Introduction to Local Projections

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AEA Continuing education 2023

Last updated: December 27, 2022

See also:
https://sites.google.com/site/oscarjorda/home/local-projections

Basic ideas

Borrowing from applied micro to draw a parallel

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Impulse responses: a comparison of two averages

$$\mathcal{R}(h) = E(E[y_{t+h}|s_t = s + \delta, \boldsymbol{x}_t] - E[y_{t+h}|s_t = s, \boldsymbol{x}_t]))$$

- y_{t+h}: outcome
- s_t: intervention
- s: baseline, e.g., $s = 0$
- δ: dose, e.g., $\delta = 1$; $\delta = var(\epsilon)^{1/2}; \ldots$
- $\boldsymbol{x}_t$: vector of exogenous and predetermined variables

Main issues to be solved

- Identification: next section
- Estimation of $E[y_{t+h}|s_t; \boldsymbol{x}_t]$
- Interpretation: multipliers
- Inference: discussed later

A trivial example

Suppose $s_t \in \{0, 1\}$ is *randomly assigned*, then:

$$\begin{aligned} \mathcal{R}(h) &= \frac{1}{N_1} \sum_{t=1}^{T-h} y_{t+h} s_t - \frac{1}{N_0} \sum_{t=1}^{T-h} y_{t+h} (1 - s_t) \\ N_1 &= \sum_{t=1}^{T-h} s_t; \quad T - h = N_1 + N_0 \end{aligned}$$

Remarks:

- inefficient (not using $\boldsymbol{x}_t$), but *consistent*
- could control for $\boldsymbol{x}_t$ with Inverse Propensity score Weighting (IPW)
- feels like the potential outcomes paradigm used in micro
- could have regressed y_{t+h} on s_t, same thing (could add $\boldsymbol{x}_t$ easily)

Estimation by Local projections

Linear case:

$$y_{t+h} = \alpha_h + \beta_h s_t + \boldsymbol{\gamma}_h \boldsymbol{x}_t + v_{t+h}; \quad \underbrace{v_{t+h} = u_{t+h} + \psi_1 u_{t+h-1} + \ldots + \psi_h u_t}_{\text{will see later why this residual MA(h)}}$$

As long as $s_t, \boldsymbol{x}_t$ exogenous w.r.t. v_t, then $\hat{\beta}_h \to \beta_h$ (identification) and then:

$$\mathcal{R}_{sy}(h) = E[y_{t+h}|s_t = s_1; \boldsymbol{x}_t] - E[y_{t+h}|s_t = s_0; \boldsymbol{x}_t] = \beta_h(s_1 - s_0)$$

General case:

$$y_{t+h} = m(s_t, \boldsymbol{x}_t; \boldsymbol{\theta}_h) + v_{t+h} \to \mathcal{R}_{sy}(h) = m(s_1, \boldsymbol{x}_t; \boldsymbol{\theta}_h) - m(s_0, \boldsymbol{x}_t; \boldsymbol{\theta}_h)$$

i.e. $m(s_t, \boldsymbol{x}_t; \boldsymbol{\theta}_h)$ can be a nonlinear function

Remarks

- **single equation estimation**: easily scales to panel, easy to extend to nonlinear specifications
- **effects 'local' to each h**: no cross-period restrictions
- **errors serially correlated**: needs fixing
- from binary to continuous treatment (dose)

Many assumptions implicit in **linear** formulation:

- **symmetry**: increase in dose same as decrease
- **scale independence**: double dose, double the effect
- **state independence**: the $\mathbf{x}_t$ don't affect $\mathcal{R}(h)$
- **treatment does not affect covariate effects**: $\boldsymbol{\gamma}_h^0 = \boldsymbol{\gamma}_h^1$
- $\delta|\mathbf{x}$ **randomly assigned**

We will analyze/generalize each of these assumptions

A STATA illustration

`LP_example.do`

- simple illustration of different variable transformations:
  - *levels vs. differences* (e.g. price index vs inflation)
  - *levels = long-differences = cumulative of differences*

$$\begin{aligned} \Delta y_{t+h} + \ldots + \Delta y_t &= y_{t+h} - y_{t+h-1} + y_{t+h-1} - y_{t+h-2} + \ldots y_t - y_{t-1} \\ &= y_{t+h} - y_{t-1} \end{aligned}$$

- shows a simple way to construct the loop and plot LPs
- maybe useful to build upon. Much left undone. Will come back to it

Relation to VARs reminder

Set aside identification discussion for now

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Propagation in an AR(1)

suppose:

$$(y_t - \mu) = \psi(y_{t-1} - \mu) + u_t$$

by recursive substitution:

$$(y_{t+h} - \mu) = \psi^{h+1}(y_{t-1} - \mu) + \underbrace{u_{t+h} + \psi u_{t+h-1} + \ldots + \psi^h u_t}_{\text{intrinsic MA residuals}}$$

suppose the intervention is $u_t = \delta$; $(u_{t+1} = \ldots = u_{t+h} = 0)$; $y_{t-1} = y^*$

$$\begin{aligned}
\mathcal{R}(h) &= E\left(E[y_{t+h}|u_t = \delta; y_{t-1} = y^*] - E[y_{t+h}|u_t = 0; y_{t-1} = y^*]\right) \\
&= E\left(\{\psi^{h+1}(y^* - \mu) + \psi^h \delta\} - \psi^{h+1}(y^* - \mu)\right) \\
&= E(\psi^h \delta) = \psi^h \delta
\end{aligned}$$

