* = 0$, and, when $x^* < ul$, $\phi^* > 0$.

Part 1 - Proof for the Slack Steady State

Given a steady state with slack collateral constraint, and as IM are competitive, it must satisfy $x^* = ul$ and $\phi^* = 0$. However, suppose that, instead, $\phi^* \neq 0$ or $x^* \neq ul$. If $\phi^* \neq 0$, because IM are not collateral-constrained, they will increase or decrease their position to arbitrage the price difference, until, in equilibrium, $\phi^* = 0$. Likewise, if $x^* < ul$ (or $x^* > ul$), from the HH's pricing, we must have $\phi^* > 0$ ($\phi^* < 0$). Again, IM will ensure $x^* = ul$ and $\phi^* = 0$.

Meanwhile, as in the slack steady state all arbitrage opportunities are eliminated. Thus, the IM's budget constraints over the long run are equivalent to:

$$C_t^{\mathrm{IM}} = F\left(K_{t-1}\right) + (1-\delta)K_{t-1} - K_t.$$

Next, combining with the first-order condition and equating $K_t = K_{t-1} = \ldots = K^*$, we obtain $K^* = K_s^* = F'^{-1}\left(\frac{1-\rho(1-\delta)}{\rho}\right)$, which is the same as that in the neoclassical growth model. Similarly, the steady state IM's consumption level is also equals the one in the frictionless neoclassical model, i.e., $C_s^* = F(K_s^*) + (1-\delta)K_s^* - K_s^* = \frac{\delta(1-\alpha\rho)+(1-\rho)(1-\delta)}{\alpha\rho}K_s^*$. Because the collateral constraints are slack, the Langrange multiplier and the shadow collateral price are zero. Also, because IM earn zero profit, their capital investment equals their savings. Thus, $\mu_s^* = 1$.

Part 2 - Proof for the Binding Steady States

Given a steady state with binding collateral constraints, we obtain $(1-\delta)K^* = x^*\phi^*$. From IM's first-order conditions, and by equating capital investment, asset positions, and spreads across periods to $(K^*,\ x^*,\ \phi^*)$, we derive $K_b^* = F'^{-1}(\delta/\rho)$. Because $\delta/\rho < \left(1-\rho(1-\delta)\right)/\rho$ and $F'^{-1}(K)$ is a decreasing function of K, we conclude that $K_b^* > F^{-1}(\frac{1-\rho(1-\delta)}{\rho})$. Moreover, from Proposition 3 and Proposition 4, we obtain $\mu_b^* = 1/\delta$, $\bar{\lambda}_b^* = 1-\rho$, and $C_b^* = (1-\alpha\rho)F(K_b^*) = \frac{\delta(1-\alpha\rho)K_b^*}{\alpha\rho}$.

Next, we prove that IM's steady state consumption level C_b^* is higher than its counterpart in the neoclassical model, $C^* = \frac{\delta(1-\alpha\rho)+(1-\rho)(1-\delta)}{\alpha\rho}F^{-1}(\frac{1-\rho(1-\delta)}{\rho})$. Therefore,

$$\frac{C^*}{C_b^*} = \frac{\frac{\delta(1-\alpha\rho)+(1-\rho)(1-\delta)}{\alpha\rho}F^{-1}(\frac{1-\rho(1-\delta)}{\rho})}{\frac{\delta(1-\alpha\rho)}{\alpha\rho}F^{-1}(\delta/\rho)} = \left(1+\frac{(1-\delta)(1-\rho)}{\delta(1-\alpha\rho)}\right)\left(1+\frac{(1-\rho)(1-\delta)}{\delta}\right)^{\frac{1}{\alpha-1}}.$$

For simplicity, we denote $v = \frac{(1-\rho)(1-\delta)}{\delta}$, and the above ratio is a function of v:

$$\frac{C^*}{C_b^*} = R(v) = \left(1+\frac{v}{(1-\alpha\rho)}\right)(1+v)^{\frac{1}{\alpha-1}}.$$

