

Conceptual Differences Between Decision Utility And Experienced Utility: A Theory for Jevons' Wish?

Orges Ormanidhi *

Department of Applied Economics and School of Mathematics
University of Minnesota
Contact Email: orman019@umn.edu

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Abstract

In the 19th century Jevons wished to capture the quantity of feeling. My theory of experienced utility aims to satisfy such wish. I define experienced utility as a function which represents an individual's hedonic experience from an activity over time. My theory is descriptive and relies on three assumptions: I) Finite number of activities; II) Choice of a single activity; and III) Rate of change of experienced utility is proportional to difference between the experienced utility and the other experienced utilities up to a positive coefficient function. The theory is presented by discussing primal, existential and functional differences between decision utility and experienced utility. I prove the existence of a unique family of experienced utilities which are expressed explicitly, take real values and are linearly independent. I view my assumptions to be weaker than those required for decision utility, specifically because my theory does not require rationality or include preferences. A concept associated with each experienced utility is what I have termed as non-experienced utilities. Non-experienced utilities are experienced utilities from activities which an individual has not chosen while spending time on the chosen activity. If used to analyze individual choice, my theory has the potential to explain both time allocation and the sequence of activities. A current area of research is analyzing how non-experienced utilities determine the switch-time from an activity to another.

I present a theory of experienced utility with the same number of assumptions as the ones needed to ensure the existence of utility in neoclassical economic theory. From hereon I denote 'utility' as 'decision utility' to distinguish the two concepts. In my view, the assumptions in my theory are weaker than those in neoclassical economic theory. Consequently, the conclusions on experienced utility are stronger than those on decision utility.

Since decision utility is the predominant concept in economic theory and its applications, I have chosen to present my theory through a discussion of the differences between these concepts. I view the differences between decision utility and experienced utility as two kinds, conceptual and applicable. By conceptual differences, I mean the differences in terms of the requirements needed to ensure the derivation of a function for each concept of utility. Therefore, the terms 'utility' and 'utility function' are used interchangeably. By applicable differences, I mean the differences in terms of the requirements needed to use the utility function in theoretical and applied analyses, for example, to describe individual behavior or explain choice. This paper is about conceptual differences, hence the word 'conceptual' in the title.

While explaining what this paper is about, it might help to point out what my theory is not about. My theory is not about individual preferences. In

fact, experienced utility in my theory does not include any preferences. On the other hand, decision utility does include preferences. Individual preferences are currently being studied in an impressive manner, most notably with the use of experiments conducted in laboratories or real-life situations. With the opportunities that these experiments present for gaining insights into preferences, scholars in this promising area of research have used decision utility to explain individual choice, especially in situations involving risk. However, my current paper does not attempt to explain choice but rather to offer an alternative *descriptive* theory of experienced utility that could be used to describe individual behavior or prescribe choice.¹

I use the same concept of experienced utility as in Kahneman et al. (1997) but my theory is fundamentally different from theirs. They present a normative theory of total experienced utility that extends decision utility whereas I present a descriptive theory of instantaneous experienced utility that is independent of decision utility. I discuss in detail the differences between these two theories. Before discussing the distinguishing features of decision utility and experienced utility, the definition for each concept is provided.

Definition. *Decision utility is a function which represents an individual's preferences over mutually exclusive alternatives.*

This definition is the same as the one in Mas-Colell et al. (1995) who define the utility function as the numerical values assigned to each element of the set of mutually exclusive alternatives an individual can choose from (Definition 1.B.2, p. 9).

Definition. *Experienced utility is a function which represents an individual's hedonic experience from an activity over time.*

This definition is based on Kahneman, Bentham, Edgeworth and Jevons. Experience in my definition is the hedonic experience. According to Oxford English Dictionary, as an adjective, hedonic means: "Of or relating to pleasure. ... In wider use, chiefly in Psychology: of, pertaining to, or involving pleasurable or painful sensations or feelings, considered as affects. Spec. hedonic tone, the degree of pleasantness or unpleasantness associated with an experience or state, esp. considered as a single quantity that can range from extreme pleasure to extreme pain." It is in this sense that hedonic experience is viewed in my paper.

¹A note on the use of words 'describe' and 'prescribe.' My theory is descriptive and it could be used to explain behavior with or without a decision rule. Without a decision rule, my theory could be used to describe behavior and with a decision rule, it could be used to prescribe behavior, which is usually the case when using decision utility to explain choice.

In Kahneman et al. (1997), experienced utility is a hedonic quality expressed as a function of time. According to Bentham (1823): “By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness” In Bentham’s view, the value that an individual assigns to pleasure or pain depends on its duration. Edgeworth (1881) conceived a psycho-physical machine (hedonimeter) as “... an ideally perfect instrument, a psychophysical machine, continually registering the height of pleasure experienced by an individual, exactly according to the verdict of consciousness, or rather diverging therefrom according to a *law of errors*. From moment to moment the hedonimeter varies; ...” More recently, reporting results from Redelmeier and Kahneman (1996), Kahneman et al. (1997) graph experiences from pain intensity for two colonoscopy patients as measures of pain intensity on a 10-point scale over minutes of time.

Jevons also viewed the intensity of pleasure or pain as a function of time and noted that the changes in such intensity from moment to moment present issues if we want to measure it. As presented by his wife in the 1888 edition of his 1879 book *The Theory of Political Economy*, according to Jevons: “Incessant variation characterises our states of mind, and this is the source of the main difficulties of the subject. Nevertheless, if these variations can be traced out at all, or any approach to method and law can be detected, it will be possible to form a conception of the resulting quantity of feeling.” An example by Jevons is shown below.

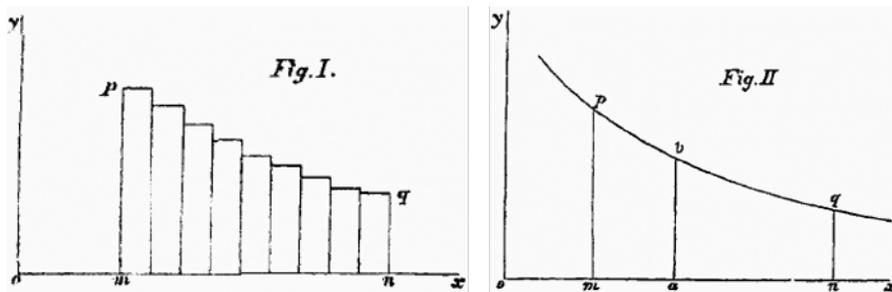


Figure 1: Jevons’ example of quantity of feeling

In Jevons’ *Fig. I* in Figure 1, he noted that the quantity of feeling during each minute may be represented by a rectangle with base one minute and height proportional to intensity of feeling during that minute. However, he also noted that it is artificial to assume that the intensity of feeling varies by such sudden steps at regular time intervals but the error becomes smaller and smaller as time intervals become shorter and shorter and is avoided when the interval becomes infinitely short. Therefore, according to Jevons, the proper

representation of the variation of feeling is by a curve as in his *Fig. II* where each point on the curve indicates the intensity of feeling at a moment in time and the whole quantity of feeling during a given time interval is measured by the area under the curve during this interval. For example, the quantity of feeling during interval mn is measured by $pmnq$ and during ma by $pmab$.

My theory of experienced utility aims to offer a method of deriving such a curve that Jevons suggested to measure the intensity of feeling at every instant of time. In an attempt not to reinvent today's use of Jevons' concept, I use the term experienced utility. In my theory, Jevons' quantity of feeling over a given period of time is termed as hedonic experience and it is the area between the curve of an experienced utility function and the axis that represents time. Such an area is the integral of the experienced utility function over a given period of time. It is not possible to know whether my theory satisfies Jevons' wish but I certainly hope so, hence the question mark (?) in the title.

The discussion of the differences between the two concepts of utility is centered on a comparison of the treatment of decision utility in neoclassical economic theory and the treatment of experienced utility in my theory. In this discussion, I also answer two questions:

1. Does decision utility have experience in it?
2. Does experienced utility have a decision or does it lead to a decision?

The conceptual differences between decision utility and experienced utility are three-fold: primal, existential and functional. Each type is discussed next, followed by comparisons with some other studies, derivation of non-experienced utilities including an example and concluding with some remarks.

1 Primal differences

By primal differences, I mean the differences in terms of the primitive for each concept of utility. The primitive for decision utility theory is the individual preference relation over a set of alternatives. It is an individual characteristic that is taken as given. The primitive for my experienced utility theory is the experienced utility itself. It is an individual experience from an activity that is also taken as given. Each primitive is known to the individual but not necessarily to others.

As terms on their own, there are no differences between 'alternative' in economic theory and 'activity' in my theory. For example, in Mas-Colell et al. (1995) the set of alternatives contains going to law school, study economics,

etc., which can be called activities in my theory as well. Also, both terms are undefined and so are the terms ‘good’ in economic theory and ‘element’ in set theory. So the term ‘activity’ is similar to the term ‘alternative’ or ‘good’ (or ‘element’). The meaning of activity depends on the individual’s environment and it becomes evident by the problem at hand, as does the meaning of alternative or good (or element of a set).

However, these terms serve different purposes for each concept of utility. In the case of decision utility, the preference relation is a binary relation on the set of alternatives and decision utility is defined in terms of this relation, not alternatives themselves. On the contrary, in the case of experienced utility, activity is the source of hedonic experience and experienced utility is defined in terms of this activity, not any preference relations between activities.

Each alternative might give hedonic experience to an individual and this experience might also be the basis for the individual preference relation. However, this experience is not represented by a decision utility function because the preference relation is an ordinal property of decision utility and all that matters is the ranking of alternatives. To answer our first question, although the alternative itself might give hedonic experience, a decision utility function does not have experience in it.

There are two reasons why a decision utility function cannot represent hedonic experience. First, since any theory takes the primitive as given and the primitive in decision utility analysis does not include individual’s experience but rather the preference relation, a decision utility function can only provide answers that pertain to the ranking of alternatives. In decision utility theory, alternatives are outcomes (Berridge and O’Doherty, 2014; Dolan and Kahneman, 2008; Kahneman et al., 1997) that are ranked, not sources of hedonic experience and a decision utility function can only represent the ranking of outcomes, not how these outcomes are formed. Second, given that an individual’s hedonic experience happens over time, since the primitive in decision utility analysis does not include time, a decision utility function is not capable to represent experience. It might be tempting to note as counterexamples the cases of choice over time and choice of time allocation when decision utility has been used. However, these cases only further illustrate why decision utility functions used in such cases cannot represent experience.

In the case of choice over time, inter-temporal behavior is analyzed with either one-shot or recursive models in which choice does not depend on time. In one-shot models, time serves as an index to distinguish between two different periods. A good x at period t is considered different from the same good x at period $t + 1$ and they are represented as two different variables, x_t and x_{t+1} . In these models, choice comes from maximizing a decision utility function of

these two variables but it does not include experience because x_t and x_{t+1} do not measure any interval of time during which the hedonic experience occurs.

In recursive models, including finite or infinite periods, although goods are indexed by period, they are considered the same in each period. However, the assumptions (a wide range of assumptions) that are needed in order to solve these models make time irrelevant and so inter-temporal choice does not depend on time. Recursive models are of different specifications but to illustrate that they do not contain experience suppose that the model is in its classic form (Stokey et al., 1989) as a sequential problem:

$$\begin{aligned} & \max_{(c_t, k_{t+1})_{t=0}^{\infty}} \sum_{t=0}^{\infty} \eta^t U(c_t) \\ \text{s.t. } & c_t + k_{t+1} \leq f(k_t), \\ & c_t, k_{t+1} \geq 0, \quad t = 0, 1, \dots, \\ & \text{given } k_0 > 0, \end{aligned}$$

where c is consumption, η a discounting factor,² k capital stock, $f(k)$ total supply of goods available per worker and $U(c_t)$ is a decision utility function that represents inter-temporal consumption preferences for a representative household.

As seen from this sequential problem, period-to-period decision utility remains constant and it is equal to discounting factor η ($U(c_{t+1})/U(c_t) = \eta$). So the decision utility value that represents the inter-temporal choice does not depend on time. Stokey et al. (1989) also show that the sequential problem leads to the so-called functional equation form, which does not include time.

Taking v as equal to the maximum value of the sequential problem above, Stokey et al. (1989) show the uniqueness of a function (called a value function) v that depends only on capital stock k : $v(k) = \max_{c,y} [U(c) + \eta v(y)]$ ($c + y \leq f(k)$, $0 \leq y \leq f(k)$, $c, y \geq 0$), which gives the solution to the recursive model. In this form variables are not indexed because the time period is irrelevant. As seen from the sequential equation form, again inter-temporal choice does not include experience because it does not depend on any period of time during which the hedonic experience occurs.

In the case of time allocation, variables do measure periods of time. However, the decision utility function maintains only the ordinal property of an individual's preference relation over a period of time and this is all that decision utility is capable of doing. A decision utility function is a function

²In their original work, Stokey et al. (1989) use β for the discounting factor but because I use β for an important concept in my analysis, I denote their discounting factor by η .

that represents only the ordering of different time allocations, not the experience that is associated with any time allocation. Decision utility is defined in terms of outcomes and not on how these outcomes are formed. In this case, the time allocations are the outcomes and although a decision utility can explain outcomes, it cannot describe how these allocations were derived. There may be many different decision utility functions that represent an individual's preference relation over time allocations but no such function is capable of representing the instant experience that occurs at each moment of time.

On the other hand, experienced utility represents an individual's experience as a function of time and the hedonic experience is the integral of experienced utility function values over a period of time. Given that experienced utility is defined as the instant experience from a single activity, each hedonic experience is associated with this activity. Even if the choice of time allocation based on calculated experiences using experienced utility is the same as the choice of time allocation based on a preference relation using decision utility, because activities occur over time, decision utility lacks the capability to capture the sequence of activities whereas experienced utility has the potential to explain sequence of activities. The calculation of experiences for an example with two activities illustrates this difference between decision utility and experienced utility in the section on non-experienced utilities. Furthermore, in both cases of choice over time and time allocation, the primitive for decision utility ignores the interaction of an individual with the environment. But experienced utility as the primitive in my theory results from the interaction of the individual with the environment, as explained in more detail in the next section.

Meanwhile, since experienced utility function in my theory is defined as a function of time spent on an activity, it does not include preferences for or a preference relation over activities. Therefore, my experienced utility can only represent the experience from an activity and not the decision on any activities. So to answer our second question, an experience utility function represents only experience and does not have a decision in it.

However, an experienced utility function in my theory has the potential to lead to a decision through what I call 'non-experienced utilities.' Non-experienced utilities represent experiences from activities in which the individual is not engaging. They are conditioned on each experienced utility because they exist while an individual is engaging in an activity and once the individual has moved onto another activity there are different activities in which the individual is not engaging and hence different non-experienced utilities.

For example, suppose that an individual has under consideration two activities, work out and watch TV, assumed not to happen at the same time. Because these activities cannot happen at the same time, while working out

the individual is gaining experienced utility from working out and simultaneously missing out on the experienced utility from watching TV and vice versa. However, time continues to go by and does not stop. So when an individual works out, it is non-experienced utility from non-watching TV given that the individual is working out that is at work and when an individual watches TV, it is non-experienced utility from non-working out given that the individual is watching TV that is at work. How non-experienced utilities could help explain individual choice is a research topic I am currently working on. After I explain the concept of non-experienced utility, I use these two activities in an example to illustrate the differences between decision utility and experienced utility when they are used to explain individual choice.

Before ending this section, it might be useful to compare the primitive for each concept of utility. From the discussion above, I claim that the primitive for decision utility takes more as given compared to the primitive for my experienced utility. An individual capable of ordering all of the outcomes from different alternatives based on a preference relation is endowed with extraordinary cognitive ability. For example, an individual that decides on a time allocation is assumed to order infinitely many time allocations. In the case of continuous variables, bounded rationality is not helpful either because the individual is still assumed to order infinitely many allocations. In my view, such an individual is more sophisticated than an individual who knows instinctively their own experience from an activity. So the primitive in my theory is more basic and borrowing labels ‘high’ and ‘low’ for levels of programming languages, the primitive in my theory is a lower level individual characteristic than the primitive in neoclassical economic theory.

2 Existential differences

By existential differences, I mean the differences in terms of the assumptions needed to ensure the existence of a function for each concept of utility. In neoclassical economic theory there are three basic assumptions required for the existence of a decision utility function:

Decision Utility Assumption I. *Finite number of goods.*

Decision Utility Assumption II. *Choice of a bundle of goods.*

Decision Utility Assumption III. *Preferences are rational and continuous.*

In my theory, there are also three assumptions required for the existence of an experienced utility function:

Experienced Utility Assumption I. *Finite number of activities.*

Experienced Utility Assumption II. *Choice of a single activity.*

Experienced Utility Assumption III. *Rate of change of experienced utility is proportional to difference between the experienced utility and the other experienced utilities up to a positive coefficient function.*

My assumption I is similar to decision utility assumption I. My assumption II has some similarity to decision utility assumption II only to the extent of the single choice. A decision utility function takes its value from every single bundle of goods and an experienced utility function takes its values from every instant of time during a single activity.