Remarks

- **iterative approach** with AR(1): from $\hat{\psi}$ obtain $\hat{\psi}^h$
- inference based on *delta method*:
  $H_0 : \psi = 0 \implies H_0 : ATE(h) = \mathcal{R}(h) = \psi^h = 0$
- **direct approach** with local projections:

$$y_{t+h} = \alpha_{h+1} + \psi_{h+1} y_{t-1} + v_{t+h}; \quad h = 0, 1, \ldots$$

- note: $v_{t+h} = u_{t+h} + \psi u_{t+h-1} + \ldots + \psi^h u_t$
- hence $E[y_{t-1}, v_{t+h}] = 0 \implies \hat{\psi}_{h+1} \xrightarrow{p} \psi^{h+1}$
- inference: correct error serial correlation (we will see how)
- $H_0 : ATE(h) = \mathcal{R}(h) = \psi_h = 0$

propagation in a VAR(2)

just to see the details

$$\underset{k\times 1}{\mathbf{y}_t} = \underset{k\times k}{A_1}\mathbf{y}_{t-1} + A_2\mathbf{y}_{t-2} + \mathbf{u}_t$$

by recursive substitution:

$$\mathbf{y}_{t+1} = (A_1^2 + A_2)\mathbf{y}_{t-1} + A_1A_2\mathbf{y}_{t-2} + \mathbf{u}_{t+1} + A_1\mathbf{u}_t$$

one more time:

$$\begin{aligned}\mathbf{y}_{t+2} =& (A_1^3 + A_2A_1 + A_1A_2)\mathbf{y}_{t-1} + (A_1^2A_2 + A_2^2)\mathbf{y}_{t-2} + \\ & \mathbf{u}_{t+2} + A_1\mathbf{u}_{t+1} + (A_1^2 + A_2)\mathbf{u}_t\end{aligned}$$

takeaway: $\mathcal{R}(h)$ a complicated function of A_1, A_2
(more on this later, an issue also raised in recent Plagborg-Møller papers)

Further exploration of the VAR—LP nexus

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A note on lag lengths

- iterated VAR-based forecasts need *correct specification*
- if not, responses will be biased
- consistency of $\mathcal{R}(h)$ only if in VAR(p) s. t. $p \to h$ as $h \to \infty$
- local projections are approximations
- no correct specification assumed
- smaller lag lengths ok for consistency under mild assumptions
- however, lag-augmentation can be very helpful for inference (later)

Some results derived more formally later

Using a VAR to construct $E[y_{t+h}|s_t, \boldsymbol{x}_t]$

Reduced-form only to explain VAR(p) vs. VAR(∞) issues

consider a VAR(p): (assume s_t and $\boldsymbol{x}_t$ in $\boldsymbol{y}_t$)

$$\underset{k\times 1}{\boldsymbol{y}_t} = \underset{k\times k}{A_1}\boldsymbol{y}_{t-1} + \ldots + A_p\boldsymbol{y}_{t-p} + \boldsymbol{u}_t; \quad E(\boldsymbol{u}_t\boldsymbol{u}_t') = \Sigma_u$$

by recursive substitution, $VMA(\infty)$:

$$\boldsymbol{y}_t = \boldsymbol{u}_t + B_1\boldsymbol{u}_{t-1} \ldots + B(\infty)\boldsymbol{y}_0;$$

$$B(\infty)\boldsymbol{y}_0 \to 0 \text{ if } |A(z)| \neq 0 \text{ for } |z| \leq 1 \quad \text{MA invertibility}$$

$B(\infty) = B(A_1, \ldots, A_p)$, e.g., see Slide 13

$\boldsymbol{y}_0$ is distant initial condition. MA invertibility $\implies B(\infty) \to 0$

Relation between $VAR(p)$ and $VMA(\infty)$

Recall the impulse response representation

$$
\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_1 B_1 + A_2 \\
\vdots &= \vdots \\
B_i &= A_1 B_{i-1} + A_2 A_{i-2} + \ldots + A_p B_{i-p}; \quad i \geq p
\end{aligned}
$$

or compactly

$$
\boxed{B_i = \sum_{j=1}^{i} B_{i-j} A_j; \quad i = 1, 2, \ldots; \quad B_0 = I_k}
$$

Constructing $E[y_{t+h}|s_t, \boldsymbol{x}_t]$ using $VMA(\infty)$

from:

$$\boldsymbol{y}_{t+h} = \boldsymbol{u}_{t+h} + \ldots + B_{h-1}\boldsymbol{u}_{t+1} + B_h\boldsymbol{u}_t + B_{h+1}\boldsymbol{u}_{t-1} + \ldots$$

then:

$$E[y_{i,t+h}|u_{j,t} = 1, \boldsymbol{u}_{t-1}, \ldots] = B_h(i,j)$$

where, $s_t = u_{j,t}$ and $\boldsymbol{x}_t = \boldsymbol{u}_{t-1}, \boldsymbol{u}_{t-2}, \ldots$ hence

$$\mathcal{R}(h) = B_h(i,j); \quad \hat{B}_h = \sum_{j=1}^{h} \hat{B}_{h-j}\hat{A}_j; \quad \hat{A}_j \text{ from VAR(p)}$$

Important: in reduced form, $E(u_{i,t}u_{l,t}) \neq 0$ for $i \neq l$, usually

hence, this is not yet a well defined experiment

Fitting a finite $VAR(p)$ to a $VAR(\infty)$ (1 of 2)

A good assumption if true DGP is VARMA (e.g. many DSGE models)