The ratio $R(v)$ is a decreasing function with respect to v, as when $v > 0$:

$$R'(v) = \frac{(1+v)^{\frac{1}{\alpha-1}}}{1-\alpha\rho} - (1+\frac{v}{1-\alpha\rho})\frac{1}{1-\alpha}(1+v)^{\frac{\alpha}{1-\alpha}} = \frac{(1+v)^{\frac{1}{\alpha-1}}}{1-\alpha\rho}\left(1 - \frac{1-\alpha\rho+v}{1-\alpha}(1+v)^{\frac{1+\alpha}{1-\alpha}}\right) < 0.$$

Thus, when $v = \frac{(1-\rho)(1-\delta)}{\delta} > 0$, we have:

$$\frac{C^*}{C_b^*} < R(0) = 1.$$

Furthermore, given the existence of $\bar{\rho}$, it is obvious that there is discountinuity in K^*, C^*, $\bar{\lambda}^*$, and μ^* around the cutoff parameter, since:

$$\lim_{\rho\to\bar{\rho}^-} K^*(\rho) = K_b^*(\bar{\rho}) = F'^{-1}(\frac{\delta}{\bar{\rho}}), \quad \lim_{\rho\to\bar{\rho}^+} K^*(\rho) = K_s^*(\bar{\rho}) = F'^{-1}(\frac{1-\rho(1-\delta)}{\bar{\rho}}).$$

Part 3 - Proof of the Existence of the Cutoff Parameter $\bar{\rho}$

We define a function $k(\rho) := F'^{-1}(\delta/\rho)$ as the potential long-run capital investment level, given that the equilibrium can support a steady state with binding collateral constraints, and when IM's discount factor is ρ.

Also, we define the following function of IM's discount factor ρ and steady state position x^* as the corresponding steady state price spread:

$$\Phi(x^*, \rho) := \frac{2\beta}{1-\beta} w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho)) - (u - x^*/l)\,\theta}\right].$$

Since if there is a binding steady state, it must satisfy the collateral constraint $(1-\delta)k(\rho) = x^*\Phi(x^*, \rho)$. Thus, we construct a product function of the steady state position and spread $G(x, \rho)$ and an auxiliary function $g(x, \rho, \theta)$:

$$G(x,\rho) := x\Phi(x,\rho) = x\frac{2\beta}{1-\beta} w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho)) - (u-x/l)\theta}\right], \; g(x,\rho,\theta) := \frac{x\theta}{w(k(\rho)) - (u-x/l)\theta}.$$

Since:

$$\frac{\partial g(x,\rho,\theta)}{\partial x} = \frac{\theta w(k(\rho)) - u\theta^2}{\left(w(k(\rho)) - (u-x/l)\theta\right)^2}, \quad \text{and} \quad \frac{\partial g^2(x,\rho,\theta)}{\partial x^2} = \frac{-2\theta^2\left(w(k(\rho)) - u\theta\right)}{l\left(w(k(\rho)) - (u-x/l)\theta\right)^3} < 0,$$

where $x \in [0, ul]$, and $g(x, \rho, \theta)$ is a strict local concave function of x over $[0, ul]$, given ρ and θ. Note here that we apply $w(k(\rho)) - u\theta > e_t^i - u\bar{\theta} > 0$.