However, my assumption II differs substantially from decision utility assumption II. Practically, decision utility assumption II has no impact on decision utility assumption I because the choice of a bundle of goods does not change the number of goods. If there are n goods, for every choice, the number of goods remains n . But my assumption II has a direct impact on my assumption I because choice of an activity changes the number of activities an individual has under consideration. If there are n activities, for every activity an individual engages in, there are $n - 1$ activities under consideration.

In my theory, non-chosen activities continue to have influence on an individual when this individual has chosen an activity. But these non-engaging activities cannot give experienced utility because from my assumption II an individual spends time on a single activity. For this reason, I use the term non-experienced utilities for experienced utilities from the other activities an individual has under consideration while engaging in an activity.

There are other differences as well. Conceptually, decision utility assumption II implies that for each bundle of goods there is a value of decision utility that represents this bundle and due to the ordinal nature of decision utility there could be many such values that represent a fixed bundle. However, experienced utility assumption II implies that for every activity there is an experienced utility *function* rather than value that represents the changing experience over time. As will be proved in the next section, for given initial conditions there is only one (cardinal) experienced utility function that represents this experience, so the function is fixed.

Experience is calculated as the integral of the experienced utility function over the period of time during which an individual engages in an activity. In my theory, activity is the source of hedonic experience and an individual's (or scholar's) working definition of activity depends on how the individual interacts with the environment. There may be many objects and surroundings,

including other individuals, in an individual's environment but it is through the individual's interaction with the environment that these objects and surroundings become relevant for the individual's experience. So the working definition of activity needs to distinguish each activity among the different activities in order to be able to capture the hedonic experience from each of them. Meanwhile, the consideration of activity as a single activity does not put any restrictions on what an individual can do during the time spent on the activity.

For example, suppose that an individual engages in the activity of 'working.' In such case, working may refer to both the activity of 'working' and the amount of working time that the individual supplies and is paid for. Also, while engaging in 'working,' the individual could consume goods (coffee, lunch, massage, etc). As seen in this case, being engaged in an activity does not mean that an individual is doing only one thing. The term 'single' means that there is one activity that serves as the source of hedonic experience. It is important to maintain a single activity because experienced utility is defined in terms of this activity and its working definition needs to make the distinction between hedonic experiences from different activities. If many activities give a certain kind of hedonic experience simultaneously, then this situation is simply a matter of labeling or a working definition because these activities could very well be redefined as a single activity for the purpose of hedonic experience.

The importance of a single activity becomes also apparent when considering the environment in which the activity takes place. In our example, while working, the individual is both a supplier and a consumer. However, if the individual consumed exactly the same goods while watching TV at home (and not working), this individual would only be a consumer. Although the same of goods were consumed, the individual would have a different experience from being both a supplier and a consumer as opposed to only being a consumer. Each experience is realized when the individual engages in the activity through the interaction with the environment, say the workplace where the individual engages in working or home where the individual "engages" in watching TV. This is why an activity needs to be construed in such a way that it is sufficiently distinct from the other activities, therefore allowing for the hedonic experience from this activity to occur.³

In choosing my example, I have tried to illustrate a real situation but even in experiments, the hedonic experience from an activity still occurs through the interaction of the individual with the environment. For example, Kahneman et al. (1997) reported measures of experienced utility from results of

³This example also hints at the kind of economic analysis one is able to do when using experienced utilities, a research topic that I will pursue in the near future.

various experiments, such as watching short film clips, discomfort from undergoing colonoscopy or discomfort from immersing hands in cold water. Each of these experiments is a single activity in the sense that it is the only source of individual's hedonic experience or experienced utility. However, notice that in each case experience is the result of an individual being engaged in the activity by interacting (watching, undergoing a procedure or immersing hands) with an environment that includes objects and surroundings (film clips, colonoscopy and possibly other medical necessities or cold water at their respective places).

Before discussing the next assumption, a note on the meaning of the word 'engage.' By engage, I mean that an individual is spending time in an activity but this does not mean that the individual is assumed to be active or even be doing something. For example, during the time spent on such activities as 'doing nothing' or 'sleeping' the individual might not feel like doing much, if anything, but for the purpose of hedonic experience the individual is still spending time on an activity and "interacting" with the environment through its objects and surroundings (TV set, sofa bed, etc.).

My assumption III has only a technical similarity with decision utility assumption III because my theory assumes integrability of rates of change and economic theory assumes continuity of preferences. Assuming that rates of change are continuous would ensure the existence of experienced utility functions too but since integrability is a weaker condition than continuity, I have chosen integrability. A benefit from integrability is that these functions are obtained even if the rates of change behave in a step-wise fashion.

However, my assumption III is conceptually different from decision utility assumption III because it does not assume rationality. Rational preferences are those that are complete and transitive (Mas-Colell et al., 1995), which in mathematical terms means that they are linearly or totally ordered. Although it is widely used, rationality has been repeatedly challenged (May, 1954; Tversky, 1969; Elster, 1979; Kahneman and Tversky, 1984; Schelling, 2006).

The lack of rationality for a preference relation is consequential because Mas-Colell et al. (1995) have proven that a decision utility function exists only if preferences are rational. So it might be helpful to look for approaches that guarantee the existence of a utility function that do not require the assumption of rationality. My theory does not assume rationality or include preferences.

In my view, the rate of change of experienced utility being positively proportional to the difference between experienced utility from an activity and the sum of experienced utilities from the other activities is a weaker condition than rationality because of its realistic appeal. In reality, as long as gaining an experience from an activity acts on an individual with greater force than the combined experience (sum of forces) from the other activities its rate of

change is positive otherwise it is negative or zero.

At the same time, I believe that rationality (or bounded rationality) is less realistic because of the enormous cognitive ability an individual is endowed with when ordering preferences over all or a subset of the possible combinations of alternatives. For example, based on the assumption of rationality (or bounded rationality), an individual who decides how to allocate time to two activities over a period of two hours is assumed to be capable of ordering infinitely many preferences over the allocations in $[0, 2] \times [0, 2]$ intervals.

Because experienced utility assumption III combined with experienced utility assumption II has some similarities with how demand and supply functions behave over time, an illustration with a market setting would be helpful. Let the demand and supply functions over time in a market be denoted by $D(t)$ and $S(t)$, respectively. The economic market equilibrium requires that quantity demanded decrease and quantity supplied increase if $D(t) > S(t)$ and quantity demanded increase and quantity supplied decrease if $D(t) < S(t)$.

Let the rates of change of demand and supply be denoted by $\dot{D} = \frac{dD}{dt}$ and $\dot{S} = \frac{dS}{dt}$, respectively. Then the dynamics that results from the market equilibrium can be expressed by the following system of equations:⁴

$$\begin{aligned}\frac{dD}{dt} &= -(D(t) - S(t)) \\ \frac{dS}{dt} &= -(-D(t) + S(t)).\end{aligned}$$

The coefficients of proportionality in the setting above are negative (here they are both -1 for simplicity) because $D(t)$ and $S(t)$ act like forces that both push their respective quantities toward the equilibrium, which occurs when $\frac{dD}{dt} = \frac{dS}{dt} = 0$ such that $D(t) = S(t)$.

In my theory, experienced utility functions act simultaneously like forces that pull an individual toward engaging in their respective activities. However, if one activity is chosen (assumption II), then the experience gained from each activity opposes the combined force of gaining experiences from the other activities at every moment in time. This combined force is simply the sum of experienced utility functions from the remaining activities.

If $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))$ denotes an experienced utility vector from n activities and t denotes time, then experienced utility:

⁴It was a similar example in Rabenstein (1992) that helped me to start conceptualizing my experienced utility assumption III.

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

is a vector-function from $\mathbb{R} \rightarrow \mathbb{R}^n$. So the dynamics from my experienced utility assumptions can be expressed by the following system:

$$\begin{aligned} du_1/dt &= \beta_1(t) (u_1(t) - u_2(t) - \dots - u_n(t)) \\ du_2/dt &= \beta_2(t) (-u_1(t) + u_2(t) - \dots - u_n(t)), \\ &\dots \\ du_n/dt &= \beta_n(t) (-u_1(t) - u_2(t) - \dots + u_n(t)) \end{aligned}$$

$$\dot{\mathbf{u}}(t) = \mathcal{B}(t)\mathbf{u}(t), \tag{1}$$

where $\dot{\mathbf{u}}(t) = (du_1/dt, du_2/dt, \dots, du_n/dt)$ denotes the rates of change and $\beta_i(t)$ is a positive function of proportionality for each activity $i = 1, 2, \dots, n$.

The above system suggests that conceptually my theory starts from a place where we do not know experienced utilities. However, we can see that an individual spends time on an activity and as time continues the individual switches to another activity. It is in this conceptual setting where experienced utilities act like pull-and-push forces. As long as for an individual the pull force from engaging in an activity is greater than the combined push forces of engaging in the other activities its rate of change is positive otherwise it is negative or zero. Therefore, in contrast with a market setting, in my theory the coefficients of proportionality are positive.

A note on the reasoning behind experienced utility assumption III. The above system includes equations but the equal sign (=) means that given that experienced utilities act like pull-and-push forces, we can assume that each rate of change is proportional to difference between the pull force and the sum of combined push forces. The equal sign does not mean that each rate of change is in fact proportional. If we know experienced utilities, there is no need for assumption III. The difficulty is that we do not know the experienced utilities and so we have to make assumptions. The assumption of proportionality is simple but history has shown that it is a very useful one. Many physical phenomena have been explained and important laws, such as Newton's laws, have been discovered using the proportionality assumption. My goal is much more modest but I do believe that assumption III is a good way to analyze the dynamics of the pull-and-push forces of individual hedonic experiences.

An important aspect of my theory is that the coefficients of proportionality are not constant but are themselves (integrable) functions of time. As Jevons noted, our minds are characterized by incessant variation and having coefficients of proportionality that change over time captures this variation. It may be the case that for a sufficiently short period of time these coefficients are constant and I analyze this case too. However, it would be too restrictive for the general case of deriving experienced utility functions to assume that they are constant. In the next section I present the experienced utility functions with changing and constant coefficients of proportionality.

3 Functional differences

By functional differences, I mean the differences in terms of functional forms for each concept of utility. The focus here is on possible implications when using decision utility and how they could be avoided if using experienced utility.

In matrix notation, the matrix $\mathcal{B}(t)$ can be written as the product of a diagonal matrix $\mathcal{B}_d(t)$ of coefficients of proportionality and a circulant matrix E with 1 in the main diagonal and -1 for the remaining elements:

$$\begin{bmatrix} \beta_1(t) & -\beta_1(t) & \cdots & -\beta_1(t) \\ -\beta_2(t) & \beta_2(t) & \cdots & -\beta_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_n(t) & -\beta_n(t) & \cdots & \beta_n(t) \end{bmatrix} = \begin{bmatrix} \beta_1(t) & 0 & \cdots & 0 \\ 0 & \beta_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n(t) \end{bmatrix} \begin{bmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 \end{bmatrix}$$

$$\mathcal{B}(t) = \mathcal{B}_d(t)E$$

Because E is a real symmetric matrix, it has a complete set, K , of orthogonal eigenvectors:

$$K = [\mathbf{k}^1 \quad \mathbf{k}^2 \quad \cdots \quad \mathbf{k}^n],$$

Given that these vectors are orthogonal, there exists the inverse of K , K^{-1} . Let $\mathbf{d} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ denote the eigenvalues of E and

$$\mathbf{w}(t) = (\omega_1(t), \omega_2(t), \cdots, \omega_n(t))$$

the eigenvalues of $\mathcal{B}(t)$. The following Lemma 1, the proof of which is in Appendix A, shows the important result that $\mathcal{B}(t)$ has the same eigenvectors as E and gives the functional form of the associated eigenvalues.

Lemma 1. $\mathcal{B}(t)$ has a complete set of eigenvectors $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^n$ and their associated eigenvalues are given by the following linear transformation:

$$\mathbf{w}(t) = K^{-1}\mathcal{B}_d(t)K\mathbf{d}, \quad (2)$$

where $\mathbf{d} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mathbf{w}(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))$. For every i , there is a one-to-one correspondence between $\mathcal{B}_d(t)\lambda_i\mathbf{k}^i$ and $\omega_i(t)\mathbf{k}^i$.

In matrix notation, the eigenvalues of $\mathcal{B}(t)$ and E are:

$$\Omega(t) = \begin{bmatrix} \omega_1(t) & 0 & \cdots & 0 \\ 0 & \omega_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n(t) \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

In order to obtain simple functional forms for experienced utility, I use the indefinite integrals of eigenvalues of matrix $\mathcal{B}(t)$:

$$\tilde{\omega}_i(t) = \int \omega_i(t)dt, \quad i = 1, 2, \dots, n$$

These indefinite integrals can be written more compactly in matrix format:

$$\tilde{\Omega}(t) = \begin{bmatrix} \tilde{\omega}_1(t) & 0 & \cdots & 0 \\ 0 & \tilde{\omega}_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\omega}_n(t) \end{bmatrix}$$

The following theorem proves the existence of a unique family of experienced utilities that are: expressed explicitly, real valued and linearly independent. Also, experienced utilities are cardinal utilities.

Theorem 1 (Family of Experienced Utilities). *Given positive coefficients of proportionality on an open interval, the rate of change of experienced utility from each activity is proportional to the difference between experienced utility from the activity and sum of experienced utilities from the other activities. Then there exists a unique family of experienced utilities which are:*

(i) expressed explicitly:

$$\mathbf{u}(t) = Ke^{\tilde{\Omega}(t)}\mathbf{c}, \quad (3)$$

where e is the exponential function and \mathbf{c} is a nonzero vector,

(ii) real valued and

(iii) linearly independent.

Also, given initial conditions, experienced utilities are cardinal utilities.

Proof. See Appendix A. □

Matrix E is a constant circulant matrix and its eigenvectors and eigenvalues are readily obtained from using mathematical and statistical software. Lemma 1 shows that matrix $\mathcal{B}(t)$ has the same eigenvectors as matrix E and it specifies the eigenvalues of matrix $\mathcal{B}(t)$ by formula (2). Then experienced utilities can be expressed explicitly by the functional forms from solution (3).

The proof of Theorem 1 is in Appendix A and this proof uses a kind of indirect way of solving system (1) with the use of the eigenvectors and eigenvalues of matrix E . It is possible to solve system (1) in a more direct way. However, in this case although experienced utilities are again expressed explicitly, their forms are more complicated. This alternative proof is in Appendix A.1.

The decision utility assumptions in the previous section guarantee the existence of a decision utility function. This function does not belong to any family of functions. On the other hand, the experienced utility assumptions guarantee the existence of a unique family of experienced utility functions.

These functions indicate that the domain and range for each concept of utility are reversed. Time is usually measured with real numbers \mathbb{R} and the quantities of goods are measured with real numbers. When the alternatives for decision utility are goods, decision utility is a function from $\mathbb{R}^n \rightarrow \mathbb{R}$, where \mathbb{R}^n are quantities of goods and the negative values often represent borrowing. On the other hand, experienced utility is a function from $\mathbb{R} \rightarrow \mathbb{R}^n$.

Assuming that preferences are known, research has been concerned on whether different decision utility specifications are capable to explain an observed choice (Loomes and Sugden, 1998; Barberis et al., 2006). However, decision utility is an unspecified function which does not belong to any family of functions and different specifications are assumed in different contexts. A more realistic case would be whether decision utility specifications capable to explain an observed choice could reveal the unknown preferences. However,

the lack of uniqueness of a decision utility function makes it difficult to know the preference relation that this function is supposed to represent.

For example, suppose that for a given period of two hours and two activities, ‘Work Out’ (x_1) and ‘Watch TV’ (x_2), an individual spends one hour on each activity. Suppose that in one case, for example a research study, the individual’s choice is studied using a specification of decision utility (DU) of the form: $DU_1 = x_1^{1/2}x_2^{1/2}$. Also, suppose that in another case, the same individual’s choice is studied using a specification from a (sufficiently) different family of functions of the form: $DU_2 = (x_1^{1/2} + x_2^{1/2})^2$. Some indifference curves for these functions are in Figure 2 below where DU_1 is represented by red curves and DU_2 by blue curves.