Suppose the DGP is:

$$\boldsymbol{y}_t = \sum_{i=1}^{\infty} A_i \boldsymbol{y}_{t-i} + \boldsymbol{u}_t \quad with \quad \sum_{i=1}^{\infty} ||A_i|| < \infty$$

hence:

$$\boldsymbol{y}_t = \sum_{i=0}^{\infty} B_i \boldsymbol{u}_{t-i}; \quad B_0 = I; \quad det\left(\sum_{i=0}^{\infty} B_i z_i\right) \neq 0$$

$$for \quad |z_i| \leq 1 \quad and \quad \sum_{i=0}^{\infty} i^{1/2} ||B_i|| < \infty$$

Fitting a finite $VAR(p)$ to a $VAR(\infty)$ (2 of 2)

Results from Lewis and Reinsel (1985), a key paper in this literature

Let p_T denote the order of the $VAR(p_T)$. If:

$$p_T \to \infty; \quad \frac{p_T^3}{T} \to 0; \quad \sqrt{T} \sum_{i=p_T+1}^{\infty} ||A_i|| \to 0 \quad as \quad T \to \infty$$

then:

$$\sqrt{T}[vec(\hat{A}_1' \ldots \hat{A}_{p_T}') - vec(A_1 \ldots A_{p_T})] \xrightarrow{d} N(\mathbf{0}, \Sigma_a^*); \quad \Sigma_a^* \neq \Sigma_a$$

where Σ_a refers to finite $VAR(p)$, and

$$\sqrt{T}[vec(\hat{B}_h') - vec(B_h)] \xrightarrow{p} N\left(\mathbf{0}, \Sigma_u \otimes \sum_{j=0}^{h-1} B_j \Sigma_u B_j'\right); h \leq p_T$$

Note: consistency not guaranteed for $h > p_T$

Takeaways and references

- $VAR(\infty)$ results in, e.g., Lütkepohl (2005, Chapter 15)
- many DSGE have VARMA reduced form or $VAR(\infty)$
- note p_T grows with T but at a slower rate
- consistency of B_h only guaranteed up to $h = p_T$
- unlike $VAR(p)$, response S.E.s $\to 0$ as $h \to \infty$
- Plagborg-Møller and Wolf (2021): for $h \leq p_T$ VARs and LPs estimate the same response
- Jordà, Singh, and Taylor (2020): for $h > p_T$ VAR responses are biased, but LPs are not (under certain conditions)

VAR vs. LP Bias in infinite lag processes

Or why LPs can be more reliable for long-horizon responses

Intuition:

- suppose D.G.P. is:

$$\boldsymbol{y}_t = \sum_{j=0}^{\infty} A_j \boldsymbol{y}_{t-j} + \boldsymbol{u}_t; \quad \sum_{j=1}^{\infty} ||A_j|| < \infty$$

- fit VAR(1)
- true vs. VAR(1) IRFs

$VAR(\infty)$	$VAR(1)$
$B_1 = A_1$	$B_1^* = A_1$
$B_2 = A_1^2 + A_2$	$B_2^* = A_1^2$
$B_3 = A_1^3 + 2A_1A_2 + A_3$	$B_3^* = A_1^3$
$B_4 = A_1^4 + 3A_1^2A_2 + 2A_1A_3 + A_4$	$B_4^* = A_1^4$

VAR bias

Consistency guaranteed up to p only for $VAR(\infty)$

objective: truncate $VAR(\infty)$ so that remaining lags are "small"

$$\frac{1}{T^{1/2}} \sum_{j=p+1}^{\infty} ||A_j|| \to 0; \quad p, T \to \infty$$

however, from the usual $VAR \to VMA$ recursion, these terms are missing for $h > p$:

$$BIAS: \; A_{p+1}B_{h-(p+1)} + \ldots + A_{h-1}B_1 + A_h; \quad h > p$$

problem: in practice VARs are truncated too early

LP bias
or lack thereof

when is the LP consistent? i,e, when is this condition met:

$$||\hat{A}_{h,1} - B_h|| \xrightarrow{p} 0; \quad p, T \rightarrow \infty$$

in the LP:

$$\boldsymbol{y}_{t+h} = A_{h,1}\boldsymbol{y}_{t-1} + \ldots + A_{h,p}\boldsymbol{y}_{t-p} + \boldsymbol{u}_{t+h}$$

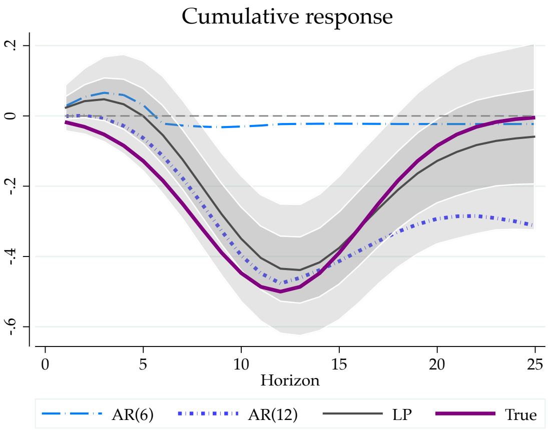
turns out same as consistency of $VAR(p)$, i.e.

$$p^{1/2}\sum_{j=0}^{\infty}||A_{k+j}|| \rightarrow 0$$

see proof in Jordà, Singh, Taylor (2020)