Similarly, since:

$$G(x,\rho) = \frac{2\beta}{1-\beta} w(k(\rho)) \int_{-\bar{\theta}}^{\bar{\theta}} g(x,\rho,\theta)p(\theta)\mathrm{d}\theta, .$$

where $p(\theta)$ is the PDF of θ, $G(x, \rho)$ is also a strict local concave function with respect to x over

$[0, ul]$ given ρ. Moreover,

$$\left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=0} = \left.\int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{(w(k(\rho)) - (u - x/l)\theta)^2} p(\theta)\mathrm{d}\theta\right|_{x=0} = \int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta}{(w(k(\rho)) - u\theta)} p(\theta)\mathrm{d}\theta > 0,$$

$$\begin{aligned}\left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=ul} &= \left.\int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{(w(k(\rho)) - (u - x/l)\theta)^2} p(\theta)\mathrm{d}\theta\right|_{x=ul} = \int_{-\bar{\theta}}^{\bar{\theta}} \frac{\theta w(k(\rho)) - u\theta^2}{(w(k(\rho)))^2} p(\theta)\mathrm{d}\theta \\ &= \int_{-\bar{\theta}}^{\bar{\theta}} \frac{-u\theta^2}{(w(k(\rho)))^2} p(\theta)d\theta < 0,\end{aligned}$$

Therefore, given ρ, $G(x,\rho)$ has a unique interior maximal value on some $\bar{x} \in (0, ul)$, where:

$$\left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x=\bar{x}} = 0.$$

On the other hand, rearranging $G(x,\rho)$, we obtain:

$$G(x,\rho) = \frac{2\beta}{1-\beta}\int_{-\bar{\theta}}^{\bar{\theta}} \frac{w(k(\rho))x\theta}{w(k(\rho)) - (u - x/l)\theta} p(\theta)\mathrm{d}\theta = \frac{2\beta}{1-\beta}\int_{-\bar{\theta}}^{\bar{\theta}} \frac{x\theta}{1 - (u - x/l)\theta/w(k(\rho))} p(\theta)\mathrm{d}\theta.$$

It is straightforward to show $\partial G(x,\rho)/\partial\rho < 0$. Thus, $G(x,\rho)$ is a decreasing function of ρ.

To prove the existence of a cut-off $\bar{\rho}$, which would determine whether IM's discount factor ρ corresponds to a slack or a binding steady state, we next define new auxiliary functions $H(\rho)$ and $Q(\rho)$ as follows:

$$H(\rho) := \max_x G(x,\rho), \quad Q(\rho) := H(\rho) - (1-\delta)k(\rho).$$

If the economy supports binding steady states, it must satisfy $Q(\rho) \geq 0$. When $Q(\rho) < 0$, it means that the collateral is sufficient to eliminate any arbitrage opportunities, and can only support steady states with slack collateral constraints. This is because, if there exist binding steady states, then, given ρ, $G(x^*,\rho) - (1-\delta)k(\rho) = 0$ should have real solutions of $x^* \in (0, ul)$. Since $G(x^*,\rho) - (1-\delta)k(\rho) < Q(\rho)$, if $Q(\rho) < 0$, then there is no solution of x^*, given this value of ρ that can support any binding steady state. $Q(\rho)$ is a continuous and strictly decreasing function of ρ, so there exists a unique cutoff value $\bar{\rho}$, such that $Q(\bar{\rho})) = 0$. That is, if ρ is above this $\bar{\rho}$, $Q(\rho) < 0$, and there only exists a slack steady state. In contrast, when $\rho \leq \bar{\rho}$, $G(x^*,\rho) - (1-\delta)k(\rho) = 0$ has real solutions on $(0, ul)$, and there exist binding steady states.

□

H Proof of Proposition 7

Proof. To prove for the existence of two distinct steady states, for any given $\rho' \in (0, \bar{\rho}]$, we define an auxiliary function $J(x,\rho') := G(x,\rho') - (1-\delta)k(\rho') \leq Q(\rho')$, where $G(x,\rho)$, $Q(\rho)$, and $k(\rho)$ are constructed in the proof of Proposition 6. Thus, $Q(\rho') \geq 0$, for $\rho' \in [0, \bar{\rho}]$. Obviously, $J(x,\rho')$ is a continuous function and only when $x = \bar{x}$, which is the global maximizer, is $J(x,\rho')$ equal to $Q(\rho')$.