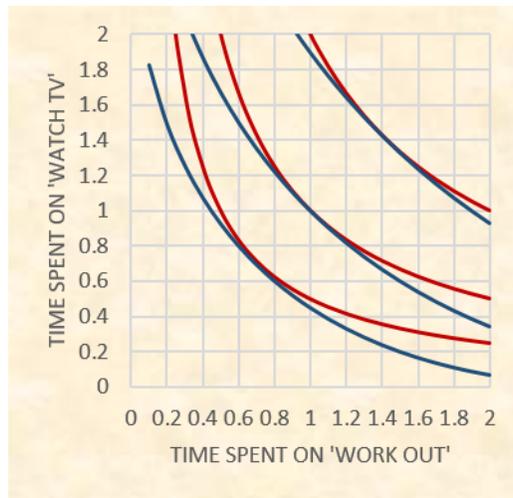


Figure 2: Indifference curves for two decision utility functions

These cases illustrate the implications when using decision utility specifications from different families of functions. Although both functions would conclude that the individual should spend one hour on each activity, it is impossible for these two decision utility functions to represent the same preference relation. For example, the individual is indifferent between bundle $(1, 1)$ and any other bundle on both red and blue indifference curves that meet at $(1, 1)$. So the individual is indifferent between all bundles on these two different indifference curves, a contradiction.

This example illustrates that when individual choice is studied using a certain decision utility function, the preference relation that this function is supposed to represent depends on the family of functional forms that this function belongs to. But the preference relation is the primitive taken as

given and a decision utility function depends on the preference relation, not the other way around. This implication is not easy to resolve. To resolve it, we would need to know either the preference relation, which is considered unknown, or the family of decision utility functions, which does not exist.

Next I discuss some differences between the two concepts in connection with discounting and uncertainty. The main difference is that the family of experienced utility functions in Theorem 1 represents hedonic experience with both discounting and uncertainty without any additional assumptions. However, additional assumptions are needed for decision utility to represent preferences with discounting and uncertainty. These assumptions are often implicit.

In case of discounting, a decision utility function is discounted at periods $t, t + 1, \dots$. As noted, the decision utility function depends on the preferences and not vice versa. Preferences are the unknown (non-observable) primitive whereas decision utility is the tool we use to analyze preferences for alternatives/goods at different periods. So decision utility belongs to the researcher studying an individual and preferences belong to the individual being studied.

Because decision utility represents preferences and not the other way around, if an individual being studied by the researcher does not discount preferences, the researcher should not discount the decision utility chosen to study the individual and if the individual discounts preferences, the researcher should discount the decision utility. When the researcher includes discounting in a study, the usually non-explicit assumption is that when decision utility is multiplied by a discounting factor, the researcher's discounted values of decision utility function chosen for a particular study represent the individual's discounted values of preference relation ordering over different periods.

For example, suppose that as in the earlier recursive model by Stokey et al. (1989), consumption c at $t = 0$ is represented by decision utility $U(c_0)$ and at $t = 1$ by discounted decision utility $\eta U(c_1)$. In their model an individual is indifferent between consumption at $t = 0$ and consumption at $t = 1$ because decision utility is discounted and not because good c is discounted. $U(c_0)$ and $\eta U(c_1)$ are two equal function values which are assumed to represent the same preference ordering for an individual indifferent between c_0 and c_1 . The implication is that when discounting decision utility instead of preferences, that is goods as the alternatives of preferences under consideration, decision utility can represent preferences only if it is a special kind of function. This finding is presented in the following lemma, which shows which kind of functions allow decision utility to represent preferences with discounting.

Lemma 2. *A researcher has a discounting factor η for decision utility DU and an individual has a discounting factor δ for preferences of goods $\mathbf{x} =$*

x^1, x^2, \dots, x^n over different periods. Then researcher's DU represents the individual's preferences only if it is a homogeneous function of degree $k = \ln \eta / \ln \delta$.

Proof. It is not known a priori if $\delta = \eta$ because DU represents preferences for x 's over different periods and not vice versa. If we denote indifference by \sim and equivalence by \Leftrightarrow , then the following holds for all $t = 0, 1, \dots$.⁵

$$\mathbf{x}_t \sim \delta \mathbf{x}_{t+1} \Leftrightarrow \frac{1}{\delta} \mathbf{x}_t \sim \mathbf{x}_{t+1} \Leftrightarrow DU(\mathbf{x}_t) = \eta DU(\mathbf{x}_{t+1}) \Leftrightarrow \frac{1}{\eta} DU(\mathbf{x}_t) = DU(\mathbf{x}_{t+1})$$

This gives:

$$\frac{1}{\eta} DU(\mathbf{x}_t) = DU\left(\frac{1}{\delta} \mathbf{x}_t\right) \Rightarrow \left(\frac{1}{\delta}\right)^k = \frac{1}{\eta} \Leftrightarrow \delta^k = \eta \Leftrightarrow k = \frac{\ln \eta}{\ln \delta}$$

This is an 'only if' result as indicated by the only right implication \Rightarrow above because not every homogeneous function of degree $\ln \eta / \ln \delta$ represents the individual's preferences. Suppose DU is a homogeneous function of degree 1. Then preferences with discounting at different periods are represented by DU only if $\delta = \eta$. This completes the proof. \square

The decision utility functions DU_1 and DU_2 mentioned earlier are both of degree 1. These decision utility functions seem to be out of fashion but as the lemma above shows they are among the few specifications that are capable to represent preferences with discounting. For example, assuming that either DU_1 or DU_2 form is known to represent preferences, then the following specifications would represent preferences with discounting:

$$DU_1 = x_1^{\alpha_1} x_2^{\alpha_2}, \dots, x_n^{\alpha_n}, \text{ where } \sum_{i=1}^n \alpha_i = \frac{\ln \eta}{\ln \delta}$$

$$DU_2 = (x_1^{1/\ln \delta} + x_2^{1/\ln \delta} + \dots + x_n^{1/\ln \delta})^{\ln \eta}$$

In my theory, experienced utility is the primitive for the individual and discounting experienced utility means that the individual's primitive is discounted.

Turning to the case of uncertainty, I would note that by uncertainty I mean that there exists some randomness for the events in question and a probability measure defined over sets of random events. Uncertainty in my theory would coincide with what Knight (1921) termed as measurable uncertainty: "... a *measurable* uncertainty, or "risk" proper, as we shall use the term, is so far

⁵Given that decision utility DU is assumed to exist, preferences are continuous.

different from an *unmeasurable* one that it is not in effect an uncertainty at all. We shall accordingly restrict the term “uncertainty” to cases of the non-quantitative type. It is this “true” uncertainty, and not risk, as has been argued, which forms the basis of a valid theory of profit”

I have decided to maintain using the term uncertainty because as seen below in my theory both discounting and uncertainty are included in the coefficients of proportionality. As for the case of uncertainty when decision utility is used, additional assumptions to the earlier decision utility assumptions are required in order to derive some structure for the form of a decision utility. So the main difference is that both discounting and uncertainty are part of experienced utility while additional assumptions are required for decision utility.

Although the earlier decision utility assumptions are sufficient to guarantee the existence of a function that represents preferences in case of uncertainty, research has been concerned with finding some structure for the decision utility function. Such research has been rewarded but always with additional assumptions. For example, in the case described as including uncertainty due to so called objective probabilities, decision utility is in the form of expected utility but with the additional assumption that alternatives or lotteries are required to be independent and in the case described as including risk due to subjective probabilities, decision utility can continue to be in the form of expected utility but with even more assumptions (Mas-Collel et al., 1995).

Other research has been concerned with finding some structure for non-expected utility functions, such as prospect theory by Kahneman and Tversky (1979) and regret theory by Loomes and Sugden (1982). The assumptions required for these functions are even stronger than those for expected utility. Non-expected utility functions used today are impressive analytically but I raise a question on their usability conceptually. My question is: If weaker assumptions on preferences for expected utility do not seem to hold, how could non-expected utility functions derived from stronger assumptions be capable to represent these preferences? In my view, if the assumptions for expected or non-expected decision utility functions do not include rationality, experiments from the area of experimental economics could provide some helpful answers.

In my theory, experienced utilities and their rates of change are functions of time. However, hedonic experience from experienced utility is instantaneous and at every moment in time neither this experience nor its rate of change depend on discounting or uncertainty. So the source of randomness in my theory is the time an individual spends on each activity.

Suppose an individual discounts experience over time by a discounting

factor $\delta(t) > 0$.⁶ Then for each activity i , the present value of experienced utility is $\delta(t)u_i(t)$ so that the pull-and-push dynamics, which determine du_i/dt , hold for $\delta(t)u_i(t)$ as opposed to $u_i(t)$. From experienced utility assumption III, for each i and $j \neq i$, the rate of change of experienced utility would be proportional to the difference between $\delta(t)u_i(t)$ and the sum of $\delta(t)u_j(t)$. As shown from the lemma below, this is in fact the case.

Next suppose that an individual is uncertain about the experience from an activity. This could be due to a range of factors related to the individual's previous and/or anticipated hedonic experience itself or the environment where the activity takes place or both. Since both experience and its rate of change are functions of time, then for each activity i , the pull-and-push dynamics are adjusted instantaneously during the time spent on the activity. So uncertainty can be measured by a function of the duration of time spent on an activity.

For example, while spending time at the beach and enjoying the sunshine, it might be difficult to convince ourselves to go to work sooner rather than later. The experience from every moment at the beach is certain but what might be uncertain is how much longer we can stay there as the pressure from not getting the work done keeps pushing us toward going to work and therefore affecting both experience at the beach and its rate of change over time.

Given uncertain duration of time s spent on an activity i , let a probability (distribution or density) function $\theta_i(s) > 0$ measure this uncertainty.⁷ The following lemma presents the result for experienced utility in the case of discounting and uncertainty.

Lemma 3. *The family of experienced utilities represents individual hedonic experience with both discounting and uncertainty.*

Proof. Suppose that the duration of time s spent on activity i starts at t_0 and ends at uncertain t . When not spending time on an activity, the experience from this activity is constant to 0. If $u_i(s) \equiv 0$, then $du_i/ds = 0$ for all $s \notin [t_0, t]$.⁸ So we can include only $s \in [t_0, t]$ in system (1). For each i , the rate of change is:

$$\frac{du_i}{ds} = \beta_i(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right)$$

⁶ δ^t is a special case, $\delta(t) = \delta^t$, $t = 0, 1, \dots$, of the general discounting considered here.

⁷Since $\theta_i(s) = 0$ means that there is 0 percent chance that an individual spends time on an activity, we can omit this case because $s = 0$.

⁸Because of the uncertainty of s , t is not fixed.

Given that discounting factor $\delta(s) > 0$ and probability function $\theta_i(s) > 0$, there exist $\alpha_i(s) > 0$ such that:

$$\beta_i(s) = \alpha_i(s)\theta_i(s)\delta(s), \quad i = 1, 2, \dots, n.$$

Then the rate of change is:

$$\begin{aligned} \frac{du_i}{ds} &= \beta_i(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right) \\ \frac{du_i}{ds} &= \alpha_i(s)\theta_i(s)\delta(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right) \\ \frac{du_i}{ds} &= \alpha_i(s)\theta_i(s) \left(\delta(s)u_i(s) - \sum_{i \neq j=1}^n \delta(s)u_j(s) \right) \end{aligned}$$

If we denote $u_i(s, \delta) = \delta(s)u_i(s)$ for each i , then:

$$\frac{du_i}{ds} = \alpha_i(s) \left(u_i(s, \delta) - \sum_{i \neq j=1}^n u_j(s, \delta) \right) \theta_i(s)$$

With $\alpha_i(s) > 0$, based on experienced utility assumptions I-III, we can create a new system such that for each i , the rate of change of $u_i(s, \delta)$ is:

$$\frac{du_i(s, \delta)}{ds} = \alpha_i(s) \left(u_i(s, \delta) - \sum_{i \neq j=1}^n u_j(s, \delta) \right)$$

Then for each i , the rate of change of $u_i(s)$ is:

$$\frac{du_i}{ds} = \beta_i(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right) = \frac{du_i(s, \delta)}{ds} \theta_i(s),$$

including both discounting $\delta(s)$ and uncertainty $\theta_i(s)$. Then:

$$\int_{t_0}^t \frac{du_i}{ds} ds = \int_{t_0}^t \beta_i(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right) ds = \int_{t_0}^t \frac{du_i(s, \delta)}{ds} \theta_i(s) ds$$

On the left hand side we obtain $u_i(t) - u_i(t_0)$. On the right hand side, since $du_i(s, \delta)/ds$ is a function of the random variable s , we obtain its expected value $E[du_i(s, \delta)/ds(t)]$ as a function of t .⁹ Then we can write:

$$u_i(t) - u_i(t_0) = \int_{t_0}^t \beta_i(s) \left(u_i(s) - \sum_{i \neq j=1}^n u_j(s) \right) ds = E\left[\frac{du_i(s, \delta)}{ds}(t)\right]$$

Since $u_i(t_0)$ is constant, by the Fundamental Theorem of Calculus:

$$\frac{du_i}{dt} = \beta_i(t) \left(u_i(t) - \sum_{i \neq j=1}^n u_j(t) \right) = \frac{dE\left[\frac{du_i(s, \delta)}{ds}(t)\right]}{dt}$$

So the family of experienced utilities is derived from a system which includes both discounting and uncertainty. This completes the proof. \square

For a sufficiently short period of time, we could assume that the coefficients of proportionality are constant. Then during this period, the discounting factor is constant, $\delta(s) = \delta$, which as noted earlier is a special case. The discounting factor could be either $\delta = 1$ for the current period with no discounting or $0 < \delta < 1$ for a more distant period. For each activity i , the uncertainty is also constant, $\theta_i(s) = \theta_i$. Then the uncertainty is measured by the uniform density function $\theta_i = 1/(t - t_0)$ over the given period. So with constant coefficients of proportionality, we have the following Remark, which is presented without proof because the result follows as a special case of Lemma 3.

Remark. *For constant coefficients of proportionality, the family of experienced utilities represents individual hedonic experience with constant discounting and uniformly distributed uncertainty.*

Experienced utilities from Theorem 1 have two other useful properties. They satisfy rationality and the axiom of choice. Although rationality is not required for their existence, experienced utilities satisfy rationality because the general solution set from solution (3) is linearly or totally ordered. Also, given an initial condition, experienced utilities satisfy the axiom of choice because for given constant c 's in solution (3), the general solution set is well-ordered.¹⁰ Satisfying these properties is important when using experienced utilities to

⁹For $s \in [t_0, t]$ (note the closed interval), the condition $\int_{t_0}^t |du_i(s, \delta)/ds| \theta_i(s) ds < \infty$ is satisfied (Rice, 1995).

¹⁰A set satisfies the axiom of choice when it is well-ordered and vice versa.

analyze individual behavior because for such a purpose, they need to satisfy at least the condition of rationality. However, for all the practical purposes maybe even more importantly, they also need to satisfy the axiom of choice.¹¹

When used to explain choice, decision utility functions are used in ways which ensure that they satisfy the axiom of choice. Becker has noted the importance of an well-ordered decision utility function: "... now everyone more or less agrees that rational behavior simply implies consistent maximization of a well-ordered function, such as a utility or profit function." (Becker, 1962, p. 1). Given that decision utility is continuous and usually real valued, it satisfies rationality. However, it does not satisfy the axiom of choice in and by itself. Decision utility usually satisfies the axiom of choice by using a function that is either assumed to take values over a compact set, such as the budget set, or defined recursively in such a way that it can attain a maximal value.

In all situations where a decision utility function is used, it is implicitly assumed to satisfy at least the assumption of rationality. This has to be the case because decision utility represents preferences that are rational. This means that the rationality of decision utility is assumed by the extension of the rationality of preferences as the primitive. On the other hand, my experienced utilities are the primitive and their rationality is not assumed but derived as a property of the family of experienced utilities. I present the results on rationality and the axiom of choice as follows.

Lemma 4. *Experienced utilities satisfy rationality.*

Proof. Let \cup denote the union of a set and \mathcal{U} the union of unions. Also let \aleph denote the cardinality for a set. Then:

$$\mathcal{U} = \cup_{t \in \mathbb{R}} \cup_{i=1}^n u_i(t)$$

With finite number of activities, $\aleph(\cup_{i=1}^n u_i(t)) < \aleph(\mathbb{R})$ for every t . Also, $\aleph(t) \leq \aleph(\mathbb{R})$ for all t . Then $\aleph(\mathcal{U}) \leq \aleph(\mathbb{R})$. But \mathbb{R} is linearly ordered, so \mathcal{U} is also linearly ordered.¹² Then experienced utilities satisfy rationality for all t . This completes the proof. \square

Given that the experienced utilities functions take values from \mathbb{R}^n , it is useful to find that they are linearly ordered and hence satisfy rationality for all time. But this does not imply that they automatically satisfy the axiom of choice on any open interval. However, given an initial condition, experienced utilities satisfy the axiom of choice.

