

Online Appendix

Asset Bubbles and Credit Constraints

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A. PROOFS OF RESULTS IN THE BASELINE MODEL

PROOF OF PROPOSITION 1

We first derive the solution in the discrete-time setup and then take the continuous-time limit. Conjecture that the value function is given by $V_t(K_t^j) = a_t K_t^j + b_t$. Substituting this conjecture and the flow-of-funds constraints (10) and (11) into the Bellman equation (9) yields

$$(A1) \quad a_t K_t^j + b_t = \max_{K_{t+\Delta}^j, K_{1t+\Delta}^j, I_t^j, L_t^j} R_t K_t^j \Delta + Q_t (1 - \delta \Delta) K_t^j + e^{-r\Delta} b_{t+\Delta} \\ + (1 - \pi \Delta) \left[-Q_t K_{t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{t+\Delta}^j \right] \\ + \pi \Delta \left[(Q_t - 1) I_t^j - Q_t K_{1t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{1t+\Delta}^j \right]$$

subject to

$$I_t^j \leq R_t K_t^j \Delta + L_t^j \leq R_t K_t^j \Delta + e^{-r\Delta} \left(a_{t+\Delta} (1 - \delta \Delta) \zeta K_t^j + b_{t+\Delta} \right).$$

The first-order condition for $K_{t+\Delta}^j$ yields

$$(A2) \quad Q_t = e^{-r\Delta} a_{t+\Delta},$$

and hence $K_{t+\Delta}^j$ and $K_{1t+\Delta}^j$ are indeterminate. This implies that firm j is indifferent between buying and selling its existing capital. Under the assumption $Q_t > 1$, the financing constraint and the credit constraint bind so that optimal investment is given by

$$(A3) \quad I_t^j = R_t K_t^j \Delta + Q_t (1 - \delta \Delta) \zeta K_t^j + B_t,$$

where we define

$$(A4) \quad B_t \equiv e^{-r\Delta} b_{t+\Delta}.$$

Substituting the investment rule back into the preceding Bellman equation and matching coefficients, we derive

$$b_t = [\pi \Delta (Q_t - 1) + 1] e^{-r\Delta} b_{t+\Delta}, \\ a_t = R_t \Delta + Q_t (1 - \delta \Delta) + \pi \Delta (Q_t - 1) [\zeta Q_t (1 - \delta \Delta) + R_t \Delta].$$

Using (A2) and (A4), we obtain

$$(A5) \quad B_t = e^{-r\Delta} B_{t+\Delta} [1 + \pi \Delta (Q_{t+\Delta} - 1)],$$

$$(A6) \quad Q_t = e^{-r\Delta} [R_{t+\Delta} \Delta + (1 - \delta\Delta) Q_{t+\Delta} + \pi \Delta (Q_{t+\Delta} - 1) (\zeta Q_{t+\Delta} (1 - \delta\Delta) + R_{t+\Delta} \Delta)].$$

Taking the continuous-time limit as $\Delta \rightarrow 0$ yields (20), (21), and (19).

We can also derive the continuous-time limit of the Bellman equation (9). Note that we can replace $e^{-r\Delta}$ with $1/(1+r\Delta)$ up to first-order approximation. Multiplying the two sides of (9) by $1+r\Delta$ gives

$$\begin{aligned} (1+r\Delta) V_t(K_t^j) &= \max (1-\pi\Delta) \left[(1+r\Delta) D_{0t}^j \Delta + V_{t+\Delta}(K_{t+\Delta}^j) \right] \\ &\quad + \pi \Delta \left[(1+r\Delta) D_{1t}^j + V_{t+\Delta}(K_{1t+\Delta}^j) \right] \\ &= \max (1-\pi\Delta) (1+r\Delta) D_{0t}^j \Delta + V_{t+\Delta}(K_{t+\Delta}^j) \\ &\quad + \pi \Delta (1+r\Delta) D_{1t}^j \\ &\quad + \pi \Delta \left[V_{t+\Delta}(K_{1t+\Delta}^j) - V_{t+\Delta}(K_{t+\Delta}^j) \right]. \end{aligned}$$

Eliminating terms of orders higher than Δ gives

$$\begin{aligned} (1+r\Delta) V_t(K_t^j) &= \max D_{0t}^j \Delta + V_{t+\Delta}(K_{t+\Delta}^j) + \pi \Delta D_{1t}^j \\ &\quad + \pi \Delta \left[V_{t+\Delta}(K_{1t+\Delta}^j) - V_{t+\Delta}(K_{t+\Delta}^j) \right]. \end{aligned}$$

Manipulating yields

$$\begin{aligned} r V_t(K_t^j) &= \max D_{0t}^j + \frac{1}{\Delta} \left[V_{t+\Delta}(K_{t+\Delta}^j) - V_t(K_t^j) \right] \\ &\quad + \pi D_{1t}^j + \pi \left[V_{t+\Delta}(K_{1t+\Delta}^j) - V_{t+\Delta}(K_{t+\Delta}^j) \right]. \end{aligned}$$

Now we take limits as $\Delta \rightarrow 0$ to obtain the continuous-time Bellman equation in (14), where we notice that

$$D_{1t}^j = Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j$$

in continuous time. Moreover, (10), (12), and (13) converge to (15), (16), and (17), respectively, as $\Delta \rightarrow 0$.

We can prove proposition 1 in continuous time directly. Given the conjecture (18), we

rewrite the dynamic programming (14) as

$$(A7) \quad rQ_t K_t^j + rB_t = \max_{I_t^j, \dot{K}_t^j, K_{1t}^j, L_t^j} R_t K_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right) \\ + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t + \pi (Q_t - 1) I_t^j \\ + \pi \left[Q_t K_t^j - Q_t K_{1t}^j + Q_t K_{1t}^j + B_t - \left(Q_t K_t^j + B_t \right) \right]$$

subject to

$$(A8) \quad I_t^j \leq L_t^j \leq \zeta Q_t K_t^j + B_t.$$

Given the assumption $Q_t > 1$, (16) and (A8) bind. We then obtain (19). Substituting this equation back into (A7) and matching coefficients, we obtain (20) and (21). By the transversality condition (6) and the form of the value function,

$$\lim_{T \rightarrow \infty} e^{-rT} \left(Q_T K_T^j + B_T \right) = 0.$$

We thus obtain (22). Q.E.D.

PROOF OF PROPOSITION 2

Using the optimal investment rule in (19), we derive the aggregate capital accumulation equation (28). The first-order condition for the static labor choice problem (7) gives $w_t = (1 - \alpha) (K_t^j / N_t^j)^\alpha$. We then obtain (8) and

$$K_t^j = N_t^j (w_t / (1 - \alpha))^{1/\alpha}.$$

Thus the capital-labor ratio is identical for all firms. Aggregating yields

$$K_t = N_t (w_t / (1 - \alpha))^{1/\alpha}$$

so that $K_t^j / N_t^j = K_t / N_t$ for all $j \in [0, 1]$. Substituting out w_t in (8) yields $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} = \alpha K_t^{\alpha-1}$ since $N_t = 1$ in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_t^j)^\alpha (N_t^j)^{1-\alpha} dj = \int (K_t^j / N_t^j)^\alpha N_t^j dj = (K_t / N_t)^\alpha \int N_t^j dj = K_t^\alpha N_t^{1-\alpha}.$$

This completes the proof. Q.E.D.