We adopt the functions $G(x,\rho)$, $Q(\rho)$ and $k(\rho)$ from the proof of Proposition 6, and construct a new auxiliary function:

$$J(x,\rho) := G(x,\rho) - (1-\delta)k(\rho) = x\Phi(x,\rho) - (1-\delta)k(\rho) \le Q\left(\rho'\right).$$

Inheriting features from $G(x,\rho)$ and $k(\rho)$, it is obvious that $J(x,\rho)$ is a strictly concave function of x, and a decreasing function of ρ. Moreover, when $J(x,\rho)=0$ has real solutions in $(0,ul)$, there exists binding steady states given ρ, and the solutions x^* correspond to the steady state market liquidity.

Also, by definition,

$$Q(\rho) = \max_x J(x,\rho).$$

From the proof in Proposition 6, we know that $Q(\rho)$ is a decreasing function of ρ, and when $Q(\rho)<0$, no binding steady states are possible. When $Q(\rho)=0$, $\rho=\bar{\rho}$. Meanwhile, when $\rho=\bar{\rho}$, if there exists any binding steady state at all, it must satisfy:

$$J(x,\bar{\rho}) = 0.$$

Thus, $J(x,\bar{\rho}) = Q(\bar{\rho})$. Given that $J(x,\rho)$ is a strictly concave function of x, and that $J(x,\rho) = Q(\rho)$ only holds when x equals to the global maximizer $\bar{x}(\bar{\rho})$, we can conclude that $x^* = \bar{x}(\bar{\rho})$ is the only solution. Therefore, when $\rho=\bar{\rho}$, there exists a unique binding steady state with $x^* = \bar{x}(\bar{\rho})$ as its market liquidity.

When $0<\rho<\bar{\rho}$, then

$$Q(\rho) = \max_x J(x,\rho) > 0.$$

We denote $x(\rho)$ as the global maximizer of $J(x,\rho)$, given ρ. By definition,

$$J(0,\rho) < 0, \quad J(ul,\rho) < 0,$$

given $J(x,\rho)$ is a strictly concave function of x, there must be two distinct solutions $x_1^* \in (0,\bar{x}(\rho))$ and $x_2^* \in (\bar{x}(\rho), ul)$ to $J(x,\rho)=0$. Thus, when $0<\rho<\bar{\rho}$, there exist two distinct binding steady states with differing levels of market liquidity, i.e. $x_1^* < x_2^*$.

Since both steady states satisfy the binding collateral constraints $x_j^*\phi_j^* = (1-\delta)K_b^*$, $j \in \{1,2\}$, where $K_b^* = k(\rho)$ is a strictly increasing function of ρ. Thus, the two steady states share the same capital investment level $K_b^* = k(\rho)$. On the other hand, if x^* is smaller in one state than the other, the steady-state price spread must be higher than the other. That is, if $x_1^* < x_2^*$, then $\phi_1^* > \phi_2^*$.

From HH's perspective, they have the same labor income and $w(k(\rho))$ in both steady states because of the common K^*. So, higher x^* reduces their consumption volatility. Thus HH's utility is therefore strictly higher in SS_2 with larger market liquidity $x_2^* > x_1^*$.

Furthermore, $J(x,\rho)$ is a decreasing function of ρ, so $J(x,\rho_1) < J(x,\rho_2)$, if $\rho_1 > \rho_2$. On the other hand,

$$\left.\frac{\partial J(x,\rho)}{\partial x}\right|_{x\in(0,\bar{x}(\rho))} = \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x\in(0,\bar{x}(\rho))} > 0, \quad \left.\frac{\partial J(x,\rho)}{\partial x}\right|_{x\in(\bar{x}(\rho),ul)} = \left.\frac{\partial G(x,\rho)}{\partial x}\right|_{x\in(\bar{x}(\rho),ul)} < 0.$$

