Online Appendix for "Simple Manipulations in School Choice Mechanisms"

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January 3, 2025

Appendix A Discussion on Proposition 1

Appendix A.1 The DA+TTC Mechanism

In Proposition 1, we show that the EADA is an efficient Pareto improvement on the DA while satisfying our weak strategy-proofness. Another well-known intuitive Pareto-improving procedure is the DA+TTC mechanism, which first runs the DA and then reallocates the allocation using Shapley and Scarf's (1974) top-trading cycles algorithm. In this subsection, we show that our criterion clearly distinguishes between these two seemingly similar mechanisms regarding their incentive properties.

To begin, we define the DA+TTC mechanism. Throughout this subsection, we assume that the choice rules C are acceptant and responsive for a linear order profile $\succ$. In the context of school choice, the top-trading cycles (TTC) algorithm operates as follows:

- Step $k (\geq 1)$. Each student in the remaining population points to the most preferable available choice. Each school points to the student with the highest priority in the population. Then, there exists at least one cycle $(i_k, s_k)_{k=1}^{K}$ such that each student i_k points to s_k and each s_k points to i_{k+1}, where subscripts are modulo K. All students in cycles are permanently assigned to the one they point to. Remove the matched students and the associated capacities.

- The algorithm ends at the step where no students are permanently assigned.

Abdulkadiroğlu and Sönmez (2003) define the this algorithm, which adapts the classic TTC algorithm to a school choice setting. They show that the matchings produced by the TTC mechanism are efficient and that the mechanism is strategy-proof.

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The DA+TTC algorithm is defined as follows, building upon the TTC algorithm:[1]

- DA Round. Run the DA. Let μ be the DA matching.

- TTC Round. For each school s, create a new priority order $\succ'_s$ from $\succ_s$ by raising all students in $\mu^{-1}(s)$ to the top, keeping other relative orderings unchanged. Run the TTC algorithm using the student preferences and the new priority orders.

By construction, the DA+TTC mechanism clearly Pareto dominates the DA. Furthermore, a standard argument involving top-trading cycles establishes its efficiency.[2] However, as discussed in the main paper, no Pareto improvement on the DA is strategy-proof, which implies that the DA+TTC mechanism also violates strategy-proofness.

Now, we are ready to present our observation. The DA+TTC mechanism does not satisfy the incentive conditions outlined in Proposition 1. Consequently, although the EADA and the DA+TTC mechanisms may seem similar, there is a significant distinction in their incentive properties.

Proposition A.1. *The DA+TTC mechanism does not admit a profitable bottom-dropping. However, it admits a profitable top-dropping.*

Proof. Let φ be the DA+TTC mechanism.

First, we show that there is no profitable bottom-dropping. Take any $R = (R_i, R_{-i})$. Let R'_i be a bottom-dropping of R_i. Define $R' = (R'_i, R_{-i})$. Suppose that i is unmatched at $\varphi^{DA}(R')$. Then, since the DA is stable and strategy-proof, we have $\varphi^{DA}(R')(i) = \varnothing$. Hence, Lemma 1 implies $\varphi(R')(i) = \varnothing$, and thus, R'_i is not profitable. Next, suppose that i is matched with some school, that is, $\varphi^{DA}(R')(i) \neq \varnothing$. Then, $\varphi^{DA}(R) = \varphi^{DA}(R')$ as the DA satisfies anti-bottom-dropping monotonicity. This implies that $\varphi(R) = \varphi(R')$ by the definition of the DA+TTC algorithm.

Second, we show by an example that there exists a profitable top-dropping in general. Suppose that there are five students $I = \{i_1, i_2, i_3, i_4, i_5\}$ and four schools $S = \{s_1, s_2, s_3, s_4\}$. Each school has a unit capacity. Then, let $\succ$ and R be defined as in the following tables:

[1] We adopt the description given in Troyan et al. (2020).

[2] See, e.g., Shapley and Scarf (1974).

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_1	i_2	i_3	i_4
i_5	i_3	i_1	i_3
i_2	i_1	i_2	i_1
$\vdots$	$\vdots$	i_4	$\vdots$
		$\vdots$	

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
s_3	$\bullet\ s_1$	$\bullet\ s_4$	$\bullet\ s_3$	s_1
s_4	s_3	s_2	$\boxed{s_4}$	$\bullet\ \boxed{\varnothing}$
$\bullet\ s_2$	$\boxed{s_2}$	$\boxed{s_3}$	$\varnothing$	$\vdots$
$\boxed{s_1}$	$\varnothing$	$\varnothing$	$\vdots$	
$\varnothing$	$\vdots$	$\vdots$		

Now, we can compute the DA+TTC matching under these preferences and priority relations. In the DA round, the DA algorithm produces a matching that is marked with boxes in the above preference list, wherein i_k is matched with s_k for each $k = 1, 2, 3, 4$, and i_5 is unmatched. Given this, note that a priority profile is unchanged in the second TTC round. Therefore, running the TTC algorithm using $\succ$ and R produces the DA+TTC matching, which is marked with bullet points. In particular, i_1 is matched with s_2.