PROOF OF PROPOSITION 3

(i) The social planner solves the following problem:

$$\max_{\{I_t\}} \int_0^\infty e^{-rt} (K_t^\alpha - \pi I_t) dt,$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t, \quad K_0 \text{ given,}$$

where K_t is the aggregate capital stock and I_t is the investment level for a firm with an investment opportunity. From this problem, we can derive the efficient capital stock K_E , which satisfies $\alpha (K_E)^{\alpha-1} = r + \delta$. The efficient output, investment and consumption levels are given by $Y_E = (K_E)^\alpha$, $I_E = \delta/\pi K_E$, and $C_E = (K_E)^\alpha - \delta K_E$, respectively.

Suppose that assumption (29) holds. We conjecture that $Q^* = Q_t = 1$ in the steady state. In this case, firm value is given by $V(K_t^j) = K_t^j$. The optimal investment rule for each firm satisfies $R_t = r + \delta = \alpha K_t^{*\alpha-1}$. Thus $K_t^* = K_E$ for $t > 0$. Given this constant capital stock for all firms, we must have $\delta K_t^* = \pi I_t^*$ for $t > 0$. Let each firm's optimal investment level satisfy $I_t^j = \delta K_t^j / \pi$. Then, when assumption (29) holds, the investment and credit constraints, $I_t^j = \delta K_t^j / \pi \leq \zeta K_t^j = V(\zeta K_t^j)$, are satisfied. We conclude that, under assumption (29), the solutions $Q_t = 1$, $K_t^* = K_E$, and $I_t^*/K_t^* = \delta/\pi$ give the bubbleless equilibrium, which also achieves the efficient allocation.

(ii) Suppose that (30) holds. Conjecture that $Q_t > 1$ in some neighborhood of the bubbleless steady state in which $B_t = 0$ for all t . We can then apply Proposition 2 and derive the steady-state equations for (21) and (28) as

$$(A9) \quad \dot{Q} = 0 = (r + \delta) Q - R - \pi \zeta Q(Q - 1),$$

$$(A10) \quad \dot{K} = 0 = -\delta K + \pi (\zeta Q K),$$

where $R = \alpha K^{\alpha-1}$. From these equations, we obtain the steady-state solutions Q^* and K^* in (31) and (32), respectively. Assumption (30) implies that $Q^* > 1$. By continuity, $Q_t > 1$ in some neighborhood of (Q^*, K^*) . This verifies our conjecture. Q.E.D.

PROOF OF PROPOSITION 4

In the bubbly steady state, (20) and (28) imply that

$$(A11) \quad 0 = rB - B\pi(Q - 1), \text{ and}$$

$$(A12) \quad 0 = -\delta K + [\zeta Q K + B]\pi,$$

where $R = \alpha K^{\alpha-1}$. Solving equations (A9), (A11), and (A12) yields equations (34), (35), and (36). By (34), $B > 0$ if and only if (37) holds. From (31) and (35), we deduce that $Q_b < Q^*$. Using condition (37), it is straightforward to check that $K_{GR} > K_E > K_b > K^*$. By the resource constraint, steady-state consumption satisfies $C = Y - \pi I = K^\alpha - \delta K$. Substituting the expressions for K_E , K_b , and K^* in Propositions 3 and 4, we can show that $C_E > C_b > C^*$. From (34), it is also straightforward to verify that the bubble-asset ratio B/K_b decreases with ζ . Q.E.D.

PROOF OF PROPOSITION 5

First, we consider the log-linearized system around the bubbly steady state (B, Q_b, K_b) . We use \hat{X}_t to denote the percentage deviation from the steady state value for any variable X_t , i.e., $\hat{X}_t = \ln X_t - \ln X$. We can show that the log-linearized system is given by

$$\begin{bmatrix} d\hat{B}_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = A \begin{bmatrix} \hat{B}_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$(A13) \quad A = \begin{bmatrix} 0 & -(r + \pi) & 0 \\ 0 & \delta + r - \zeta(2r + \pi) & [(1 - \zeta)r + \delta](1 - \alpha) \\ \pi B/K_b & \zeta(r + \pi) & -\pi B/K_b \end{bmatrix}.$$

We denote this matrix by

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix},$$

where we deduce from (A13) that $a < 0$, $c > 0$, $d > 0$, $e > 0$, and $f < 0$. Since $\zeta < \frac{\delta}{r + \pi}$, we have $b = (1 - \zeta)r + \delta - \zeta(r + \pi) > 0$. The characteristic equation for the matrix A is

$$(A14) \quad F(x) \equiv x^3 - (b + f)x^2 + (bf - ce)x - acd = 0.$$

We observe that $F(0) = -acd > 0$ and $F(-\infty) = -\infty$. Thus, there exists a negative root to the above equation, denoted by $\lambda_1 < 0$. Let the other two roots be λ_2 and λ_3 . We rewrite $F(x)$ as

$$(A15) \quad \begin{aligned} F(x) &= (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) \\ &= x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \end{aligned}$$

Matching terms in equations (A14) and (A15) yields $\lambda_1\lambda_2\lambda_3 = acd < 0$ and

$$(A16) \quad \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0.$$

We consider two cases. (i) If λ_2 and λ_3 are two real roots, then it follows from $\lambda_1 < 0$ that λ_2 and λ_3 must have the same sign. Suppose $\lambda_2 < 0$ and $\lambda_3 < 0$. We then have $\lambda_1\lambda_2 > 0$ and $\lambda_1\lambda_3 > 0$. This implies that $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$, which contradicts equation (A16). Thus we must have $\lambda_2 > 0$ and $\lambda_3 > 0$.

(ii) If either λ_2 or λ_3 is complex, then the other must also be complex. Let

$$\lambda_2 = a_1 + a_2i \text{ and } \lambda_3 = a_1 - a_2i,$$

where a_1 and a_2 are some real numbers and $i = \sqrt{-1}$. We can show that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2a_1\lambda_1 + a_1^2 + a_2^2.$$

Since $\lambda_1 < 0$, the above equation and equation (A16) imply that $a_1 > 0$.