That is, for $x \in (0, \bar{x}(\rho))$, $J(x, \rho)$ is an increasing function of x. And, for $J(x_1, \rho_1) = J(x_1', \rho_2) = 0$, where $x_1,\ x_1' \in (0, \bar{x}(\rho))$, $x_1 > x_1'$ must hold. The market liquidity in SS_1 increases with ρ. Similarly, x_2^* in SS_2 decreases with ρ, i.e., $x_2' > x_2$. Because $k(\rho)$ is an increasing function of ρ, $K_b^*(\rho_1) > K_b^*(\rho_2)$. From binding collateral constraints, $x_2\phi_2 = (1-\delta)K_b^*(\rho_1)$, and $x_2'\phi_2' = (1-\delta)K_b^*(\rho_2)$, we can conclude that $\phi_2 > \phi_2'$. It is thus simple to verify that

$$\frac{\partial \Phi(x,\rho)}{\partial x} < 0, \qquad \frac{\partial \Phi(x,\rho)}{\partial \rho} < 0,$$

and we can also derive $\phi_1 < \phi_1'$.

In terms of welfare, IM are indifferent between the two regimes, because their utility and consumption depend solely on the capital investment K_b^*, which is independent of the position size x^* or the price difference ϕ^*. By contrast, for HH, the two steady states carry significantly different welfare implications. Given a fixed level of K_b^*, HH in the two steady states receive the same amount of labor income. However, there is less market liquidity in the first steady state SS_1, which would expose HH to more unhedged consumption risk from endowment shocks. Thus, HH would strictly prefer SS_2 with more liquid financial markets.

□

I Proof of Proposition 8

Proof. We modify the definition of functions $\Phi(x,\rho)$, $G(x,\rho)$, $g(x,\rho,\theta)$, $J(x,\rho)$, $H(\rho)$, and $Q(\rho)$ in the proof of Propositions 6 and 7 by extending them as functions of the shock intensity u. We obtain:

$$\begin{aligned}
\Phi(x,\rho,u) &:= \phi^*(x^*,K^*) = \frac{2\beta}{1-\beta} w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho)) - (u - x/l)\theta}\right], \\
G(x,\rho,u) &:= x\Phi(x,\rho,u) = x\frac{2\beta}{1-\beta} w(k(\rho))\mathbb{E}\left[\frac{\theta}{w(k(\rho)) - (u - x/l)\theta}\right], \\
g(x,\rho,u,\theta) &:= \frac{x\theta}{w(k(\rho)) - (u - x/l)\theta}, \quad J(x,\rho,u) := G(x,\rho,u) - (1-\delta)k(\rho), \\
H(\rho,u) &:= \max_x G(x,\rho,u), \quad Q(\rho,u) := H(\rho,u) - (1-\delta)k(\rho).
\end{aligned}$$

In particular,

$$\frac{\partial g(x,\rho,u,\theta)}{\partial u} = \frac{x\theta^2}{(w(k(\rho)) - (u - x/l)\theta)^2} \geq 0.$$

Similarly, $\partial G(x,\rho,u)/\partial u > 0$, $\partial J(x,\rho,u)/\partial u > 0$, for $x \in (0, ul]$. It follows naturally that $\partial H(\rho,u)/\partial u > 0$, and $\partial Q(\rho,u)/\partial u > 0$. Thus, if $u_1 < u_2$, then $Q(\rho,u_1) < Q(\rho,u_2)$.

The cutoff value $\bar{\rho}$ for a given u is determined by ρ', such that $Q(\rho',u) = 0$. Because $Q(\rho,u)$

is a decreasing function of ρ and an increasing function of u, then to satisfy:

$$Q(\bar{\rho}_1, u_1) = 0, \quad Q(\bar{\rho}_2, u_2) = 0.$$

we must have $\bar{\rho}_1 < \bar{\rho}_2$ if $u_1 < u_2$. The cutoff value thus increases with u.