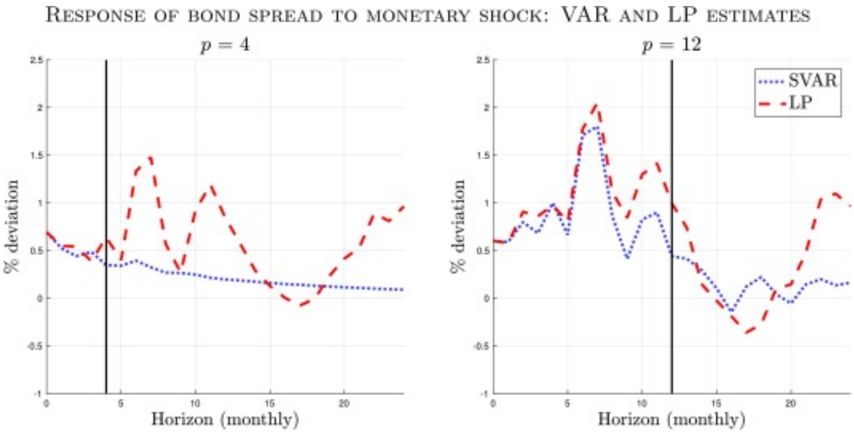
Illustration of VAR vs. LP bias

Based on MA(24) model



Another example

Figure 2 in Palgborg-Møller and Wolf (2021, ECTA)



Multipliers and Counterfactuals

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Two models, same response, different conclusions

Alloza, Gonzalo, Sanz (2020)

$$(a)\begin{cases}\Delta y_t &= \beta \Delta s_t + u_t^y \\ \Delta s_t &= \rho \Delta s_{t-1} + u_t^s\end{cases};\quad (b)\begin{cases}\Delta y_t &= \beta \Delta s_t + \rho \Delta y_{t-1} + u_t^y \\ \Delta s_t &= u_t^s\end{cases};\ \boldsymbol{u}_t \sim D(\mathbf{0}, I)$$

Note: $\mathcal{R}_{sy}^a(h) = \beta\rho^h = \mathcal{R}_{sy}^b(h)$. Both can be estimated with the LP:

$$\Delta y_{t+h} = \gamma_h \Delta s_t + \psi_h \Delta y_{t-1} + v_{t+h}$$

Propagation in (a), due to correlated treatment, in (b) correlated outcome. Consider augmenting LP with treatment leads:

$$\Delta y_{t+h} = \gamma_h \Delta s_t + \psi_h \Delta y_{t-1} + \sum_{i=1}^{h} \phi_i \Delta s_{t+i} + v_{t+h};$$

$\tilde{\mathcal{R}}_{sy}^a(h) = \beta$; $\quad \tilde{\mathcal{R}}_{sy}^b(h) = \beta\rho^h$

What is going on?

- in both cases, Δs_t is strictly exogenous. Leads are allowed in the LP
- in model (a), including leads removes the effect from future potential treatments (due to treatment serial correlation)
- in model (b), on average, there is no expectation of additional treatment. The leads do not matter
- what is the effect of a single treatment? In (a) β, in (b) $\beta\rho^h$
- think of the LP MA(h) residual structure. In general, the MA would have terms in u^y_{t+i} and u^s_{t+i}. But in model (b) coeffs on u^s_{t+i} are all zero
- another way to think about these effects is using multipliers

From previous example

Consider the following model (model (a) earlier):

$$\begin{cases} \Delta y_t & = \beta \Delta s_t + u_t^y \\ \Delta s_t & = \rho \Delta s_{t-1} + u_t^s \end{cases} ; \quad \boldsymbol{u}_t \sim D\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_s \end{pmatrix} \right)$$

Trivially: $\mathcal{R}_{sy}(h) = \beta \rho^h$; $\mathcal{R}_{ss}(h) = \rho^h$

The cumulative impact, $\mathcal{C}_{ij}(h) = \sum_{k=0}^{h} \mathcal{R}_{ij}(k)$ can be directly estimated from:

$$\begin{aligned} y_{t+h} - y_{t-1} = \Delta_h y_{t+h} &= \theta_h \Delta s_t + v_{t+h}^y; \quad v_{t+h}^y \sim MA(h) \\ s_{t+h} - s_{t-1} = \Delta_h s_{t+h} &= \psi_h \Delta s_t + v_{t+h}^s; \quad v_{t+h}^s \sim MA(h) \end{aligned}$$

with $\mathcal{C}_{sy}(h) = \theta_h = \beta \sum_{k=0}^{h} \rho^k$; $\mathcal{C}_{ss}(h) = \psi_h = \sum_{k=0}^{h} \rho^k$

Calculating the multiplier

Define:

$$m_h = \frac{\mathcal{C}_{sy}(h)}{\mathcal{C}_{ss}(h)} = \frac{\beta \sum_{k=0}^{h} \rho^k}{\sum_{k=0}^{h} \rho^k} = \beta; \text{cum. change in } y \text{ due to cum. change in } s$$

Suppose Δz_t is a valid instrument for Δs_t then:

$$\begin{aligned} E(\Delta_h y_{t+h}, \Delta z_t) &= \theta_h E(\Delta s_t \Delta z_t) \\ E(\Delta_h s_{t+h}, \Delta z_t) &= \psi_h E(\Delta s_t \Delta z_t) \end{aligned}$$

hence m_h can be directly estimated from the IV projection:

$$\Delta_h y_{t+h} = m_h \Delta_h s_{t+h} + \eta_{t+h}; \quad \text{instrumented with } \Delta z_t$$

References

- Ramey, Valerie A. 2016. Macroeconomic shocks and their propagation. In *Handbook of Macroeconomics*, Vol. 2, ed. John Taylor and Harald Uhlig. Elsevier, 71–162. Chapter 2.
- Ramey, Valerie A. and Sarah Zubairy. 2018. Government spending multipliers in good times and in bad: Evidence from U.S. historical data. *Journal of Political Economy*, 126(2):850–901.
- Stock, James H. and Mark Watson. 2018. Identification and estimation of dynamic causal effects in macroeconomics using external instruments. *Economic Journal*, 128(610): 917–948.