Next, consider the following preference profile. All students except for i_1 have the same preferences as before. Note that R'_{i_1} is a top-dropping of R_{i_1}.

R'_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
$\bullet\ s_4$	$\bullet\ s_1$	s_4	$\bullet\ s_3$	s_1
s_2	$\boxed{s_3}$	$\bullet\ \boxed{s_2}$	$\boxed{s_4}$	$\bullet\ \boxed{\varnothing}$
$\boxed{s_1}$	s_2	s_3	$\varnothing$	$\vdots$
$\varnothing$	$\varnothing$	$\varnothing$	$\vdots$	
s_3	$\vdots$	$\vdots$		

Likewise, we can compute the DA matching, which is marked with boxes. Here, in the second TTC round, we create new priority orders for schools s_2 and s_3 by raising i_3 and i_2 to the top, respectively. Priority orders for schools s_1 and s_4 are unchanged. Then, running the TTC algorithm using the new priority, we get a matching marked with bullet points in the above preference list. In particular, i_1 is matched with s_4, which is strictly preferred over s_2 under the original preference R_{i_1}. Therefore, R'_{i_1} is a profitable manipulation of R_{i_1} and the DA+TTC admits a profitable top-dropping. □

The intuition behind this latter result is as follows: Because the DA satisfies weak Maskin monotonicity, a top-dropping transformation improves the initial round of DA matching. However, during the subsequent TTC algorithm, it may alter a student to which each school points. This alteration can lead to the formation of a different trading cycle, thereby producing a distinct outcome. Notably, this misreporting student may end up in a more favorable position, as illustrated in the example above.

Appendix A.2 The Converse of Proposition 1

A natural theoretical question arising from Proposition 1 is whether the EADA is the only mechanism possessing these properties. The following example illustrates that, in general, there are other efficient Pareto improvements over the DA that also satisfy our weak strategy-proof conditions. However, the main theorem indicates that the EADA is a unique mechanism when we impose the additional requirement that no simple collective top-droppings or bottom-droppings are profitable.

Example A.1. Consider a situation in which there are four students $I = \{i_1, i_2, i_3, i_4\}$ and two schools $S = \{s_1, s_2\}$. School s_1 has a unit capacity and school s_2 has two seats. Let C^* and R^* be acceptant responsive choice rules with the following order profile $\succ^*$ and the preferences of the following forms, respectively.

$\succ^*_{s_1}$	$\succ^*_{s_2}$
i_4	i_1
i_1	i_2
i_2	i_3
i_3	i_4

$R^*_{i_1}$	$R^*_{i_2}$	$R^*_{i_3}$	$R^*_{i_4}$
$\underline{s_1}$	$\bullet\ s_1$	s_2	$\bullet\ \underline{s_2}$
$\bullet\ s_2$	$\underline{s_2}$	s_1	s_1
$\varnothing$	$\varnothing$	$\bullet\ \underline{\varnothing}$	$\varnothing$

Let μ^* be the matching that is marked with bullet points in the above preference list: $\mu^*(i_1) = s_2$, $\mu^*(i_2) = s_1$, $\mu^*(i_3) = \varnothing$, and $\mu^*(i_4) = s_2$. A computation shows that μ^* is an efficient matching that Pareto dominates the DA matching at R^* and C^*. We also have $\mu^* \neq \varphi^{EDA}(R^*, C^*)$, where the EADA matching is underlined above.

We are now ready to provide a counterexample. Consider a mechanism φ such that $\varphi(R^*, C^*) = \mu^*$ and $\varphi(R, C) = \varphi^{EDA}(R, C)$ for all other pairs of R and C. It follows from the definition that $\varphi \neq \varphi^{EDA}$ is an efficient Pareto improvement on the DA. It remains to show that φ has no profitable top-dropping/bottom-dropping.

Take any i and a pair R and C. If $C \neq C^*$ holds, then φ coincides with φ^{EDA}, and thus, Proposition 1 implies i has no profitable top-dropping/bottom-dropping. Hence, assume $C = C^*$. Let $R'_i \neq R_i$ be either a top-dropping or a bottom-dropping of R_i. Write $s = \varphi(R, C)(i)$ and $s' = \varphi(R'_i, R_{-i}, C)(i)$. There are three cases to consider.

First, suppose $s \neq \varphi^{EDA}(R, C)(i)$ and $s' = \varphi^{EDA}(R'_i, R_{-i}, C)(i)$. This implies that if the manipulation is profitable, we must have $i = i_1$ because i_2 matches with the most preferred choice under R. If R'_i is a bottom-dropping of R_i, then a calculation shows $s' = \varnothing$, which is thus unprofitable. If R'_i is a top-dropping of R_i, then we must have $s' = s_2$, which again means the manipulation is not profitable.