From the property of $G(x, \rho, u)$, it is apparent that $J(x, \rho, u)$ is a strictly concave function of x. By definition,

$$\left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=0} > 0; \quad \left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=ul} < 0; \quad \left.\frac{\partial J(x,\rho,u)}{\partial x}\right|_{x=\bar{x}} = 0.$$

where $\partial J(x,\rho,u)/\partial x$ is a continuous function of x, so we can conclude that $\partial J(x,\rho,u)/\partial x > 0$ for $x \in (0, \bar{x})$, and $\partial J(x,\rho,u)/\partial x < 0$ for $x \in (\bar{x}, ul)$. The steady state x^* in both $x \in (0, \bar{x})$ and $x \in (\bar{x}, ul)$ is determined by the requirement that they must satisfy $J(x^*, \rho, u) = 0$.

Previously, we know that $\partial J(x,\rho,u)/\partial u > 0$, so if $u_1 < u_2$, for a given x, $J(x, \rho, u_1) < J(x, \rho, u_2)$. Because $J(x,\rho,u)$ is an increasing function of x at $x_1^*[u_j]$, $j \in \{1, 2\}$, then, to satisfy $J(x_1[u_1], \rho, u_1) = 0$ and $J(x_1[u_2], \rho, u_2) = 0$ simultaneously, we must have $x_1^*[u_1] > x_1^*[u_2]$. The similar also applies for SS_2, where the opposite holds, $x_2^*[u_1] < x_2^*[u_2]$.

The steady state capital investment level $K_b^* = k(\rho)$ is only determined by ρ, and it remains independent of u. Thus, in the two economies with the same discount factor ρ of IM, $K_b^*[u_1] = K_b^*[u_2] = k(\rho)$.

□

J Proof of Corollary 1.1

Proof. Following the initial shock in capital investment K_{t-1} or IM's wealth W_t from Proposition 4, the shadow collateral price $\bar{\lambda}$ drops given the price level ϕ_t and ϕ_t/ϕ_{t+1} remain the same or increase. From Proposition 3 and the first-order condition Equation (4), we obtain $K_t < K^*$, and hence $x_t < x^*$, following the collateral constraints. Next, we show that the collective unwinding indeed moves the equilibrium price gap level ϕ_t and arbitrage profitability ϕ_t/ϕ_{t+1} upward.

From Proposition 1 and Lemma 1, we have:

$$\phi_t = \frac{2w\left[K_{t-1}^*\right]}{\left(\sum_{j=1}^{\infty} \beta^j \mathbb{E}\left[\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{B}}}\right]\right)^{-1} + \left(x_t - x_{t-1}^*\right)/l} = \frac{2w\left[K_{t-1}^*\right]}{M_t + \left(x_t - x_{t-1}^*\right)/l}.$$

Compared to the steady-state price gap, we can easily prove from HH's budget constraints that $M_t < M^*$. Because $x_t - x^* < 0$, the immediate price spread $\phi_t > \phi^*$. Similarly, we show that the ratio ϕ_t/ϕ_{t+1} also decreases with K_t and x_t.

The long-term effects follow naturally from Proposition 6.

□

K Proof of Corollary 1.2

Proof. We start by comparing two alternative thought scenarios:

1. The same shock hits the economy at the bad steady state (K^*, x_b^*, ϕ_b^*), and, afterward, the economy reverts to the same pre-shock state.
2. The same shock hits the economy at the good steady state $\left(K^*,\ x_g^*,\ \phi_g^*\right)$ and, afterward, the economy reverts to the same pre-shock state.

From Corollary 1.1, we know that IM's liquidation repayments immediately after the shock in the two scenarios are $x_b^*\phi_{b,t}$ and $x_g^*\phi_{g,t}$. In addition, $\phi_{b,t} > \phi_{g,t}$, which follows from the pricing formulas of ϕ_t and Proposition 7.