¹¹A set that is well-ordered is also linearly ordered but not vice versa, for example, \mathbb{R} .

¹²This result follows from Folland (1999, p. 9, Proposition 0.14).

Lemma 5. *Given an initial condition, experienced utilities satisfy the axiom of choice.*

Proof. Let t_0 be an initial condition and \mathcal{I} an open interval where experienced utilities are defined. So the values $u_i(t)$ of experienced utilities are for $t \in [t_0, \infty) \cap \mathcal{I} \neq \emptyset$, where \cap denotes intersection and \emptyset the empty set. Then:

$$\mathcal{U}_{t_0} = \cup_{t \in [t_0, \infty)} \cup_{i=1}^n u_i(t),$$

where \mathcal{U}_{t_0} means that the union of unions is for $t \in [t_0, \infty)$. With finite number of activities, $\aleph(\cup_{i=1}^n u_i(t)) < \aleph([t_0, \infty))$ for every t . Also, $\aleph(t) \leq \aleph([t_0, \infty))$ for all $t \in [t_0, \infty)$. Then $\aleph(\mathcal{U}_{t_0}) \leq \aleph([t_0, \infty))$. But $[t_0, \infty)$ is well-ordered, so \mathcal{U}_{t_0} is also well-ordered. Then experienced utilities satisfy the axiom of choice for a given initial condition at t_0 . This completes the proof. \square

Given that for an initial condition, experienced utilities satisfy the axiom of choice, they can be used to analyze behavior. How this can be achieved is a topic of my current research. The use of experienced utilities will be presented by discussing applicable differences between decision utility and experienced utility similarly with the way I am discussing their conceptual differences.

Experienced utility functions can be used to analyze individual behavior over any period of time. If this period is sufficiently short, as noted earlier, we can assume that the coefficients of proportionality are constant. These constant coefficients are shown below without the variable t :

$$\mathcal{B} = \begin{bmatrix} \beta_1 & -\beta_1 & \cdots & -\beta_1 \\ -\beta_2 & \beta_2 & \cdots & -\beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_n & -\beta_n & \cdots & \beta_n \end{bmatrix}$$

Because the matrix of the coefficients of proportionality \mathcal{B} has constant elements, its eigenvalues found in Lemma 1 are also constant. These constant eigenvalues are also shown below without the variable t :

$$\Omega = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$

In order to obtain special cases of the functional forms for experienced utility from Theorem 1 with constant coefficients of proportionality, I use the product of eigenvalues of matrix \mathcal{B} and the variable t : $\omega_i t$, $i = 1, 2, \dots, n$.

These products can be written more compactly in matrix format:

$$\Omega t = \begin{bmatrix} \omega_1 t & 0 & \cdots & 0 \\ 0 & \omega_2 t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n t \end{bmatrix}$$

The form of experienced utilities as special cases of solution (3) can be derived using the same steps as those in the proof of Theorem 1 with $\omega_i t$ for $\int \omega_i(t) dt$, $i = 1, 2, \dots, n$. Their form is in following corollary.

Corollary 1.1. *With constant coefficients of proportionality, the family of experienced utilities is expressed as:*

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = c_1 \begin{bmatrix} k_1^1 \\ k_2^1 \\ \vdots \\ k_n^1 \end{bmatrix} e^{\omega_1 t} + c_2 \begin{bmatrix} k_1^2 \\ k_2^2 \\ \vdots \\ k_n^2 \end{bmatrix} e^{\omega_2 t} + \cdots + c_n \begin{bmatrix} k_1^n \\ k_2^n \\ \vdots \\ k_n^n \end{bmatrix} e^{\omega_n t}$$

In the next section I continue discussing the differences between decision utility and experienced utility with comparisons of findings from other studies.

4 Comparisons

The implications of the lack of a family of decision utility functions for the preference relation as a primitive can also be seen in the findings from other studies. For instance, research continues to find that often decision utility functions are not capable to represent the underlying preference relation.

As noted in the previous section, in cases that include uncertainty (or risk), the existence of decision utility requires additional assumptions. Furthermore, in specific studies that include uncertainty, the evaluation of different specifications of decision utility has been achieved with even more assumptions related to the specifications themselves. Meanwhile, as Lemma 3 shows, experienced utility functions from Theorem 1 include both discounting and uncertainty.

One case where different decision utility specifications have been tested is by Loomes and Sugden (1998). They test three different specifications of risky choice, the Harless-Camerer (after Harless and Camerer, 1994), the Hey-Orme (after Hey and Orme, 1994) and a random preference model proposed by Loomes and Sugden (1995) as a generalization of Becker et al. (1962).

Loomes and Sugden (1998) conducted their tests through an experiment and their findings illustrate the inability of decision utility functions to represent the underlying individual preference relation. For example, they report that the observed rate of dominance is lower than what is predicted by either the Harless-Camerer or the Hey-Orme specification but high enough to contradict the random preference model, which predicts no-dominance.

Another case is by Barberis et al. (2006) whose results show that a wide range of utility specifications, including expected utility and non-expected recursive utility functions, are not capable to explain risk-averse preferences for a small independent gamble, even when the gamble is actuarially favorable. They suggest that narrow-framing, an assumption which assumes that individuals isolate the risk of a single option from their overall risk (usually risk for overall wealth), could be an important factor to explain individual behavior.¹³

It is interesting to explore the role of narrow-framing, as well as more broadly bounded rationality. However, as long as their role is evaluated through a decision utility function, the evaluation will have to assume rationality, or a refined version thereof, otherwise there exists no such a function that can be used for the evaluation. While I continue the critique of rationality, one might what wonder what would happen if decision utility functions were evaluated in cases when the rationality assumption is actually satisfied.

Such cases have been provided by the work on (decision) utility functions using artificial agents, which are programmed to make choices in strictly defined environments. For example, while trying to remedy issues with the utility functions used for these agents, Hibbard (2012) used an artificial agent known as AIXI, which was created by Hutter (2005).¹⁴ The part ‘AI’ stands for ‘Artificial Intelligence’ and ‘XI’ for the Greek letter ξ , which is a prior probability of agent’s history.

This artificial agent is defined in terms of ξ , its (decision) utility function, a discount factor and its interaction with the environment over time (Hibbard, 2012, p. 2). Meanwhile, in my theory a real life agent: has an experienced utility function for each activity; gains experience from engaging in an activity through the interaction with the environment; has a discount factor δ ; and, for every activity i , has a probability function $\theta_i(s)$, which could depend on the history of previous experience (if any), anticipation of future experience as well as external environmental factors.

According to Hibbard (2012), Hutter (2005) showed that AIXI does in

¹³Narrow-framing (Kahneman and Lovallo, 1993) is part of the broader notion of bounded rationality (Kahneman and Tversky, 1981 and Kahneman, 2003).

¹⁴According to Hibbard (2012), Hutter (2005) significantly advanced the mathematical theory of rational agents with the definition of AIXI.

fact maximizes expected utility but the construction of AIXI is computationally impossible. Incidentally, this result confirms my claim that rationality (or bounded rationality) assumes an enormous cognitive ability for an individual. However, in my opinion, the issue with AIXI in explaining human behavior is not that it is impossible to construct but rather that it implies that we know both the individual's preference relation and the decision utility function.

A case where individual's decision utility is not assumed to be known by the analyst who studies individual behavior is the agent created by Frick et al. (2019). Their agent has similarities with the AIXI and while AIXI was found to be computationally impossible to construct, Frick et al.' agent is endowed with extraordinary cognitive ability by their numerous definitions and axioms. At the same time, in my view, their agent is a limited agent because the agent's choice behavior is deterministic. The conditions that Frick et al. impose in order to derive a dynamic random utility are presented in terms of elaborate definitions and self evident truths (axioms)¹⁵ for a decision utility that is unknown to the analyst who sees choice behavior to appear as stochastic due to information asymmetry and a realized utility (their term) that is known to the individual whose choice behavior is considered deterministic.

In a way, the researcher has full control over which conditions to impose in order to obtain a theory but the conditions in Frick et al. (2019) make the decision utility assumptions mentioned so far look rather weak. However, I believe that the stronger the conditions are the less likely it is for a decision utility function to represent preferences. As for the deterministic choice behavior, although the choice itself after it has been made is deterministic, the individual's behavior could still include a random process, which was shown in Lemma 3. Based on neoclassical economic theory we know neither the preferences nor the decision utility whereas based on my theory we know the family of experienced utility functions and we do not require to know the preferences.

Turning back to Hibbard (2012), the effort was in fact to remedy two other issues related only to AIXI, self-delusion and self-modification (identified by Orseau and Ring, 2011a and 2011b; and Dewey, 2011). Self-delusion occurs when an (artificial) agent modifies behavior to create the illusion of maximizing utility (Hibbard, 2012, pp. 3-5) and self-modification occurs when such agent modifies the (decision) utility function itself even when it is programmed to prevent self-delusion (Hibbard, 2012, pp. 16-18). Hibbard (2012) proposed the formulation of (decision) utility function in two steps: first, infer a model of the environment from the agent's history of its interaction with the environment

¹⁵According to Google's definition, axiom is "a statement or proposition which is regarded as being established, accepted, or self-evidently true."

and second, define (decision) utility as a function of this model.

Although Hibbard's proposal was based on examples (rather than proof), it might provide some insights for economic analysis because, to my knowledge, behavioral issues that arise from the interaction between economic (real life) agents and their environment are not part of the definitions of decision utility functions. For example, 'framing' as presented by Kahneman and Tversky (1981) would, in my opinion, be an environmental factor because it is present when an agent's decision depends on the way in which alternatives are presented. Kahneman and Tversky attributed choices that do not conform to transitivity of preferences to the way in which alternatives were presented. As far as I know, definitions of decision utility functions that have been used in economic analysis include only outcomes, not environmental factors.

In a case that also includes experienced utility, I compare my theory of experienced utility with the theory presented by Kahneman et al. (1997). Before looking at the details of each theory, the main difference is that their theory is a normative theory of total experienced utility of temporally extended outcomes whereas my theory is a descriptive theory of instantaneous experienced utility. Kahneman et al.'s experienced utility as a function of time extends the definition of decision utility whereas my definition of experienced utility as a function of time does not rely on the definition of decision utility.

According to Kahneman et al. (1997), decision utility is the weight of an outcome and experienced utility is a hedonic quality. They view decision utility as the outcomes' or attributes' weight in the decision of individual choice and experienced utility as the hedonic quality or pleasure/displeasure attributes of each moment of experience of temporally extended outcomes, discussed in more detail below. Their outcomes are measured in two ways, as remembered utility from previous outcomes or total utility from normative profiles of outcomes. These are both derived from instant utility, which is considered as a measure of the intensity of hedonic experience at every point in time similar to Bentham (1823) and Edgeworth (1881). In contrast, my definition of experienced utility includes hedonic experience as descriptive quality.

As far as I know, Kahneman et al. (1997) were the first ones to prove the existence of experienced utility but, in contrast with my theory, they derive its existence with axioms that are stronger than the requirements for the existence of decision utility. Specifically, they note that decision utility is a measure on temporally extended outcomes (TEOs), which are outcomes inferred from choices of episodes. An episode is a mapping from a time interval to a set of outcomes, such as commodity bundles, health states, etc. A TEO is a mapping from finite disjoint union of intervals of episodes to the set of outcomes. Then Kahneman et al. (1997) assign an instant utility value from

direct measurements in experiments to each point of time and use this value to assign to every TEO a time-dependent function called a utility profile.

A profile is a function taking the instant utility value at each point of the domain of the TEO. Since a TEO can include different episodes, the utility profile of a TEO is the concatenation of the utility profiles of its episodes. Total utility is an increasing transform of an integral of instant utility and it is a normative concept because it is a measure on TEOs that can be used to evaluate TEOs based on their total utility values. From experimental results of measurements on subjects, Kahneman et al. (1997) reported that their observed preferences do not maximize experienced utility, meaning that their measures of experienced utility and decision utility do not coincide.

Kahneman et al. prove the existence of experienced utility using stopwatch time instead of calendar time restarting the interval of time at the beginning of each profile at time 0 and restrict the functions of their utility profiles to start from 0 as well. In my theory, I use calendar time and my functions of experienced utilities are not restricted to start at 0 at any given time.

Before pointing out other differences, it needs to be noted that Kahneman et al. (1997) have the undeniable advantage of using empirical measurements of instant utility. However, the comparison with my theory is not based on actual measurements but on the analytic framework and conditions required for the existence of experienced utility. Kahneman et al. prove the existence of unspecified forms of integrals of instant utility whereas my theory proves the existence of a family of functions of experienced (or instant) utilities. The theory by Kahneman et al. requires that instant utilities and hence experiences remain constant over individuals' age whereas in my theory coefficients of proportionality for each activity change over time to reflect differences in experiences of individuals at different ages. Also, Kahneman et al. assume zero discounting and their theory cannot be modified to include discounting or uncertainty, but my theory can include both discounting and uncertainty.

Furthermore, the profiles in Kahneman et al. (1997) are required to satisfy three axioms, Axiom I of sup norm continuous weak ordering of utility profiles, Axiom II of monotonicity of instant utility and Axiom III of monotonicity of total utility, from which they derive ordered profiles based on integrals of instant utilities over a period of time. An important point to note is on the role of neutral utility profiles, which Kahneman et al. (1997) define as instant utilities that are hedonically neutral, in other words states that are neither good nor bad. They play a crucial role in their theory because their Axiom I is on the concatenation of neutral utility profiles so that total utility of a profile does not change when the profile is concatenated or extended over continuous time intervals with another profile with neutral utility.

This neutral utility would result from a neutral state with the value of zero in the measurement scale of instant utility: “A standard demonstration involves two pails of warm water, one warmer than the other. A subject who immerses one hand in each of these pails will eventually report that both hands feel alike, neither hot nor cold. The neutral value provides a natural zero point for bipolar dimensions” (Kahneman et al., 1997, p. 380). We see a discussion about such a state again in Kahneman (2000). However, Read (2007) doubts the existence of a hedonic neutral state.

Read (2007) interprets Kahneman et al.s zero-point also as a stop/go signal where ‘stop’ is used to interrupt an activity that causes pain and ‘go’ is used to continue an activity that gives satisfaction. However, by further detailing an example with cold and hot water, Read illustrates that if the zero point existed, then it would be at the point where the water that is too cold is at the same time too hot. Therefore, he notes that: “... we cannot measure, or even conceptualise, the zero-point in the absence of specific choice options” (Read, 2007, p. 56). But if Read’s argument holds, Axiom I in Kahneman et al. (1997) cannot be taken as a self evident truth from which reliable conclusions on choice can be drawn and as such it cannot be truly an axiom.

In summary, the existence of experienced utility in Kahneman et al. (1997) is derived from axioms that are even stronger than the assumptions required for decision utility and these axioms might have questionable validity. In the next section, I present non-experienced utilities and an example.

5 Non-experienced utilities

In the example when an individual would like to spend time at the beach but at the same time feels the pressure from having to go to work, there are two activities and so there are two experienced utilities. Based on experienced utility assumption II, an individual engages in a single activity and gains hedonic experience from spending time only on this activity. Because the hedonic experience at every moment in time is from spending time on a chosen activity, experienced utilities from non-chosen activities cannot be realized.

I use the term non-experienced utilities for experienced utility functions from non-chosen activities while an individual spends time on a chosen activity. While an individual spends time on an activity, if there were no other forces that push the individual towards engaging in other activities, the individual would continue spending time on the same activity without having to choose any other activities. So non-experienced utilities are important because they could help to explain the switch from one activity onto another so that the

individual's choice of a time allocation to an activity could be determined. Although choice is not a topic of this paper, non-experienced utilities from the non-chosen activities are discussed because they exist and do not just disappear once the individual starts to engage in an activity.

Non-experienced utilities are the forces that pull¹⁶ an individual to their own activities while spending time on the chosen activity. It is through these forces that the switch to another activity occurs so that a time allocation to an activity could be explained. Hence it is useful to know their functional form.

In reality an individual always engages in an activity even when the individual does not seem to be doing anything. So non-experienced utilities are always there because experienced utility is always there. However, we cannot expect non-experienced utilities to be of the same form as the functions in Theorem 1 because they are the experienced utility functions from engaging, rather than from non-engaging, in an activity. At every moment in time non-experienced utilities depend on which activity an individual is spending time on. Given n activities, because an individual engages in a single activity, there are $n(n - 1)$ non-experienced utilities. Experienced utility functions are unique and there are n such functions. For each of them there are $n - 1$ non-experienced utility functions and which group of these $n - 1$ functions is influencing the individual depends on which activity the individual has chosen.