Second, suppose $s = \varphi^{EDA}(R, C)(i)$ and $s' \neq \varphi^{EDA}(R'_i, R_{-i}, C)(i)$. Note that R'_i cannot be a bottom-dropping of R_i and is thus a top-dropping of R_i. Additionally, either $i = i_1$ or $i = i_2$ holds, which implies that either s_2 or $\varnothing$ is the most preferred choice under R_i. Then, a calculation using $R_{-i} = R^*_{-i}$ shows that i must match with the most preferred choice before the manipulation, and thus there is no incentive for the manipulation.

Finally, for the remaining case where both s and s' match with the assignments under the EADA, Proposition 1 implies that the manipulation is unprofitable. Therefore, there exists another efficient Pareto improvement on the DA that has no profitable top-dropping/bottom-dropping. □

Appendix A.3 Other Simple Manipulations

In Proposition 1, we find that the EADA mechanism represents an efficient Pareto improvement over the DA that allows for no profitable top-dropping or bottom-dropping. In contrast, as indicated in the subsequent proposition, all efficient Pareto improvements on the DA admit a profitable anti-bottom-dropping. This suggests that it may be theoretically challenging to require that all simple manipulations be unprofitable.

Proposition A.2. *All mechanisms that Pareto dominate the DA admit a profitable anti-bottom-dropping.*

Proof. Afacan et al.'s (2022) proof of their Proposition 1 works directly as a proof of this statement. Since their original statement is different, however, we provide a proof for completeness. Let a mechanism φ Pareto dominate the DA. Then, there exist R and C such that $\varphi(R, C) R \varphi^{DA}(R, C)$ and $\varphi(R, C)(i) P_i \varphi^{DA}(R, C)(i)$ for some i. We write $s = \varphi(R, C)(i)$ and $s' = \varphi^{DA}(R, C)(i)$ to abbreviate notation.

Now, let R'_i be a preference that truncates from R_i all schools that are strictly less preferred to s, as defined in the proof of Proposition 6. Then, we have $\varnothing P'_i s'$ by construction, and therefore, one can see that $\varphi^{DA}(R'_i, R_{-i}, C)(i) = \varnothing$ by the definition of the DA algorithm. Then, since the outside option is always underdemanded under the DA, Lemma 1 in the Appendix implies that $\varphi(R'_i, R_{-i}, C)(i) = \varnothing$. Finally, note that $s P'_i \varnothing$ by construction. Moreover, R_i is an anti-bottom-dropping of R'_i by construction. Therefore, R_i is a profitable anti-bottom-dropping of R'_i. □

Appendix B Discussion on Theorem 1

Appendix B.1 On Top-dropping Monotonicity

In Remark 1, we observe that top-dropping monotonicity implies a stronger condition, specifically, invariance to the upper-manipulation described by Afacan et al. (2022). Below, we provide a formal discussion of this statement. An upper-manipulation is defined as follows.

Definition B.1. A preference R'_i is an *upper-manipulation* of R_i at s, if R'_i is a monotonic transformation of R_i at s, and $s' R'_i s''$ implies $s' R_i s''$ for all s' and s'' with $s P_i s', s''$.

We say that a preference profile R' is an upper-manipulation of R at a matching μ if each R_i is an upper-manipulation of R_i at $\mu(i)$. A mechanism is *upper-manipulation-proof* if for any R, any upper-manipulation of R_i at $\varphi(R, C)(i)$ is not profitable.

A key conceptual distinction between top-dropping and upper-manipulation is that our definition of top-dropping operates independently of the assignment resulting from truthful reporting. This feature makes top-dropping much easier to describe in plain language, which is crucial for the comprehension of real-world participants.

The formal argument presented in Remark 1 is stated as follows. In the proof, we represent a preference as a vector: $R_i = (s_1, s_2, \ldots, s_{|S|+1})$, where the l-th element s_l represents the l-th preferred option under R_i.

Proposition B.3. *Suppose that a mechanism φ satisfies top-dropping monotonicity. Then, if R' is an upper-manipulation of R at $\varphi(R, C)$, we have $\varphi(R', C) = \varphi(R, C)$.*

Proof. Take any student i. It is enough to prove for the case $R'_j = R_j$ for all $j \neq i$. Suppose $R'_i = (s_1, s_2, \ldots, s_{|S|+1})$. Let $s_k \equiv \varphi(R', C)(i)$. The definition of upper-manipulation implies that $s_1, \ldots, s_{k-1}$ are ranked above s_k under R_i as well. Moreover, relative rankings among those s with $s_k R_i s$ are the same between the two preferences R_i and R'_i.

Consider the following algorithm, which outputs a manipulation of an input R_i: Let s_l be the most preferred choice. If $l \leq k-2$, other ordering being equal, place s_l to the point right before s_k in the preference. If $l \geq k-1$, place s_l to the point right before s_{l+1}. Repeat this until s_{k-1} is placed right before s_k, which ends within $k-1$ steps.