From the above analysis, we conclude that the matrix A has one negative eigenvalue and the other two eigenvalues are either positive real numbers or complex numbers with a positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state $(0, Q^*, K^*)$. We linearize B_t around zero and log-linearize Q_t and K_t and obtain the following linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = \begin{bmatrix} r - \pi(Q^* - 1) & 0 & 0 \\ 0 & a & b \\ \frac{\pi}{K^*} & c & d \end{bmatrix} \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{R^*}{Q^*} - \zeta\pi Q^*, \quad b = \frac{R^*}{Q^*}(1 - \alpha) > 0, \\ c &= \pi\zeta Q^* > 0, \quad d = 0. \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix in the preceding linearized system. One eigenvalue, denoted by λ_1 , is equal to $r - \pi(Q^* - 1) < 0$ and the other two, denoted by λ_2 and λ_3 , satisfy

$$(A17) \quad \lambda_2\lambda_3 = ad - bc = 0 - bc < 0.$$

It follows from (A17) that λ_2 and λ_3 must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

PROOF OF PROPOSITION 6

The discrete-time Bellman equation is given by

$$\begin{aligned} V_t(K_t^j) &= \max (1 - \theta\Delta)(1 - \pi\Delta) \left[D_{0t}^j \Delta + e^{-r\Delta} V_{t+\Delta}(K_{t+\Delta}^j) \right] \\ &\quad + (1 - \theta\Delta)\pi\Delta \left[D_{1t}^j + e^{-r\Delta} V_{t+\Delta}(K_{1t+\Delta}^j) \right] + \theta\Delta V_t^*(K_t^j). \end{aligned}$$

As in the proof of Proposition 1, taking the continuous-time limit as $\Delta \rightarrow 0$ and substituting the flow-of-funds constraints yield the Bellman equation in Section IV.C. Substi-

tuting the conjectured value function $V_t(K_t^j) = Q_t K_t^j + B_t$ into this equation yields

$$\begin{aligned} r(Q_t K_t^j + B_t) &= \max_{I_t^j, \dot{K}_t^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t \\ &\quad + \pi (Q_t - 1) I_t^j + \theta [Q_t^* K_t^j - (Q_t K_t^j + B_t)] \end{aligned}$$

subject to

$$I_t^j \leq \zeta Q_t K_t^j + B_t.$$

When $Q_t > 1$, optimal investment is given by $I_t^j = \zeta Q_t K_t^j + B_t$. Substituting this rule back into the preceding Bellman equation and matching coefficients yield (38) and (39). Equation (28) follows from aggregation and the market-clearing condition. Q.E.D.

PROOF OF PROPOSITION 7

By (38), we can show that

$$(A18) \quad Q_s = \frac{r + \theta}{\pi} + 1.$$

Since $Q_s > 1$, we can apply Proposition 6 in some neighborhood of Q_s . Equation (39) implies that

$$(A19) \quad 0 = (r + \delta + \theta)Q_s - \theta G(K) - R - \pi(Q_s - 1)\zeta Q_s,$$

where $R = \alpha K^{\alpha-1}$. The solution to this equation gives K_s . Once we have obtained K_s and Q_s , we use equation (28) to determine B_s .

The difficult part is to solve for K_s since $G(K)$ is not an explicit function. To show the existence of K_s , we define θ^* as

$$\frac{r + \theta^*}{\pi} + 1 = \frac{\delta}{\pi \zeta} = Q^*.$$

That is, θ^* is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

Let $Q(\theta)$ be the expression on the right-hand side of equation (A18). We then use this equation to rewrite equation (A19) as

$$\alpha K^{\alpha-1} - (r + \delta + \theta)Q(\theta) + \theta G(K) + (r + \theta)\zeta Q(\theta) = 0.$$

Define the function $F(K; \theta)$ as the expression on the left-hand side of the equation above. Notice that $Q(\theta^*) = Q^* = G(K^*)$ by definition and $Q(0) = Q_b$ where Q_b is given in (35). Condition (37) ensures the existence of the bubbly steady-state value Q_b and the bubbleless steady-state values Q^* and K^* .

Define

$$K_{\max} = \max_{0 \leq \theta \leq \theta^*} \left[\frac{(r + \delta + \theta - (r + \theta)\zeta)Q(\theta) - \theta Q^*}{\alpha} \right]^{\frac{1}{\alpha-1}}.$$

By (36), we can show that

$$K_b = \left[\frac{(r + \delta - r\zeta)Q(0)}{\alpha} \right]^{\frac{1}{\alpha-1}}.$$

Thus we have $K_{\max} \geq K_b$ and hence $K_{\max} > K^*$. We want to prove that

$$F(K^*; \theta) > 0, \quad F(K_{\max}; \theta) < 0,$$

for $\theta \in (0, \theta^*)$. If this is true, then it follows from the intermediate value theorem that there exists a solution K_s to $F(K; \theta) = 0$ such that $K_s \in (K^*, K_{\max})$.

First, notice that

$$F(K^*; 0) = \alpha K^{*\alpha-1} - r(1 - \zeta)Q_b - \delta Q_b > \alpha K_b^{\alpha-1} - r(1 - \zeta)Q_b - \delta Q_b = 0,$$

and $F(K^*; \theta^*) = 0$. We can verify that $F(K; \theta)$ is concave in θ for any fixed K . Thus, for all $0 < \theta < \theta^*$,

$$\begin{aligned} F(K^*; \theta) &= F\left(K^*, \left(1 - \frac{\theta}{\theta^*}\right)0 + \frac{\theta}{\theta^*}\theta^*\right) \\ &> \left(1 - \frac{\theta}{\theta^*}\right)F(K^*, 0) + \frac{\theta}{\theta^*}F(K^*, \theta^*) > 0. \end{aligned}$$

Next we can derive

$$\begin{aligned} F(K_{\max}; \theta) &= \alpha K_{\max}^{\alpha-1} - (r + \delta + \theta)Q(\theta) + \theta G(K_{\max}) + (r + \theta)\zeta Q(\theta) \\ &< \alpha K_{\max}^{\alpha-1} - (r + \delta + \theta)Q(\theta) + \theta G(K^*) + (r + \theta)\zeta Q(\theta) < 0, \end{aligned}$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping by inspecting the phase diagram for (K_t, Q_t) so that $G(K_{\max}) < G(K^*)$, and the second inequality follows from the definition of K_{\max} and the fact that $G(K^*) = Q^*$.

Finally, note that $Q(\theta) < Q^*$ for $0 < \theta < \theta^*$. We use equation (A12) and $K_s > K^*$ to deduce that

$$\frac{B_s}{K_s} = \frac{\delta}{\pi} - \zeta Q(\theta) > \frac{\delta}{\pi} - \zeta Q^* = 0.$$

This completes the proof of the existence of a stationary equilibrium with stochastic bubbles (B_s, Q_s, K_s) .

