SUPPLEMENTAL APPENDIX FOR "INFORMATION SPILLOVER IN MULTI-GOOD ADVERSE SELECTION" (FOR ONLINE PUBLICATION ONLY)

BINGCHAO HUANGFU⋆ AND HENG LIU†

In this supplemental appendix, we first state and prove five technical lemmas (Lemmas S.1–S.5), which are stated in the Appendix of the paper and used to prove the results in the main paper. Then we prove the results in Section 6 of the main paper.

TECHNICAL LEMMAS

Let μ_t^i be the probability of high quality for good $i = 1,2$ in period t. Let $\mu_t^i(h)$ be the probability of high quality for the remaining good $i = 1,2$ in period t if the seller's action is $h \in \{ra, ar, rr\}$ in period $t-1$. Let μ_τ^t be the probability of the seller type $\tau \in \{HH, HL, LH, LL\}$ in period t.

For any $k \in \{rr, ar, ra, aa\}$, let p_k be the probability that type LL seller chooses k in period one. Let p_{HL} be the probability that type HL seller chooses rr. Let p_{LH} be the probability that type LH seller chooses rr. The following expressions describe the updated beliefs in period two when trade happens with positive probability in period one.

$$\frac{\mu_{HL}p_{HL} + \mu_{HH}}{\mu_{HL}p_{HL} + \mu_{LH}p_{LH} + p_{rr}\mu_{LL} + \mu_{HH}} = \frac{\mu_{LH}p_{LH} + \mu_{HH}}{\mu_{LH}p_{LH} + \mu_{HL}p_{HL} + p_{rr}\mu_{LL} + \mu_{HH}} = \mu^*. \tag{1}$$

.

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL}) + p_{ra}\mu_{LL}} = \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH}) + p_{ar}\mu_{LL}} = \mu^*. \tag{2}$$

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL}) + p_{ra}\mu_{LL}} \geq \mu^*, \ \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH}) + p_{ar}\mu_{LL}} \geq \mu^*. \tag{3}$$

The following five lemmas are originally stated in the Appendix of the main paper (pages 22–23). Here we list them for completeness.

Lemma S.1. (i) If $\mu^* \leq \frac{1}{2}$, there are solutions to (1) and (2). (ii) If $\mu^* > \frac{1}{2}$, then there are solutions to (1) and (2) if and only if $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$. (iii) If $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$, there is a solution to (1) and (3), in which $p_{rr} = 0$.

⋆INSTITUTE FOR SOCIAL AND ECONOMIC RESEARCH, NANJING AUDIT UNIVERSITY

†DEPARTMENT OF ECONOMICS, UNIVERSITY OF MICHIGAN

E-mail addresses: hfbcalan@gmail.com, hengliu29@gmail.com.

Date: December 17, 2020.

Lemma S.2. Under refinement D1, suppose $\mu^* > \frac{1}{2}$ and Assumption 3 holds, if the belief in period $t = 1$ satisfies Assumption 1 or the belief in period $t \geq 2$ satisfies $(\mu_t^1, \mu_2^2) \in \mathcal{B}$, then $\mu_{t+1}^1(rr) \leq \mu^*$ and $\mu_{t+1}^2(rr) \leq \mu^*$.

Lemma S.3. Under refinement D1, suppose $\mu^* \leq \frac{1}{2}$ and $\delta > \bar{\delta}$, if the initial belief in period $t = 1$ satisfies Assumption 1 or the belief in period $t \geq 2$ satisfies $(\mu_t^1, \mu_2^2) \in \mathcal{B}$, then $\mu_{t+1}^1(rr) \leq \mu^*$ and $\mu_{t+1}^2(rr) \leq \mu^*$.

Lemma S.4. Suppose $\delta > \frac{v_L - c_L}{c_H - c_L}$, if there are three types: HH, HL and LH in period $t \geq 2$ and the belief satisfies $(\mu_t^1, \mu_t^2) \in \mathcal{B}$ in period $t \geq 2$, then the equilibrium continuation payoff of LH and HL in period t is $v_L - c_L + \delta(v_H - c_H)$.

Lemma S.5. Suppose Assumptions 1-3 hold. Let $n + 1$ be the first period in which LL does not choose rr. If $n \geq 1$, then we have the following:

(1) $n < +\infty$;
(2) if $n \geq 2$, all types of the seller choose rr in period t, where $2 \leq t \leq n$;
(3) $(\mu_{t+1}^1(rr), \mu_{t+1}^2(rr)) \in \mathcal{B}$ for $1 \leq t \leq n$;
(4) under refinement D1, $\mu_{t+1}^1(ra) = \mu_{t+1}^2(ar) = \mu^*$ for $1 \leq t \leq n$;
(5) LL does not choose aa in period 1.

Proof of Lemma S.1

Proof. **Step 1:** There are solutions to (1) and (2) if $\mu^* \leq \frac{1}{2}$.

We start with the case that $\mu^* < \frac{1}{2}$. Since $\mu^* < \frac{1}{2}$, $\mu^* \geq \mu_{HH} + \mu_{HL}$ and $\mu^* \geq \mu_{HH} + \mu_{LH}$, then $1 > 2\mu^* \geq 2\mu_{HH} + \mu_{HL} + \mu_{LH}$. Therefore, $\mu_{HH} < \mu_{LL}$. Note that (2) implies that

$$p_{ra} = \frac{\mu_{HL}}{\mu_{LL}} \frac{1 - \mu^*}{\mu^*} (1 - p_{HL}). \tag{4}$$

$$p_{ar} = \frac{\mu_{LH}}{\mu_{LL}} \frac{1 - \mu^*}{\mu^*} (1 - p_{LH}). \tag{5}$$

In addition, (1) implies that

$$p_{HL} = \frac{\mu^*}{1 - 2\mu^*} \frac{\mu_{LL}}{\mu_{HL}} p_{rr} - \frac{1 - \mu^*}{1 - 2\mu^*} \frac{\mu_{HH}}{\mu_{HL}}. \tag{6}$$

$$p_{LH} = \frac{\mu^*}{1 - 2\mu^*} \frac{\mu_{LL}}{\mu_{LH}} p_{rr} - \frac{1 - \mu^*}{1 - 2\mu^*} \frac{\mu_{HH}}{\mu_{HL}}. \tag{7}$$

Finally, (4), (5), (6) and (7) imply that

$$p_{ra} = \frac{\mu_{HL}}{\mu_{LL}} \frac{1 - \mu^*}{\mu^*} - \frac{1 - \mu^*}{1 - 2\mu^*} p_{rr} + \frac{(1 - \mu^*)^2}{\mu^*(1 - 2\mu^*)} \frac{\mu_{HH}}{\mu_{LL}}. \tag{8}$$

$$p_{ar} = \frac{\mu_{LH}}{\mu_{LL}} \frac{1 - \mu^*}{\mu^*} - \frac{1 - \mu^*}{1 - 2\mu^*} p_{rr} + \frac{(1 - \mu^*)^2}{\mu^*(1 - 2\mu^*)} \frac{\mu_{HH}}{\mu_{LL}}. \tag{9}$$

As $p_{ar} \geq 0$, $p_{ra} \geq 0$, $p_{HL} \geq 0$ and $p_{LH} \geq 0$, then (8), (9), (6) and (7) imply that

$$\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

$$\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above two equations have a solution p_{rr} since $\mu^* < \frac{1}{2}$.

As $p_{ar} + p_{ra} + p_{rr} \leq 1$, then (8), (9) imply that

$$p_{rr} \geq (\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}}.$$

We need to find $p_{rr} \in [0,1]$ to satisfy all above three inequalities. We first check that there exists p_{rr} to satisfy all above three inequalities. It is equivalent to show that

$$\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1 < \frac{\min\{\mu_{HL}, \mu_{LH}\}}{\mu_{LL}\mu^*} - \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above inequality holds if $\mu^* \geq \mu_{LH} + \mu_{HH}$ and $\mu^* \geq \mu_{HL} + \mu_{HH}$, which are true. Next, we prove that $p_{rr} \in [0,1]$, which is equivalent to show that the lower bound of p_{rr} is less than 1: $(\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}} \leq 1$ and $\frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} < 1$. The first inequality is equivalent to $(1-\mu^*)(\mu^* - \frac{\mu_{HL}+\mu_{LH}}{2} - \mu_{HH}) > 0$, thus $\mu^* \geq \frac{\mu_{HL}+\mu_{LH}}{2} + \mu_{HH}$, which is true. The second inequality is equivalent to $\mu^* > \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}}$. By the fact that $\mu_{HH} < \mu_{LL}$, we have $\mu_{HH} + \frac{\mu_{HL}+\mu_{LH}}{2} - \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}} = \frac{\mu_{HL}+\mu_{LH}}{2}\frac{\mu_{LL}-\mu_{HH}}{\mu_{HH}+\mu_{LL}} > 0$. Therefore, $\mu^* > \mu_{HH} + \frac{\mu_{HL}+\mu_{LH}}{2} > \frac{\mu_{HH}}{\mu_{HH}+\mu_{LL}}$.