Now consider the scenario in Corollary 1.2 (that is, starting from the good steady state and shifting to the bad one). Compared with Scenario 2, we can derive $\phi_t > \phi_{b,t}$, because the corollary scenario only differs in the pre-shock position size x_g^*. For the same after-shock price $\phi_{b,t}$, IM will incur larger obligated repayments $x_g^*\phi_{b,t} > x_b^*\phi_{b,t}$. Thus, IM's wealth losses are amplified to a greater extent, compared to Scenario 2 case. From the proof of Corollary 1.1, K_t and x_t also drop more dramatically: $K_t < K_{b,t}$, $x_t < x_{b,t}$ and $\phi_t > \phi_{b,t}$.

Similarly, compared with Scenario 1, because $\phi_t > \phi_{b,t} > \phi_{g,t}$, the obligated repayment at t is larger, $x_g^*\phi_t > x_g^*\phi_{b,t}$. Thus, K_t and x_t also drop further: $K_t < K_{g,t}$, $x_t < x_{b,t} < x_{g,t}$.

The long-term convergence feature follows from Proposition 6.

□

L Proof of Corollary 1.3

Proof. According to Proposition 8, the new good steady state features lower market liquidity and a larger price spread than the pre-shock one. Thus, the immediate short-run price reaction is that ϕ_t surges in order to match the dynamics that converge to the new steady-state level. This can be seen from the following pricing formula:

$$\frac{\phi_t}{C_t^{\mathrm{A}} + C_t^{\mathrm{B}}} = \lim_{j\to\infty} (\beta\mathbb{E}\frac{\theta_{t+1}}{C_{t+1}^{\mathrm{A}} + C_{t+1}^{\mathrm{B}}} + \beta^2\mathbb{E}\frac{\theta_{t+2}}{C_{t+2}^{\mathrm{A}} + C_{t+2}^{\mathrm{B}}} + \cdots + \beta^j\mathbb{E}\frac{\theta_{t+j}}{C_{t+j}^{\mathrm{A}} + C_{t+j}^{\mathrm{B}}}).$$

Because $u_1 - x_{t+j} \to u_1 - x_{n,g}^* < u - x_g^*$, the RHS of the above is higher than its counterpart before the shock. Hence, the LHS also increases. This only happens, however, when ϕ_t increases above ϕ_g^*, given that x_t remains x_g^* or decreases.

Therefore, IM suffer additional losses at t after the shock, because they still carry the pre-shock large position x_g^*. From the proof of Corollaries 1.1 and 1.2, IM also have to cut down x_t and K_t as a result.

The long-run effects follow from Propositions 6 and 8.

□

M Proof of Proposition 9

Proof. Before we compare the post-shock welfare, we first prove the following lemma.

Lemma 2. *IM's indirect utility function at $t+1$ follows the form: $V(W_t) = \frac{\alpha}{1-\rho\alpha}\log(W_t) + D_t$, where D_t is a deterministic function of t.*

Proof of Lemma 2. We conjecture that IM's indirect function after t follows the form: $V(W_t) = z\log W_t + D_t$, where z is a constant. Since $V(W_t) = \log(C_{t+1}) + \rho V(W_{t+1})$, and from Proposition 3, we have $C_{t+1} = (1-\alpha\rho)W_{t+1} = (1-\alpha\rho)F(K_t) = (1-\alpha\rho)F(\alpha\rho\mu_t W_t)$, $V(W_t)$ follows:

$$\begin{aligned} V(W_t) =& \log((1-\alpha\rho)W_{t+1}) + \rho V(W_{t+1}) \\ =& \log((1-\alpha\rho)W_{t+1}) + \rho z\log(W_{t+1}) + \rho D_{t+1}. \end{aligned}$$