When $\theta = 0$, the bubble never bursts and hence $K_s = K_b$. When θ is sufficiently small, K_s is close to K_b by continuity. Since K_b is smaller than the golden rule capital stock K_{GR} , $K_s < K_{GR}$ when θ is sufficiently small. Since $K^\alpha - \delta K$ is increasing for all

$K < K_{GR}$, we deduce that $K_s^\alpha - \delta K_s > K^{*\alpha} - \delta K^*$. This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

B. PROOFS OF RESULTS IN SECTION V

B1. Endogenous Credit Constraints

PROOF OF PROPOSITION 8

As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$(B1) \quad r V_t \left(K_t^j, M_t^j \right) = \max_{\dot{M}_t^j, \dot{K}_t^j, M_{1t}^j, K_{1t}^j, I_t^j, L_t^j} D_{0t}^j + \dot{V}_t \left(K_t^j, M_t^j \right) \\ + \pi \left[D_{1t}^j + V_t \left(K_{1t}^j, M_{1t}^j \right) - V_t \left(K_t^j, M_t^j \right) \right]$$

subject to (41),

$$(B2) \quad D_{0t}^j = R_t K_t^j - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right),$$

$$(B3) \quad D_{1t}^j = P_t \left(M_t^j - M_{1t}^j \right) + Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j,$$

$$(B4) \quad I_t^j \leq P_t \left(M_t^j - M_{1t}^j \right) + L_t^j.$$

When an investment opportunity arrives with the Poisson rate π , firm j 's asset holdings jump to $M_{1t} \geq 0$ and its value function changes from $V_t \left(K_t^j, M_t^j \right)$ to $V_t \left(K_{1t}^j, M_{1t}^j \right)$. This explains the Bellman equation in (B1). The interpretations of constraints are similar to those in Section II. In particular, equation (B4) is the financing constraint. Firm j can sell assets $(M_t^j - M_{1t}^j)$ and borrow L_t^j to finance investment. According to the collateral constraint (41), firm j uses capital as collateral only.

Substituting the conjectured value function in (45) and the flow-of-funds constraints (B2) and (B3) into the dynamic programming problem (B1) yields

$$r \left(Q_t K_t^j + P_t M_t^j \right) \\ = \max_{\dot{M}_t^j, \dot{K}_t^j, M_{1t}^j, K_{1t}^j, I_t^j, L_t^j} R_t K_t - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right) \\ + Q_t \dot{K}_t^j + K_t^j \dot{Q}_t + \dot{P}_t M_t^j + P_t \dot{M}_t^j \\ + \pi \left[P_t \left(M_t^j - M_{1t}^j \right) + Q_t I_t^j - I_t^j + Q_t K_t - Q_t K_{1t} \right] \\ + \pi \left[Q_t K_{1t} + P_t M_{1t}^j - \left(Q_t K_t + P_t M_t^j \right) \right]$$

subject to (B4) and

$$L_t^j \leq \zeta Q_t K_t^j.$$

Thus $P_t \dot{M}_t^j$ and $Q_t \dot{K}_t^j$ cancel themselves out in the Bellman equation so that firm j is indifferent between buying and selling any amount of the intrinsically useless asset and indifferent between buying and selling any amount of capital, when no investment opportunity arrives. Moreover, $Q_t K_{1t}^j$ also cancels itself out and hence K_{1t}^j is indeterminate.

When an investment opportunity arrives with Poisson rate π , under the assumption $Q_t > 1$, it is profitable to invest as much as possible. In this case firm j sells all its asset holdings to non-investing firms, i.e., $M_{1t}^j = 0$, and borrows as much as possible so that $L_t^j = \zeta Q_t K_t^j$. The optimal investment level is

$$I_t^j = \zeta Q_t K_t^j + P_t M_t^j.$$

Substituting this solution back into the preceding Bellman equation and matching coefficients, we obtain equations (21) and (47).

It follows from (47) that $r P_t > \dot{P}_t$. Thus households will not hold the bubble asset and their short-sale constraints bind. This means that the market-clearing condition for the asset is given by $\int M_t^j dj = 1$. By a law of large numbers, aggregate capital satisfies

$$\dot{K}_t = \delta K_t + \pi \left(\zeta Q_t K_t + P_t \int M_t^j dj \right).$$

We then obtain (46). Since the equilibrium system is the same as that in Proposition 2 once we set $P_t = B_t$, we can use Proposition 4 to study the steady state with a bubble $P > 0$. Thus the existence condition is (37). Note that $\zeta = 0$ also permits the existence of a bubble. Q.E.D.

PROOF OF PROPOSITION 9

The proof follows from that of Proposition 11 in online Appendix B.4 by setting $X_t = 0$, $\zeta = 1$, and $g = 0$. We omit the details. Q.E.D.

B2. Liquidity Mismatch

We now relax the liquidity mismatch assumption and suppose that at most a fraction λ of the proceeds from the sale of old capital is available to finance investment. Then the financing constraint in continuous time becomes

$$(B5) \quad I_t^j \leq L_t^j + Q_t (K_t^j - K_{1t}^j),$$

and K_{1t}^j satisfies

$$(B6) \quad K_{1t}^j \geq (1 - \lambda) K_t^j.$$

Firm j 's decision problem is given by the Bellman equation (14) subject to (15), (17), (B5), and (B6). We conjecture that the value function takes the form $V_t(K_t^j) = Q_t K_t^j + B_t$. Substitute this conjecture into the Bellman equation. When an investment opportunity arrives, under the assumption $Q_t > 1$, firm j wants to invest as much as possible so that the financing constraint and the credit constraint bind. Moreover, the firm chooses $K_{1t}^j = (1 - \lambda) K_t^j$ and optimal investment is given by

$$I_t^j = (\zeta + \lambda) Q_t K_t^j + B_t.$$

Substituting these decision rules into the Bellman equation and matching coefficients, we deduce that B_t still satisfies equation (20), and Q_t satisfies

$$(B7) \quad \dot{Q}_t = (r + \delta) Q_t - R_t - \pi (\zeta + \lambda) Q_t (Q_t - 1).$$

Aggregate investment is given by

$$\pi I_t = \pi [(\zeta + \lambda) Q_t K_t + B_t],$$

and aggregate capital satisfies

$$(B8) \quad \dot{K}_t = -\delta K_t + \pi [(\zeta + \lambda) Q_t K_t + B_t].$$

The equilibrium system for (Q_t, K_t, B_t) is given by (B7), (B8) and (20). Thus the analysis in Sections III and IV still applies except that ζ is replaced by $\zeta + \lambda$. In particular, by Proposition 4, the bubbly and bubbleless steady states coexist if and only if

$$0 < \zeta + \lambda < \frac{\delta}{r + \pi}.$$

This implies that as long as λ is sufficiently small, a bubbly equilibrium exists.

B3. Equity Issues

PROOF OF PROPOSITION 10

As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$\begin{aligned} r V_t(K_t^j) = & \max_{\dot{K}_t^j, K_{1t}^j, I_t^j, L_t^j, S_{0t}^j, S_{1t}^j} D_{0t}^j - S_{0t}^j + \dot{V}_t(K_t^j) + \pi (D_{1t}^j - S_{1t}^j) \\ & + \pi [V_t(K_{1t}^j) - V_t(K_t^j)] \end{aligned}$$

subject to (17),

$$(B9) \quad D_{0t}^j = R_t K_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right) + S_{0t}^j - \frac{\varphi (S_{0t}^j)^2}{2 K_t^j},$$

$$(B10) \quad D_{1t}^j + I_t^j + L_t^j = Q_t I_t^j + L_t^j + Q_t K_t^j - Q_t K_{1t}^j + S_{1t}^j - \frac{\varphi (S_{1t}^j)^2}{2 K_t^j},$$

$$(B11) \quad I_t^j \leq L_t^j + S_{1t}^j.$$

Substituting (B10) into the Bellman equation yields

$$\begin{aligned} r V_t \left(K_t^j \right) &= \max_{\dot{K}_t^j, K_{1t}^j, I_t^j, L_t^j, S_{0t}^j, S_{1t}^j} D_{0t}^j - S_{0t}^j + \dot{V}_t \left(K_t^j \right) \\ &\quad + \pi \left[(Q_t - 1) I_t^j - \frac{\varphi (S_{1t}^j)^2}{2 K_t^j} \right] \\ &\quad + \pi \left[Q_t K_t^j - Q_t K_{1t}^j + V_t \left(K_{1t}^j \right) - V_t \left(K_t^j \right) \right]. \end{aligned}$$

Conjecture that V_t is given by (18). Using (B9), we can show that $S_{0t}^j = 0$.