Next, we prove that there exist solutions to (1) and (2) if $\mu^* = \frac{1}{2}$. Since $\mu^* = \frac{1}{2}$ implies that $\mu^* > \mu_{HH} + \mu_{HL}$ and $\mu^* > \mu_{HH} + \mu_{LH}$, then $1 = 2\mu^* > 2\mu_{HH} + \mu_{HL} + \mu_{LH}$, and thus $\mu_{HH} < \mu_{LL}$.

By calculation, $p_{rr} = \frac{\mu_{HH}}{\mu_{LL}} < 1$, $p_{ra} = \frac{\mu_{HL}}{\mu_{LL}}(1-p_{HL})$, $p_{ar} = \frac{\mu_{LH}}{\mu_{LL}}(1-p_{LH})$, $\mu_{HL}p_{HL} = \mu_{LH}p_{LH}$. In order to satisfy $p_{ra} + p_{ar} + p_{rr} \leq 1$, we need $\frac{1}{2} \leq \mu_{LL} + \mu_{HL}p_{HL}$. Assume without loss of generality that $\mu_{HL} \leq \mu_{LH}$, then let $p_{HL} = 1$ and $p_{LH} = \frac{\mu_{HL}}{\mu_{LH}}$. Then, we only need to show that $\frac{1}{2} \leq \mu_{LL} + \mu_{HL}$, which holds since $\frac{1}{2} = \mu^* > \mu_{HH} + \mu_{LH}$. To summarize, we construct a solution: $p_{rr} = \frac{\mu_{HH}}{\mu_{LL}}$, $p_{ra} = 0$, $p_{ar} = \frac{\mu_{LH}-\mu_{HL}}{\mu_{LL}}$, $p_{HL} = 1$ and $p_{LH} = \frac{\mu_{HL}}{\mu_{LH}}$.

Step 2: If $\mu^* > \frac{1}{2}$, then there exist solutions to (1) and (2) if and only if $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$.

Since $p_{ar} \geq 0$, $p_{ra} \geq 0$, $p_{HL} \geq 0$ and $p_{LH} \geq 0$, then (6), (7), (8) and (9) implies that

$$\frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

$$\frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq p_{rr} \leq \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

As $\mu^* > \frac{1}{2}$, the above two equations make sense. By $p_{ar} + p_{ra} + p_{rr} \leq 1$, (8) and (9) implies that

$$p_{rr} \leq (\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}}$$

We need to find $p_{rr} \in [0,1]$ to satisfy all above three inequalities. First, there is p_{rr} to satisfy the above three inequalities. It is equivalent to show that

$$\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1 < \frac{\min\{\mu_{HL},\mu_{LH}\}}{\mu_{LL}}\frac{1}{\mu^*} - \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}.$$

The above two equations hold if $\mu^* \geq \mu_{LH} + \mu_{HH}$ and $\mu^* \geq \mu_{HL} + \mu_{HH}$, which are true.

Next, there exists $p_{rr} \in [0,1]$. We first prove that the lower bound of p_{rr} is less than 1: $\frac{\mu_{HL}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} \leq 1$ and $\frac{\mu_{LH}}{\mu_{LL}}\frac{1-2\mu^*}{\mu^*} + \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}} < 1$. It is equivalent to $\mu^* > \frac{\mu_{HH}+\mu_{HL}}{\mu_{LL}+\mu_{HH}+2\mu_{HL}}$ and $\mu^* > \frac{\mu_{HH}+\mu_{LH}}{\mu_{LL}+\mu_{HH}+2\mu_{LH}}$. Assume without loss of generality that $\mu_{HL} \geq \mu_{LH}$. Then, $\mu_{LL} + \mu_{HH} + 2\mu_{HL} > 1$, so $\mu^* \geq \mu_{HH} + \mu_{HL} > \frac{\mu_{HH}+\mu_{HL}}{\mu_{LL}+\mu_{HH}+2\mu_{HL}} \geq \frac{\mu_{HH}+\mu_{LH}}{\mu_{LL}+\mu_{HH}+2\mu_{LH}}$. The second inequality is implied by $\mu_{LL} > \mu_{HH}$ and $\mu_{LH} \leq \mu_{HL}$. Next, we show that the upper bound of p_{rr} is not less than 0: $(\frac{1-\mu^*}{\mu^*}\frac{1-\mu_{LL}-\mu_{HH}}{\mu_{LL}} - 1)(1-2\mu^*) + \frac{(1-\mu^*)^2}{\mu^*}\frac{2\mu_{HH}}{\mu_{LL}} \geq 0$, which is equivalent to $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} \geq 1$. To summarize, there exists $p_{rr} \in [0,1]$.

Since $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$ implies $p_{rr} < 0$, there is no solution to (1) and (2).

Step 3: There exist solutions to (1) and (3), if $\mu^* > \frac{1}{2}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

Let $p_{rr} = 0$. Given $p_{rr} = 0$ and (1),

$$p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}}, \; p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}}.$$

From the above equations, (3) and $p_{ar} + p_{ra} = 1$, we can show that $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

Note that $p_{HL} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{HL}} < 1$ and $p_{LH} = \frac{1-\mu^*}{2\mu^*-1}\frac{\mu_{HH}}{\mu_{LH}} < 1$, which is implied by $\mu^* > \mu_{HL} + \mu_{HH}$, $\mu^* > \mu_{LH} + \mu_{HH}$ and $\frac{\mu_{HH}}{2\mu^*-1} + \frac{\mu_{LL}}{1-\mu^*} < 1$.

□

PROOF OF LEMMA S.2

Proof. We argue by contradiction that $\mu^1_{t+1}(rr) \leq \mu^*$. Assume the contrary that $\mu^1_{t+1}(rr) > \mu^*$. Assume that period $t+k$ is the first period such that both goods remain untraded and HH or HL accepts the offer for good 1 with positive probability, where $k \geq 1$. Therefore, if $k \geq 2$, the belief of good 1 is larger than μ^* in period $t+1, \cdots, t+k-1$. In the following steps except step 1, if not specified, the actions of sellers and buyers are taken in period $t+k-1$. The proof is broken into following 6 steps.

Step 1: In period $t+k$, the offer for good 1 is at least c_H (we assume that the actions of the players are taken in period $t+k$ in this step).

Since HH or HL accepts the offer for good 1 with positive probability, by skimming properties in Lemma 3, LH and LL accepts the offer for good 1 for sure. Then, HH and HL choose pure strategy with respect to good 1, since otherwise buyer 1 can increases the offer a little bit to make a profit. There are four cases to consider: (1) HH chooses rr with positive probability and HL accepts the offer for good 1. HH is the only type to choose rr, and LH will deviate to rr, instead of accepting the offer for good 1. (2) HH chooses ra and HL accepts the offer for good 1. HL gets $\delta(v_H - c_H)$ from good 1 by choosing ra. By skimming properties in Lemma 3, HL chooses aa with positive probability in period $t+k$. Therefore, the offer for good 1 in period $t+k$ is at least c_H, since otherwise HL gets negative profit from good 1 by choosing aa, and consequently ra dominates aa for HL, a contradiction. (3) HH chooses aa with positive probability and HL rejects the offer for good 1. By skimming properties in Lemma 3, HL chooses ra with positive probability. Since HH strictly prefers aa to ra, then HL also strictly prefers aa to ra, a contradiction. (4) HH chooses ar and HL rejects the offer for good 1. Since $\mu^1_{t+1}(rr) = 1$, then by choosing rr, HH can guarantee at least $\delta(v_H - c_H)$. Therefore, the offer for good 1 in period $t+k$ is at least c_H, since otherwise HH gets a negative payoff from good 1, so HH gets less than $\delta(v_H - c_H)$ by choosing ar, and consequently rr is a profitable deviation for HH, a contradiction. In all, the offer for good 1 in period $t+k$ is at least c_H.

Step 2: If $k \geq 2$, show that there is a losing offer for good 1 in period $t+k-1$, and both HH and LH choose rr for sure in period $t+k-1$.

First, we prove that $\mu^2_{t+k-1} \leq \mu^*$. Otherwise, by Lemma 4, $\mu^2_{t+k-1} > \mu^*$ and $\mu^1_{t+k-1} > \mu^*$ imply that all seller types choose aa, a contradiction to the definition of $t+k-1$.

