The converse envelope theorem

Ludvig Sinander
Northwestern University

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Envelope theorem: optimal decision-making $\Longrightarrow$ $\boxtimes$ formula.

Textbook intuition: $\boxtimes$ formula is consequence of FOC.

Modern envelope theorem of MS02:[1] almost no assumptions.

$\hookrightarrow$ FOC ill-defined, so need different intuition.

My theorem: with almost no assumptions,
$\boxtimes$ formula equivalent to generalised FOC.

- an envelope theorem: FOC $\Longrightarrow$ $\boxtimes$
- *a converse:* $\boxtimes$ $\Longrightarrow$ FOC.

Application to mechanism design.

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[1] Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, *70*(2), 583–601. doi:10.1111/1468-0262.00296

Environment

Agent chooses action x from a set $\mathcal{X}$
to maximise objective $f(x,t)$, where $t \in [0,1]$ is a parameter.

No assumptions on $\mathcal{X}$, almost none on f:

(1) $f(x,\cdot)$ is differentiable for each $x \in \mathcal{X}$

(2) $f(x,\cdot)$ is 'not too erratic'. (definition: slide 12)

Decision rule: a map $X : [0,1] \to \mathcal{X}$.

Associated *value function:* $V_X(t) \coloneqq f(X(t),t)$.

Envelope theorem

X satisfies the $\boxtimes$ formula iff

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0,1].$$

Equivalently: V_X is absolutely continuous and

$$V_X'(t) = f_2(X(t), t) \quad \text{for a.e. } t \in [0,1].$$

X is *optimal* iff for every t, $X(t)$ maximises $f(\cdot, t)$.

Modern envelope theorem (MS02).
Any optimal decision rule satisfies the $\boxtimes$ formula.

Textbook intuition

Differentiation identity:

$$V_X'(t) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} f(X(t+m),t)\Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{f_2(X(t),t)}_{\text{'direct effect'}}.$$

$$\begin{aligned} V_X'(t) &= \text{direct effect} && (\boxtimes \text{ formula}) \\ \iff \quad \text{indirect effect} &= 0 && (\text{FOC}). \end{aligned}$$

Problem: 'indirect effect' (hence FOC) ill-defined!

- $f(\cdot,t)$ & X need not be differentiable.
- actions $\mathcal{X}$ need have no convex or topological structure.

The outer first-order condition

Disjuncture: in general, $\boxtimes$ formula $\not\Leftrightarrow$ FOC.

- one solution: add strong ‘classical’ assumptions. (slide 13)
- my solution: find the correct FOC!

Decision rule X *satisfies the outer FOC* iff

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0,1).$$

‘Integrated’ version of classical FOC.

- always well-defined
- equiv’nt to classical FOC when latter well-defined. (slide 13)

Theorem

Envelope theorem & converse.

For a decision rule $X : [0,1] \to \mathcal{X}$, the following are equivalent:

(1) X satisfies the oFOC

$$\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s \bigg|_{m=0} = 0 \quad \text{for all } r, t \in (0,1),$$

and $V_X(t) \coloneqq f(X(t), t)$ is absolutely continuous.

(2) X satisfies the ⌧ formula

$$V_X(t) = V_X(0) + \int_0^t f_2(X(s), s)\mathrm{d}s \quad \text{for every } t \in [0,1].$$

(proof idea: slide 14)

Mechanism design application: environment

Agent with preferences $f(y, p, t)$ over physical outcome $y \in \mathcal{Y}$ and payment $p \in \mathbf{R}$.

- type $t \in [0, 1]$ is agent's private info
- assume single-crossing.

What's new:

- outcome space $\mathcal{Y}$ is an abstract partially ordered set
- preferences not assumed quasi-linear in payment.

A *physical allocation* is $Y : [0, 1] \to \mathcal{Y}$.

Y is *implementable* iff $\exists$ payment rule $P : [0, 1] \to \mathbf{R}$ s.t. (Y, P) is incentive-compatible.

$$\Big(\text{viz.} \quad f(Y(t), P(t), t) \geq f(Y(r), P(r), t) \quad \text{for all } r, t.\Big)$$

Mechanism design application: theorem

Implementability theorem.
Under weak regularity assumptions,
any increasing physical allocation is implementable.

Argument:

- fix an increasing physical allocation $Y : [0,1] \to \mathcal{Y}$
- choose a payment rule P so that $\boxtimes$ holds
- then by *converse envelope theorem*, oFOC holds $\iff$ mechanism (Y,P) is locally IC.
- finally, local IC $\implies$ global IC by single-crossing.

Mechanism design application: example

Monopolist selling information.

Physical allocations $\mathcal{Y}$:
distributions of posterior beliefs, ordered by Blackwell.

By the implementability theorem, any information allocation that gives higher types Blackwell-better signals can be implemented.

Thanks!

Definition of 'not too erratic'

A family $\{\phi_x\}_{x\in\mathcal{X}}$ of functions $[0,1]\to\mathbf{R}$ is *uniformly absolutely continuous (UAC)* iff the family

$$\left\{ t \mapsto \sup_{x\in\mathcal{X}} \left| \frac{\phi_x(t+m)-\phi_x(t)}{m} \right| \right\}_{m>0}$$

is uniformly integrable.

'$f(x,\cdot)$ not too erratic' (slide 3)
means precisely that $\{f(x,\cdot)\}_{x\in\mathcal{X}}$ is UAC.

– a sufficient condition (maintained by MS02):
  $t \mapsto \sup_{x\in\mathcal{X}}|f_2(x,t)|$ dominated by an integrable function.

– a stronger sufficient condition: f_2 bounded.

↪ back to environment (slide 3)

Classical assumptions

Classical assumptions:

– $\mathcal{X}$ is a convex subset of $\mathbf{R}^n$
– action derivative f_1 exists & is bounded
– only Lipschitz continuous decision rules X are considered.

(Bad for applications. Especially the Lipschitz restriction!)

Classical FOC: $$\frac{\mathrm{d}}{\mathrm{d}m} f(X(t+m),t)\Big|_{m=0} = 0 \quad \text{for a.e. } t.$$

Classical envelope theorem and converse.
Under the classical assump'ns, classical FOC $\iff$ ⌧ formula.

Housekeeping lemma.
oFOC $\iff$ classical FOC whenever the latter is well-defined.

↪ back to oFOC (slide 6)

Proof idea

Textbook intuition was based on differentiation identity:

$$V_X'(s) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} f(X(s+m), s)\Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{f_2(X(s), s)}_{\text{'direct effect'}},$$

or (integrating)

$$V_X(t) - V_X(r) = \int_r^t \frac{\mathrm{d}}{\mathrm{d}m} f(X(s+m), s)\Big|_{m=0} \mathrm{d}s + \int_r^t f_2(X(s), s)\mathrm{d}s.$$

I prove that the 'outer' version is always valid:

$$V_X(t) - V_X(r) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}m} \int_r^t f(X(s+m), s)\mathrm{d}s\Big|_{m=0}}_{\text{'indirect effect'}} + \underbrace{\int_r^t f_2(X(s), s)\mathrm{d}s}_{\text{'direct effect'}}.$$

The rest is easy:

$$\begin{aligned} & V_X(t) - V_X(r) = \text{direct effect} && (\boxtimes \text{ formula}) \\ \iff \quad & \text{indirect effect} = 0 && (\text{oFOC}). \end{aligned}$$

↪ back to theorem (slide 7)