Optimal Disclosure of Value Distribution Information in All-Pay Auctions

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June 28, 2019

All-pay auctions and information disclosure

- All-pay auctions
  - Examples: R&D races, bidding for procurement contracts, lawsuits/litigation, policy debates, legislative, lobbying, electoral campaigns and sports, etc..
  - In many contexts, an important goal is to elicit effort, expenditure, etc..
- In an auction with incomplete information, the organizer can manipulate players' beliefs and thus their bidding behaviors by information disclosure.
  - e.g., job promotions: candidates' abilities

Our research

- We consider a 2-player all-pay auction
  - Binary private values; two possible value distributions (states)
  - The organizer commits to a public disclosure policy
    - Bayesian persuasion approach
    - Discloses a signal contingent on the state
- Main finding:
  - If the two private values are sufficiently different, a monotone equilibrium always exists. An uninformative disclosure policy is optimal;
  - If the two private values are sufficiently close, there exists two beliefs that separate beliefs generating monotone and non-monotone equilibrium:
    - if the prior induces a monotone equilibrium, an uninformative disclosure policy is optimal;
    - if the prior induces a non-monotone equilibrium, a partial disclosure which generates a posterior distribution over the two separate beliefs is optimal.

Literature review

- All-pay auctions with complete information:
  - Hillman and Riley (1989); Baye, Kovenock and de Vries (1993, 1996); Barut and Kovenock (1998), etc.
- All-pay auctions with incomplete information:
  - Continuous types: Amann and Leininger (1996); Krishna and Morgan (1997); Lu and Parreiras (2017)
  - Discrete types: Siegel (2014); Rentschler and Turocy (2016); Liu and Chen (2016), Chi, Murto and Valimaki (2019)
- Information disclosure in all-pay auctions/contests:
  - Zhang and Zhou (2016); Serena (2017); Lu, Ma, and Wang (2018); Kuang, Zhao, and Zheng (2019),

Environment

- Two players: $i \in \{1, 2\}$
  - ex ante symmetric, risk neutral
  - private value: $v_i \in \{v_l, v_h\}$, $(v_h > v_l > 0)$
    - v_1 and v_2 are identically and independently drawn from distribution $p(v|\omega)$;
    - $\omega \in \Omega = \{G, B\}$: a common unknown state of world;
    - common prior of state: $(P_G, 1 - P_G)$, $0 \leq P_G \leq 1$;
    - $p(v_h|G) = \alpha$ and $p(v_l|B) = \beta$.
    - $1 \geq \alpha > 1 - \beta \geq 0$: G is a good stage, higher chance for higher type
- The auction organizer:
  - can disclose information about the state ω
  - in particular, discloses public signal s to players according to policy

$$\pi = \{p(s|\omega)\}_{s \in \mathcal{S}, \omega \in \Omega}$$

  - maximize the ex ante expected total bids

Environment

- The time line:
  - The organizer commits to policy π;
  - State ω is realized, and signal s is disclosed;
  - Players observes their private values and the signal;
  - Players places their bids.

Belief updating

Upon receiving signal $s \in \mathcal{S}$, player $i \in \{1, 2\}$ has

- posterior μ_s and private value v_i
- belief about opponent v_{-i}:

$$p_s(v|v_i) = \frac{\sum_{\omega\in\Omega} p(v|\omega)p(v_i|\omega)\mu_s(\omega)}{\sum_{\omega\in\Omega} p(v_i|\omega)\mu_s(\omega)}, \ \forall v \in \{v_l, v_h\}.$$

Claim 1

In the posterior all-pay auction game, players' private values are affiliated, i.e.,

$$p_s(v_i|v_i) \geq p_s(v_i|v_j).$$

Monotonicity condition

Condition M: For $i \in \{1,2\}$, $v_i p_s(v|v_i)$ increases in v_i for every $v \in \{v_h, v_l\}$.

- Let $v = v_h / v_l$, define

$$\begin{aligned}&\phi(\mu_s(G))\\ =& v \cdot \underbrace{\frac{\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))}{\alpha\mu_s(G) + (1-\beta)(1-\mu_s(G))}}_{p_s(v_l|v_h)} - \underbrace{\frac{(1-\alpha)^2\mu_s(G) + \beta^2(1-\mu_s(G))}{(1-\alpha)\mu_s(G) + \beta(1-\mu_s(G))}}_{p_s(v_l|v_l)}\end{aligned}$$

- Condition M is equivalent to requiring $\phi(\mu_s(G)) \geq 0$.