Next, we argue that LH rejects the offer for good 1. Assume the contrary that LH accepts the offer for good 1 with positive probability. Therefore, in period $t+k-1$, only the low type seller accepts the offer for good 1 and consequently the offer in period $t+k-1$ is v_L by zero profit condition of buyer 1. By choosing aa, LH gets at most $v_L - c_L$ since $\mu^2_{t+k-1} \leq \mu^*$; by choosing ar, LH gets at most $(v_L - c_L) + \delta(v_H - c_H)$; by choosing rr, LH gets at least $\delta(c_H - c_L)$, by Step 1. $\delta > \frac{v_L - c_L + \delta(v_H - c_H)}{c_H - c_L}$ implies that LH prefers rr to ar and aa, a contradiction to the assumption that LH accepts the offer for good 1 with positive probability.

We next prove that LL rejects the offer for good 1. We have shown that all three types other than LL choose to reject the offer for good 1. If LL chooses to accept the offer for good 1 with positive probability, then this will reveal LL's type and is the worst possible strategy for LL, and thus LL will deviate to rejecting the offer for good 1, a contradiction.

Finally, we prove that HH and LH choose rr for sure. Assume to the contrary that HH or LH chooses ra with positive probability. Skimming properties A and B.2 imply that HL

and LL also choose ra for sure. If only ra is on the equilibrium path in period $t+k-1$, then by the result of one-good model, there is a winning offer for good 1, a contradiction to the definition of k. If rr is the equilibrium path in period $t+k-1$, then $\mu^2_{t+k}(rr)=1$. Moreover, HH and LH can only choose pure strategy: rr or ra, since otherwise it is a profitable deviation for buyer 2 by increasing the offer for good 2 a little bit to attract HH or LH to accept the offer for sure. If HH chooses rr for sure, then LH also chooses rr, and hence, the offer for good 2 in period $t+k-1$ is v_L. Then, HL gets a payoff at most $v_L-c_L+\delta(v_H-c_H)$. It is a profitable deviation for HL to choose rr: $\delta(v_H-c_L)>v_L-c_L+\delta(v_H-c_H)$, a contradiction. If HH chooses ra for sure, then LH also chooses ra to get at least $\delta(c_H-c_L)$, which is higher than the payoff from choosing rr: $\delta(v_L-c_L+\delta(v_H-c_H))$. In all, all four types choose ra for sure, a contradiction to the assumption that rr is on the equilibrium path in period $t+k-1$.

Step 3: If $k\geq 2$, then HL mixes between rr and ra and LL is indifferent between ra and rr in period $t+k-1$.

First, we prove that $\mu^2_{t+k}(rr)\leq\mu^*$. Assume by contradiction that $\mu^2_{t+k}(rr)>\mu^*$. Since $\mu^1_{t+k}>\mu^*$, then Lemma 4 implies that there is a winning offer for good 2 in period $t+k$, which is larger than c_H. Therefore, in period $t+k-1$, by choosing rr, HL gets at least $\delta(c_H-c_L)$; by choosing ra, HL gets at most $(v_L-c_L)+\delta(v_H-c_H)$. Since $\delta(c_H-c_L)>(v_L-c_L)+\delta(v_H-c_H)$, then HL chooses rr for sure in period $t+k-1$. Also, LL chooses rr for sure in period $t+k-1$. Bayes rule implies that $\mu^2_{t+k}(rr)\leq\mu^2_{t+k-1}\leq\mu^*$, a contradiction to $\mu^2_{t+k}(rr)>\mu^*$. A corollary is that in period $t+k$ with two goods, LH and HH get zero profit from good 2.

We next argue by contradiction that HL mixes between rr and ra. Assume the contrary. We consider the following two cases. First, HL chooses ra for sure in period $t+k-1$. It is straightforward to show that LL also chooses ra with positive probability. Therefore, $\mu^2_{t+k}(rr)=1$, and consequently, LL deviates to rr to make a higher profit, a contradiction. Second, HL chooses rr for sure in period $t+k-1$. Then LL also chooses rr for sure, since otherwise by choosing ra, LL would reveals its type and gets a lower payoff than rr. In all, there are two losing offers in period $t+k-1$. Given rr in period $t+k-1$, there is no belief updating: $\mu^1_{t+k}(rr)=\mu^1_{t+k-1}$. Therefore, in period $t+k-1$, buyers of good 1 can deviate to make an offer $V(\mu^1_{t+k-1})-\epsilon$ (small enough ϵ) so that all type would prefer ar to rr, for the following two reasons: (i) for good 1, all types of seller would accept the offer $V(\mu^1_{t+k-1})-\epsilon$ in period $n+k-1$ instead of waiting for one more period and get an offer $V(\mu^1_{t+k-1})$; (ii) $\mu^2_{t+k}(rr)\leq\mu^*$ implies that rr gives LH and HH zero profit for good 2, so ar is a strictly better choice than rr, for LH and HH. However, ar is not a strictly better choice for HL and LL if $\mu^2_{t+k}(ar)=0$. Therefore, refinement D1 implies that $\mu^2_{t+k}(ar) > \mu^2_{t+k}(rr) = \mu^2_{t+k-1}$, and thus all four types strictly prefers ar to rr. In all, buyer 1 in period $t+k-1$ gets a positive profit ϵ by making an offer $V(\mu^1_{t+k-1}) - \epsilon$, a contradiction.

We thus have proved that HL mixes between ra and rr in period $t+k-1$. Then, LL is also indifferent between ra and rr in period $t+k-1$. This is because HL and LL get the same payoff from good 2 by choosing either rr or ra, and for good 1, the payoff difference between choosing rr and ra is $\delta(V(\mu^1_{t+k}(rr)) - V(\mu^1_{t+k}(ra))$, for both HL and LL.

Step 4: If $k \geq 2$, it is not possible that there are three types: HH, HL and LH in period $t+k-1$.

Assume by contradiction that there are three types: HH, HL and LH in period $t+k-1$. By Step 1, HH and LH choose rr for sure. Bayes rule implies that $\mu^1_{t+k}(rr) \leq \mu^1_{t+k-1}$. In period $t+k-1$, buyer 1 is willing to offer $V(\mu^1_{t+k-1}) - \epsilon$ (small enough ϵ) so that HH and LH chooses ar instead of rr, and HL chooses aa instead of ar, because the new choices bring all three types a weakly higher payoff from good 2, and a strictly higher payoff from good 1. This is a profitable deviation for buyer 1, a contradiction to buyer 1's zero profit condition.

Step 5: Show that $k = 1$.

We argue by contradiction that $k \geq 2$. Now, we first show that HL chooses rr for sure in period $t+k-2$. Assume the contrary HL chooses ra with positive probability in period $t+k-2$. Note that we have $\mu^1_{t+k-1}(ra) > \mu^*$, since otherwise LL strictly prefers rr and thus $\mu^1_{t+k-1}(ra) = 1$, a contradiction. By skimming property B.1, LL strictly prefers ra to rr in period $t+k-2$. Therefore, in period $t+k-1$ with two goods, there are at most three types: HH, HL and LH, a contradiction to Step 4. Next, we prove that LH chooses rr for sure in period $t+k-2$. Assume the contrary LH chooses ar with positive probability in period $t+k-2$. If $\mu^2_{t+k-1}(ar) > \mu^*$, then by skimming property B.3, LL strictly prefers ar to rr in period $t+k-2$, which also reaches a contradiction to Step 3. If $\mu^2_{t+k-1}(ar) \leq \mu^*$, then LH gets $v_L - c_L$ by choosing ar in period $t+k-2$; LH gets at least $\delta^2(c_H - c_L)$ by chooses rr in period $t+k-2$. Since $\delta > (\frac{v_L - c_L}{c_H - c_L})^{\frac{1}{2}}$, then LH chooses rr in period $t+k-2$, a contradiction. It follows that all four types choose rr in period $t+k-2$.

An immediate conclusion is that $\mu^1_{t+k-1}(rr) = \mu^1_{t+k-2}$. If $k=2$, then $\mu^1_{t+1}(rr) = \mu^1_t < \mu^*$, a contradiction to $\mu^1_{t+1}(rr) > \mu^*$. If $k \geq 3$, then we will show that buyer 1 has a profitable deviation in period $t+k-2$. In period $t+k-2$, buyer 1 can deviate to make an offer $V(\mu^1_{t+k-2}) - \epsilon$ (small enough ϵ) so that all types chooses ar in period $t+k-2$, for the following two reasons: all four types get higher payoff from good 1 by accepting offer for good 1 immediately in period $t+k-2$; ar brings all four types higher payoff than rr from good 2 in period $t+k-2$. This is because, for HH and LH, ar dominates rr in period $t+k-2$, but not for HL and LL, then refinement D1 implies that $\mu^2_{t+k-1}(ar)=1$. In all, buyer 1 gets a profit $\epsilon>0$ in period $t+k-2$ by making the offer $V(\mu^1_{t+k-2})-\epsilon$, a contradiction to buyer 1's zero profit condition.