When an investment opportunity arrives, under the assumption $Q_t > 1$, firm j invests as much as possible so that the credit constraint (17) and the financing constraint (B11) bind. Using the first-order condition for S_{1t}^j , we derive

$$S_{1t}^j = \frac{1}{\varphi} (Q_t - 1) K_t^j, \quad I_t^j = \zeta Q_t K_t^j + B_t + \frac{1}{\varphi} (Q_t - 1) K_t^j.$$

Substituting the conjectured value function $V_t \left(K_t^j \right) = Q_t K_t^j + B_t$ and the above decision rules into the Bellman equation and matching coefficients, we obtain (20) and

$$(B12) \quad \dot{Q}_t = (r + \delta) Q_t - R_t - \pi \left[\zeta Q_t + \frac{1}{2\varphi} (Q_t - 1) \right] (Q_t - 1).$$

Aggregate capital satisfies

$$(B13) \quad \dot{K}_t = -\delta K_t + \pi \left(Q_t \zeta K_t + B_t + \frac{1}{\varphi} (Q_t - 1) K_t \right).$$

As in the proof of Proposition 1, we can show that $R_t = \alpha K_t^{\alpha-1}$.

In the bubbly steady state, we use equation (20) to derive

$$Q_b = 1 + \frac{r}{\pi} > 1.$$

Thus $Q_t > 1$ in a neighborhood of the bubbly steady state. Using (B13), we derive

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \zeta Q_b - \frac{1}{\varphi}(Q_b - 1).$$

Given the condition in the proposition we have $B > 0$. Finally, we use (B12) to derive

$$\begin{aligned} R_b &= \alpha (K_b)^{\alpha-1} = (r + \delta) Q_b - \pi \left[\zeta Q_b + \frac{1}{2\varphi}(Q_b - 1) \right] (Q_b - 1) \\ &= [(1 - \zeta)r + \delta] \left(\frac{r}{\pi} + 1 \right) - \frac{1}{2\varphi} \frac{r^2}{\pi}. \end{aligned}$$

Given the condition in the proposition we can check that $R_b > 0$. From the proof above we can see that the condition is also necessary. Q.E.D.

B4. Additional Asset with Exogenous Rents

PROOF OF PROPOSITION 11

With technical progress, firm j 's static labor choice problem is

$$(B14) \quad R_t K_t^j = \max_{N_t^j} (K_t^j)^\alpha (A_t N_t^j)^{1-\alpha} - w_t N_t^j,$$

where w_t is the wage rate and R_t is given by

$$(B15) \quad R_t = \alpha \left(\frac{w_t / A_t}{1 - \alpha} \right)^{\frac{\alpha-1}{\alpha}}.$$

Firm j 's dynamic programming problem in continuous time is given by (B1) subject to (B3), (B4), (48), (53), and

$$D_{0t}^j = R_t K_t^j + X_t M_t^j - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right).$$

Since one unit of the asset pays X_t rents, $X_t M_t^j$ enters the above flow-of-funds constraint.

Conjecture that the value function takes the following form:

$$V_t \left(K_t^j, M_t^j \right) = Q_t K_t^j + P_t M_t^j + B_t.$$

Substituting this conjectured function and the flow-of-funds constraints into the dynamic

programming problem (B1) yields

$$\begin{aligned}
& r \left(Q_t K_t^j + P_t M_t^j + B_t \right) \\
= & \max_{\dot{M}_t^j, \dot{K}_t^j, \dot{M}_{1t}^j, \dot{K}_{1t}^j, \dot{I}_t^j, \dot{L}_t^j} R_t K_t + X_t M_t^j - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right) \\
& + Q_t \dot{K}_t^j + K_t^j \dot{Q}_t + \dot{P}_t M_t^j + P_t \dot{M}_t^j + \dot{B}_t \\
& + \pi \left[P_t \left(M_t^j - M_{1t}^j \right) + Q_t I_t^j - I_t^j + P_t \left(M_{1t}^j - M_t^j \right) \right]
\end{aligned}$$

subject to (B4), (53), and

$$(B16) \quad L_t^j \leq \zeta Q_t K_t^j + B_t.$$

Thus $P_t \dot{M}_t^j$ and $Q_t \dot{K}_t^j$ cancel themselves out in the Bellman equation so that firm j is indifferent between buying and selling any amount of the intrinsically useless asset and indifferent between buying and selling any amount of capital, when no investment opportunity arrives. Moreover, $Q_t K_t^j$ cancels itself out and hence K_{1t}^j is indeterminate.

When an investment opportunity arrives with Poisson rate π , under the assumption $Q_t > 1$, the firm will invest as much as possible. It follows from (B4), (53), and (B16) that $M_{1t}^j = (1 - \zeta) M_t^j$ and optimal investment is given by

$$I_t^j = \zeta Q_t K_t^j + \zeta P_t M_t^j + B_t.$$

Substituting this solution back into the preceding Bellman equation and matching coefficients, we obtain equations

$$(B17) \quad \dot{P}_t = r P_t - X_t - \pi (Q_t - 1) \zeta P_t,$$

$$(B18) \quad \dot{B}_t = r B_t - \pi (Q_t - 1) B_t,$$

$$(B19) \quad \dot{Q}_t = (r + \delta) Q_t - R_t - \pi (Q_t - 1) Q_t \zeta.$$

It follows from (B17) that $r P_t > \dot{P}_t + X_t$. Thus households will not hold the asset and their short-sale constraints bind. This means that the market-clearing condition for the asset is given by $\int M_t^j dj = 1$. By a law of large numbers, aggregate capital satisfies

$$\dot{K}_t = \delta K_t + \pi \left(\zeta Q_t K_t + P_t \zeta \int M_t^j dj + B_t \right).$$

We then obtain

$$\dot{K}_t = -\delta K_t + \pi (Q_t \zeta K_t + \zeta P_t + B_t).$$

As in the proof of Proposition 2, the labor-market clearing condition gives $R_t = \alpha (K_t/A_t)^{\alpha-1}$ and $Y_t = K_t^\alpha A_t^{1-\alpha}$.