Equilibrium

- Strategy $F_i^s(x|v)$: the probability that player i bids at most x when his value is v and belief is μ_s
  - mixed strategy
  - $\text{supp}[F_i^s(\cdot|v_i)] \in [0, v_i]$
- Given a strategy profile $F^s = (F_1^s, F_2^s)$, player i's expected payoff is

$$u^s(v_i) = \int_0^{v_i} \left\{ v_i \underbrace{\left[p_s(v_h|v_i)F_{-i}^s(x|v_h) + p_s(v_l|v_i)F_{-i}^s(x|v_l)\right]}_{\text{expected winning probability}} - x \right\} dF_i^s(x|v_i)$$

- *Symmetric* equilibria: $F_i^s = F^s = (F^s(\cdot|v_h), F^s(\cdot|v_l))$
- Equilibrium is *monotone* if and only if for any $x \in \text{supp}[F_i^s(\cdot|v_h)]$ and $y \in \text{supp}[F_i^s(\cdot|v_l)]$, we have $y \leq x$. Otherwise, it's non-monotone.

Equilibrium

Proposition 2.1

In the posterior all-pay auction game with distribution of value distribution μ_s, there exists a unique symmetric equilibrium. Specifically,

1. *if $\phi(\mu_s(G)) \geq 0$, the equilibrium is monotone, and players' equilibrium strategies are*

$$F^{s,m}(x|v_l) = \frac{x}{v_l p_s(v_l|v_l)} \text{ on } [0, v_l p_s(v_l|v_l)],$$

$$F^{s,m}(x|v_h) = \frac{x - v_l p_s(v_l|v_l)}{v_h p_s(v_h|v_h)} \text{ on } [v_l p_s(v_l|v_l), v_l p_s(v_l|v_l) + v_h p_s(v_h|v_h)];$$

Monotonic equilibrium

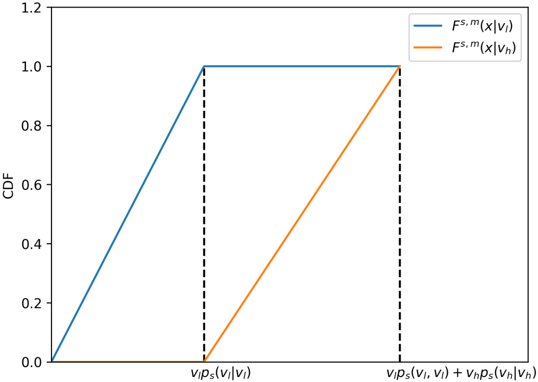

Figure 1: Monotone equilibrium when $\phi(\mu_s(G)) \geq 0$

Equilibrium

Proposition 2.2

In the posterior all-pay auction game with distribution of value distribution μ_s, there exists a unique symmetric equilibrium. Specifically,

1. *if $\phi(\mu_s(G)) < 0$, the equilibrium is non-monotone, and players' equilibrium strategies are*

$$F^{s,nm}(x|v_l) = x \cdot \frac{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]} \text{ on } [0, \underline{x}(s)],$$

$$F^{s,nm}(x|v_h) = \begin{cases} x \cdot \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]} & \text{on } [0, \underline{x}(s)] \\ \frac{x - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h)} & \text{on } [\underline{x}(s), v_h], \end{cases}$$

where $\underline{x}(s) = \frac{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}$.