Step 6: We reach a contradiction to $\mu^1_{t+1}(rr)>\mu^*$.

By Lemma 5, in period t, HH and HL rejects the offer for good 1, and any serious offer for good 1 in period t is v_L. As a result, ar gives LH at most $v_L-c_L+\delta(v_H-c_H)$ in period t. By $k=1$ (see Step 5), LH gets at least $\delta(c_H-c_L)$ by choosing rr in period t. By the assumption that $\delta>\frac{v_L-c_L+\delta(v_H-c_H)}{c_H-c_L}$, LH strictly prefer rr to ar in period t. Also, LL does not choose aa or ar in period t, since either choice reveals LL's type and leads to a payoff lower than choosing rr. Next, LL does not choose rr for sure, since otherwise $\mu^1_{t+1}(rr)\le\mu^1_t\le\mu^*$, a contradiction. As a result, LL chooses ra with positive probability in period t. Notice that $\mu^1_{t+1}(ra)\ge\mu^*$, since otherwise LL would strictly prefers aa to ra, a contradiction. However, since only rr and ra are on the equilibrium path in period t, then $\mu^1_{t+1}(ra)\ge\mu^*$ and $\mu^1_{t+1}(rr)>\mu^*$ violate Bayes rule. □

PROOF OF LEMMA S.3

Proof. It is without loss of generality to check the belief updating in period 1. We need to show that $\mu^1_2(rr)\le\mu^*$.

We first prove that it is impossible that $\mu^1_2(rr)>\mu^*$ and $\mu^2_2(rr)>\mu^*$. Assume the contrary. In period 1, LL that does not choose aa, since rr dominates aa for LL. If ra is on the equilibrium path in period 1, then $\mu^1_2(ra)\ge\mu^*$, since otherwise aa dominates ra for LL. Also, LH chooses ar with positive probability in period 1, since otherwise only rr and ra can be on the equilibrium path, and it is impossible that $\mu^1_2(rr)>\mu^*$ and $\mu^1_2(ra)\ge\mu^*$. Therefore, LH weakly prefers ar to rr in period 1: $v_L-c_L+\delta(V(\mu^2_2(ar))-c_H)\ge\delta(V(\mu^1_2(rr))+V(\mu^2_2(rr))-c_L-c_H)$.

It follows that

$$\mu^1_2(rr)+\mu^2_2(rr)-\mu^2_2(ar)\le\frac{1-\delta}{\delta}\frac{v_L-c_L}{v_H-v_L}. \tag{10}$$

Moreover, we have $\mu^1_2(rr)>\mu^*$, $\mu^2_2(rr)>\mu^*$ and $\mu^2_2(ar)=\mu^1_2(ra)>\mu^*$:

$$\frac{\mu_{HL}p_{HL}+\mu_{HH}}{\mu_{HL}p_{HL}+\mu_{LH}p_{LH}+p_{rr}\mu_{LL}+\mu_{HH}}>\mu^*,\ \frac{\mu_{LH}p_{LH}+\mu_{HH}}{\mu_{LH}p_{LH}+\mu_{HL}p_{HL}+p_{rr}\mu_{LL}+\mu_{HH}}>\mu^*,$$

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL})+p_{ra}\mu_{LL}}=\frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH})+p_{ar}\mu_{LL}}>\mu^*.$$

The above conditions imply that

$$(p_{HL}\mu_{HL}+p_{LH}\mu_{LH})(1-2\mu^*)>\frac{2\mu^*}{1-2\mu^*}\mu_{LL}p_{rr}-\frac{2(1-\mu^*)}{1-2\mu^*}\mu_{HH},$$

$$p_{HL}\mu_{HL} + p_{LH}\mu_{LH} < \frac{\mu^*}{1-\mu^*}\mu_{LL}p_{rr} + 1 - \mu_{HH} - \frac{\mu_{LL}}{1-\mu^*}.$$

If there exists a solution, we can show that

$$\mu_2^1(rr) + \mu_2^2(rr) - \mu_2^2(ar) > \frac{2\mu^* - 1 - \mu_{HH} + \mu_{LL}}{1 - \frac{\mu_{HH}}{\mu^*}}, \tag{11}$$

and the upper bound is attained when $p_{rr} \to \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}$ and $p_{HL}\mu_{HL} + p_{LH}\mu_{LH} \to 0$. However, when $\delta > (1 + \frac{2\mu^*-1-\mu_{HH}+\mu_{LL}}{\mu^*-\mu_{HH}}\frac{c_H-v_L}{v_L-c_L})^{-1}$, (10) and (11) cannot hold simultaneously.

Next, we argue that it is impossible that $\mu_2^1(rr) > \mu^*$ and $\mu_2^2(rr) \leq \mu^*$. Assume the contrary.

The first observation is that LH gets at least $\delta(c_H - c_L)$ by choosing rr in period 1. If the seller of high quality good 1 accepts the offer for good 1 with positive probability in period 2, then the offer for good 2 in period 2 is at least c_H. Then, LH can guarantee a payoff of $\delta(c_H - c_L)$ by choosing rr in period 1. If the seller of high quality good 1 rejects the offer for good 1 in period 2, then we continue our proof by two cases:

Case 1: LL chooses rr with positive probability in period 1.

Assume that LH chooses rr in period 1, then LL chooses rr for sure in period 1, a contradiction to $\mu_2^1(rr) > \mu^*$. Therefore, LH chooses ar with positive probability in period 1. If LL chooses rr for sure in period 1, then $\mu_2^2(ar) = 1$, and thus LH gets a payoff $v_L - c_L + \delta(v_H - c_H) > \delta(c_H - c_L)$ (by $\delta > \frac{c_H-v_L}{v_H-c_H}$) in period 1. The remaining case is that LL mixes between rr and ar. In order that LH choosing ar with positive probability, skimming property B.1 shows that both LH and LL choose to accept the offer for good 2 for sure in period 2. If $\mu_2^2(rr) = \mu^*$, then there is a winning offer c_H for good 2, and there is a winning offer $V(\mu_2^1(rr)) > c_H$ for good 1. Then, LH gets at least $\delta(c_H - c_L)$ in period 2. If $\mu_2^2(rr) < \mu^*$, then HH chooses rr in period 2, since otherwise HH chooses ra with positive probability in period 2, then skimming property A shows that HL chooses ra for sure, and thus the offer for good 2 is less than c_H, by $\mu_2^2(rr) < \mu^*$. Therefore, HH would rather choose rr, a contradiction. Therefore, $\mu_3^1(rr) = 1$. LH can guarantee a payoff $\delta(v_H - c_L) > c_H - c_L$ (by $\delta > \frac{c_H-c_L}{v_H-c_L}$) in period 2, and thus LH can guarantee a payoff of $\delta(c_H - c_L)$ by choosing rr in period 1.

Case 2: LL does not choose rr in period 1.

In period 2, there are at most three types HH, HL and LH. By choosing ar in period 2, LH guarantees a payoff $v_L - c_L + \delta(v_H - c_H) > c_H - c_L$ (by $\delta > \frac{c_H-v_L}{v_H-c_H}$) in period 2, and thus a payoff of $\delta(c_H - c_L)$ by choosing rr in period 1.

The next observation is that LH chooses ar with positive probability in period 1. Otherwise, HH and LH choose rr in period 1, and only rr and ra are on the equilibrium path in period 1. If ra is on the equilibrium path, then $\mu_2^1(ra) \geq \mu^*$. However, $\mu_2^1(ra) \geq \mu^*$ and $\mu_2^1(rr) > \mu^*$ violate Bayes rule.

Combining the above two observations, we get $v_L - c_L + \delta(V(\mu_2^2(ar)) - c_H) \geq \delta(c_H - c_L)$. As a result,

$$\mu_2^2(ar) \geq 2\mu^* - \frac{1-\delta}{\delta}\frac{v_L - c_L}{v_H - v_L}. \tag{12}$$

Bayes rule implies that

$$\frac{\mu_{HL}p_{HL} + \mu_{HH}}{\mu_{HL}p_{HL} + \mu_{LH}p_{LH} + p_{rr}\mu_{LL} + \mu_{HH}} > \mu^*.$$

$$\frac{\mu_{HL}(1-p_{HL})}{\mu_{HL}(1-p_{HL}) + p_{ra}\mu_{LL}} = \frac{\mu_{LH}(1-p_{LH})}{\mu_{LH}(1-p_{LH}) + p_{ar}\mu_{LL}} > \mu^*.$$

By $p_{LH} \geq 0$, the above conditions imply that

$$\frac{\mu^*}{1-\mu^*}\mu_{LL}p_{rr} - \mu_{HH} < p_{HL}\mu_{HL} < \frac{\mu^*}{1-\mu^*}\mu_{LL}p_{rr} + 1 - \mu_{HH} - \frac{\mu_{LL}}{1-\mu^*}.$$

If $\mu_{LL} \geq 1-\mu^*$, there is no solution. If $\mu_{LL} < 1-\mu^*$, then there is a solution and

$$\mu_2^2(ar) < \mu^*\frac{1-\mu_{HH}-\mu_{LL}}{\mu^*-\mu_{HH}}, \tag{13}$$

where the upper bounded is attained if $p_{rr} \to \frac{1-\mu^*}{\mu^*}\frac{\mu_{HH}}{\mu_{LL}}$ and $p_{HL} \to 0$.