As in the case of experienced utilities, $u_i(t)$ denotes experienced utility from activity i . Non-experienced utility from activity j is denoted by $u_{j|i}(t)$, $j \neq i$, where the condition denoted by '| ' indicates that activity j is not chosen given that activity i is chosen because non-experienced utilities depend on which activity is chosen. The dynamics are the same as for experienced utilities because the individual is under the influence of pull-and-push forces and the same three experienced utility assumptions apply. However, in this case the pull force of each non-experienced utility from a non-chosen activity has to withstand the combined push force of both non-experienced utilities from the other non-chosen activities and experienced utility from the chosen activity.¹⁷

The setting is similar to the setting in Theorem 1 except that the number of activities under consideration is $n - 1$ but the number of forces is n . For any chosen activity i , the dynamics from experienced utility assumptions are:

¹⁶A force is a pull force if it attracts the individual towards its own activity. Although non-experienced utilities never happen, each non-experienced utility is a pull force for its own non-chosen activity and a push force for all of the other activities, including the chosen activity where the individual is currently spending time on.

¹⁷Experienced utility from the chosen activity is a pull force only for the activity and a push force for all of the other activities.

$$\begin{aligned}
\dot{u}_{1|i} &= \beta_1(t) (u_{1|i}(t) \cdots - u_{i-1|i}(t) - u_{i+1|i}(t) - \cdots - u_{n|i}(t)) - \beta_1(t)u_i(t) \\
&\quad \dots \\
\dot{u}_{i-1|i} &= \beta_{i-1}(t) (-u_{1|i}(t) \cdots + u_{i-1|i}(t) - u_{i+1|i}(t) \cdots - u_{n|i}(t)) - \beta_{i-1}(t)u_i(t) \\
\dot{u}_{i+1|i} &= \beta_{i+1}(t) (-u_{1|i}(t) \cdots - u_{i-1|i}(t) + u_{i+1|i}(t) \cdots - u_{n|i}(t)) - \beta_{i+1}(t)u_i(t) \\
&\quad \dots \\
\dot{u}_{n|i} &= \beta_n(t) (-u_{1|i}(t) \cdots - u_{i-1|i}(t) - u_{i+1|i}(t) \cdots + u_{n|i}(t)) - \beta_n(t)u_i(t) \\
\dot{\mathbf{u}}_i(t) &= \mathcal{B}_i(t)\mathbf{u}_i(t) + \mathbf{b}_i(t), \tag{4}
\end{aligned}$$

where

$$\begin{aligned}
\dot{\mathbf{u}}_i(t) &= (\dot{u}_{1|i}, \dots, \dot{u}_{i-1|i}, \dot{u}_{i+1|i}, \dots, \dot{u}_{n|i}) = \\
&= (du_{1|i}/dt, \dots, du_{i-1|i}/dt, du_{i+1|i}/dt, \dots, du_{n|i}/dt)
\end{aligned}$$

denotes the rates of change and

$$\mathbf{u}_i(t) = (u_{1|i}(t), \dots, u_{i-1|i}(t), u_{i+1|i}(t), \dots, u_{n|i}(t))$$

denotes non-experienced utilities.

Also, $\beta_j(t)$, $j \neq i$, is the coefficient of proportionality for each activity $j = 1, \dots, n-1, n+1, \dots, n$ and

$$\mathbf{b}_i(t) = (-\beta_1(t)u_i(t), \dots, -\beta_{i-1}(t)u_i(t), -\beta_{i+1}(t)u_i(t), \dots, -\beta_n(t)u_i(t))$$

denotes the products $-\beta_j(t)u_i(t)$. In matrix notation, when an individual spends time on activity i , the coefficients of proportionality can be written as:

$$\mathcal{B}_i(t) = \begin{bmatrix} \beta_1(t) & \cdots & -\beta_1(t) & -\beta_1(t) & \cdots & -\beta_1(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{i-1}(t) & \cdots & \beta_{i-1}(t) & -\beta_{i-1}(t) & \cdots & -\beta_{i-1}(t) \\ -\beta_{i+1}(t) & \cdots & -\beta_{i+1}(t) & +\beta_{i+1}(t) & \cdots & -\beta_{i+1}(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n(t) & \cdots & -\beta_n(t) & -\beta_n(t) & \cdots & \beta_n(t) \end{bmatrix}$$

In system (4) we know experienced utility from activity i where the individual is currently spending time on and we don't know non-experienced utilities.¹⁸ While an individual spends time on an activity, non-experienced

¹⁸The individual might know instinctively own non-experienced utilities but the researcher would not necessarily know the individual's non-experienced utilities.

utilities are conditioned on this activity. Knowing their functional form would help to determine when the individual switches to another activity.

In this setting non-experienced utilities act like pull-and-push forces and again experienced utility from the chosen activity is a push force. As long as the pull force from a non-chosen activity for an individual is greater than the combined push force from the other non-chosen activities and the push force from the chosen activity, its rate of change is positive. So again the coefficients of proportionality are positive.

System (4) is non-homogeneous and contains the homogeneous part:

$$\dot{\mathbf{u}}_{|i}(t) = \mathcal{B}_{|i}(t)\mathbf{u}_{|i}(t)$$

and the nonzero part: $\mathbf{b}_{|i}(t)$.

When solving for the homogeneous part of system (4) it is possible to use the proof in Appendix A or A.1. Because of the simpler functional form, I use the proof and its solution from Appendix A. For experienced utility $u_i(t)$ in the nonzero part of system (4), it is also possible to use the form from solution (3) or (A.8). Because of the simpler functional form, I use solution (3).

System (4) is similar to system (1) but with $n - 1$ activities. As in the case of experienced utilities, let $\lambda_{1|i}, \dots, \lambda_{i-1|i}, \lambda_{i+1|i}, \dots, \lambda_{n|i}$ be the eigenvalues of $E_{(n-1) \times (n-1)}$ of size $(n - 1) \times (n - 1)$. Given coefficients of proportionality in system (4), the eigenvalues and eigenvectors of $\mathcal{B}_{|i}(t)$ are from Lemma 2. They are denoted by $\omega_{j|i}(t)$ and $\mathbf{k}_{|i}^j$, $j \neq i$, $j = 1, \dots, i - 1, i + 1, \dots, n$, respectively. The eigenvectors can be written in matrix format:

$$K_{|i} = \begin{bmatrix} \mathbf{k}_{|i}^1 & \dots & \mathbf{k}_{|i}^{i-1} & \mathbf{k}_{|i}^{i+1} & \dots & \mathbf{k}_{|i}^n \end{bmatrix}$$

The indefinite integrals of eigenvalues of matrix $\mathcal{B}_{|i}(t)$ are:

$$\tilde{\omega}_{j|i}(t) = \int \omega_{j|i}(t) dt, \quad j = 1, \dots, i - 1, i + 1, \dots, n$$

The results from Theorem 1 are useful to solve system (4). When an individual engages in activity i with experienced utility $u_i(t)$, other activities $j = 1, \dots, i - 1, i + 1, \dots, n$ have non-experienced utilities $u_{j|i}(t)$. Experienced utilities from all activities in solution (3) are expressed in terms of eigenvectors and eigenvalues of matrix $\mathcal{B}(t)$ and the nonzero constant vector \mathbf{c} . Except for experienced utility from activity i , all of the other experienced utilities from solution (3) are solutions to system (4).

The $j \neq i$ values for eigenvectors of K in solution (3) are denoted by K_{-i} and the values for \mathbf{c} are the same as in solution (3). So these values are the

values of K excluding the values in row i for K_{-i} and hence the ‘ $-i$ ’ subscript but they are exactly the same values for \mathbf{c} as in solution (3) and hence there is no subscript. These values can be seen more easily from solution (A.6), Appendix A. Then the values of \mathbf{c} are predetermined from solving system (1), they are not determined from solving system (4). With this notation, non-experienced utilities are given by the following theorem.

Theorem 2 (Non-experienced Utilities). *Given positive coefficients of proportionality on an open interval, the rate of change of non-experienced utility from each non-chosen activity is proportional to the difference between non-experienced utility from the activity and sum of non-experienced utilities from the other non-chosen activities as well as experienced utility from the chosen activity. Then there exist unique non-experienced utilities which are:*

(i) *expressed explicitly:*

$$\begin{aligned} \mathbf{u}_{|i}(t) = & K_{|i} \text{diag} (e^{\tilde{\omega}_{1|i}(t)}, \dots, e^{\tilde{\omega}_{i-1|i}(t)}, e^{\tilde{\omega}_{i+1|i}(t)}, \dots, e^{\tilde{\omega}_{n|i}(t)}) \mathbf{c}_{|i} \\ & + K_{-i} \text{diag} (e^{\tilde{\omega}_1(t)}, \dots, e^{\tilde{\omega}_{i-1}(t)}, e^{\tilde{\omega}_i(t)}, e^{\tilde{\omega}_{i+1}(t)}, \dots, e^{\tilde{\omega}_n(t)}) \mathbf{c}, \end{aligned} \quad (5)$$

where ‘diag’ stands for diagonal matrix and $\mathbf{c}_{-|i}$ is a vector of constants,

(ii) *real valued and*

(iii) *linearly independent.*

Proof. See Appendix B. □

If we know the functional forms for non-experienced utilities, we are able to use them to analyze their influence on the individual’s experienced utility and derive useful conclusions about individual choice. As in the case of experienced utilities, for a sufficiently short period of time we can assume again that the coefficients of proportionality are constant. These constant coefficients are shown below without the variable t :

$$\mathcal{B}_{-|i} = \begin{bmatrix} \beta_1 & \cdots & -\beta_1 & -\beta_1 & \cdots & -\beta_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{i-1} & \cdots & \beta_{i-1} & -\beta_{i-1} & \cdots & -\beta_{i-1} \\ -\beta_{i+1} & \cdots & -\beta_{i+1} & +\beta_{i+1} & \cdots & -\beta_{i+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n & \cdots & -\beta_n & -\beta_n & \cdots & \beta_n \end{bmatrix}$$

The eigenvectors of $\mathcal{B}_{-|i}$ are also the same as eigenvectors of $E_{(n-1) \times (n-1)}$. Their values are denoted by:

$$\mathbf{k}_{|i}^j = \left(k_{1|i}^j, \dots, k_{i-1|i}^j, k_{i+1|i}^j, \dots, k_{n|i}^j \right),$$

$j \neq i, j = 1, \dots, i-1, i+1, \dots, n$. The eigenvectors of \mathcal{B} are the same as eigenvectors of $E_{n \times n}$ as well. However, when used to solve system (4), their values do not include those for activity i . Their values are denoted by:

$$\mathbf{k}_{-i}^j = \left(k_1^j, \dots, k_{i-1}^j, k_{i+1}^j, \dots, k_n^j \right),$$

$j = 1, \dots, i-1, i, i+1, \dots, n$, including $j = i$. The form of non-experienced utilities as special cases of solution (5) can be derived using the same steps as those in the proof of Theorem 2 with $\omega_{j|i}t$ for $\int \omega_{j|i}(t)dt, i = 1, 2, \dots, n$. With $\mathbf{c}_{|i} = (c_{1|i}, \dots, c_{i-1|i}, c_{i+1|i}, \dots, c_{n|i})$, their form is in following corollary.

Corollary 2.1. *With constant coefficients of proportionality, non-experienced utilities are expressed as:*

$$\begin{aligned} \begin{bmatrix} u_{1|i}(t) \\ \vdots \\ u_{i-1|i}(t) \\ u_{i+1|i}(t) \\ \vdots \\ u_{n|i}(t) \end{bmatrix} &= c_{1|i} \begin{bmatrix} k_{1|i}^1 \\ \vdots \\ k_{i-1|i}^1 \\ k_{i+1|i}^1 \\ \vdots \\ k_{n|i}^1 \end{bmatrix} e^{\omega_{1|i}t} + \dots + c_{i-1|i} \begin{bmatrix} k_{1|i}^{i-1} \\ \vdots \\ k_{i-1|i}^{i-1} \\ k_{i+1|i}^{i-1} \\ \vdots \\ k_{n|i}^{i-1} \end{bmatrix} e^{\omega_{i-1|i}t} \\ &+ c_{i+1|i} \begin{bmatrix} k_{1|i}^{i+1} \\ \vdots \\ k_{i-1|i}^{i+1} \\ k_{i+1|i}^{i+1} \\ \vdots \\ k_{n|i}^{i+1} \end{bmatrix} e^{\omega_{i+1|i}t} + \dots + c_{n|i} \begin{bmatrix} k_{1|i}^n \\ \vdots \\ k_{i-1|i}^n \\ k_{i+1|i}^n \\ \vdots \\ k_{n|i}^n \end{bmatrix} e^{\omega_{n|i}t} \\ &+ c_1 \begin{bmatrix} k_1^1 \\ \vdots \\ k_{i-1}^1 \\ k_{i+1}^1 \\ \vdots \\ k_n^1 \end{bmatrix} e^{\omega_1 t} + \dots + c_i \begin{bmatrix} k_1^i \\ \vdots \\ k_{i-1}^i \\ k_{i+1}^i \\ \vdots \\ k_n^i \end{bmatrix} e^{\omega_i t} + \dots + c_n \begin{bmatrix} k_1^n \\ \vdots \\ k_{i-1}^n \\ k_{i+1}^n \\ \vdots \\ k_n^n \end{bmatrix} e^{\omega_n t} \end{aligned}$$

5.1 Example

With the specifications from Corollary 1.1 and Corollary 2.1, I illustrate the results with an example of two activities, $n = 2$. Suppose that an individual has under consideration to work out and watch TV over a period of two hours. Let experienced utilities be $u_1(t)$ from working out and $u_2(t)$ from watching TV. Also, suppose that $\beta_1(t) = \beta_2(t) = 1$ and that at $t_0 = 0$ the initial condition is $\mathbf{u}(0) = (30, 10)$. Then based on Corollary 1.1, from the eigenvalues and eigenvectors,¹⁹ we obtain the experienced utility functions:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 20 + 10e^{2t} \\ 20 - 10e^{2t} \end{bmatrix}$$

Also, suppose that over the two hour period, the individual chooses to watch TV during the first hour and to work out during the second hour.

If my experienced utility extends decision utility, because $u_1(t) > u_2(t)$ for all $t \in [0, 2]$, based on the assumption of rationality and utility maximization as the decision rule, the individual would choose to work out for two hours and not watch any TV at all. We can calculate the hedonic experience that the individual gets from each activity: $\int_0^2 (20 + 10e^{2t}) dt$ from working out and $\int_0^2 (20 - 10e^{2t}) dt$ from watching TV. Graphically, these hedonic experiences are shown in the following figure, where the red area represents experience from working out and the blue area from watching TV:

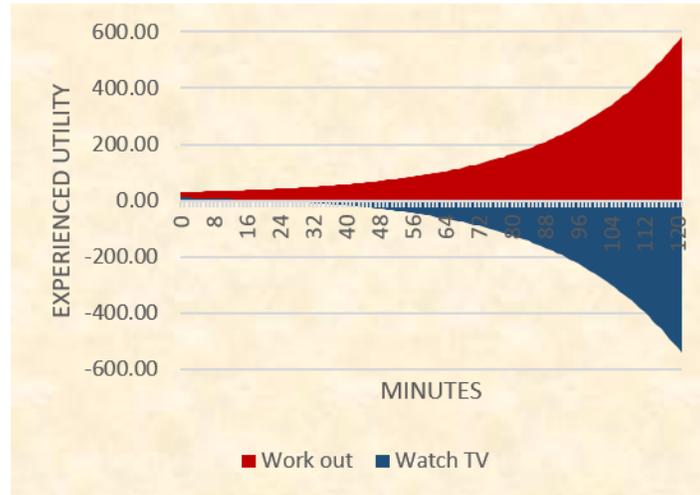


Figure 3: Hedonic experience from two activities

¹⁹The calculations are not shown but are provided upon request.

However, my theory does not assume rationality and as the example states, the individual chooses to allocate one hour to each activity. So the question is whether my theory of experienced utility is capable of explaining the observed time allocation. The present answer is that it has the potential to explain both time allocation and the sequence of activities. In order to explore this potential, we need to look at the dynamics of experienced utilities as pull-and-push forces in conjunction with the role of non-experienced utilities.