Then aggregate capital K_t , the asset price P_t , and the stock price bubble B_t will all grow at the rate g in the steady state. However, the capital price Q_t and the rental rate R_t

will not grow. The detrended equilibrium system becomes

$$\begin{aligned}\dot{k}_t &= -(\delta + g)k_t + \pi(Q_t \xi k_t + \zeta p_t + b_t), \\ \dot{p}_t &= (r - g)p_t - x - \pi(Q_t - 1)\zeta p_t, \\ \dot{b}_t &= (r - g)b_t - \pi(Q_t - 1)b_t, \\ \dot{Q}_t &= (r + \delta)Q_t - \alpha k_t^{\alpha-1} - \pi(Q_t - 1)Q_t \xi,\end{aligned}$$

where $k_t = K_t/A_t$, $p_t = P_t/A_t$, $b_t = B_t/A_t$, and $x = X_t/A_t$. In the bubbly steady state these variables and Q_t are all constant over time. Suppressing the time subscript in the steady state gives

$$(B20) \quad 0 = -(\delta + g)k + \pi(Q\xi k + \zeta p + b),$$

$$(B21) \quad 0 = (r - g)p - x - \pi(Q - 1)\zeta p,$$

$$(B22) \quad 0 = (r - g)b - \pi(Q - 1)b,$$

$$(B23) \quad 0 = (r + \delta)Q - \alpha k^{\alpha-1} - \pi(Q - 1)Q\xi.$$

In the bubbly steady state $b > 0$, we can use (B22) to compute

$$Q_b = \frac{r - g}{\pi} + 1.$$

Assume that $r > g$ so that $Q_b > 1$ and hence $Q_t > 1$ in the neighborhood of the bubbly steady state. Using (B21) and (B23), we can compute

$$\begin{aligned}p &= \frac{x}{(r - g)(1 - \zeta)}, \\ R &= \alpha k^{\alpha-1} = [(r + \delta) - (r - g)\xi] \left(\frac{r - g}{\pi} + 1 \right).\end{aligned}$$

Thus the bubbly steady-state detrended capital stock is given by

$$k_b = \left\{ \frac{1}{\alpha} [(r + \delta) - (r - g)\xi] \left(\frac{r - g}{\pi} + 1 \right) \right\}^{\frac{1}{\alpha-1}}.$$

After solving for Q_b , k_b , and p , we use equation (B20) to solve for b described in the proposition. We need $b > 0$. We then have the second inequality in condition (56). For $x > 0$ in (56), we need

$$\frac{\delta + g}{\pi} - \left(\frac{r - g}{\pi} + 1 \right) \xi > 0.$$

We then obtain the condition in (55). This condition also implies that $(r + \delta) - (r - g)\xi > 0$ so that $k_b > 0$. The conditions in the propositions are also necessary. Q.E.D.

B5. Intertemporal Debt

PROOF OF PROPOSITION 12

We first derive the discrete-time solution and then take the continuous-time limit. Conjecture that the value function takes the form

$$V_t(K_t^j, L_t^j) = a_t K_t^j - a_t^L L_t^j + b_t.$$

Substituting this conjecture and the flow-of-funds constraints (57) and (58) into the Bellman equation yields

$$\begin{aligned} & a_t K_t^j - a_t^L L_t^j + b_t \\ = & \max_{I_t^j, K_{t+\Delta}^j, K_{1t+\Delta}^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} R_t K_t^j \Delta - L_t^j + Q_t (1 - \delta \Delta) K_t^j + e^{-r\Delta} b_{t+\Delta} \\ & + (1 - \pi \Delta) \left[e^{-r_{ft}\Delta} L_{t+\Delta}^j - Q_t K_{t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{t+\Delta}^j - e^{-r\Delta} a_{t+\Delta}^L L_{t+\Delta}^j \right] \\ & + \pi \Delta \left[e^{-r_{ft}\Delta} L_{1t+\Delta}^j - Q_t K_{1t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{1t+\Delta}^j - e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j \right] \\ & + \pi \Delta (Q_t - 1) I_t^j \end{aligned}$$

subject to

$$(B24) \quad I_t^j \leq R_t K_t^j \Delta + e^{-r_{ft}\Delta} L_{1t+\Delta}^j - L_t^j,$$

$$(B25) \quad a_{t+\Delta}^L L_{1t+\Delta}^j \leq b_{t+\Delta} + a_{t+\Delta} \zeta (1 - \delta \Delta) K_t^j,$$

where (B25) is the credit constraint derived from (60) using the conjectured value function.

By the linear property of the Bellman equation above, the first-order conditions for $L_{t+\Delta}^j$ and $K_{t+\Delta}^j$ yield

$$(B26) \quad e^{-r_{ft}\Delta} = e^{-r\Delta} a_{t+\Delta}^L, \quad Q_t = e^{-r\Delta} a_{t+\Delta}.$$

and hence $L_{t+\Delta}^j$, $K_{t+\Delta}^j$, and $K_{1t+\Delta}^j$ are indeterminate. This implies that firm j is indifferent between saving and borrowing when no investment opportunity arrives, and is also indifferent between buying and selling capital. When $Q_t > 1$, it is profitable for firm j to invest as much as possible so that the financing constraint (B24) and the credit constraint (B25) bind. Thus optimal investment is given by

$$I_t^j = R_t K_t^j \Delta + B_t + Q_t \zeta (1 - \delta \Delta) K_t^j - L_t^j,$$

where we have used (B26) and defined

$$(B27) \quad B_t \equiv e^{-r\Delta} b_{t+\Delta}.$$

Substituting the investment rule back into the Bellman equation and matching coefficients, we derive

$$a_t = R_t \Delta + Q_t (1 - \delta \Delta) + \pi \Delta (Q_t - 1) (R_t \Delta + Q_t \zeta (1 - \delta \Delta)),$$

$$a_t^L = 1 + \pi \Delta (Q_t - 1),$$

$$b_t = e^{-r\Delta} b_{t+\Delta} + \pi \Delta (Q_t - 1) B_t.$$

Using (B26) and (B27) and the preceding three equations, we derive

(B28)

$$Q_t = e^{-r\Delta} [R_{t+\Delta} \Delta + Q_{t+\Delta} (1 - \delta \Delta) + \pi \Delta (Q_{t+\Delta} - 1) (R_{t+\Delta} \Delta + Q_{t+\Delta} \zeta (1 - \delta \Delta))],$$

(B29)

$$e^{-r_{ft}\Delta} = e^{-r\Delta} [1 + \pi \Delta (Q_{t+\Delta} - 1)],$$

(B30)

$$B_t = e^{-r\Delta} [1 + \pi \Delta (Q_{t+\Delta} - 1)] B_{t+\Delta}.$$

Taking the continuous-time limit as $\Delta \rightarrow 0$ yields the equations in Proposition 12.