As long as $\delta > (1 + \frac{2\mu^*-1-\mu_{HH}+\mu_{LL}}{\mu^*-\mu_{HH}}\frac{c_H-v_L}{v_L-c_L})^{-1}$, (12) and (13) do not hold simultaneously. □

PROOF OF LEMMA S.4

Proof. By Lemma 5, the seller of high-quality good $i = 1,2$ rejects the offer for good i in period t. Therefore, HH chooses rr, LH chooses ar or rr, and HL chooses ra or rr in period t. Assume without loss of generality that $\mu_t^1 < \mu^*$.

We first prove that LH chooses ar with positive probability in period t. Assume the contrary that LH chooses rr for sure in period t. Bayes rules show that $\mu_{t+1}^1(rr) \leq \mu_t^1 < \mu^*$. Assume that $t + k_1$ is the first period that LH chooses ar with positive probability, where $k_1 \geq 1$. In period $n + k_1$, $\mu_{t+k_1}^1(rr) \leq \mu_t^1 < \mu^*$. Therefore, LH gets at most $v_L - c_L + \delta(v_H - c_H)$ in period $t + k_1$. However, by deviating to ar in period t, LH guarantee a payoff $v_L - c_L + \delta(v_H - c_H)$ in period t, since D1 implies that $\mu_{t+1}^2(ar) = 1$. In all, ar is a profitable deviation for LH in period t, a contradiction.

Next, we argue that HL chooses ra with positive probability in period t. Assume by contradiction that HL chooses rr for sure in period t. Since LH chooses ar with positive probability in period t, then Bayes rules show that $\mu_{t+1}^1(rr) < \mu_t^1 \leq \mu^*$. Assume that $t + k_2$ is the first period that HL chooses ra with positive probability, where $k_2 \geq 1$. In period

$n+k_2$, $\mu^1_{t+k_2}(rr) < \mu^1_t \le \mu^*$. Therefore, HL gets at most $v_L - c_L + \delta(v_H - c_H)$ in period $t+k_2$. However, by deviating to ra in period t, HL guarantee a payoff $v_L - c_L + \delta(v_H - c_H)$ in period t, since D1 implies that $\mu^2_{t+1}(ra) = 1$. In all, ra is a profitable deviation for HL in period t, a contradiction.

To summarize, LH (HL) is the only type to choose ar (ra) in period t, and thus $LH(HL)$ gets a payoff $v_L - c_L + \delta(v_H - c_H)$ in period t. □

PROOF OF LEMMA S.5

Proof. Define $n+1$ as the first period in which LL does not choose rr. In this lemma, we study the case that $n \ge 1$. The proof is broken into the following eight steps.

Step 1: LL does not choose aa in period $2 \le t \le n+1$.

If $n \ge 1$, then LL does not choose aa with positive probability in period $2 \le t \le n+1$, since otherwise LL would rather choose aa instead of rr in period 1, a contradiction to the fact there is a positive probability that LL remains in period $n+1$.

Step 2: In any period $t \ge 1$, if $\mu^1_{t+1}(rr) = \mu^2_{t+1}(rr) = \mu^*$ and $(\mu^1_t, \mu^2_t) \in \mathcal{B}$, then LL does not choose rr in period t.

In this step, actions are taken in period t, if not specified.

First, LL chooses ar with positive probability. Assume the contrary, then in period t, ar is off the equilibrium path, because otherwise LL can deviate to ar to get a profit. Therefore, only rr and ra is on the equilibrium path. Because $\mu^1_{t+1}(rr) = \mu^*$ and $\mu^1_t < \mu^*$, then Bayes rule implies that $\mu^1_{t+1}(ra) < \mu^*$, which means that ra is dominated by aa, a contradiction.

Second, LL chooses ra with positive probability. Assume the contrary, then in period t, ra is is off the equilibrium path, because otherwise LL can deviate to ra to get a profit. Therefore, only rr and ar is on the equilibrium path. Because $\mu^2_{t+1}(rr) = \mu^*$ and $\mu^2_t \le \mu^*$, then $\mu^2_{t+1}(ar) \le \mu^*$, and thus LH gets $v_L - c_L$ in period t. Since $t \ge 1$, then LH would rather choose ar instead of rr in period 1. Therefore, in period $t+1$, there are at most two seller types that remain: HH and LH, a contradiction to Lemma S.2.

Next, HL chooses ra with positive probability, since otherwise by choosing ra, LL reveals its type and gets a profit lower than $2(v_L - c_L)$, which is the payoff of choosing aa in period t, a contradiction. Similarly, LH chooses ar with positive probability.

Finally, $\mu^1_{t+1}(ra) = \mu^2_{t+1}(ar) > \mu^*$. If $\mu^1_{t+1}(ra) < \mu^*$, then aa dominates ra for LL in period t, a contradiction. It follows that $\mu^1_{t+1}(ra) \ge \mu^*$, and similarly, $\mu^2_{t+1}(ar) \ge \mu^*$. However, it is impossible that $\mu^1_{t+1}(ra) = \mu^2_{t+1}(ar) = \mu^*$, since otherwise $\mu^1_{t+1}(rr) = \mu^2_{t+1}(rr) = \mu^1_{t+1}(ra) = \mu^2_{t+1}(ar) = \mu^*$ violates Bayes' rule.

To summarize, by skimming property B.1 in Lemma 3, the facts that (i) $\mu^1_{t+1}(ra) = \mu^2_{t+1}(ar) > \mu^*$, (ii) HL (LH) weakly prefers ra (ar) to rr in period t, and (iii) LL does not choose aa in period $t+1$(step 1), imply that LL strictly prefers ra (ar) to rr in period t. We reach a conclusion that LL does not choose rr in period t.

Step 3: Show that $n < +\infty$.

Assume by contradiction that $n = \infty$, which means that LL chooses rr with positive probability in any period $t \geq 1$. An implication of step 2 is that $(\mu^1_t, \mu^2_t) \in \mathcal{B}$ for any period t, since otherwise there exists $t^* \geq 1$ such that $\mu^1_{t^*+1}(rr) = \mu^2_{t^*+1}(rr) = \mu^*$ and $(\mu^1_{t^*}, \mu^2_{t^*}) \in \mathcal{B}$, and thus LL does not choose rr in period t^*, a contradiction.

Since $(\mu^1_t, \mu^2_t) \in \mathcal{B}$ for any period t, then by Lemma 5, the offer for each good is v_L. In period t, the only reason that LL chooses rr with positive probability is that LL expects to choose ar or ra and enjoy a high payoff in a future period $t+k$, which equals to $v_L - c_L + \delta(V(\mu^2_{t+k+1}(ar)) - c_L)$ or $v_L - c_L + \delta(V(\mu^1_{t+k+1}(ra)) - c_L)$. Denote the supremum of $\mu^2_{t+1}(ar)$ and $\mu^1_{t+1}(ra)$ for all $t \geq 1$ as $\bar{\mu}$. For any $\epsilon > 0$, there exists a period $\bar{t}$ in which $\mu^2_{\bar{t}+1}(ar) > \bar{\mu} - \epsilon$ or $\mu^1_{\bar{t}+1}(ra) > \bar{\mu} - \epsilon$. Assume without loss of generality that $\mu^2_{\bar{t}+1}(ar) > \bar{\mu} - \epsilon$. In period $\bar{t}$, ar brings a payoff at least $v_L - c_L + \delta(V(\bar{\mu} - \epsilon) - c_L)$ and rr brings a payoff at most $\delta(v_L - c_L + \delta(V(\bar{\mu}) - c_L)$. For small $\epsilon > 0$, ar dominates rr for LL in period $\bar{t}$, a contradiction to the assumption that LL chooses rr with positive probability in any period $t \geq 1$.

Step 4: Show that $(\mu^1_{n+1}(rr), \mu^2_{n+1}(rr)) \in \mathcal{B}$.

Assume the contrary that $\mu^1_{n+1}(rr) = \mu^2_{n+1}(rr) = \mu^*$. Also assume without loss of generality that $(\mu^1_n, \mu^2_n) \in \mathcal{B}$. Step 2 have shown that LL does not choose rr in period n, a contradiction to the definition of $n+1$.