Now, we get a preference $R^1_i = (\ldots, s_{k-1}, s_k, \ldots)$. Since R^1_i is derived from R_i by an iteration of top-droppings at $\varphi(R, C)(i)$, top-dropping monotonicity implies $\varphi(R, C) = \varphi(R^1_i, R_{-i}, C)$. Next, starting from R^1_i, run the above algorithm where k is replaced with $k-1$. Then, we get $R^2_i = (\ldots, s_{k-2}, s_{k-1}, s_k, \ldots)$. By the same rationale, we have $\varphi(R^1_i, R_{-i}, C) = \varphi(R^2_i, R_{-i}, C)$, hence $\varphi(R, C) = \varphi(R^2_i, R_{-i}, C)$.

Repeating this procedure, we get R^{k-1}, where the first k coordinates of R_i^{k-1} and R_i' coincide. Recall that ranking among those s with $s_k R_i s$ are the same between the two preferences R_i and R_i'. Therefore, together with the definition of the algorithm, we have $R_i^{k-1} = R_i'$. Hence, $\varphi(R, C) = \varphi(R_i^k, R_{-i}, C) = \varphi(R', C)$. □

We say that a mechanism φ satisfies *upper-manipulation monotonicity* if for any R', R, and C, we have $\varphi(R, C) = \varphi(R', C)$ whenever R' is an upper-manipulation of R at $\varphi(R, C)$. Proposition B.3 shows that Theorem 1 remains true if we replace top-dropping monotonicity with upper-manipulation monotonicity.

Corollary B.1. *A mechanism φ satisfies upper-manipulation monotonicity, and anti-bottom-dropping monotonicity, and respects top-top pairs, if and only if $\varphi = \varphi^{EDA}$.*

Appendix B.2 Independence of Axioms

In this section, we establish the logical independence of the three axioms presented in Theorem 1. Specifically, we demonstrate that for each axiom, there exists a mechanism that does not satisfy that particular axiom while still satisfying the other two. The examples below illustrate this point.

Example B.2. In this example, we see that anti-bottom-dropping monotonicity and respecting top-top pairs do not imply top-dropping monotonicity in general.

Let us consider the DA mechanism φ^{DA}. As φ^{DA} is a stable mechanism, it respects top-top pairs. Moreover, φ^{DA} satisfies anti-bottom-dropping monotonicity, which one can see by the definition of the DA algorithm.

Meanwhile, φ^{DA} fails to satisfy top-dropping monotonicity. To illustrate, suppose that there are three students $I = \{i, j, k\}$ and two schools $S = \{s_1, s_2\}$ with capacities $q_{s_1} = q_{s_2} = 1$. Assume acceptant responsive choice rules C_{s_1} and C_{s_2} for the following orders. Also, consider the following list of student preferences.

$\succ_{s_1}$	$\succ_{s_2}$
k	i
j	j
i	k

R_i	R_j	R_j'	R_k
$\underline{s_1}$	s_1	$\underline{\varnothing}$	$\underline{s_2}$
$\boxed{s_2}$	$\boxed{\varnothing}$	s_1	$\boxed{s_1}$
$\varnothing$	s_2	s_2	$\varnothing$

Now, if we run the DA at $R = (R_i, R_j, R_k)$, it produces the matching $\varphi^{DA}(R, C)$ marked with boxes in the above preference list. Note that R_j' is a top-dropping of R_j, and that j does not match with the student's most preferred school s_1 under $\varphi^{DA}(R, C)$. However, for $R' = (R_i, R_j', R_k)$, the matching $\varphi^{DA}(R', C)$ underlined above does not match with $\varphi^{DA}(R, C)$, meaning φ^{DA} does not satisfy top-dropping monotonicity. □

Example B.3. The following example shows that top-dropping monotonicity and respecting top-top pairs do not imply anti-bottom-dropping monotonicity in general.

Consider the *school-proposing DA mechanism*, whose outputs are given by the following algorithm for each input R and C:

- Step 1. Every school $s \in S$ applies to the students $C_s(I)$. Students tentatively accept the most preferred school among acceptable applicants and reject the rest.
- Step $k(\geq 2)$. Every school $s \in S$ applies to the students $C_s(N)$, where N is the set of students who have not rejected s in the earlier steps. Students tentatively accept the most preferred school among acceptable applicants as well as the previously tentatively accepted school and reject the rest.

The algorithm ends at the step when no school is rejected by a student. Each school tentatively accepted by a student in the last step matches with the student. All remaining students match with the outside option.

Let φ^{SDA} be the school-proposing DA mechanism. It respects top-top pairs because it is a stable mechanism (See, e.g., Roth and Sotomayor (1992)).

Now, we show that φ^{SDA} satisfies top-dropping monotonicity. Let R_i' be a top-dropping of R_i, and suppose that i does not match with their most preferred choice s at R and C. Hence, $s \neq \varnothing$ because φ^{SDA} is individually rational. Moreover, the school s has never applied to i during the algorithm. Therefore, under the inputs $R' = (R_i', R_{I\setminus\{i\}})$ and C, the algorithm runs in the same manner as under R and C. Thus, we have $\varphi^{SDA}(R', C) = \varphi^{SDA}(R, C)$. Hence, φ^{SDA} satisfies top-dropping monotonicity.