Panel data applications

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LPs in panels

The set-up

$$y_{i,t+h} = \alpha_i + \delta_t + s_{i,t}\beta_h + \boldsymbol{x}_{i,t}\boldsymbol{\gamma}_h + v_{i,t+h}; \quad i = 1, \ldots, n; \quad t = 1, \ldots, T$$

- α_i unit-fixed effects
- δ_t time-fixed effects
- $\boldsymbol{x}_{i,t}$ exogenous and pre-determined variables
- $s_{i,t}$ treatment variable
- β_h response coefficient of interest

Sample code: LP_example_panel.do

Panel-LPs

Remarks: usual panel data issues appear here too

- LP is costly in short-panels (lost time dimension cross-sections)
- but cross-section brings more power
- incidental parameter issues (fixed effects):
  - beware of high autocorr and low T (Alvarez and Arellano, 2003 ECTA)
  - will need Arellano-Bond or similar estimator
- inference
  - n, T large $\rightarrow$ two-way clustering helps MA(h) and heteroscedasticity
  - n large, T small $\rightarrow$ cluster by unit helps with MA(h)
  - T large, n small $\rightarrow$ cluster by time helps heteroscedasticity
  - else, Driscoll-Kraay is like Newey-West for panel data
  - when clustering with small n, T, may need bootstrap.
    See papers here and here.
    See also `summclust` and `boottest` STATA ado files

Cointegration

A brief detour

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What is cointegration?

Idea: two variables can be I(1) but their linear combination is I(0). Example:

$$\begin{cases} y_{1,t} & = \gamma y_{2,t} + u_{1,t} \\ y_{2,t} & = y_{2,t} + u_{2,t} \end{cases} ; \quad y_{1,t}, y_{2,t} \sim I(1) \quad but \quad z_t = y_{1,t} - \gamma y_{2,t} \sim I(0)$$

In general:

$$\boldsymbol{y}_t = \alpha + \Phi_1 \boldsymbol{y}_{t-1} + \ldots + \Phi_p \boldsymbol{y}_{t-p} + \boldsymbol{u}_t$$

cointegration means:

$$\Phi(1) \equiv I - \Phi_1 - \ldots - \Phi_p \quad then \quad rank(\Phi(1)) = g < n$$

that is, the system has $n - g$ unit roots and g cointegrating vectors, s.t. $\Phi(1) = BA'$ with A, B $n \times g$ matrices, and $A'\boldsymbol{y}_t = z_t$ cointegrating vectors

The VECM representation

Using general representation of a VAR(p)

$$\begin{aligned} \boldsymbol{y}_{t+1} &= \Phi_1 \boldsymbol{y}_t + \ldots + \Phi_{p+1} \boldsymbol{y}_{t-p} + \boldsymbol{\alpha} + \boldsymbol{u}_{t+1} \\ \boldsymbol{y}_{t+1} &= \Psi_1 \Delta \boldsymbol{y}_t + \ldots + \Psi_p \Delta \boldsymbol{y}_{t-p+1} + \Pi \boldsymbol{y}_t + \boldsymbol{\alpha} + \boldsymbol{u}_{t+1} \end{aligned}$$

with $\Psi_j = -[\Phi_{j+1} + \ldots + \Phi_{p+1}]$; for $j = 1, \ldots, p$ and $\Pi = \sum_{j=1}^{p+1} \Phi_j$
subtracting $\boldsymbol{y}_t$ on both sides:

$$\Delta \boldsymbol{y}_{t+1} = \Psi_1 \Delta \boldsymbol{y}_t + \ldots + \Psi_p \Delta \boldsymbol{y}_{t-p+1} + \Psi_0 \boldsymbol{y}_t + \boldsymbol{\alpha} + \boldsymbol{u}_{t+1}$$

Note: $\Psi_0 = -\Phi(1) = BA'$ when there is cointegration, and $\boldsymbol{z}_t = A' \boldsymbol{y}_t$

VECM

$$\Delta \boldsymbol{y}_{t+1} = \Psi_1 \Delta \boldsymbol{y}_t + \ldots + \Psi_p \Delta \boldsymbol{y}_{t-p+1} - B \boldsymbol{z}_t + \boldsymbol{\alpha} + \boldsymbol{u}_{t+1}$$

How does cointegration affect impulse responses?

Remarks

- responses from levels VAR **always** correct
- responses from differenced VAR **only** correct if no cointegration
- cointegration improves efficiency …
- … but estimation and inference more dificult
- responses often not used to investigate LR equilibrium relationships but should
- useful to impose LR exclusion identification restrictions

Cointegrated systems in state-space form

notice:

$$\Psi_0 = \Pi - I = -\Phi(1);$$

if rank $(\Psi_0) < n \rightarrow \Phi(1) = BA'$; cointegrating vector: $\boldsymbol{z}_t = A'\boldsymbol{y}_t$

$$\begin{bmatrix} \boldsymbol{z}_{t+1} \\ \Delta\boldsymbol{y}_{t+1} \\ \Delta\boldsymbol{y}_t \\ \vdots \\ \Delta\boldsymbol{y}_{t-p+1} \end{bmatrix} = \begin{bmatrix} A'\Pi & A'\Psi_1 & \dots & A'\Psi_{p-1} & A'\Psi_p \\ -B & \Psi_1 & \dots & \Psi_{p-1} & \Psi_p \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_t \\ \Delta\boldsymbol{y}_t \\ \Delta\boldsymbol{y}_{t-1} \\ \vdots \\ \Delta\boldsymbol{y}_{t-p} \end{bmatrix} + \begin{bmatrix} A'\boldsymbol{u}_{t+1} \\ \boldsymbol{u}_{t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Z_{t+1} = \boldsymbol{\Psi} Z_t + V_{t+1}$$