Non-monotonic equilibrium

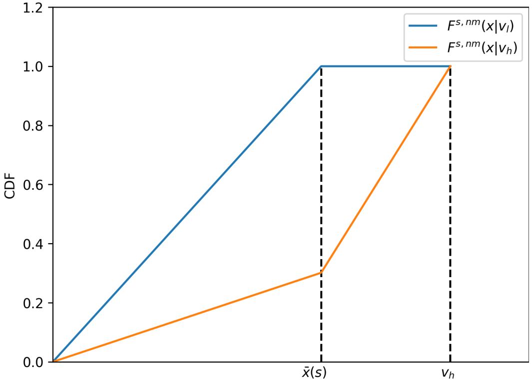

Figure 2: Non-monotone equilibrium when $\phi(\mu_s(G)) < 0$

Equilibrium

Corollary 2.3

In the posterior all-pay auction game with μ_s,

1. *if $\phi(\mu_s(G)) \geq 0$, the expected total bids in equilibrium is*

$$R^m(\mu_s) = v_l p_s(v_l|v_l) + \big(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)\big) \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

   The low value type makes zero payoff. The high value type's expected payoff is $v_l \phi(\mu_s(G)) = v_h p_s(v_l|v_h) - v_l p_s(v_l|v_l)$.

2. *if $\phi(\mu_s(G)) < 0$, the expected total bids in equilibrium is*

$$R^{nm}(\mu_s) = \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

   Both value types make zero payoff.

Information disclosure

The organizer's problem is

$$\max_{\tau} \quad \sum_{\mu_s} \tau(\mu_s) R(\mu_s)$$
$$\text{s.t.} \quad \sum_{\mu_s} \tau(\mu_s) \mu_s(\omega) = \mu_0(\omega).$$

- if $\phi(\mu_s(G)) \geq 0$, then $R(\mu_s) = R^m(\mu_s)$;
- if $\phi(\mu_s(G)) < 0$, then $R(\mu_s) = R^{nm}(\mu_s)$.

Information disclosure

Lemma 3.1

Define $v_0 = 1 + \frac{(\sqrt{\alpha} - \sqrt{1-\beta})^2}{(1-\alpha)\beta}$. *Given posterior* μ,

1. *if* $v \geq v_0$, $\phi(\mu_s(G)) \geq 0$ *for* $\forall \mu_s(G) \in [0, 1]$, *that is, for an all-pay auction with any* μ_s, *the equilibrium is always monotone.*
2. *if* $v < v_0$, *there exists an interval* $(\mu_1^v(G), \mu_2^v(G)) \subset [0, 1]$ *such that* $\phi(\mu_s(G)) < 0$ *for* $\forall \mu_s(G) \in (\mu_1^v(G), \mu_2^v(G))$. *That is, for an all-pay auction with* $\mu_s(G) \in (\mu_1^v(G), \mu_2^v(G))$, *the equilibrium must be non-monotone; otherwise it is monotone.*

Information disclosure

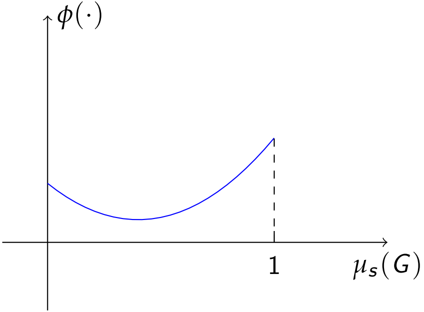

Figure 3: $v \geq v_0$

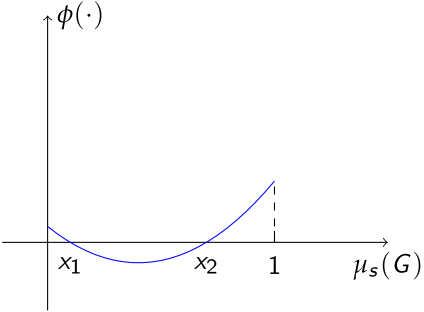

Figure 4: $v < v_0$

Sufficiently different types: $v \geq v_0$

The organizer's problem can be formulated as

$$\begin{aligned} \max_{\tau} \quad & \hat{R}(\tau) = E_{\tau} R^m(\mu_s) \\ \text{s.t.} \quad & \sum_{\mu} \tau(\mu_s)\mu_s(\omega) = \mu_0(\omega), \forall \omega. \end{aligned} \tag{1}$$

Lemma 3.2

$R^m(\mu_s(G))$ is concave in $\mu_s(G)$.

Proposition 3.3

If the two value types are sufficiently different, i.e., $v \geq v_0$, the optimal signal is uninformative.