Step 5: If the updated belief is $\mu^1_{n+2}(rr) = \mu^2_{n+2}(rr) = \mu^*$ in period $n+2$, then the equilibrium payoff of LH and HL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)$ and the equilibrium payoff of LL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)$.

Assume without loss of generality that $\mu^1_{n+1} < \mu^*$ and $\mu^2_{n+1} \leq \mu^*$. By the same logic as in step 2, we can prove that $\mu^1_{n+2}(ra) = \mu^2_{n+2}(ar) > \mu^*$. Therefore, the equilibrium payoff of LH and HL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)$ and the equilibrium payoff of LL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)$.

Step 6: If the update belief satisfies $(\mu^1_{n+2}(rr), \mu^2_{n+2}(rr)) \in \mathcal{B}$ in period $n+2$, then the equilibrium payoff of LH and HL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)$ and the equilibrium payoff of LL in period $n+1$ is $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)$.

Assume without loss of generality that LL chooses ar with positive probability in period $n+1$. We prove that LH chooses ar with positive probability in period $n+1$, since otherwise $\mu^2_{n+2}(ar) = 0$, and then aa dominated ar for LL in period $n+1$, a contradiction.

We next prove that LH chooses rr with positive probability in period $n+1$, since otherwise there are at most two types in period $n+2$: HL and HH, and thus $\mu^1_{n+2}(rr) = 1$, a contradiction. Similarly, HL chooses rr with positive probability in period $n+1$.

To summarize, both HL and LH choose rr in period $n+1$, and gets $\delta(v_L - c_L + \delta(v_H - c_H))$, since Lemma S.4 shows that in period $n+2$ with three types HH, HL and LH, the equilibrium continuation payoff in period $n+2$ is $v_L - c_L + \delta(v_H - c_H)$. Moreover, since LH is indifferent between ar and rr in period $n+1$, then both HL and LH get $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)$ in period $n+1$. Consequently, LL chooses ar in period $n+1$ and gets $v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)$.

Step 7: If $n \geq 2$, all types choose rr in period $2 \leq t \leq n$, and LL does not choose aa in period 1.

Assume by contradiction that LH chooses ar with positive probability in period $2 \leq t \leq n$. Observe that $\mu^2_{t+1}(ar) > \mu^*$. Otherwise $\mu^2_{t+1}(ar) \leq \mu^*$, and LH gets $v_L - c_L$ in period t, which means that LH would rather choose ar instead of rr in period 1. Therefore, given rr in period $n+1$, only HL and HH remain in period $n+2$, a contradiction to Lemma S.2. Under $\mu^2_{t+1}(ar) > \mu^*$, then the payoffs of LH and LH by choosing ar and rr are as follows: $V^t_{LH}(ar) = v_L - c_L + \delta(V(\mu^2_{t+1}(ar)) - c_H)$ and $V^t_{LL}(ar) = v_L - c_L + \delta(V(\mu^2_{t+1}(ar)) - c_L)$; $V^t_{LH}(rr) = \delta^{n+1-t}[v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)]$ and $V^t_{LH}(rr) = \delta^{n+1-t}[v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)]$. Therefore, $V^t_{LL}(ar) - V^t_{LH}(ar) > V^t_{LL}(rr) - V^t_{LH}(rr)$. As a result, if LH weakly prefers ar to rr in period t, then LL strictly prefers ar to rr in period t, a contradiction that LL remains in period $n+1$ with positive probability.

We have proved that LH chooses rr for sure in period t, and similarly, HL chooses rr for sure in period t. Therefore, LL does not choose ar or ra in period t, since otherwise it will reveal its type, which is dominated by choosing aa in period t.

Finally, we will show that LL does not choose aa in period 1. Since LH weakly prefers rr to ar in period 1 and chooses rr for sure in period $2 \leq t \leq n$, then $V^t_{LH}(ar) = v_L - c_L \leq V^t_{LH}(rr) = \delta^n[v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_H)]$. By $\delta > \frac{v_L - c_L + \delta(v_H - c_H)}{c_H - c_L}$ and $V(\mu^2_{n+2}(ar)) \leq v_H$, we have $v_L - c_L < \delta^{n+1}[c_H - c_L]$. Summing up the above two inequalities, we get $2(v_L - c_L) < \delta^n[v_L - c_L + \delta(V(\mu^2_{n+2}(ar)) - c_L)]$, which means that LL strictly prefers rr to aa in period 1.

Step 8: $\mu^2_{t+1}(ar) = \mu^1_{t+1}(ra) = \mu^*$ for period $1 \leq t \leq n$.

In period $2 \leq t \leq n$, we know that ar is off the equilibrium path. If $\mu^2_{t+1}(ar) < \mu^*$, then $V^t_{LL}(ar) - V^t_{LH}(ar) = v_L - c_L < \delta^{n+1-t}(c_H - c_L) = V^t_{LL}(rr) - V^t_{LH}(rr)$, which means that LH prefers ar to rr more than LL in period t. By D1, $\mu^2_{t+1}(ar) = 1$, a contradiction to $\mu^2_{t+1}(ar) < \mu^*$. If $\mu^2_{t+1}(ar) > \mu^*$, then $V^t_{LL}(ar) - V^t_{LH}(ar) = c_H - c_L > \delta^{n+1-t}(c_H - c_L) =$

$V_{LL}^t(rr) - V_{LH}^t(rr)$. By refinement D1, $\mu_{t+1}^2(ar) = 0$, a contradiction to $\mu_{t+1}^2(ar) > \mu^*$. Therefore, $\mu_{t+1}^2(ar) = \mu^*$ for $2 \le t \le n$.

In period 1, if ar is off the equilibrium path, then by same argument in the previous paragraph, we get $\mu_2^2(ar) = \mu^*$. If ar is on the equilibrium path in period 1, then LH chooses ar with positive probability in period 1. If $\mu_2^2(ar) < \mu^*$, then LL also chooses ar with positive probability in period 1. Moreover, LH prefers ar to rr more than LL in period 1, which means that LH does not choose rr in period 1. Therefore, only HL and HH remain in period $n+2$, a contradiction to Lemma S.2. If $\mu_2^2(ar) > \mu^*$, then LL prefers ar to rr more than LH in period 1, contradicting the fact that LL chooses rr in period 1.

Hence, we have $\mu_{t+1}^2(ar) = \mu^*$ and, by symmetry, $\mu_{t+1}^1(ra) = \mu^*$ for $t = 1, \ldots, n$. □

Proofs of the Results in Section 6

In this section, we prove the results in Section 6 of the main paper, regarding the robustness of the main insight. The notations in this section follow those in the main paper.

Proof of Proposition 4.

Proof. In any period $m+1$ where $0 \le m < K$, the equilibrium play is that there is no trade. By symmetry, we only prove that in period $m+1$, it is not profitable for buyer 1 to deviate to a serious offer. There are two types of deviations for buyer 1:

Case 1: The first deviation is to make an offer for good 1 that all types of the seller accept.

Since ar is off the equilibrium path in period $m+1$, we consider the most pessimistic belief: $\mu_{m+2}^2(ar) = 0$. For buyer 1, in period $m+1$, a profitable deviating offer for good 1 is $V(\tau) - \epsilon$, for some $\tau > 0$ and small $\epsilon > 0$. To prevent HH from accepting the offer $V(\tau) - \epsilon$ in period $m+1$, we need $V(\tau) - \epsilon - c_H < \delta^{K-m}(V(\tau) - c_H + V(\tau) - c_H)$. The right hand side of this inequality is the continuation payoff of HH by choosing rr in period $m+1$. To prevent HL from accepting the offer $V(\tau) - \epsilon$ in period $m+1$, we need $V(\tau) - \epsilon - c_H + v_L - c_L < \delta^{K-m}(V(\tau) - c_H + V(\tau) - c_L)$, where the right hand side is the continuation payoff of HL by choosing rr in period $m+1$. By the definition of K, that is, $\delta^K \ge \frac{1}{2}$, and $c_H - c_L > 2(v_L - c_L)$, which holds since $\mu^* > \frac{1}{2}$ and $v_L - c_L < v_H - c_H$, it is straightforward to verify that both inequalities hold for any $\epsilon > 0$.

Case 2: The second deviation is to make an offer for good 1 that only the low type seller accepts.

Since ar is off the equilibrium path in period $m+1$, we again consider the most pessimistic belief: $\mu_{m+2}^2(ar) = 0$. For buyer 1, in period $m+1$, a profitable deviating offer for good 1 is $v_L - \epsilon$, since only low-quality good 1 is sold. To prevent LH from accepting the new offer $v_L - \epsilon$ in period $m+1$, we need $v_L - \epsilon - c_L < \delta^{K-m}(V(\tau) - c_L + V(\tau) - c_H)$. To prevent LL from accepting the new offer $v_L - \epsilon$ in period $m+1$, we need $v_L - \epsilon - c_L + v_L - c_L < \delta^{K-m}(V(\tau) - c_L + V(\tau) - c_L)$. Since $\delta^K \geq \frac{1}{2}$ and $c_H - c_L > 2(v_L - c_L)$, which follows from $\mu^* > \frac{1}{2}$ and $v_L - c_L < v_H - c_H$, both inequalities hold for any $\epsilon > 0$.