Usefulness of state-space representation

Calculating impulse responses through recursive substitution

long-run dynamics:

$$\boldsymbol{z}_{t+h} = \boldsymbol{\Psi}^h_{[1,1]}\boldsymbol{z}_t + \boldsymbol{\Psi}^h_{[1,2]}\Delta y_t + \sum_{j=3}^{p-2} \boldsymbol{\Psi}^h_{[1,j]}\Delta y_{t-j+2} + \boldsymbol{\nu}_{t+h}$$

$$\boldsymbol{\nu}_{t+h} = A'\boldsymbol{u}_{t+h} + A'(I+\Gamma_1)U_{t+h-1} + \ldots + A'(I+\Gamma_1+\ldots+\Gamma_{h-1})U_{t+1}$$

short-run dynamics:

$$\Delta \boldsymbol{y}_{t+h} = \boldsymbol{\Psi}^h_{[2,1]}\boldsymbol{z}_t + \boldsymbol{\Psi}^h_{[2,2]}\Delta \boldsymbol{y}_t + \sum_{j=3}^{p-2} \boldsymbol{\Psi}^h_{[2,j]}\Delta \boldsymbol{y}_{t-j+2} + \boldsymbol{\upsilon}_{t+h}$$

$$\boldsymbol{\upsilon}_{t+h} = \boldsymbol{u}_{t+h} + \Gamma_1\boldsymbol{u}_{t+h-1} + \ldots + \Gamma_{h-1}\boldsymbol{u}_{t+1}$$

where

$$\Delta \boldsymbol{y}_t = \sum_{j=0}^{\infty} \Gamma_j \boldsymbol{u}_{t-j}$$

Responses to equilibrium shocks

equilibrium dynamics, short- vs. long-run effects:

$$\mathcal{R}_z(h; A'u_{t+1} = 1) = (I + \Gamma_1 + \ldots + \Gamma_h)A = \underbrace{\boldsymbol{\Psi}^h_{[1,1]}}_{LR} + \underbrace{\boldsymbol{\Psi}^h_{[1,2]}A}_{SR}$$

short-run dynamics, short- vs long-run effects:

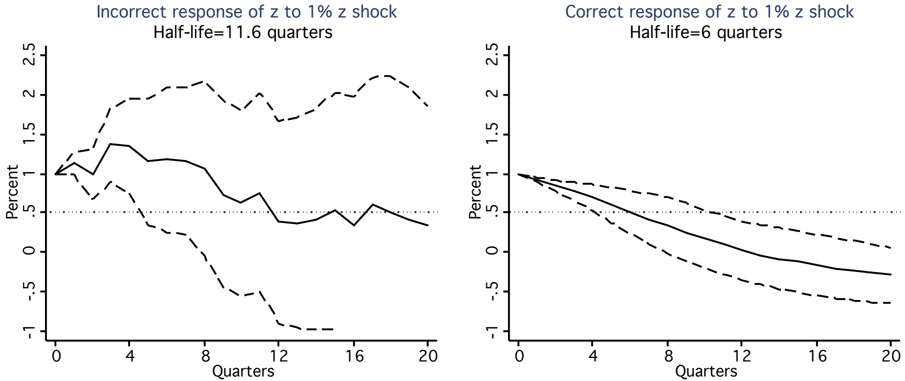
$$\mathcal{R}_{\Delta y}(h; A'u_{t+1} = 1) = \Gamma_h A = \underbrace{\boldsymbol{\Psi}^h_{[2,1]}}_{LR} + \underbrace{\boldsymbol{\Psi}^h_{[2,2]}A}_{SR}$$

remarks:

- note shock cointegrating vector, $\boldsymbol{z}$, not a variable
- each response, 2 parts:
  1. return to equilibrium (LR)
  2. short-run frictions (SR)

Application

Chong, Yanping, Òscar Jordà, and Alan M. Taylor. 2012. The Harrod-Balassa-Samuelson Hypothesis: Real Exchange Rates and their Long-Run Equilibrium. *International Economic Review*, 53(2): 609—634



Variance decompositions

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intuition

two important recent references:

- Gorodnichenko, Yuriy and Byoungchan Lee. 2020. Forecast error variance decompositions with local projections. Journal of Business and Economics Statistics
- Plagborg Møller, Mikkel and Christian K. Wolf. 2022. Instrumental variable identification of dynamic variance decompositions. Journal of Political Economy.

can always write $y_{t+h} = \hat{E}_t(y_{t+h}) + \hat{v}_{t+h}$

then R^2 of regression of $\hat{v}_{t+h}$ on $\epsilon_{j,t+h}, \ldots, \epsilon_{j,t}$ measures percent of FEV explained by j-shock

assumes structural shock $\epsilon_{j,t}$ available

Smooth local projections

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Smoothing

relevant references:

- Barnichon, Regis and Christian Brownlees. 2018. Impulse response estimation by smooth local projections. Available at: https://sites.google.com/site/regisbarnichon/research
- Barnichon, Regis and Christian Matthes. 2018. Functional approximations of impulse responses (FAIR). *Journal of Monetary Economics*, forthcoming.