Finally, the following example shows that φ does not satisfy anti-bottom-dropping monotonicity. Suppose $I = \{i, j\}$ and $S = \{s_1, s_2\}$, each with capacity 1. Assume acceptant responsive choice rules C_{s_1} and C_{s_2} for orders $\succ_{s_1}$ and $\succ_{s_2}$ such that $j \succ_{s_1} i$ and $i \succ_{s_2} j$. Also, consider the following lists of student preferences:

R_i	R_i'	R_j
$\boxed{s_1}$	s_1	$\boxed{s_2}$
$\varnothing$	$\underline{s_2}$	$\underline{s_1}$
s_2	$\varnothing$	$\varnothing$

If we run the school-proposing DA under the preference profile $R = (R_i, R_j)$, we obtain the matching $\varphi^{SDA}(R, C)$ marked with boxes in the above preference list. Now, $\varphi^{SDA}(R, C)(i) \neq \varnothing$, and R_i' is an anti-bottom-dropping of R_i. Yet, for an anti-bottom-dropping $R' = (R_i', R_j)$ of the profile R at $\varphi^{SDA}(R, C)$, we get $\varphi^{SDA}(R', C)$, underlined in the above list. Hence, this example shows that φ^{SDA} does not satisfy anti-bottom-dropping monotonicity. □

Example B.4. Consider a null mechanism $\varphi^{\varnothing}$ that always assigns the matching μ such that $\mu(i) = \varnothing$ for each student $i \in I$. This mechanism satisfies top-dropping/anti-bottom-dropping monotonicity. Meanwhile, $\varphi^{\varnothing}$ does not respect top-top pairs: In a setting $I = \{i\}$ and $S = \{s\}$ with $q_s = 1$, for example, if s is acceptable under R, then respecting top-top pairs requires $\varphi^{\varnothing}(R, C)(i) = s$ for any C. □

Appendix B.3 The EADA with Partial Consent

In the paper, we focus on the EADA mechanism where all students consent to their priorities being violated. However, Kesten (2010) define a general mechanism that incorporates a consenting constraint. In this context, we present a result that generalizes Theorem 1 to accommodate scenarios involving partial consent.

First, we introduce the EADA mechanism under partial consent. Fix a subset $J \subset I$ of students, which we refer to as a consenting constraint. An interpretation is that a student i consents to his/her priority being violated if and only if $i \in J$. Then, Ehlers and Morrill's (2020) simplified EADA mechanism with consenting constraint, denoted by $\varphi^{EDA(J)}$, outputs the matching obtained by the following algorithm for each input R and C:[3]

- Step 1. Run the DA at (R, C). For each underdemanded $s \in S \cup \{\varnothing\}$ and each $i \in J$ assigned to s, permanently assign i to s and remove both i and s.

- Step $k(\geq 2)$. Run the DA at (R, C) on the remaining population. For each underdemanded $s \in S \cup \{\varnothing\}$ and each $i \in J$ assigned to s, permanently assign i to s and remove both i and s.

The algorithm terminates when all students i matched with an underdemanded alternative do not consent to their priority violations, that is, $i \notin J$. Intuitively, in each step, students who do not consent can remain in the market, ensuring that their priorities are not violated in subsequent steps. Note that $\varphi^{EDA} = \varphi^{EDA(I)}$. It is straightforward to see that the mechanism $\varphi^{EDA(J)}$ weakly Pareto dominates the DA.

It turns out that the EADA with a consenting constraint $J \neq I$ does not satisfy top-dropping monotonicity.[4] Consequently, we propose the following weaker axiom, defined in the spirit of Kojima and Manea's (2010) weak Maskin monotonicity.

[3]To the best of our knowledge, no study defines the EADA with partial consent under general choice rules beyond responsive ones. Here, we extend Ehlers and Morrill's (2020) algorithm incorporating consenting constraints in a straightforward manner.

[4]Note that Proposition 4 shows that the EADA under full consent is the only mechanism that weakly Pareto dominates the DA and satisfies top-dropping monotonicity.

Definition B.2. We say that a mechanism φ satisfies weak top-dropping monotonicity if for any top-dropping R' of R at $\varphi(R, C)$, we have $\varphi(R', C)R'\varphi(R, C)$.

The following result can be viewed as an analogous counterpart to Theorem 1 that characterizes the EADA under full consent with top-dropping monotonicity, anti-bottom-dropping monotonicity, and respecting top-top pairs.

Proposition B.4. *The mechanism $\varphi^{EDA(J)}$ satisfies weak top-dropping monotonicity, anti-bottom-dropping monotonicity, and respects top-top pairs.*

Proof. We show that $\varphi^{EDA(J)}$ is weakly top-dropping monotonic. The proof for the other two properties is a trivial modification of that for the EADA with full consent.[5] For simplifying notation, we fix C and abbreviate it throughout.