Sufficiently different types: $v \geq v_0$

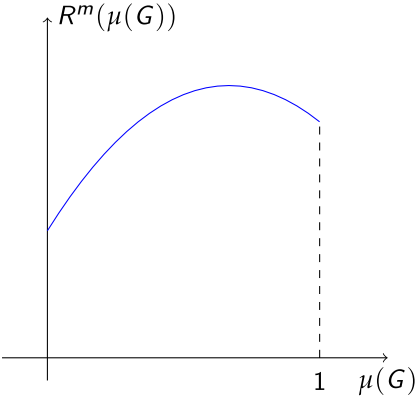

Figure 5: Expected revenue in posterior game: $v \geq v_0$

Sufficiently close types: $v \leq v_0$

The organizer's expected revenue from a posterior game induced by μ_s is

$$R(\mu_s(G)) = \begin{cases} R^{nm}(\mu_s(G)) & \text{if } \phi(\mu_s(G)) < 0; \\ R^{m}(\mu_s(G)) & \text{if } \phi(\mu_(G)) \geq 0. \end{cases}$$

Lemma 3.4

For the μ_s such that $\phi(\mu_s(G)) = 0$, $R^{nm}(\mu_s) = R^m(\mu_s)$.

Lemma 3.5

For any μ_s such that $\phi(\mu_s) \leq 0$,

$$R^{nm}(\mu_s(G)) \leq v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right].$$

The equality holds if and only if $\phi(\mu_s(G)) = 0$

Sufficiently close types: $v \leq v_0$

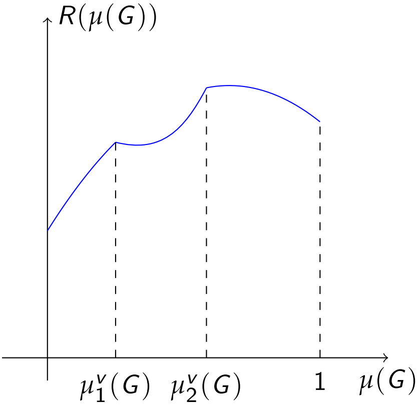

Figure 6: Expected revenue in posterior game: $v < v_0$

Sufficiently close types: $v \leq v_0$

Lemma 3.6

Define $\tilde{R} : [0,1] \to [0,+\infty)$ *as:*

$$\tilde{R}(\mu_s(G)) = \begin{cases} v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right] & \text{if } \phi(\mu_s(G)) < 0; \\ R^m(\mu_s(G)) & \text{if } \phi(\mu_s(G)) \geq 0. \end{cases}$$

$\tilde{R}$ *is the concave closure of* R.

Sufficiently close types: $v \leq v_0$

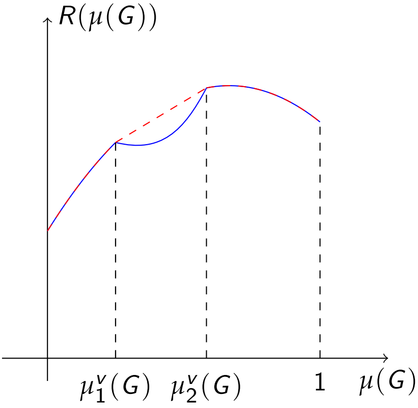

Figure 7: Concave closure $\tilde{R}$: $v < v_0$

Sufficiently close types: $v \leq v_0$

Proposition 3.7

When the two types are relatively close, i.e., $v < v_0$,

1. *if $\phi(\mu_0(G)) \geq 0$, that is, no disclosure induces a monotone equilibrium, the organizer's optimal signal is uninformative, i.e., no disclosure.*
2. *if $\phi(\mu_0(G)) < 0$, that is, no disclosure induces a non-monotone equilibrium, the organizer's optimal signal generates μ_1^v and μ_2^v.*

Corollary 3.8

When $v \geq p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, no disclosure is optimal; when $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, the partial disclosure which generates a posterior distribution μ_1^v and μ_2^v is optimal.

Concluding Remarks

- We consider a two-player all-pay auction model with binary private values.
- Two possible value distributions.
- The problem for the organizer is to design a revenue-maximizing disclosure policy of value distribution
- A Bayesian Persuasion approach is adopted, while focusing on public signals
- When the two private values are sufficiently different, it's optimal to choose uninformative disclosure policy. Otherwise, an informative partial disclosure policy is optimal.

Thank you very much!