Rearranging, we have:

$$\begin{aligned} z\log(W_t) + D_t =& (1+\rho z)\log(W_{t+1}) + \log((1-\alpha\rho) + \rho D_{t+1} \\ &= (1+\rho z)\log(F(\alpha\rho\mu_t W_t)) + \log(1-\alpha\rho) + \rho D_{t+1} \\ &= (1+\rho z)\log(a\alpha(\alpha\rho\mu_t W_t)^{\alpha}L^{\gamma}) + \log(1-\alpha\rho) + \rho D_{t+1} \\ &= (1+\rho z)\alpha\log(W_t) + (1+\rho z)\log(a\alpha(\alpha\rho\mu_t)^{\alpha}L^{\gamma}) + \log(1-\alpha\rho) + \rho D_{t+1}. \end{aligned}$$

Equating the coefficients of $\log(W_t)$ on both sides: $z = (1+\rho z)\alpha$. Thus,

$$\begin{aligned} &z = \frac{\alpha}{1-\alpha\rho}, \\ &D_t = (1+\rho z)\log(a\alpha(\alpha\rho\mu_t)^{\alpha}L^{\gamma}) + \log(1-\alpha\rho) + \rho D_{t+1}. \end{aligned}$$

Hence, after the shock, IM's utility increases with their post-shock wealth. □

From Proposition 3, we know K_{t+j} and x_{t+j} also increase with W_{t+j}, $\forall j \in [0,\infty)$, which in turn is a nondecreasing function of W_t. As HH's post-shock utility increases with both K_{t+j} (through labor income effects) and x_{t+j} (through risk-sharing), $\forall j \in [0,\infty)$, their welfare is also a nondecreasing function of W_t.

Next, without loss of generality, denote x^*_B, x^*_G, and x^*_N as the liquidity supply in the pre-shock bad regime, the pre-shock good regime and the given post-shock regime, respectively. We prove the proposition in the following cases:

- Case 1: $x^*_N < x^*_B < x^*_G$;
- Case 2: $x^*_B \le x^*_N < x^*_G$;
- Case 3: $x^*_B < x^*_G \le x^*_N$.

Case 1: In this case, IM in both pre-shock regimes suffered arbitrage losses. Because $W_t = F(K^*) + (1-\delta)K^* - x^*\phi_t$, and from Equation (9), $\frac{\partial\phi_t}{\partial x^*} > 0$, W_t decreases with pre-shock regime liquidity x^*. Hence, IM's wealth of the bad pre-shock regime is greater than that of the good pre-shock regime. Similar logic also applies for x_{t+j} and K_{t+j}, $\forall j \in [0,\infty)$. Thus, in this case, in the economy with the bad pre-shock regime, the welfare, liquidity supply, and capital are higher than those in the counterpart economy.

Case 2: In the economy with the bad pre-shock regime, x_t and ϕ_t immediately converge to x_N^* and ϕ_N^*; IM get a windfall profits from reducing liability at t, as $\phi_N^* \leq \phi_B^*$. In the economy with good initial regime, $x_t < x_N^*$ and $\phi_t > \phi_N^*$; IM suffer arbitrage losses from increasing liability at t, since $\phi_N^* \geq \phi_G^*$. Thus, IM's wealth of the bad pre-shock regime is greater than that of the good pre-shock regime before converging to the new steady state. Similar logic also applies for x_{t+j} and K_{t+j},$\forall j \in [0, \infty)$. Thus, in this case, in the economy with the bad pre-shock regime, the welfare, liquidity supply, and capital are higher than those in the counterpart economy.

Case 3: x_t and ϕ_t in two economies both immediately converge to x_N^* and ϕ_N^*, and IM get a windfall profits from reducing liability at t. As $x_G^* \geq x_B^*$, IM's wealth of the bad pre-shock regime is greater than that of the good pre-shock regime. $x_{t+j} = x_N^*$ and $K_{t+j} = K_N^*$ $(\forall j \in [0, \infty))$ are identical. Thus, in this case, IM's welfare is higher in the economy with the bad pre-shock regime; HH's post-shock welfare in the two economies are identical. Also, the two economies have the same post-shock liquidity and capital. □