Since $n = 2$, for each experienced utility there is one non-experienced utility. So based on Corollary 2.1, non-experienced utilities are:

$$\begin{aligned} u_{1|2}(t) &= c_{1|2}e^t + 20 + 10e^{2t} \\ u_{2|1}(t) &= c_{2|1}e^t + 20 - 10e^{2t} \end{aligned} ,$$

where $u_{1|2}(t)$ is non-experienced utility from non-working out given that the individual is spending time on watching TV and $u_{2|1}(t)$ is non-experienced utility from non-watching TV given that the individual is working out. If for each activity the switch to the other activity is at one hour, we can actually compute the constants. So the non-experienced utility functions are:

$$\begin{aligned} u_{1|2} &= (-20e) e^t + 20 + 10e^{2t} \\ u_{2|1} &= (20e) e^t + 20 - 10e^{2t} \end{aligned}$$

With switch-time at one hour, based on my theory, we have two candidate solutions that contain time allocation (A) and sequence of activities (S):

$$\begin{aligned} S_1 &= \{work\ out; watch\ TV\}, & A_1 &= \{1, 1\} \\ S_2 &= \{watch\ TV; work\ out\}, & A_2 &= \{1, 1\} \end{aligned} ,$$

Note that because my theory uses experienced utility as a function of time, it might be possible to analyze both time allocation and sequence of activities. On the other hand, it is not possible to make conclusions about the sequence when using decision utility. For example, our earlier decision utility function $DU_1 = x_1^{1/2} x_2^{1/2}$ can explain that the individual allocates one hour to each activity but cannot figure out the sequence of activities. Graphically, the hedonic experiences from these sequences are shown in the next figure:

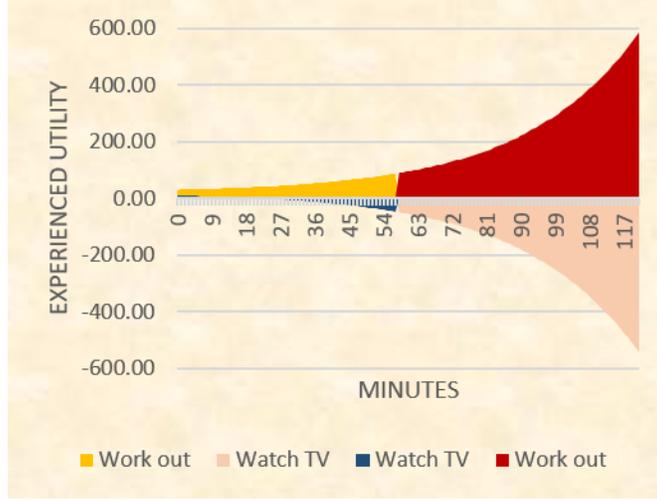


Figure 4: Hedonic experience from sequences of activities

If my experienced utility is independent of decision utility, it serves as a descriptive theory and either solution can explain time allocation. In this case, because the individual is not required to be rational and hence does not necessarily maximize experienced utility, any of the two sequences is plausible.

If decision utility extends my experienced utility, it serves as a normative theory. In this case, because the individual is required to be rational and to maximize experienced utility, only the second sequence is optimal. With the experienced utility functions, we can calculate the hedonic experience as total utility (TU) from time allocation and sequence of activities:

$$\begin{aligned}
 TU_1 &= \int_0^1 (20 + 10e^{2t}) dt + \int_1^2 (20 - 10e^{2t}) dt = 35 + 10e^2 - 5e^4 \approx -164 \\
 TU_2 &= \int_0^1 (20 - 10e^{2t}) dt + \int_1^2 (20 + 10e^{2t}) dt = 45 - 10e^2 + 5e^4 \approx 244
 \end{aligned}$$

Since $TU_2 > TU_1$, from a normative perspective, the individual chooses:

$$S_2 = \{watch\ TV; work\ out\}, \quad A_2 = \{1, 1\}$$

I have chosen this example to illustrate that for a time allocation and a sequence of activities, my theory can be used as a normative theory. With the assumption of experienced utility maximization, my theory has the potential to explain both time allocation and sequence of activities and in this case it would prescribe individual behavior. If the individual does not maximize experienced utility, then my theory can still be used as a descriptive theory. With

the use of switch-time, my theory would continue to explain time allocation and in this case it would only describe individual behavior.

Meanwhile, decision utility can only explain time allocation when used as a normative theory, with the assumption that the individual maximizes decision utility as a result of rationality. A normative theory prescribes individual behavior. If the individual does not maximize decision utility, decision utility is not capable to explain behavior. The potential advantage of experienced utility is due to the fact that it is obtained from a descriptive theory with weaker assumptions than those required for decision utility.

At the beginning of the paper I provided a rationale on why decision utility is not capable to represent hedonic experience even when the alternatives are time allocations. The essence was that although decision utility is capable to explain outcomes, it cannot explain how they are formed. The example in this section illustrates what decision utility misses and what experienced utility has the potential to capture. Based on this example, the outcomes are the time allocations to activities and they may be formed by different sequences of activities. Decision utility represents the ordering of outcomes but misses the sequences whereas experienced utility represents the hedonic experience and has the potential to explain both the outcomes and sequences.

More broadly, this example illustrates the potential benefit when using a theory that relies on weaker assumptions. As Becker (1962) noted, rationality implies consistent maximization of a (decision) utility function. Without rationality my theory is descriptive and has the potential to explain time allocation and with rationality it is normative and has the potential to explain both time allocation and sequence of activities. So far I have been careful to use the word ‘potential’ as often as needed because how this can be achieved is a current area of research. In the next section I conclude with some remarks.

6 Concluding remarks

I have presented a theory of experienced utility with three assumptions: I) Finite number of activities; II) Choice of a single activity; III) The rate of change of experienced utility is positively proportional to the difference between experienced utility from each activity and sum of experienced utilities from the other activities. Jevons wished to form a conception of the quantity of feeling. Hopefully my theory would have satisfied such wish.

Since the concept of utility in neoclassical economic theory is the predominant tool used in economic analysis, as Kahneman has done in numerous papers, I use the term decision utility for what is usually called utility to dis-

tinguish it from experienced utility in my theory. I present my theory through a discussion of conceptual differences between decision utility and experienced utility. I define experienced utility as a differentiable function of time that represents an individual's experience from a single activity.

Jevons viewed the quantity of feeling by a curve that measures its intensity at every moment in time and in my theory this curve is the experienced utility from an activity. Also, Jevons' quantity of feeling over a period of time is measured in my theory by the integral of experienced utility over a given period and this integral measures the so-called hedonic experience.

For the discussion of this paper, I define decision utility as a function that represents an individual's preference relation over mutually exclusive alternatives. This is essentially the same definition found in Mas-Collel et al. (1995), albeit in both theoretical and applied work decision utility usually is not defined in this way and it is specified in such complicated forms that may make my definition of decision utility look quite simplistic.

However, in an effort to maintain the minimal assumptions needed for the existence of decision utility, I have used the simplest form of its definition. So the conclusions of this paper on the conceptual differences between decision utility and experienced utility are reached without the additional assumptions and conditions needed for the existence of decision utility specifications used today. In my discussion, I also answer two questions:

1. Does decision utility have experience in it?
2. Does experienced utility have a decision or does it lead to a decision?

The answer to the first question is that decision utility does not have experience in it. The answer to the second question is that experienced utility does not have a decision in it either but it could lead to a decision. The way in which experience utility could help to explain choice is an area of my current research whose results, in my opinion, are best presented through a discussion of what I call applicable differences between the two concepts of utility because of the use of each concept to analyze individual behavior.

I have identified three types of conceptual differences between decision utility, which I call primal, existential and functional. By primal differences, I mean the differences in terms of their primitives. The primitive in decision utility theory is an individual's preference relation and the primitive in my theory is an individual's experienced utility. I view experienced utility as a more basic primitive than the preference relation.

By existential differences, I mean the differences in terms of the assumptions that are required for the existence of a function for each concept of utility.

In my theory, experienced utilities act like simultaneous pull-and-push forces. Each experienced utility from an activity is a force that pulls an individual to engage in this activity and the other experienced utilities are the forces that push the individual to engage in the other activities. In my view, the assumptions required for decision utility are stronger than the assumptions required for my experienced utility. In particular, my assumptions do not require rationality or include a preference relation.

Rationality is necessary for the existence of decision utility because if preferences are not rational, there is no decision utility function that can represent these preferences. Since rationality is one of the assumptions that has been repeatedly questioned, it is useful to have a theory that does not rely on rationality. Another important part of my theory is that the coefficients of proportionality are functions of time and so they are not restricted to be constant, although for a sufficiently short period of time they might be.

By functional differences, I mean the differences in terms of functional forms for each concept of utility. Decision utility has no specific form whereas the three experienced utility assumptions guarantee the existence of a unique family of experienced utility functions. These functions are: expressed explicitly, real valued and linearly independent. For a sufficiently short period of time, assuming constant coefficients of proportionality, experienced utilities can be expressed in a simpler form.

The lack of uniqueness for decision utility creates difficulties for its different specifications to represent the underlying preference relation and research continues to find that decision utility functions are not capable to represent preferences. Also, for cases that include discounting and uncertainty, decision utility requires additional assumptions whereas my experienced utility includes both discounting and uncertainty. Furthermore, although rationality is not required to derive the family of experienced utility functions, these functions satisfy rationality and for an initial condition the axiom of choice. Research studies that use artificial agents show that rationality is not attainable.

Kahneman et al. (1997) were the first ones to prove the existence of experienced utility but their theory is different from my theory. Kahneman et al.'s theory is a normative theory that relies on and extends decision utility whereas my theory is a descriptive theory that is independent of and in my view more basic than decision utility. Their theory includes much stronger assumptions than decision utility, for example, it does not allow for discounting or uncertainty. Meanwhile, I believe that my theory has weaker assumptions than decision utility theory and it includes both discounting and uncertainty. Furthermore, the axioms in Kahneman et al. (1997) might lack veracity.

Another concept related to experienced utility in my theory is the concept

of non-experienced utilities, which are experienced utilities from the activities in which an individual is not engaging while spending time on an activity. These non-experienced utilities are conditioned on the current activity and continue to have influence for as long as the individual spends time on the chosen activity. The same three experienced utility assumptions guarantee the existence of non-experienced utilities. They are: expressed explicitly, real valued and linearly dependent. Also, for a sufficiently short period of time, assuming constant coefficients of proportionality, non-experienced utilities can be expressed in a simpler form.

Using an example of two activities, I have illustrated that my theory of experienced utility has the potential to explain an individual's decision on both time allocation to activities and sequence of these activities. Since this would be done through the effect of non-experienced utilities, my current research aims to find how non-experienced utilities determine the switch-time.

Once the switch-time is determined, there are two ways in which my theory could be used. When used as a normative theory, it could explain both time allocation and sequence of activities. When used as descriptive theory, it could explain only time allocation. Because my theory is descriptive, the conclusion on time allocation would not require the assumption of rationality. With the knowledge on time allocation, the next step would be to study an individual's economic behavior through the individual's dual role as a consumer and a producer during a given time allocation.

I would like to end with an example including three metaphors, the machine by Professor Jay Coggins, rivers by Professor C. Ford Runge and cascades by myself. Suppose that we want to study a given number of rivers that originate from the same source and form cascades as they flow. Let's assume that their rates of flow are proportional to the difference between the force of their own flows and sum of forces of the other flows.

A current tool available called decision utility allows us to measure the volume from every river by standing at the bottom of each cascade where we can see the water coming down the cascade but not the water flowing. Imagine now if we had a machine called the family of experienced utilities that allows us to measure the volume from every river by standing at the top of each cascade where we can see both the water coming down the cascade and the water flowing. I personally prefer the machine.

Appendices

A Proof of Theorem 1

The system of differential equations is:

$$\begin{aligned}
 du_1/dt &= \beta_1(t) (u_1(t) - u_2(t) - \cdots - u_n(t)) \\
 du_2/dt &= \beta_2(t) (-u_1(t) + u_2(t) - \cdots - u_n(t)), \\
 &\dots \\
 du_n/dt &= \beta_n(t) (-u_1(t) - u_2(t) - \cdots + u_n(t))
 \end{aligned}$$

$$\dot{\mathbf{u}}(t) = \mathcal{B}(t)\mathbf{u}(t) \tag{A.1}$$

Let $\mathcal{B}_d(t)$ denote the diagonal matrix of coefficients of proportionality

$$\beta_1(t), \beta_2(t), \dots, \beta_n(t)$$

and E the circulant matrix consisting of 1's for its main diagonal entries and -1 's for the remaining entries. Then we can write:

$$\begin{bmatrix} \beta_1(t) & -\beta_1(t) & \cdots & -\beta_1(t) \\ -\beta_2(t) & \beta_2(t) & \cdots & -\beta_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_n(t) & -\beta_n(t) & \cdots & \beta_n(t) \end{bmatrix} =$$

$$\begin{bmatrix} \beta_1(t) & 0 & \cdots & 0 \\ 0 & \beta_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n(t) \end{bmatrix} \begin{bmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 \end{bmatrix} \quad \text{or}$$

$$\mathcal{B}(t) = \mathcal{B}_d(t)E$$

If $\mathbf{k}^i = (k_1^i, k_2^i, \dots, k_n^i)$ is an eigenvector of E and λ_i the respective eigenvalue, $i = 1, 2, \dots, n$, because E is a real symmetric matrix, it has a complete set K of orthogonal eigenvectors

$$K = [\mathbf{k}^1 \quad \mathbf{k}^2 \quad \cdots \quad \mathbf{k}^n],$$

Since K has orthogonal eigenvectors, it has a unique inverse K^{-1} . Then I combine K , K^{-1} and $\lambda_1, \lambda_2, \dots, \lambda_n$ with $\mathcal{B}_d(t)$ to derive the useful result that $\mathcal{B}(t)$ has the same eigenvectors as E and find the eigenvalues of $\mathcal{B}(t)$. This

result is crucial to express experienced utilities with simple functional forms. The following lemma shows why \mathbf{k}^i , $i = 1, 2, \dots, n$, is an eigenvector of $\mathcal{B}(t)$ and how to find its associated eigenvalue $\omega_i(t)$ for every t on an open interval.

Lemma 1. $\mathcal{B}(t)$ has a complete set of eigenvectors $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^n$ and their associated eigenvalues are given by the following linear transformation:

$$\mathbf{w}(t) = K^{-1}\mathcal{B}_d(t)K\mathbf{d}, \quad (\text{A.2})$$

where $\mathbf{d} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mathbf{w}(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))$. Also, for every i , there is a one-to-one correspondence between $\mathcal{B}_d(t)\lambda_i\mathbf{k}^i$ and $\omega_i(t)\mathbf{k}^i$.

Proof. If $\mathcal{B}(t)$ has a complete set of eigenvectors $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^n$, for every t , there exist n associated eigenvalues $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$ that are functions of t . But $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^n$ are also the eigenvectors of E with associated eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then all of the following relations must hold:

$$\begin{array}{ll} \mathcal{B}(t)\mathbf{k}^1 \equiv \omega_1(t)\mathbf{k}^1 & E\mathbf{k}^1 = \lambda_1\mathbf{k}^1 \\ \mathcal{B}(t)\mathbf{k}^2 \equiv \omega_2(t)\mathbf{k}^2 & E\mathbf{k}^2 = \lambda_2\mathbf{k}^2 \\ \dots & \dots \\ \mathcal{B}(t)\mathbf{k}^n \equiv \omega_n(t)\mathbf{k}^n & E\mathbf{k}^n = \lambda_n\mathbf{k}^n, \end{array} \quad (\text{A.3})$$

where \equiv means equivalence so that the relations on the left side hold for all t and hence they are identities. Summing over the relations on the left side and using $\mathcal{B}(t) = \mathcal{B}_d(t)E$ and the relations on the right side, we have:

$$\begin{aligned} \mathcal{B}(t)\mathbf{k}^1 + \mathcal{B}(t)\mathbf{k}^2 + \dots + \mathcal{B}(t)\mathbf{k}^n &\equiv \omega_1(t)\mathbf{k}^1 + \omega_2(t)\mathbf{k}^2 + \dots + \omega_n(t)\mathbf{k}^n \\ \mathcal{B}_d(t)E\mathbf{k}^1 + \mathcal{B}_d(t)E\mathbf{k}^2 + \dots + \mathcal{B}_d(t)E\mathbf{k}^n &\equiv \omega_1(t)\mathbf{k}^1 + \omega_2(t)\mathbf{k}^2 + \dots + \omega_n(t)\mathbf{k}^n \\ \mathcal{B}_d(t)\lambda_1\mathbf{k}^1 + \mathcal{B}_d(t)\lambda_2\mathbf{k}^2 + \dots + \mathcal{B}_d(t)\lambda_n\mathbf{k}^n &\equiv \omega_1(t)\mathbf{k}^1 + \omega_2(t)\mathbf{k}^2 + \dots + \omega_n(t)\mathbf{k}^n \\ \mathcal{B}_d(t)(\lambda_1\mathbf{k}^1 + \lambda_2\mathbf{k}^2 + \dots + \lambda_n\mathbf{k}^n) &\equiv \omega_1(t)\mathbf{k}^1 + \omega_2(t)\mathbf{k}^2 + \dots + \omega_n(t)\mathbf{k}^n \\ \mathcal{B}_d(t)K\mathbf{d} &\equiv K\mathbf{w}(t), \end{aligned} \quad (\text{A.4})$$

where $\mathbf{d} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mathbf{w}(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))$.