Let $\bar{R}$ be a top-dropping (t.d.) of R at $\varphi^{EDA(J)}(R)$. As in the proof of Lemma 5, let two sequences $R^1, \ldots, R^K$ and $\bar{R}^1, \ldots, \bar{R}^{\bar{K}}$ be induced by the EADA algorithm with consenting constraint J under R and $\bar{R}$, respectively. If $K > \bar{K}$, define $\bar{R}^k \equiv \bar{R}^{\bar{K}}$ for each k with $\bar{K} < k \leq K$. Then, for each k, let J^k and $\bar{J}^k$ be those in J who match with underdemanded choices at $(R^k, \varphi^{EDA(J)}(R^k))$ and $(\bar{R}^k, \varphi^{EDA(J)}(\bar{R}^k))$, respectively.

First, we prove by mathematical induction that $\bar{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$. Define $R^0 \equiv R$ and $\bar{R}^0 \equiv \bar{R}$. Then, the argument is true at $k = 0$. Suppose that it is true that $k-1$ with $k \geq 1$. Then, weak Maskin monotonicity (Kojima and Manea (2010)) implies that

$$\varphi^{DA}(\bar{R}^{k-1})R^{k-1}\varphi^{DA}(R^{k-1}).$$

Thus, from Lemma 4, $J^{k-1} \subset \bar{J}^{k-1}$.

Now, we prove the argument at k. Define $\hat{R}^k \equiv (\bar{R}^k_{J^{k-1}}, \bar{R}^{k-1}_{I \setminus J^{k-1}})$. To begin with, we show that $\hat{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$. Take any $i \in I$. If $i \in J^{k-1}$, Lemma 1 implies

$$\varphi^{DA}(\bar{R}^{k-1})(i) = \varphi^{DA}(R^{k-1})(i) = \varphi^{DA}(R^k)(i) \equiv s.$$

Therefore, both $\bar{R}^k_i$ and R^k_i are top-droppings of $\bar{R}^{k-1}_i$ and R^{k-1}_i that rank s at the top. Hence, $\hat{R}^k_i = \bar{R}^k_i$ t.d. R^k_i at $\varphi^{DA}(R^k)$. If $i \notin J^{k-1}$, $R^k_i = R_i$ and $\hat{R}^k_i = \bar{R}_i$. Hence, $\hat{R}^k_i$ t.d. R^k_i at $\varphi^{EDA(J)}(R)$ and thus at $\varphi^{DA}(R^k)$. Summarizing, $\hat{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$.

Next, we show that $\bar{R}^k$ t.d. $\hat{R}^k$ at $\varphi^{DA}(\hat{R}^k)$. To prove this, we first show that $s\bar{R}^k_i\varphi^{DA}(\hat{R}^k)(i)$ implies $s\bar{R}^{k-1}_i\varphi^{DA}(\hat{R}^k)(i)$. Note that it is straightforward when $i \notin \bar{J}^{k-1}$. If $i \in \bar{J}^{k-1}$, then note that $\hat{R}^k$ t.d. $\bar{R}^{k-1}$ at $\varphi^{DA}(\bar{R}^{k-1})$ by the construction of $\hat{R}^k$. Hence, weak Maskin monotonicity and Lemma 1 implies $\varphi^{DA}(\hat{R}^k)(i) = \varphi^{DA}(\bar{R}^{k-1})(i)$. Thus, we must have $s = \varphi^{DA}(\hat{R}^k)(i)$ by the construction of $\bar{R}^k$, and thus, $s\bar{R}^{k-1}_i\varphi^{DA}(\hat{R}^k)(i)$. Now,

[5] In Lemma 5 and Lemma 6, we rely much on the efficiency of φ^{EDA} in proving top-dropping monotonicity. Consequently, we need to modify the proof in a slightly non-trivial manner.

weak Maskin monotonicity implies $\varphi^{DA}(\bar{R}^k)\bar{R}^k\varphi^{DA}(\hat{R}^k)$. To prove that $\bar{R}^k$ t.d. $\hat{R}^k$ at $\varphi^{DA}(\hat{R}^k)$, take any $i \in I$. If $i \in J^{k-1}$, then $\bar{R}_i^k = \hat{R}_i^k$. If $i \notin \bar{J}^{k-1}$, then $\bar{R}_i^k = \bar{R}_i^{k-1} = \hat{R}_i^k$. Finally, let $i \notin J^{k-1}$ and $i \in \bar{J}^{k-1}$. Then, we have $\hat{R}_i^k = \bar{R}_i^{k-1}$. As $\bar{R}_i^k$ t.d. $\hat{R}_i^k = \bar{R}_i^{k-1}$ at $\varphi^{DA}(\bar{R}^k)(i) = \varphi^{DA}(\bar{R}^{k-1})(i)$ by definition, so is at $\varphi^{DA}(\hat{R}^k)(i)$ by $\varphi^{DA}(\bar{R}^k)\bar{R}^k\varphi^{DA}(\hat{R}^k)$. In summary, $\bar{R}^k$ t.d. $\hat{R}^k$ at $\varphi^{DA}(\hat{R}^k)$.