Finally, by Lemma 4, in period $K+1$ the beliefs are $\tau > \mu^*$ for both goods and the game has an equilibrium in which trade happens immediately with offers for both goods equal to $V(\tau)$. □

Proof of Proposition 5.

Proof. The proof consists of four steps.

Step 1: Continuation payoffs.

If $\mu_2^1(rr) = \mu_2^2(rr) = \tau > \mu^*$, then the continuation payoff of LH and HL is $V_{HL} = V_{LH} = \delta^K(V(\tau) - c_L + V(\tau) - c_H)$, where K is any integer satisfying $\delta^K \geq \frac{1}{2}$.

Step 2: The updated belief in period 2.

In period 2, $\mu_2^1(rr) = \mu_2^2(rr) = \tau > \mu^*$ and $\mu_2^1(ra) = \mu_2^2(ar) = \tilde{\mu}$. Bayes' rule shows that

$$\frac{\mu_{HH} + \mu_{HL}p_{HL}}{\mu_{HH} + \mu_{HL}p_{HL} + \mu_{LH}p_{LH}} = \frac{\mu_{HH} + \mu_{LH}p_{LH}}{\mu_{HH} + \mu_{HL}p_{HL} + \mu_{LH}p_{LH}} = \tau > \mu^*,$$

$$\frac{\mu_{LH}(1 - p_{LH})}{\mu_{LH}(1 - p_{LH}) + \mu_{LL}p_{ar}} = \frac{\mu_{HL}(1 - p_{HL})}{\mu_{HL}(1 - p_{HL}) + \mu_{LL}p_{ra}} = \tilde{\mu}.$$

where $p_{HL}(p_{HL})$ is the probability of rr chosen by $HL(LH)$, and $p_{ar}(p_{ra})$ is the probability of $ar(ra)$ chosen by LL. Simple calculation shows that $\tilde{\mu} > \hat{\mu} \equiv 1 - \frac{(2\mu^*-1)\mu_{LL}}{2\mu^*-1-\mu_{HH}} > \mu^*$.

Step 3: The seller's equilibrium behavior in period 1.

To satisfy the belief updating in Step 2, LH (HL) is indifferent between rr and ar (ra). By choosing ar, LH (HL) gets a payoff $v_L - c_L + \delta(V(\tilde{\mu}) - c_H)$, where $\tilde{\mu} > \hat{\mu} > \mu^*$. By choosing rr, LH (HL) gets a payoff $\delta^K(V(\tau) - c_L + V(\tau) - c_H)$, where $\tau > \mu^*$. Therefore,

$$v_L - c_L + \delta(V(\tilde{\mu}) - c_H) = \delta^K(V(\tau) - c_L + V(\tau) - c_H).$$

From the belief updating, we have $\frac{2\mu_{HH}+x}{\mu_{HH}+x} = 2\tau$ and $\frac{\mu_{LL}}{1-\mu_{HH}-x} = 1 - \tilde{\mu}$, where $x = p_{HL}\mu_{HL} + p_{LH}\mu_{LH}$. Therefore, $\frac{\mu_{HH}}{2\tau-1} + \frac{\mu_{LL}}{1-\tilde{\mu}} = 1$.

Step 4: There exists a solution $(\tau, \tilde{\mu})$ such that $\tau > \mu^*$ and $\tilde{\mu} > \hat{\mu}$.

If $v_L - c_L + \delta(V(\hat{\mu}) - c_H) > \frac{1}{2}(c_H - c_L)$, then let $\tau = \mu^*+$ and then $v_L - c_L + \delta(V(\tilde{\mu}) - c_H) = \delta^K(c_H - c_L)$. There exists $\tilde{\mu} = \hat{\mu}+$ and $\delta^K > \frac{1}{2}$ such that the above equation holds.

In the above equilibrium, the payoff of each seller's type is: $V_{LL} = v_L - c_L + \delta(V(\tilde{\mu}) - c_L)$, $V_{HL} = V_{LH} = v_L - c_L + \delta(V(\tilde{\mu}) - c_L)$, $V_{HH} = 0$. Since we have shown that $\hat{\mu} < \tilde{\mu}$, then this equilibrium delivers weakly higher payoff for each seller type than the beneficial spillover equilibrium. □

Proof of Proposition 6.

Proof. First, HH gets zero profit, which is the least payoff that HH can get. Second, LH gets $v_L - c_L$. This is also the least payoff that LH can get. If not, then in period 1, there is a losing offer for good 1 and LH gets a payoff $V_{LH} < v_L - c_L$. However, buyer 1 can offer $v_L - \epsilon$ so that $v_L - \epsilon - c_L > V_{LH}$ so that LH is willing to accept, consequently, buyer 1 can guarantee a positive profit in period 1, a contradiction to buyers' zero profit condition. Similarly, we prove that $v_L - c_L$ is the least payoff that HL can guarantee in any equilibrium.

Finally, we need to prove that LL gets the least payoff in delay equilibrium N. In order for LL to get the least payoff, the initial delay before there is any trade should reach its maximum. Assume that there is some trade in period $N+1$. Then LH (HL) chooses *ar* (*ra*) and the best payoff that LH (HL) can get is $v_L - c_L + \delta(v_H - c_H)$ in period $N+1$. Therefore, the longest delay N must satisfy $v_L - c_L < \delta^N(v_L - c_L + \delta(v_H - c_H))$. In delay equilibrium N, the payoff of LH in period 1 is $V_{LH} = v_L - c_L = \delta^N(v_L - c_L + \delta(V(\hat{\mu}') - c_H))$, where $\hat{\mu}' < 1$. Consequently, the payoff of LL in period 1 satisfies $V_{LL} = \delta^N(v_L - c_L + \delta^N(V(\hat{\mu}') - c_L)) = v_L - c_L + \delta^N(c_H - c_L)$, which is the least payoff for LL since N reaches its maximum. □

Proof of Proposition 7.

Proof. In period 2, there are only two seller types: M and H. Assume that M rejects the offer with probability α in period 1. Then Bayes rule implies that the probability of H seller in period 2 is $\frac{\mu_{HH}}{\alpha(\mu_{HL}+\mu_{LH})+\mu_{HH}}$. Define a threshold belief level μ_2^* as $2v_H\mu_2^* + (v_H + v_L)(1-\mu_2^*) = 2c_H$. Then, we have $\mu_2^* = \frac{2c_H - v_L - v_H}{v_H - v_L} = 2\mu^* - 1$. In equilibrium, the probability of H seller equals the threshold μ_2^*:

$$\frac{\mu_{HH}}{\alpha(\mu_{HL}+\mu_{LH})+\mu_{HH}} = \mu_2^*. \qquad (\star)$$

Hence, we have $\alpha = \frac{1-\mu_2^*}{\mu_2^*}\frac{\mu_{HH}}{\mu_{HL}+\mu_{LH}} \in (0,1)$, which is guaranteed by $\mu_{LL} + \frac{\mu_{HH}}{2\mu^*-1} < 1$.

In period 1, there are two seller types, L and M, who accept the offer with positive probabilities. Conditional on accepting the offer in period 1, the probability of M seller in period 1 is $\frac{(1-\alpha)(\mu_{HL}+\mu_{LH})}{(1-\alpha)(\mu_{HL}+\mu_{LH})+\mu_{LL}}$. Define μ_1^* as the threshold level such that the expected valuation of the buyer is exactly the reservation value of M type in period 1. Therefore, we have $(v_H + v_L)\mu_1^* + (2v_L)(1-\mu_1^*) = c_H + c_L$ and $\mu_1^* = \frac{c_H + c_L - 2v_L}{v_H - v_L}$. In equilibrium, to guarantee that M seller accepts the offer with a positive probability, we need the probability of M seller to be larger than or equal to μ_1^*, that is,

$$\frac{(1-\alpha)(\mu_{HL}+\mu_{LH})}{(1-\alpha)(\mu_{HL}+\mu_{LH})+\mu_{LL}} > \mu_1^*. \qquad (\star\star)$$

Equation ($\star$) implies that the probability of M in period 1 is $\hat{\mu} \equiv 1 - \frac{\mu_{LL}}{1-\frac{\mu_{HH}}{2\mu^*-1}}$ and Assumption 4 guarantees that ($\star\star$) holds.