As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$(B31) \quad r V_t (K_t^j, L_t^j) = \max_{D_{0t}^j, D_{1t}^j, I_t^j, L_{1t}^j} D_{0t}^j + \dot{V}_t (K_t^j, L_t^j) \\ + \pi \left[D_{1t}^j + V_t (K_{1t}^j, L_{1t}^j) - V_t (K_t^j, L_t^j) \right]$$

subject to

$$(B32) \quad \dot{L}_t^j = r_{ft} L_t^j + D_{0t}^j - R_t K_t^j + Q_t (\dot{K}_t^j + \delta K_t^j),$$

$$(B33) \quad D_{1t}^j = Q_t I_t^j + L_{1t}^j - L_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j,$$

(B34)

$$I_t^j \leq L_{1t}^j - L_t^j,$$

(B35)

$$V_t (K_{1t}^j, L_{1t}^j) \geq V_t (K_{1t}^j, 0) - V_t (\zeta K_t^j, 0).$$

Conjecture that the value function takes the form

$$(B36) \quad V_t(K_t^j, L_t^j) = Q_t K_t^j - L_t^j + B_t.$$

Substituting this conjecture into the Bellman equation yields

$$\begin{aligned} r(Q_t K_t^j - L_t^j + B_t) &= \max \dot{L}_t^j - r_{ft} L_t^j + R_t K_t^j - Q_t \dot{K}_t^j - Q_t \delta K_t^j \\ &\quad + Q_t \dot{K}_t^j + \dot{Q}_t K_t^j - \dot{L}_t^j + \dot{B}_t \\ &\quad + \pi \left[(Q_t - 1) I_t^j + L_{1t}^j - L_t^j + Q_t K_t^j - Q_t K_{1t}^j \right] \\ &\quad + \pi \left[Q_t K_{1t}^j - L_{1t}^j + B_t - (Q_t K_t^j - L_t^j + B_t) \right]. \end{aligned}$$

Thus \dot{K}_t^j and \dot{L}_t^j cancel themselves out so that firm j is indifferent between saving and borrowing and between buying and selling capital, when no investment opportunity arrives. Moreover, $Q_t K_{1t}^j$ also cancels itself out so that firm j is indifferent between buying and selling capital when an investment opportunity arrives. Simplifying yields

$$(B37) \quad r(Q_t K_t^j - L_t^j + B_t) = \max -r_{ft} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j + \dot{B}_t + \pi (Q_t - 1) I_t^j.$$

Given the conjectured value function, the credit constraint (B35) becomes

$$L_{1t}^j \leq Q_t \xi K_t^j + B_t.$$

Using the financing constraint (B34), we obtain

$$I_t^j \leq L_{1t}^j - L_t^j \leq \xi Q_t K_t^j + B_t - L_t^j.$$

When an investment opportunity arrives, under the assumption $Q_t > 1$, it is profitable for firm j to invest as much as possible so that both the financing and credit constraints bind. We then have

$$I_t^j = \xi Q_t K_t^j + B_t - L_t^j.$$

Substituting this investment rule back into the Bellman equation (B37) and matching coefficients, we derive the equations for Q_t , B_t , and r_{ft} given in Proposition 12.

We now compute

$$I_t = \int I_t^j dj = \xi Q_t K_t + B_t - \int L_t^j dj.$$

Since $r_{ft} < r$, households short-sale constraints bind so that $L_t^h = 0$ and the bond

market-clearing condition becomes $\int L_t^j dj = 0$. Thus

$$(B38) \quad I_t = \xi Q_t K_t + B_t.$$

Substituting (B38) into the law of motion for aggregate capital yields the equation for K_t given in Proposition 12. Finally, we can use the same procedure as in the proof of Proposition 2 to derive $R_t = \alpha K_t^{\alpha-1}$. Q.E.D.

PROOF OF PROPOSITION 13

The proof follows from those of Propositions 3 and 4. Since $r_{f_t} = \dot{B}_t/B_t$ in the bubbly equilibrium, $r_f = 0$ in the bubbly steady state as $\dot{B}_t = 0$.

In the bubbleless steady state in which $B = 0$, we have $Q^* = \delta/(\pi\xi)$ and

$$r_f^* = r - \pi(Q^* - 1) = r + \pi - \delta/\xi < 0,$$

where the inequality follows from condition (37). Q.E.D.

C. SELF-ENFORCING DEBT CONTRACTS

Consider a type of credit constraint which is popular in the self-enforcing debt literature (see, e.g., Bulow and Rogoff (1989), Kehoe and Levine (1993), Alvarez and Jermann (2000), Albuquerque and Hopenhayn (2004), Kocherlakota (2008), and Hellwig and Lorenzoni (2009)).²² There is no collateral. Suppose that the only penalty on the firm for defaulting is that it will be excluded from the financial market forever. Since internal funds $R_t K_t^j$ come as flows, the firm has no funds with which to make a lumpy investment I_t^j . Denote by $V_t^a(K_t^j)$ the autarky value of firm j that cannot access the financial market. $V_t^a(K_t^j)$ satisfies the Bellman equation

$$rV_t^a(K_t^j) = \max_{\dot{K}_t^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \dot{V}_t^a(K_t^j).$$

This is a standard dynamic programming problem and no bubble can exist in V_t^a by the usual transversality condition. Conjecture that $V_t^a(K_t^j) = Q_t^a K_t^j$. Substituting this conjecture into the Bellman equation above yields

$$rQ_t^a K_t^j = \max_{\dot{K}_t^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \dot{Q}_t^a K_t^j + Q_t^a \dot{K}_t^j.$$

Optimizing with respect to \dot{K}_t^j , we deduce $Q_t = Q_t^a$. Matching the coefficients of K_t^j gives

$$(C1) \quad \dot{Q}_t = (r + \delta) Q_t - R_t.$$

²²Kocherlakota (2008) and Hellwig and Lorenzoni (2009) show that a bubble can exist with self-enforcing debt constraints while leaving consumption allocation unchanged in a pure exchange economy.

We now turn to firm j 's decision problem before defaulting. Firm value $V_t(K_t^j)$ satisfies the Bellman equation

$$(C2) \quad rV_t(K_t^j) = \max_{\dot{K}_t^j, I_t^j, K_{1t}^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \dot{V}_t(K_t^j) + \pi \left[Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j + V_t(K_{1t}^j) - V_t(K_t^j) \right]$$

subject to the financing constraint $I_t^j \leq L_t^j$ and the following credit constraint

$$(C3) \quad -L_t^j + V_t(K_{1t}^j) \geq V_t^a(K_{1t}^j).$$

This credit constraint is an incentive constraint which can be interpreted as follows. Write the discrete-time approximation to (C3) as

$$(C4) \quad -L_t^j + e^{-r\Delta} V_{t+\Delta}(K_{1t+\Delta}^j) \geq e^{-r\Delta} V_{t+\Delta}^a(K_{1t+\Delta}^j).$$

When an investment opportunity arrives at time t , firm j takes on debt L_t^j to finance investment I_t^j . At the end of period $[t, t + \Delta]$, the firm's capital sales $Q_t I_t^j$ are realized. If it repays the debt, its continuation value is given by the expression on the left-hand side of (C4). If it defaults on the debt, it will be excluded from the financial market forever and its continuation value is given by the expression on the right-hand side of (C4). Inequality (C4) ensures that the firm has no incentive to default. The constraint (C3) is the continuous time limit as $\Delta \rightarrow 0$.