Therefore, we have $\hat{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$ and $\bar{R}^k$ t.d. $\hat{R}^k$ at $\varphi^{DA}(\hat{R}^k)$. Thus, the weak Maskin monotonicity of the DA implies that

$$\varphi^{DA}(\bar{R}^k)\bar{R}^k\varphi^{DA}(\hat{R}^k)\hat{R}^k\varphi^{DA}(R^k).$$

Hence, $\varphi^{DA}(\bar{R}^k)\hat{R}^k\varphi^{DA}(R^k)$. As $\hat{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$, we get $\varphi^{DA}(\bar{R}^k)R^k\varphi^{DA}(R^k)$.

Now, we show that $\bar{R}^k$ t.d. R^k at $\varphi^{DA}(R^k)$. Take any $i \in I$. If $i \in J^{k-1}$, then, as seen earlier, $\bar{R}_i^k = \hat{R}_i^k$ t.d. R_i^k at $\varphi^{DA}(R^k)(i)$. If we instead have $i \notin J^{k-1}$, then, $R_i^k = R_i$. Then, there are two sub-cases. If $i \notin \bar{J}^{k-1}$, then $\bar{R}_i^k = \bar{R}_i$. Since $\bar{R}_i$ t.d. R_i at $\varphi^{EDA(J)}(R)$, so is at $\varphi^{DA}(R^k)$. If $i \in \bar{J}^{k-1}$, $\bar{R}_i^k$ is obtained from $\bar{R}_i^{k-1}$ by dropping schools preferred to $\varphi^{DA}(\bar{R}^{k-1})(i)$ under $\bar{R}_i^{k-1}$. Therefore, using Lemma 1, $\bar{R}_i^k$ t.d. $\bar{R}_i^{k-1}$ at $\varphi^{DA}(\bar{R}^k)(i)$. Since we have $\bar{R}_i^{k-1}$ t.d. R_i^{k-1} at $\varphi^{DA}(R^{k-1})$ and $\varphi^{DA}(\bar{R}^k)R^k\varphi^{DA}(R^k)$, we can conclude that $\bar{R}_i^k$ t.d. $R_i^k = R_i^{k-1} = R_i$ at $\varphi^{DA}(R^k)$.

Finally, we prove that $\varphi^{EDA(J)}$ satisfies weak top-dropping monotonicity. The above argument at $k = K$ shows that $\bar{R}^K$ t.d. R^K at $\varphi^{DA}(R^K)$. Therefore, from the weak Maskin monotonicity of the DA,

$$\varphi^{EDA(J)}(\bar{R})\bar{R}\varphi^{DA}(\bar{R}^K)\bar{R}^K\varphi^{DA}(R^K) = \varphi^{EDA(J)}(R),$$

where the first and the last relations follow by the construction of the sequences. Note that $\bar{R}^K$ is obtained from $\bar{R}$ by dropping some schools that are preferred to $\varphi^{DA}(\bar{R}^K)$. Therefore, from the second relation, $\bar{R}^K$ t.d. $\bar{R}$ at $\varphi^{DA}(\bar{R}^K)$, which implies $\varphi^{DA}(\bar{R}^K)\bar{R}\varphi^{DA}(R^K)$. Thus, we eventually have the desired relation. □

The converse is not true, and therefore, this proposition is not a characterization for the EADA with partial consent. To see this, consider a mechanism whose output is the matching obtained by the EADA algorithm (with full consent $J = I$) that stops in two steps. An analogous proof shows that all three conditions are satisfied under this alternative mechanism.

Appendix C Discussion on Section 6

Appendix C.1 Discussion on Proposition 8

In this section, we demonstrate the independence of the three axioms presented in Proposition 8. Theorem 1 implies that the EADA satisfies anti-bottom-dropping monotonicity and respects top-top pairs, but it is not strategy-proof. The null mechanism $\varphi^{\varnothing}$, which always outputs a matching wherein all students are unmatched, is a simple example of mechanisms that satisfy strategy-proofness and anti-bottom-dropping monotonicity while not respecting top-top pairs. The next example shows that strategy-proofness and respecting top-top pairs do not imply anti-bottom-dropping monotonicity in general.

Example C.5. Consider a mechanism φ that outputs the following matching for each R and each C: Take any $i \in I$. If we have $i \notin C_s(I)$ and $sR_j\varnothing$ for all $s \in S$ and all $j \neq i$, then set $\varphi(R, C)(i) \equiv \varnothing$. If not, set $\varphi(R, C)(i) \equiv \varphi^{DA}(R, C)(i)$. Note that $\varphi(R, C)$ is a well-defined matching because the number of students assigned to each school never exceeds the number of students assigned under the DA algorithm.