Many solutions. A simple one: Gaussian Basis Functions

Intuition: impose some cross-horizon discipline to smooth LP wiggles. Can improve efficiency

Other options: bayesian shrinkage
see, e.g. Miranda-Agrippino and Rico. 2018. Bayesian Local Projections

A general approach to smoothing

GMM provides local projection estimates of the response $\hat{\mathcal{R}}$ given by $\hat{\gamma}$ and $\hat{\Sigma}_{\hat{\gamma}}$

a natural solution is minimum distance

let $\boldsymbol{\psi}(\hat{\boldsymbol{\gamma}}, \boldsymbol{\theta})$ be a function that returns a smoothed estimate of $\hat{\boldsymbol{\gamma}}$ based on auxiliary parameters $\boldsymbol{\theta}$, then:

$$\min_{\boldsymbol{\theta}} \left[\hat{\boldsymbol{\gamma}} - \boldsymbol{\psi}(\boldsymbol{\theta})\right]' \hat{\Sigma}_{\gamma} \left[\hat{\boldsymbol{\gamma}} - \boldsymbol{\psi}(\boldsymbol{\theta})\right]$$

delivering, $\hat{\boldsymbol{\theta}}, \hat{\Sigma}_{\hat{\theta}}$ and if $dim(\boldsymbol{\gamma}) > dim(\boldsymbol{\theta})$, a test of overidentifying restrictions for $\boldsymbol{\psi}(\boldsymbol{\theta})$

Smoothing with Gaussian Basis Functions

suppose no controls to simplify

$$\mathcal{R}(h; a, b, c) = \psi(h) = ae^{-\left(\frac{h-b}{c}\right)^2}$$

Using GMM set-up, two estimators: direct v. 2-step
Direct estimator:

$$\min_{a,b,c} \left[\sum_{t=1}^{T} Z_t'(\boldsymbol{y}_{t,H} - S_t\psi(h))\right]' \hat{W} \left[\sum_{t=1}^{T} Z_t'(\boldsymbol{y}_{t,H} - S_t\psi(h))\right]$$

2-step: Step-1 is usual LP, get $\hat{\gamma}, \hat{\Sigma}_\gamma$, then min. distance

$$\min_{a,b,c} \left[\hat{\gamma} - \phi(h)\right]' \hat{\Sigma}_\gamma \left[\hat{\gamma} - \phi(h)\right]$$

GBF–GMM

Remarks

- direct method requires NL estimation techniques
- however, problem is reasonably well behaved
- 2-step method provides useful intuition
- note H-period LP, but 3 parameters so $(H+1)-3$ overidentifying restrictions
- regardless of method, J-test natural specification test
- considerable gain in parsimony $\implies$ efficiency
- GBF approximation works well with ”single humps”
- multiple ”humps” require more basis functions $\implies$ GBF approach no longer as practical

approximation using gaussian basis functions

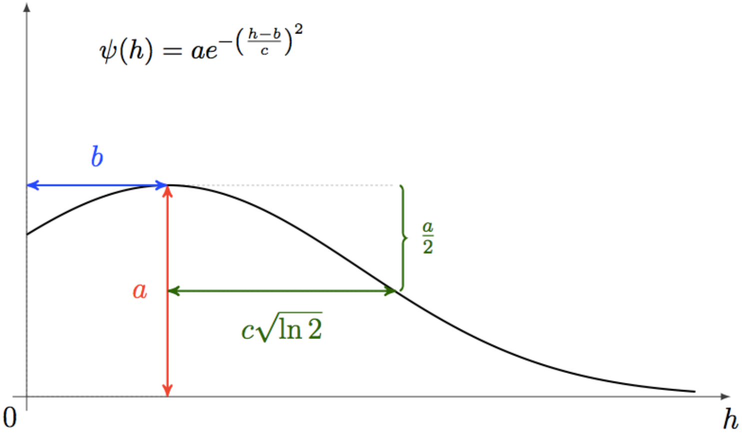
recall:

$$\mathcal{R}(h) = \psi(h) = ae^{-\left(\frac{h-b}{c}\right)^2}$$

what does each parameter do?

- a scales the entire response
- b dates the peak effect
- c measures the half-life

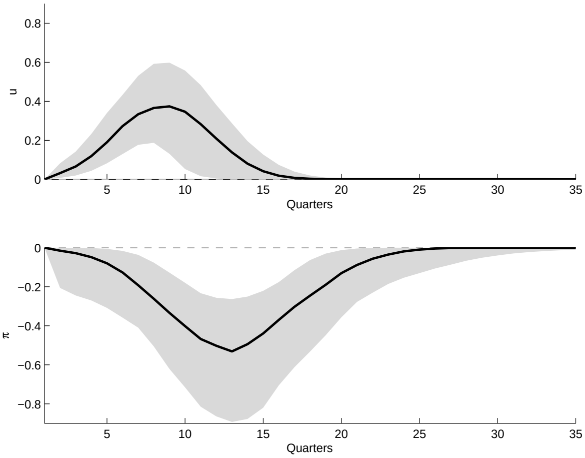
gaussian basis functions
the picture



Sample code: LP_GBF.do

GBF–GMM example

unemployment v. inflation response to monetary policy shock



Nonlinearities and other potential extensions

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The principle

What we are after:

$$\mathcal{R}_{sy}(h) = E[y_{t+h}|s_t = s_0 + \delta; \mathbf{x}_t] - E[y_{t+h}|s_t = s_0; \mathbf{x}_t]$$

No reason to assume the conditional expectation is linear

Example:

$$\begin{aligned} y_{t+h} &= \gamma_{1h} s_t + \gamma_{2h} s_t^2 + \boldsymbol{\gamma} \mathbf{x}_t + v_{t+h} \quad \rightarrow \\ \mathcal{R}_{sy}(h) &= \gamma_{1h}(s_0 + \delta) + \gamma_{2h}(s_0 + \delta)^2 + \boldsymbol{\gamma} \mathbf{x}_t - (\gamma_{1h} s_0 + \gamma_{2h} s_0^2 + \boldsymbol{\gamma} \mathbf{x}_t) \\ &= \gamma_{1h} + \gamma_{2h}(\delta^2 + 2 s_0 \delta) \end{aligned}$$