Given that $\mathcal{B}_d(t)$ is a diagonal matrix with nonzero elements in its main diagonal for all t , it has a unique inverse $\mathcal{B}_d^{-1}(t)$. So for each t , we have a system of linear equations in the unknowns $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$:

$$K^{-1}\mathcal{B}_d^{-1}(t)K\mathbf{w}(t) = \mathbf{d}$$

But K and K^{-1} contain n linearly independent vectors and for every i , $\beta_i(t) > 0$. So the determinant (\det) of the matrix of this system is different from 0 because

$$\det K^{-1}\mathcal{B}_d^{-1}(t)K = \det K^{-1} \det \mathcal{B}_d^{-1} \det K \neq 0$$

for all t . So for each t , this system has exactly one and only one solution. Hence $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$ are uniquely determined as in (A.2):

$$\mathbf{w}(t) = K^{-1}\mathcal{B}_d(t)K\mathbf{d}$$

The derivation of this solution is based on identities and the relations hold for all t . Hence the relations on the left side of (A.3) are satisfied and $\mathcal{B}(t)$ has eigenvectors $\mathbf{k}^1, \mathbf{k}^2, \dots, \mathbf{k}^n$ with associated eigenvalues $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$.

Also the identities in system (A.4) mean that the relations hold for all t . Then for every i , $\mathcal{B}_d(t)\lambda_i\mathbf{k}^i = \omega_i(t)\mathbf{k}^i$. Given that this relation is uniquely determined, there is a one-to-one correspondence between $\mathcal{B}_d(t)\lambda_i\mathbf{k}^i$ and $\omega_i(t)\mathbf{k}^i$.

The eigenvalues can be written as diagonal matrices:

$$\Omega(t) = \begin{bmatrix} \omega_1(t) & 0 & \cdots & 0 \\ 0 & \omega_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n(t) \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Using these matrices and the fact that $KK^{-1} = \mathcal{B}_d(t)\mathcal{B}_d^{-1}(t) = I$, where I is the identity matrix, the results of this lemma are as follows:

$$\mathcal{B}(t)K = \mathcal{B}_d(t)EK = \mathcal{B}_d(t)K\Lambda = \mathcal{B}_d(t)KK^{-1}\mathcal{B}_d^{-1}(t)K\Omega(t) = K\Omega(t)$$

This completes the proof. □

The following proof of Theorem 1 utilizes the (constant) eigenvectors of matrix $\mathcal{B}(t)$ and the indefinite integrals of their associated eigenvalues:

$$\tilde{\omega}_i(t) = \int \omega_i(t)dt, \quad i = 1, 2, \dots, n$$

These indefinite integrals can be written more compactly in matrix format:

$$\tilde{\Omega}(t) = \begin{bmatrix} \tilde{\omega}_1(t) & 0 & \cdots & 0 \\ 0 & \tilde{\omega}_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\omega}_n(t) \end{bmatrix}$$

Theorem 1 (Family of Experienced Utilities). *Given positive coefficients of proportionality on an open interval, the rate of change of experienced utility from each activity is proportional to the difference between experienced utility from the activity and sum of experienced utilities from the other activities. Then there exists a unique family of experienced utilities which are:*

(i) *expressed explicitly:*

$$\mathbf{u}(t) = Ke^{\tilde{\Omega}(t)}\mathbf{c}, \quad (\text{A.5})$$

where e is the exponential function and \mathbf{c} is a nonzero vector,

(ii) *real valued and*

(iii) *linearly independent.*

Also, given initial conditions, experienced utilities are cardinal utilities.

Proof.

(i) System (A.1) has non-constant coefficients and its solution would likely be given by approximations that require numerical or infinite series methods. However, using Lemma 1, it is possible to express its general solution explicitly.

To prove this part, I look for a fundamental set of solutions containing the vector functions: $\mathbf{v}^i(t) = \mathbf{k}^i e^{\tilde{\omega}_i(t)}$, $i = 1, 2, \dots, n$, where $\mathbf{k}^i = (k_1^i, k_2^i, \dots, k_n^i)$. Substituting $\dot{\mathbf{u}}(t) = \dot{\mathbf{v}}^i(t)$ and $\mathbf{u}(t) = \mathbf{v}^i(t)$ in system (A.1), we obtain:

$$\begin{aligned} \omega_i(t)k_1^i e^{\tilde{\omega}_i(t)} &= \beta_1(t) (k_1^i - k_2^i - \dots - k_n^i) e^{\tilde{\omega}_i(t)} \\ \omega_i(t)k_2^i e^{\tilde{\omega}_i(t)} &= \beta_2(t) (-k_1^i + k_2^i - \dots - k_n^i) e^{\tilde{\omega}_i(t)} \\ &\dots \\ \omega_i(t)k_n^i e^{\tilde{\omega}_i(t)} &= \beta_n(t) (-k_1^i - k_2^i - \dots + k_n^i) e^{\tilde{\omega}_i(t)}, \\ \omega_i(t)\mathbf{k}^i &= \mathcal{B}(t)\mathbf{k}^i \end{aligned}$$

From Lemma 1, this is an identity. Then every function $\mathbf{v}^i(t)$ is a solution to system (A.1). It remains to show that these solutions constitute a fundamental set of solutions to system (A.1). The matrix $V(t) = [\mathbf{v}^1 \quad \mathbf{v}^2 \quad \dots \quad \mathbf{v}^n]$ contains all of the solutions of the form $\mathbf{k}^i e^{\tilde{\omega}_i(t)}$, $i = 1, 2, \dots, n$. Then we can write:

$$V(t) = Ke^{\tilde{\Omega}(t)}$$

But K contains n linearly independent vectors and for every i , $e^{\tilde{\omega}_i(t)} > 0$. So the determinant of $V(t)$ is different from 0 because

$$\det V(t) = \det Ke^{\tilde{\Omega}(t)} = \det K \det e^{\tilde{\Omega}(t)} \neq 0$$

for all t . This means that the vector functions $\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^n$ are n linearly independent solutions to (A.1) and so they form a fundamental set of solutions. Given a nonzero vector of constants $\mathbf{c} = (c_1, c_2, \dots, c_n)$, the general solution to system (A.1) is given by equation (A.5):

$$\mathbf{u}(t) = V(t)\mathbf{c} = c_1\mathbf{v}^1(t) + c_2\mathbf{v}^2(t) + \dots + c_n\mathbf{v}^n(t) = Ke^{\tilde{\Omega}(t)}\mathbf{c}$$

Because this is the general solution, functions $\mathbf{u}(t) = (u_1(t), u_2(t), \dots, u_n(t))$ constitute a unique family of functions that represent experienced utilities. The specifications of experienced utilities are:

$$\begin{aligned} u_1(t) &= c_1k_1^1e^{\tilde{\omega}_1(t)} + c_2k_1^2e^{\tilde{\omega}_2(t)} + \dots + c_nk_1^ne^{\tilde{\omega}_n(t)} \\ u_2(t) &= c_1k_2^1e^{\tilde{\omega}_1(t)} + c_2k_2^2e^{\tilde{\omega}_2(t)} + \dots + c_nk_2^ne^{\tilde{\omega}_n(t)}, \\ &\dots \\ u_n(t) &= c_1k_n^1e^{\tilde{\omega}_1(t)} + c_2k_n^2e^{\tilde{\omega}_2(t)} + \dots + c_nk_n^ne^{\tilde{\omega}_n(t)} \end{aligned}$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = c_1 \begin{bmatrix} k_1^1 \\ k_2^1 \\ \vdots \\ k_n^1 \end{bmatrix} e^{\tilde{\omega}_1(t)} + c_2 \begin{bmatrix} k_1^2 \\ k_2^2 \\ \vdots \\ k_n^2 \end{bmatrix} e^{\tilde{\omega}_2(t)} + \dots + c_n \begin{bmatrix} k_1^n \\ k_2^n \\ \vdots \\ k_n^n \end{bmatrix} e^{\tilde{\omega}_n(t)} \quad (\text{A.6})$$

This completes the part on the existence of a unique family of experienced utility functions and part (i) of the theorem.

(ii) The eigenvectors and eigenvalues of $\mathcal{B}(t)$ are real valued. So all quantities in (A.5) are real and experienced utility functions are real valued. This completes part (ii) of the theorem.

(iii) Next, the linear independence of experienced utility functions is proved by way of contradiction.

Assume that experienced utility functions are linearly dependent and that $u_1(t)$ is a linear combination of the other experienced utility functions. So there exist constants $\gamma_2, \dots, \gamma_n$ such that for all t :

$$u_1(t) = \gamma_2u_2(t) + \dots + \gamma_nu_n(t)$$

and

$$\dot{u}_1(t) = \gamma_2\dot{u}_2(t) + \dots + \gamma_n\dot{u}_n(t)$$

If we let

$$\Gamma = \begin{bmatrix} 0 & \gamma_2 & \cdots & \gamma_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

then for all t we should have $\mathbf{u}(t) \equiv \Gamma \mathbf{u}(t)$ and $\dot{\mathbf{u}} \equiv \Gamma \dot{\mathbf{u}}$. Substituting $\Gamma \mathbf{u}(t)$ for $\mathbf{u}(t)$ in system (A.1), $\dot{\mathbf{u}}(t) = \mathcal{B}(t)\mathbf{u}(t)$, we have:

$$\dot{\mathbf{u}}(t) = \mathcal{B}(t)\Gamma \mathbf{u}(t)$$

Also substituting $\mathcal{B}(t)\mathbf{u}(t)$ for $\dot{\mathbf{u}}(t)$ in $\dot{\mathbf{u}} \equiv \Gamma \dot{\mathbf{u}}$, we have:

$$\dot{\mathbf{u}}(t) = \Gamma \mathcal{B}(t)\mathbf{u}(t)$$

Denoting $L_1 = \mathcal{B}(t)\Gamma$ and $L_2 = \Gamma \mathcal{B}(t)$ the above linear transformations, given that $\mathcal{B}(t)\Gamma \mathbf{u}(t) = \Gamma \mathcal{B}(t)\mathbf{u}(t)$ for every \mathbf{u} , then $L_1 = L_2$:

$$\begin{bmatrix} 0 & \gamma_2\beta_1(t) - \beta_1(t) & \cdots & \gamma_n\beta_1(t) - \beta_1(t) \\ 0 & -\gamma_2\beta_2(t) + \beta_2(t) & \cdots & -\gamma_n\beta_2(t) - \beta_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\gamma_2\beta_n(t) - \beta_n(t) & \cdots & -\gamma_2\beta_n(t) + \beta_n(t) \end{bmatrix} = \begin{bmatrix} -\sum_{i=2}^n \gamma_i\beta_i(t) & \gamma_2\beta_2(t) - \sum_{i=3}^n \gamma_i\beta_i(t) & \cdots & \sum_{i=2}^{n-1} \gamma_i\beta_i(t) + \gamma_n\beta_n(t) \\ -\beta_2(t) & \beta_2(t) & \cdots & -\beta_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_n(t) & -\beta_n(t) & \cdots & \beta_n(t) \end{bmatrix}$$

So $\beta_2(t) = \cdots = \beta_n(t) = 0$, a contradiction. Therefore experienced utility functions are linearly independent, which completes part (iii) of the theorem.

Given initial conditions, the nonzero vector \mathbf{c} is uniquely determined so that experienced utilities are also uniquely determined. Then experienced utility functions are cardinal utilities. This completes the proof. \square

A.1 Alternative Proof of Theorem 1

Theorem 1 (Family of Experienced Utilities). *Given positive coefficients of proportionality on an open interval, the rate of change of experienced utility from each activity is proportional to the difference between experienced utility from the activity and sum of experienced utilities from the other activities. Then there exists a unique family of experienced utilities which are:*

(i) expressed explicitly,

(ii) real valued and

(iii) linearly independent.

Also, given initial conditions, experienced utilities are cardinal utilities.

Proof. For any t , let $V(t) = e^{\int_{t_0}^t \mathcal{B}(s)ds}$, where e is for exponential function. Then:

$$V'(t) = \mathcal{B}(t)e^{\int_{t_0}^t \mathcal{B}(s)ds} = \mathcal{B}(t)V(t)$$

Hence the column vectors of:

$$V(t) = [\mathbf{v}^1(t) \quad \mathbf{v}^2(t) \quad \cdots \quad \mathbf{v}^n(t)]$$

are solutions to (A.1). But for $t = t_0$, $e^{\int_{t_0}^t \mathcal{B}(s)ds} = e^O$, where O is the zero matrix. Then $\det V(t_0) = 1 \neq 0$ for at least one t . Hence vectors $\mathbf{v}^1(t), \mathbf{v}^2(t), \dots, \mathbf{v}^n(t)$ are n linearly independent solutions to (A.1) and they form a fundamental set of solutions. So the general solution to system (A.1) consists of the vector functions:

$$\mathbf{u}(t) = V(t)\mathbf{c} = c_1\mathbf{v}^1(t) + c_2\mathbf{v}^2(t) + \cdots + c_n\mathbf{v}^n(t), \quad (\text{A.7})$$

where $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is a nonzero vector of constants. Given that this is the general solution, the functions $u_1(t), u_2(t), \dots, u_n(t)$ constitute a unique family of functions that represent experienced utilities. This completes the part of the theorem on the existence of a unique family of experienced utilities.

(i) If $\tilde{\mathcal{B}}(t)$ and $\tilde{B}_d(t)$ denote the integrals of $\mathcal{B}(t)$ and $B_d(t)$ respectively, then:

$$\tilde{\mathcal{B}}(t) = \int_{t_0}^t \mathcal{B}(s)ds = \int_{t_0}^t B_d(s)E ds = \int_{t_0}^t B_d(s)ds E = \tilde{B}_d(t)E$$

Because $\tilde{B}_d(t)$ and E are real symmetric matrices, each of them has a complete set of orthogonal real eigenvectors. $\tilde{B}_d(t)$ is also positive definite with a unique $[\tilde{B}_d(t)]^{1/2}$ and a unique $[\tilde{B}_d(t)]^{-1/2}$. Using these matrices, we find that:

$$[\tilde{B}_d(t)]^{-1/2}\tilde{\mathcal{B}}(t)[\tilde{B}_d(t)]^{1/2} = [\tilde{B}_d(t)]^{-1/2}\tilde{B}_d(t)E[\tilde{B}_d(t)]^{1/2} = [\tilde{B}_d(t)]^{1/2}E[\tilde{B}_d(t)]^{1/2}$$

Then $\tilde{\mathcal{B}}(t)$ is similar to the real and symmetric matrix $[\tilde{B}_d(t)]^{1/2}E[\tilde{B}_d(t)]^{1/2}$. So $\tilde{\mathcal{B}}(t)$ is diagonalizable and has a complete set of real eigenvectors as well

as real eigenvalues. Let $\tilde{K}(t)$ be a matrix with n orthogonal eigenvectors of $\tilde{\mathcal{B}}(t)$ as columns and $\tilde{D}(t)$ a diagonal matrix with associated eigenvalues in its main diagonal. Because $\tilde{K}(t)$ has a (unique) inverse, $[\tilde{K}(t)]^{-1}$, we can write:

$$\tilde{\mathcal{B}}(t) = \tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1}$$

With this notation, solution (A.7) can be written as:

$$\mathbf{u}(t) = e^{\int_{t_0}^t \mathcal{B}(s)ds} \mathbf{c} = e^{\tilde{\mathcal{B}}(t)} \mathbf{c} = e^{\tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1}} \mathbf{c}$$

Using the definition of the exponential matrix function, we can write:

$$e^{\tilde{\mathcal{B}}(t)} = I + \sum_{k=1}^{\infty} \frac{1}{k!} [\tilde{\mathcal{B}}(t)]^k,$$

where I is the identity matrix. Also, using the earlier result for $\tilde{\mathcal{B}}(t)$, we can express $[\tilde{\mathcal{B}}(t)]^k$ in terms of its eigenvectors and eigenvalues:

$$\begin{aligned} [\tilde{\mathcal{B}}(t)]^k &= [\tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1}]^k = \\ &= \left(\tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1} \right) \left(\tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1} \right) \cdots \left(\tilde{K}(t)\tilde{D}(t)[\tilde{K}(t)]^{-1} \right) = \\ &= \tilde{K}(t)\tilde{D}(t) \left([\tilde{K}(t)]^{-1}\tilde{K}(t) \right) \tilde{D}(t) \left([\tilde{K}(t)]^{-1}\tilde{K}(t) \right) \tilde{D}(t) \cdots \tilde{D}(t)[\tilde{K}(t)]^{-1} \\ &= \tilde{K}(t)[\tilde{D}(t)]^k[\tilde{K}(t)]^{-1} \end{aligned}$$

for every k . Then the exponential matrix function can be written as:

$$\begin{aligned} e^{\tilde{\mathcal{B}}(t)} &= \tilde{K}(t)I[\tilde{K}(t)]^{-1} + \sum_{k=1}^{\infty} \frac{1}{k!} [\tilde{K}(t)[\tilde{D}(t)]^k[\tilde{K}(t)]^{-1}] \\ &= \tilde{K}(t) \left(I + \sum_{k=1}^{\infty} \frac{1}{k!} [\tilde{D}(t)]^k \right) [\tilde{K}(t)]^{-1} \\ &= \tilde{K}(t)e^{\tilde{D}(t)}[\tilde{K}(t)]^{-1} \end{aligned}$$

Let $\tilde{\lambda}_1(t), \tilde{\lambda}_2(t), \dots, \tilde{\lambda}_n(t)$ denote the eigenvalues of $\tilde{\mathcal{B}}(t)$. Given that

$$e^{\tilde{D}(t)} = \text{diag} \left(e^{\tilde{\lambda}_1(t)}, e^{\tilde{\lambda}_2(t)}, \dots, e^{\tilde{\lambda}_n(t)} \right),$$

where ‘diag’ stands for diagonal matrix, the general solution (A.7) can be expressed explicitly in terms of the eigenvectors and eigenvalues of the integral of the matrix of coefficient of proportionality functions. The solution is:

$$\begin{aligned} \mathbf{u}(t) &= \tilde{K}(t)e^{\tilde{D}(t)}[\tilde{K}(t)]^{-1}\mathbf{c} = \\ &\tilde{K}(t)\text{diag}\left(e^{\tilde{\lambda}_1(t)}, e^{\tilde{\lambda}_2(t)}, \dots, e^{\tilde{\lambda}_n(t)}\right)[\tilde{K}(t)]^{-1}\mathbf{c} \end{aligned} \quad (\text{A.8})$$

This completes part (i) of the theorem.