First, φ respects top-top pairs because $\varphi(R, C)(i) = \varphi^{DA}(R, C)(i) = s$ if (i, s) is a top-top pair. Second, we show that φ is strategy-proof. If i's assignment coincides with the assignment under the DA, then i does not gain by any misreport because the DA is strategy-proof and individually rational. Otherwise, i matches with the outside option no matter what preference i submits. Therefore, φ is strategy-proof.

Third, φ violates anti-bottom-dropping monotonicity. Suppose $I = \{i, j\}$ and $S = \{s_1, s_2\}$, each with capacity 1. Let the schools have acceptant responsive choice rules C_{s_1} and C_{s_2} for a common order $i \succ j$. Then, consider the following preferences:

R_i	R'_i	R_j
s_1	s_1	s_2
$\varnothing$	s_2	$\varnothing$
s_2	$\varnothing$	s_1

Under $R = (R_i, R_j)$, the output of φ equals that of the DA, which assigns s_1 to i and s_2 to j. Thus, the profile $R' = (R'_i, R_j)$ is an anti-bottom-dropping of R at $\varphi(R, C)$. However, the mechanism φ matches j with the outside option $\varnothing \neq s_2$ at R' by definition, which violates the condition of anti-bottom-dropping monotonicity. □

Appendix C.2 Discussion on Proposition 9

In this section, we provide examples of mechanisms to which Proposition 9 applies. One notable example is a family of mechanisms called *application-rejection mechanisms*, which is introduced in Chen and Kesten (2017). Throughout this discussion, we assume a fixed profile of acceptant responsive choice rules corresponding to an order profile $\succ$. For each parameter $e \in \mathbb{N}$, the application-rejection mechanism $\varphi^e : \mathcal{R} \to \mathcal{M}$ outputs a matching according to the following algorithm for each profile R:

- Round $t = 0, 1, \ldots$:
  - Step 1. Each unassigned student from the previous round applies to their $(te + 1)$th choice at R. Each school tentatively accepts students from the applicants following their order $\succ$ up to their remaining capacity. The remaining applicants are rejected. Proceed to the Final step if either $e = 1$ or no student is rejected. Otherwise, go to the next step 2.
  - Step $k \geq 2$. The rejected students in the previous step apply to their next preferred choice at R. Each school tentatively accepts students among the pool of the new applicants and the tentatively accepted students following their order $\succ$ up to their remaining capacity. The remaining students are rejected. Proceed to the Final step if for each student i, either i is assigned to a choice or i has been rejected by all his/her first $(te + e)$ choices. Otherwise, go to the next step $k + 1$.
  - Final Step. The round t ends, and tentatively accepted students are permanently matched with the alternatives. Proceed to the next round, $t + 1$.
- The algorithm terminates when all students are assigned to some alternative.

The algorithm described above resembles the DA algorithm, but it differs in that the assignment is finalized at the end of each round. The list length considered in each round is parameterized by $e \in \mathbb{N}$. This family of mechanisms encompasses various well-known mechanisms, including the Boston mechanism φ^1, the *Shanghai mechanism* φ^2, the *Chinese parallel mechanism* φ^e for $2 \leq e < \infty$, and the DA mechanism $\varphi^\infty = \varphi^{DA}$.

For each $e \in \mathbb{N}$, the application-rejection algorithm exhibits two key properties: First, if there is a top-top pair, they will be permanently matched during the first step of the first round. Second, outputs are always individually rational, ensuring that all relative rankings below the students' assignments are never considered.

These two properties together imply that application-rejection mechanisms respect top-top pairs and satisfy anti-bottom-dropping monotonicity under all acceptant responsive choice rules. Consequently, Proposition 9 indicates that these mechanisms yield stable matchings if and only if they are strategy-proof. Kumano (2013) provides a theorem that includes this conclusion, focusing on the Boston mechanism φ^1.

Additional examples can be derived from a generalization of application-rejection mechanisms that account for real-world constraints. As noted by Abdulkadiroğlu et al. (2005), students often face restrictions on the number of schools to which they can apply. Under these constraints, even the DA can result in allocations that are neither stable nor strategy-proof. The theoretical analysis of these issues has been explored by Haeringer and Klijn (2009), Pathak and Sönmez (2013), and Decerf and Van der Linden (2021), for instance.

Considering this scenario, for a positive integer $k > 0$, we define φ^e_k to be the mechanism such that $\varphi^e_k(R) \equiv \varphi^e(R(k))$, where $R(k)$ truncates from R all schools that are ranked strictly lower than the kth choice. Essentially, φ^e_k is the application-rejection mechanism φ^e, where students can apply to at most k schools. The same two observations regarding respect for top-top pairs and anti-bottom-dropping monotonicity apply to φ^e_k. Therefore, the equivalence between stability and strategy-proofness holds for these limited list mechanisms as well. Similarly, one can see that under the EADA with a limited list length, denoted φ^{EDA}_k, stability and strategy-proofness remain equivalent.

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