Hence, $\mathcal{R}_{sy}(h)$ depends on δ and s_0, just like NL regression

Binary dependent variable

Example: response probability of financial crisis to today's credit shock

$$\mathcal{R}_{sy}(h) = P(y_{t+h} = 1 | s_t = s_0 + \delta; \boldsymbol{x}_t) - P(y_{t+h} = 1 | s_t = s_0; \boldsymbol{x}_t)$$

Remarks:

- logit/probit $\rightarrow \mathcal{R}_{sy}(h)$ depends on s_0, δ and $\boldsymbol{x}_t$
- can estimate a linear probability model. But crises are tail events

Another example: Text-based recession probabilities

Ferrari Minesso, M., Lebastard, L. & Le Mezo, H. Text-Based Recession Probabilities. IMF Econ Rev (2022).

Response of recession probability

Marginal effect of 1% increase in newspaper-based index

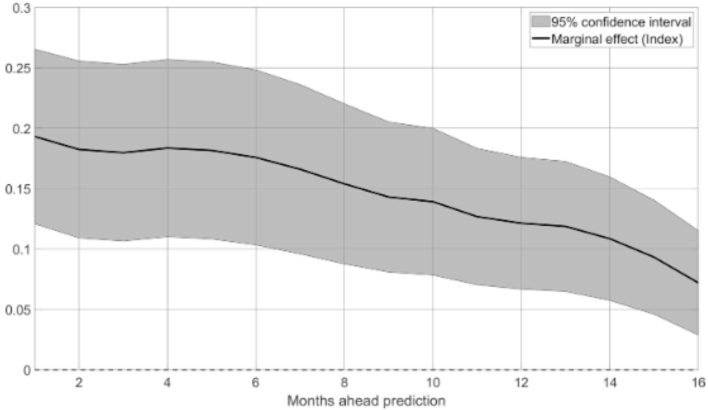


Fig. 2 Marginal effects from Eq. (4.1). Notes: Marginal effects $\left(\frac{\partial P(Recession_{t+k}=1|t)}{\partial Index_t}\right)$ from the probit regression for a 1% increase in the newspaper-based index (i.e., a 1% increase in the share of newspaper articles discussing a recession in the USA). Grey shaded areas report 95% confidence intervals

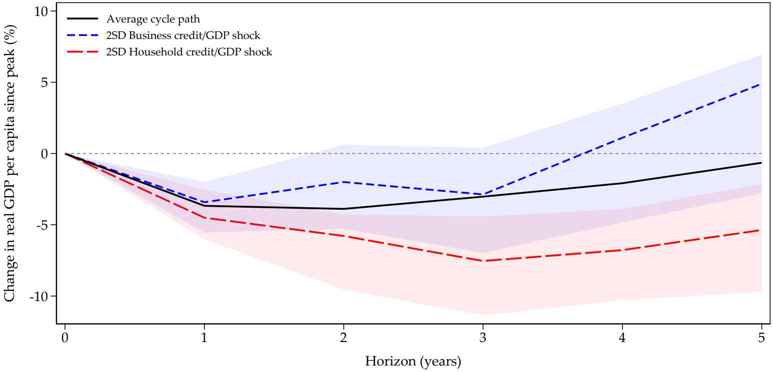
Quantile LPs

Example: does high corporate debt increase risk of left tail GDP draws? Does it depend on legal bankruptcy framework?

$$
\begin{aligned}
\hat{\gamma}_{h,\tau} \;=\; & \underset{\gamma_{h,\tau}}{\text{argmin}} \sum_{1}^{t(P)} \Big(\tau\, \mathbf{1}(\Delta_h y_{it(p)+h} \geq s_{it(p)}\gamma_{h,\tau}) \big| \Delta_h y_{it(p)+h} - s_{it(p)}\gamma_{h,\tau} \big| \\
+ \; & (1-\tau)\, \mathbf{1}(\Delta_h y_{it(p)+h} < s_{it(p)}\gamma_{h,\tau}) \big| \Delta_h y_{it(p)+h} - s_{it(p)}\gamma_{h,\tau} \big| \Big)
\end{aligned}
$$

Jordà, Kornejew, Schularick, and Taylor. 2022. Zombies at large? Corporate debt overhang and the macroeconomy. Review of Financial Studies

Figure A.4: *Business and household debt, responses at 20th percentile of real GDP per capita growth*



Notes: Figures show the predictive effects on growth of a two-SD business/household debt buildup in the five years preceding the recession based on a LP series of quantile regressions. Business credit booms shown in the left-hand side panel and household debt booms shown in the right-hand side panel. Shaded areas denote the 95% confidence interval based on bootstrap replications. See text.

Factor models

Idea: control for many covariates using factor model. Suppose:

$$\begin{cases} \underset{k\times 1}{\boldsymbol{x}_t} & = \lambda(L) \underset{q\times 1}{\boldsymbol{f}_t} + \boldsymbol{e}_t \\ \boldsymbol{f}_t & = \pi(L)\boldsymbol{f}_{t-1} + \boldsymbol{\eta}_t \end{cases} ; \quad k >> q;\ E(\boldsymbol{e}_t) = E(\boldsymbol{\eta}_t) = 0;\ E(\boldsymbol{e}_t\boldsymbol{\eta}'_{t-j}) = 0\ \forall j$$

Then LP can be specified as:

$$y_{t+h} = \beta_h s_t + \sum_{j=0}^{p} \boldsymbol{\gamma}_j \boldsymbol{f}_{t-j} + v_{t+h}$$