(ii) The eigenvectors and eigenvalues of $\tilde{\mathcal{B}}(t)$ are real-valued. Then all quantities in (A.8) are real and this completes part (ii) of the theorem.

Part (iii) on the linear independence of experienced utilities and the part on cardinal utilities are proved as in Appendix A and this completes the proof. \square

B Proof of Theorem 2

The system of differential equations is:

$$\begin{aligned} \dot{u}_{1|i} &= \beta_1(t) (u_{1|i}(t) \cdots - u_{i-1|i}(t) - u_{i+1|i}(t) - \cdots - u_{n|i}(t)) - \beta_1(t)u_i(t) \\ &\quad \dots \\ \dot{u}_{i-1|i} &= \beta_{i-1}(t) (-u_{1|i}(t) \cdots + u_{i-1|i}(t) - u_{i+1|i}(t) \cdots - u_{n|i}(t)) - \beta_{i-1}(t)u_i(t) \\ \dot{u}_{i+1|i} &= \beta_{i+1}(t) (-u_{1|i}(t) \cdots - u_{i-1|i}(t) + u_{i+1|i}(t) \cdots - u_{n|i}(t)) - \beta_{i+1}(t)u_i(t) \\ &\quad \dots \\ \dot{u}_{n|i} &= \beta_n(t) (-u_{1|i}(t) \cdots - u_{i-1|i}(t) - u_{i+1|i}(t) \cdots + u_{n|i}(t)) - \beta_n(t)u_i(t) \\ \dot{\mathbf{u}}_i(t) &= \mathcal{B}_i(t)\mathbf{u}_i(t) + \mathbf{b}_i(t) \end{aligned} \quad (\text{B.1})$$

Similar to the proof of Theorem 1 in Appendix A, the following proof of Theorem 2 utilizes the constant eigenvectors of matrix $\mathcal{B}_i(t)$. These eigenvectors are denoted by $\mathbf{k}_{|i}^j$, $j = 1, \dots, i-1, i+1, \dots, n$, with

$$\mathbf{k}_{|i}^j = (k_{1|i}^j, \dots, k_{i-1|i}^j, k_{i+1|i}^j, k_{n|i}^j).$$

The indefinite integrals of their associated eigenvalues are:

$$\tilde{\omega}_{j|i}(t) = \int \omega_{j|i}(t)dt, \quad j = 1, \dots, i-1, i+1, \dots, n$$

These indefinite integrals can be written more compactly in matrix format:

$$\tilde{\Omega}_{|i}(t) = \begin{bmatrix} \tilde{\omega}_{1|i}(t) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\omega}_{(i-1)|i}(t) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & +\tilde{\omega}_{(i+1)|i}(t) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \tilde{\omega}_{n|i}(t) \end{bmatrix}$$

Theorem 2 (Non-experienced Utilities). *Given positive coefficients of proportionality on an open interval, the rate of change of non-experienced utility from each non-chosen activity is proportional to the difference between non-experienced utility from the activity and sum of non-experienced utilities from the other non-chosen activities as well as experienced utility from the chosen activity. Then there exist unique non-experienced utilities which are:*

(i) *expressed explicitly:*

$$\begin{aligned} \mathbf{u}_{|i}(t) &= K_{|i} \text{diag} (e^{\tilde{\omega}_{1|i}(t)}, \dots, e^{\tilde{\omega}_{i-1|i}(t)}, e^{\tilde{\omega}_{i+1|i}(t)}, \dots, e^{\tilde{\omega}_{n|i}(t)}) \mathbf{c}_{|i} \\ &+ K_{-i} \text{diag} (e^{\tilde{\omega}_1(t)}, \dots, e^{\tilde{\omega}_{i-1}(t)}, e^{\tilde{\omega}_i(t)}, e^{\tilde{\omega}_{i+1}(t)}, \dots, e^{\tilde{\omega}_n(t)}) \mathbf{c}, \end{aligned} \quad (\text{B.2})$$

where ‘diag’ stands for diagonal matrix and $\mathbf{c}_{|i}$ is a vector of constants,

(ii) *real valued and*

(iii) *linearly independent.*

Proof.

(i) System (B.1) is non-homogeneous and similarly with system (A.1) it also has non-constant coefficients and its solution would likely be given by approximations that require numerical or infinite series methods. However, using Theorem 1, it is possible to express its general solution explicitly.

For the homogeneous part of system (B.1), I use the fundamental set of solutions as in Theorem 1. So the vector functions $\mathbf{v}_{|i}^j(t) = \mathbf{k}_{|i}^j e^{\tilde{\omega}_{j|i}(t)}$, $j = 1, \dots, i-1, i+1, \dots, n$, are solutions to the homogeneous part:

$$\dot{\mathbf{u}}_{|i}(t) = \mathcal{B}_{|i}(t) \mathbf{u}_{|i}(t)$$

of system (B.1). From Theorem 1, they constitute a fundamental set of solutions. The matrix $V_{|i}(t) = \begin{bmatrix} \mathbf{v}_{|i}^1 & \cdots & \mathbf{v}_{|i}^{i-1} & \mathbf{v}_{|i}^{i+1} & \cdots & \mathbf{v}_{|i}^n \end{bmatrix}$ contains these solutions. With $K_{|i} = \begin{bmatrix} \mathbf{k}_{|i}^1 & \cdots & \mathbf{k}_{|i}^{i-1} & \mathbf{k}_{|i}^{i+1} & \cdots & \mathbf{k}_{|i}^n \end{bmatrix}$, we can write:

$$V_{|i}(t) = K_{|i} e^{\tilde{\Omega}_{|i}(t)}$$

Here $K_{|i}$ contains $n - 1$ linearly independent vectors and for every j , $j \neq i$, $e^{\tilde{\omega}_{j|i}(t)} > 0$. So the determinant of $V_{|i}(t)$ is different from 0 because

$$\det V_{|i}(t) = \det K_{|i} e^{\tilde{\Omega}_{|i}(t)} = \det K_{|i} \det e^{\tilde{\Omega}_{|i}(t)} \neq 0$$

for all t . This means that the vector functions $\mathbf{v}_{|i}^1, \dots, \mathbf{v}_{|i}^{i-1}, \mathbf{v}_{|i}^{i+1}, \dots, \mathbf{v}_{|i}^n$ are $n - 1$ linearly independent solutions to: $\dot{\mathbf{u}}_{|i}(t) = \mathcal{B}_{|i}(t)\mathbf{u}_{|i}(t)$. So they form a fundamental set of solutions for the homogeneous part of system (B.1).

If $\mathbf{c}_{|i} = (c_{1|i}, \dots, c_{i-1|i}, c_{i+1|i}, \dots, c_{n|i})$ is a vector of constants and $\mathbf{u}_p(t)$ is a particular solution to system (B.1), then its general solution is:

$$\begin{aligned} \mathbf{u}_{|i}(t) &= V_{|i}(t)\mathbf{c}_{|i} + \mathbf{u}_p(t) = \\ &= \mathbf{v}_{|i}^1 c_{1|i} + \dots + \mathbf{v}_{|i}^{i-1} c_{i-1|i} + \mathbf{v}_{|i}^{i+1} c_{i+1|i} + \dots + \mathbf{v}_{|i}^n c_{n|i} + \mathbf{u}_p(t) = \\ &= K_{|i} \text{diag} (e^{\tilde{\omega}_{1|i}(t)}, \dots, e^{\tilde{\omega}_{i-1|i}(t)}, e^{\tilde{\omega}_{i+1|i}(t)}, \dots, e^{\tilde{\omega}_{n|i}(t)}) \mathbf{c}_{|i} + \mathbf{u}_p(t) \end{aligned}$$

A particular solution $\mathbf{u}_p(t)$ is found by inspection of solutions to systems (A.1) and (B.1). Suppose $j < i - 1$. Then the rate of change for experienced utility $u_j(t)$ from activity j in (A.1) is:

$$\begin{aligned} du_j/dt &= \\ &= \beta_j(t) (-u_1(t) - \dots + u_j(t) - \dots - u_{i-1}(t) - u_i(t) - u_{i+1}(t) - \dots - u_n(t)) \end{aligned}$$

The rate of change for non-experienced utility $u_{j|i}$ from activity j in (B.1) is:

$$\begin{aligned} \dot{u}_{j|i} &= \\ &= \beta_j(t) (-u_{1|i}(t) - \dots + u_{j|i}(t) - \dots - u_{i-1|i}(t) - u_{i+1|i}(t) - \dots - u_{n|i}(t)) \\ &\quad - \beta_j(t) u_i(t) = \\ &= \beta_j(t) (-u_{1|i}(t) - \dots + u_{j|i}(t) - \dots - u_{i-1|i}(t) - u_i(t) - u_{i+1|i}(t) - \dots - u_{n|i}(t)) \end{aligned}$$

The above expressions for du_j/dt and $\dot{u}_{j|i}$ hold for all $j \neq i$. If we look at row $j \neq i$ of solution (A.6) to system (A.1), by inspection of the rate of change of experienced utility function $u_j(t)$ we can see that it is the same for du_j/dt and $\dot{u}_{j|i}$. So we have the important result that for every $j \neq i$:

$$du_j/dt = \dot{u}_{j|i}(t).$$

Then a particular solution $\mathbf{u}_p(t)$ to system (B.1) is obtained from solution (A.6) for all experienced utilities except for $u_i(t)$:

$$\mathbf{u}_p(t) = c_1 \begin{bmatrix} k_1^1 \\ \vdots \\ k_{i-1}^1 \\ k_{i+1}^1 \\ \vdots \\ k_n^1 \end{bmatrix} e^{\tilde{\omega}_1(t)} + \dots + c_i \begin{bmatrix} k_1^i \\ \vdots \\ k_{i-1}^i \\ k_{i+1}^i \\ \vdots \\ k_n^i \end{bmatrix} e^{\tilde{\omega}_i(t)} + \dots + c_n \begin{bmatrix} k_1^n \\ \vdots \\ k_{i-1}^n \\ k_{i+1}^n \\ \vdots \\ k_n^n \end{bmatrix} e^{\tilde{\omega}_n(t)}$$

If $\mathbf{k}_{-i}^j = (k_1^j, \dots, k_{i-1}^j, k_{i+1}^j, \dots, k_n^j)$, $j = 1, \dots, i-1, i, i+1, \dots, n$, including $j = i$, then the values of K excluding the values in row i denoted by K_{-i} are:

$$K_{-i} = [\mathbf{k}_{-i}^1 \quad \dots \quad \mathbf{k}_{-i}^{i-1} \quad \mathbf{k}_{-i}^i \quad \mathbf{k}_{-i}^{i+1} \quad \dots \quad \mathbf{k}^n]$$

With this notation, the general solution to system (B.1) is expressed explicitly:

$$\begin{aligned} \mathbf{u}_{|i}(t) &= K_{|i} \text{diag} (e^{\tilde{\omega}_{1|i}(t)}, \dots, e^{\tilde{\omega}_{i-1|i}(t)}, e^{\tilde{\omega}_{i+1|i}(t)}, \dots, e^{\tilde{\omega}_{n|i}(t)}) \mathbf{c}_{|i} \\ &+ K_{-i} \text{diag} (e^{\tilde{\omega}_1(t)}, \dots, e^{\tilde{\omega}_{i-1}(t)}, e^{\tilde{\omega}_i(t)}, e^{\tilde{\omega}_{i+1}(t)}, \dots, e^{\tilde{\omega}_n(t)}) \mathbf{c} \end{aligned}$$

Because this is the general solution, these are unique functions of non-experienced utilities. This completes uniqueness and part (i) of the theorem.

(ii) All quantities in (B.2) are real and so non-experienced utility functions are real valued. This completes part (ii) of the theorem.

(iii) Next, the linear independence of non-experienced utility functions is proved by way of contradiction. Suppose that non-experienced utilities are linearly dependent. Then there exist constants $\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n$ not all zero such that for all t :

$$\gamma_1 u_{1|i}(t) + \dots + \gamma_{i-1} u_{i-1|i}(t) + \gamma_{i+1} u_{i+1|i}(t) + \dots + \gamma_n u_{n|i}(t) = 0 \quad (\text{B.3})$$

If $\mathbf{u}^h(t) = V_{|i}(t) \mathbf{c}_{|i}$, then the vector functions:

$$\mathbf{u}^h(t) = (u_1^h(t), \dots, u_{i-1}^h(t), u_{i+1}^h(t), \dots, u_n^h(t))$$

are solutions the homogeneous part of system (B.1): $\dot{\mathbf{u}}_{|i}(t) = \mathcal{B}_{|i}(t) \mathbf{u}_{|i}(t)$, where ‘ h ’ indicates that the solution is for the homogeneous part. From Theorem 1, these are the experienced utilities when there are $n - 1$ activities and so they

are linearly independent. Also the particular solution to system (B.1) includes the vector functions:

$$\mathbf{u}_p(t) = (u_1(t), \dots, u_{i-1}(t), u_{i+1}(t), \dots, u_n(t))$$

which are the $n - 1$ experienced utilities except for experienced utility u_i from solution (A.2). From Theorem 1, these are $n - 1$ experienced utilities when there are n activities and so they also are linearly independent. Using this notation, the general solution to system (B.1) can be written as:

$$\mathbf{u}_i(t) = \mathbf{u}^h(t) + \mathbf{u}_p(t)$$

The vector functions $u_j^h(t)$ in $\mathbf{u}^h(t)$, $j = 1, \dots, i - 1, i + 1, \dots, n$, span the solution space of the homogeneous part of system (B.1) when there are $n - 1$ activities. The vector functions $u_j(t)$ in $\mathbf{u}_p(t)$, $j = 1, \dots, i - 1, i + 1, \dots, n$, span the solution subspace of dimension $n - 1$ of system (A.1) when there are n activities. So there is no function in $\mathbf{u}^h(t)$ which can be a linear combination of functions in $\mathbf{u}_p(t)$ and vice versa.

Then equation (B.3) holds if and only if both of the following hold:

$$\begin{aligned} \gamma_1 u_1^h(t) + \dots + \gamma_{i-1} u_{i-1}^h(t) + \gamma_{i+1} u_{i+1}^h(t) + \dots + \gamma_n u_n^h(t) &= 0 \\ \gamma_1 u_1(t) + \dots + \gamma_{i-1} u_{i-1}(t) + \gamma_{i+1} u_{i+1}(t) + \dots + \gamma_n u_n(t) &= 0 \end{aligned}$$

These imply that functions $u_1^h(t), \dots, u_{i-1}^h(t), u_{i+1}^h(t), \dots, u_n^h(t)$ are linearly dependent and also functions $u_1(t), \dots, u_{i-1}(t), u_{i+1}(t), \dots, u_n(t)$ are linearly dependent, a contradiction. So non-experienced utilities are linearly independent. This completes part (iii) of the theorem and the proof. \square

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