Since the probability of M in period 1 is $\hat{\mu}$, then the zero profit condition of the buyer guarantees that the offer in period 1 is $p^* = v_L + V(\hat{\mu})$. The trading rate λ from period 2 onward is such that M type is indifferent between accepting the offer in period 1 and period 2. Finally, it is straightforward to verify that L strictly prefers to accept the offer in period 1 and H strictly prefers to reject the offer in period 1.

□

Proof of Corollary 1.

Proof. Denote V_k the payoff of seller type $k \in \{HH, HL, LH, LL\}$ in the equilibrium described by Theorem 1. We have $V_{HL} = V_{LH} = v_L - c_L + \delta(V(\hat{\mu}) - c_H)$, $V_{LL} = v_L - c_L + \delta(V(\hat{\mu}) - c_L)$, and $V_{HH} = 0$.

Denote U_k as the payoff of seller type $k \in \{HH, HL, LH, LL\}$ in the equilibrium described by Theorem 4. We have $U_{HL} = U_{LH} = v_L - c_L$, $U_{LL} = 2(v_L - c_L)$, and $U_{HH} = 0$.

Denote W_k as the payoff of seller type $k \in \{HH, HL, LH, LL\}$ in the equilibrium described by Proposition 7. We have $W_{HL} = W_{LH} = p^* - c_L - c_H = v_L - c_L + (V(\hat{\mu}) - c_H)$, $W_{LL} = p^* - 2c_L = v_L - c_L + (V(\hat{\mu}) - c_L)$, and $W_{HH} = 0$.

If $\delta = 1$ and Assumption 2 holds, then we have $V_k = W_k$ for any $k \in \{HH, HL, LH, LL\}$. If $\delta = 1$ and Assumption 2 does not hold, then we have $U_k < W_k$ for any $k \in \{HL, LH, LL\}$ and $U_{HH} = W_{HH}$. Therefore, the result holds. □

Proof of Proposition 8.

Proof. We first construct the equilibrium. If the updated belief in period 2 is not (μ^*, μ^*), which means that $(\mu_2^1, \mu_2^2) \in \mathcal{B}$. Therefore, Lemma S.4 implies that the equilibrium payoff of LH and HL in period 2 is $v_L - c_L + \delta(v_H - c_H)$. By deviating to rr in period 2, LL can get $\delta(v_L - c_L + \delta(v_H - c_L))$ in period 1.

It is straightforward that rr dominates aa for LL in period 1. Therefore, LL chooses ar or ra in period 1. Assume without loss of generality that LL mixes between ar and ra in period 1. Then in period 1, LH chooses ar with a positive probability and HL chooses ra with a positive probability, since otherwise LL would reveal its type. Also, HL and LH choose rr with positive probabilities, since otherwise the updated belief in period 2 is such that one of the two goods is high type for sure, a contradiction. Thus, in period 1, LH mixes between rr and ar, and HL mixes between rr and ra.

Since LH (HL) is indifferent between rr and ar (ra), then $v_L - c_L + \delta(V(\kappa) - c_H) = \delta(v_L - c_L + \delta(v_H - c_H))$, where $\mu_2^1(ra) = \mu_2^2(ar) = \kappa$. Bayes' rule shows that

$$\frac{\mu_{HH} + \mu_{HL}p_{HL}}{\mu_{HH} + \mu_{HL}p_{HL} + \mu_{LH}p_{LH}} = \frac{\mu_{HH} + \mu_{LH}p_{LH}}{\mu_{HH} + \mu_{HL}p_{HL} + \mu_{LH}p_{LH}} \leq \mu^*,$$

$$\frac{\mu_{LH}(1 - p_{LH})}{\mu_{LH}(1 - p_{LH}) + \mu_{LL}p_{ar}} = \frac{\mu_{HL}(1 - p_{HL})}{\mu_{HL}(1 - p_{HL}) + \mu_{LL}p_{ra}} = \kappa,$$

where $p_{HL}(p_{HL})$ is the probability that $HL(LH)$ chooses rr, and $p_{ar}(p_{ra})$ is the probability that LL chooses $ar(ra)$. Then, we have $\kappa \leq \hat{\mu} \equiv 1 - \frac{(2\mu^*-1)\mu_{LL}}{2\mu^*-1-\mu_{HH}}$. Since $\delta < \frac{v_L - c_L + \delta(V(\hat{\mu}) - c_H)}{v_L - c_L + \delta(v_H - c_H)}$, then there exists $\kappa \in (\mu^*, \hat{\mu})$ such that $v_L - c_L + \delta(V(\kappa) - c_H) = \delta(v_L - c_L + \delta(v_H - c_H))$.

Finally, we verify that the equilibrium constructed above is Pareto dominated by the beneficial spillover equilibrium. In the above equilibrium, the payoff of each seller's type is: $V_{LL} = v_L - c_L + \delta(V(\kappa) - c_L)$, $V_{HL} = V_{LH} = v_L - c_L + \delta(V(\kappa) - c_L)$, $V_{HH} = 0$. Since we have shown that $\hat{\mu} > \kappa$, the result holds. □

Proof of Theorem 5.

Proof. We know from Lemma 5 that the high-type seller of each good always rejects any offer in the first period. Therefore, no buyer offers more than v_L in period 1.

In period 1, the equilibrium offer for each good is v_L, and hence the buyer of each good earns zero profit. We first prove that it is not profitable for the buyer of each good to make an offer less than v_L. Assume that buyer 1 deviates to an offer $p_1 < v_L$ in period 1. Since the equilibrium offer v_L of good 1 makes LH indifferent between rr and ar, then with private offer (which means that the seller's future continuation value by choosing rr and ar remains constant), a lower offer p_1 of good 1 make LH strictly prefer rr to ar. Since the offer v_L of good 1 makes LL indifferent between ra and ar, then with private offer, a lower offer p_1 of good 1 make LL strictly prefer ra to ar. Therefore, p_1 is rejected by all four types, and hence p_1 is not a profitable deviation.

We then prove that all four seller types choose the optimal strategy in period 1. For LL, both ar and ra deliver a payoff $v_L - c_L + \delta(c_H - c_L)$; both aa and rr deliver $2(v_L - c_L)$, which is less than $v_L - c_L + \delta(c_H - c_L)$. Thus, it is optimal for LL to mix between ar and ra in period 1. For HL, ra and rr deliver a payoff $v_L - c_L$; ar and aa bring $v_L - c_H + v_L - c_L$, which is less than $v_L - c_L$. Therefore, it is optimal for HL to mix between ra and rr in period 1. By symmetry, it is optimal for LH to mix between ar and rr in period 1.

In period 2, if both goods remain untraded, then Bayes' rule implies that the updated belief is $\mu_2^1(rr) = \mu_2^2(rr) = \mu^*$. The equilibrium strategy of each buyer is to mix between a winning offer c_H and a losing offer, and hence each buyer gets zero profit. We next prove that there is not a profitable deviation for each buyer to make an offer with positive profit. Assume that buyer 1 deviates to offer p_1. If $p_1 > c_H$, then p_1 is also a winning offer since c_H is a winning offer, but buyer 1 earns a negative profit since the expected valuation of good 1 for the buyer is $\mu^* v_H + (1-\mu^*)v_L = c_H < p_1$. If $p_1 < c_H$, then the seller with a high-quality good 1 rejects the offer p_1, since otherwise she would get a negative profit. If buyer 1 makes a positive profit, then we have $p_1 < v_L$ and $p_1 - c_L < v_L - c_L < \delta(\lambda c_H + (1-\lambda)v_L - c_L)$. Thus, the seller with a low-quality good 1 also rejects the offer p_1. That is, any offer $p_1 < c_H$ is a losing offer. Therefore, it is optimal for buyer 1 to mix between a winning offer c_H and a losing offer in period 2.

We next prove that c_H is a winning offer for each good $i = 1, 2$ in period 2. By rejecting offer c_H, the seller with high-quality good i can only get zero profit in the future, and hence it is optimal for her to accept c_H. By rejecting the offer c_H, the seller with a low-quality good i can get a continuation payoff $\delta(\lambda c_H + (1-\lambda)v_L - c_L) = v_L - c_L$, but she can guarantee a payoff $c_H - c_L$ by accepting c_H in period 2, and hence it is optimal for her to accept c_H.

In period 2, if only one good is traded, then $\mu_2^1(ra) \geq \mu^*$ and $\mu_2^2(ar) \geq \mu^*$ are consistent with the Bayes' rule. Moreover, Bayes' rule also implies that it is impossible that $\mu_2^1(ra) = \mu_2^2(ar) = \mu^*$. In the one-good model without severe adverse selection, it is optimal for each buyer to offer c_H. Notice that even if $\mu_2^1(ra) = \mu^*$ or $\mu_2^2(ar) = \mu^*$, the buyer of the remaining good cannot mix between c_H and a losing offer since LL is indifferent between ar and ra in period 1. Finally, the proof of the optimality condition in period $t \geq 3$ is the same as that in period 2. $\square$