Conjecture that

$$(C5) \quad V_t(K_t^j) = Q_t K_t^j + B_t.$$

Then (C3) becomes $L_t^j \leq B_t$. This constraint is similar to that in Martin and Ventura (2012). Substituting (C5) into (C2) yields

$$(C6) \quad rQ_t K_t^j + rB_t = \max_{I_t^j, \dot{K}_t^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \pi (Q_t - 1) I_t^j + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t$$

subject to

$$(C7) \quad I_t^j \leq B_t.$$

When $Q_t > 1$, the optimal investment level is $I_t^j = B_t$. Substituting this investment rule back into the Bellman equation and matching coefficients, we obtain (C1) and

$$(C8) \quad rB_t = \dot{B}_t + \pi (Q_t - 1) B_t.$$

The law of motion for aggregate capital is

$$(C9) \quad \dot{K}_t = -\delta K_t + \pi B_t, \quad K_0 \text{ given.}$$

The equilibrium system is given by three differential equations (C1), (C8), and (C9) for (Q_t, B_t, K_t) together with the usual transversality condition.

This equilibrium system is the same as that for the baseline model in Section II when $\xi = 0$. Thus the analysis in Sections III and IV for $\xi = 0$ applies here. Both bubbleless and bubbly equilibria exist and their steady states are unique.

D. RISK-AVERSE HOUSEHOLDS

We replace risk-neutral households with risk-averse households in the baseline model. Suppose that the representative household has the following utility function:

$$\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt,$$

where ρ is the subjective discount rate and γ is the risk aversion parameter. The household faces the budget constraint (4) subject to the no-Ponzi-game condition. Then we derive the consumption Euler equation

$$(D1) \quad \frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} (r_t - \rho),$$

where r_t is equal to the return on any stock j in the absence of aggregate uncertainty and is also called the discount rate. Equation (5) holds where r is replaced by r_t . Firm j solves the following dynamic programming problem:

$$(D2) \quad r_t V_t(K_t^j) = \max_{D_t^j, I_t^j} D_t^j + \dot{V}_t(K_t^j) + \pi \left[(L_t^j - I_t^j) + (Q_t I_t^j - L_t^j) \right]$$

subject to (15), (16), and (17). For tractability, we assume that capital does not jump at the time when an investment opportunity arrives. As we show earlier, this assumption is without loss of generality due to the liquidity mismatch assumption.

The aggregate state variables of the economy are B_t , Q_t , and K_t , where B_t represents the aggregate size of the bubble. The discount rate r_t is a function of the aggregate state variables. Conjecture that

$$V_t(K_t^j) = Q_t K_t^j + B_t^j,$$

where B_t^j is the bubble component in firm j 's stock price. Substituting this conjecture

into the preceding dynamic programming problem yields

$$(D3) \quad r_t Q_t K_t^j + r_t B_t^j = \max_{I_t^j, \dot{K}_t^j} R_t K_t^j - Q_t (\dot{K}_t^j + \delta K_t^j) + \pi (Q_t - 1) I_t^j \\ + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t^j,$$

subject to

$$(D4) \quad I_t^j \leq \zeta Q_t K_t^j + B_t^j.$$

When $Q_t > 1$, the constraint (D4) binds so that the optimal investment level is $I_t^j = \zeta Q_t K_t^j + B_t^j$. Substituting this rule back into the Bellman equation and matching the coefficients of K_t^j , we obtain

$$(D5) \quad \dot{Q}_t = (r_t + \delta) Q_t - R_t - \pi \zeta Q_t (Q_t - 1),$$

$$(D6) \quad \dot{B}_t^j = r_t B_t^j - B_t^j \pi (Q_t - 1).$$

The usual transversality conditions must hold.

Since $B_t = \int B_t^j dj$, it follows from (D6) that the aggregate bubble satisfies

$$(D7) \quad \dot{B}_t = r_t B_t - B_t \pi (Q_t - 1).$$

The law of motion for aggregate capital still satisfies (28). The resource constraint is given by

$$(D8) \quad C_t + \pi (\zeta Q_t K_t + B_t) = Y_t.$$

The equilibrium system consists of five equations (28), (D1), (D5), (D7), and (D8) for five aggregate variables $(C_t, r_t, K_t, Q_t, B_t)$. The transversality condition also holds

$$(D9) \quad \lim_{T \rightarrow \infty} e^{-\int_0^T r_s ds} Q_T K_T = 0, \quad \lim_{T \rightarrow \infty} e^{-\int_0^T r_s ds} B_T = 0.$$

Note that an equilibrium only determines the size B_t of the aggregate bubble, but an individual firm's bubble size B_t^j is indeterminate. Thus it is possible that some firms have no bubbles, while others have bubbles of different sizes.

We use a variable without the time subscript to denote its steady state value. Then (D1) implies $r = \rho$ and hence the steady-state system is the same as that in the baseline model of Section II. Our analysis of steady states in Sections III and IV still applies to the case of risk-averse households. We are unable to derive analytical results for local dynamics because the equilibrium system contains five equations, but it is straightforward to derive numerical solutions.

E. GENERAL SHORT-SALE CONSTRAINTS

In Section V.E we have assumed that households cannot short intertemporal bonds, or effectively they cannot borrow. We now relax this assumption and allow households to borrow a proportion of their labor income.

ASSUMPTION 9: *The representative household can borrow or short intertemporal bonds up to a proportion χ of its wage income, i.e., $L_t^h \geq -\chi w_t$, $\chi \geq 0$. Firms cannot hold or trade each other's stocks.*

We follow the same steps as before to derive the equilibrium system. From the firm's decision problem we show that the value function takes the form

$$(E1) \quad V_t(K_t^j, L_t^j) = Q_t K_t^j - L_t^j + B_t.$$

When $Q_t > 1$, optimal investment is given by

$$I_t^j = \zeta Q_t K_t^j + B_t - L_t^j.$$

We can also show that the equations for Q_t , B_t , and r_{ft} are given in Proposition 12. We need to derive the law of motion for aggregate capital.

Since $r_{ft} < r$, households will borrow by short-selling bonds until their short-sale constraints bind, i.e.,

$$L_t^h = -\chi w_t = -\chi(1 - \alpha)K_t^\alpha.$$

The last equality follows from the wage equation in equilibrium. By the bond market-clearing condition

$$\int L_t^j dj = L_t^h = -(1 - \alpha)\chi Y_t.$$

Aggregating the law of motion for an individual firm's capital, we obtain

$$(E2) \quad \begin{aligned} \dot{K}_t &= -\delta K_t + \pi \left(\zeta Q_t K_t + B_t - \int L_t^j dj \right) \\ &= -\delta K_t + \pi \left(\zeta Q_t K_t + B_t + (1 - \alpha)\chi K_t^\alpha \right). \end{aligned}$$

We now derive the bubbly steady state. Using equations for Q_t , K_t and r_{ft} in Proposition 12, we can show that

$$Q_b = \frac{r + \pi}{\pi} > 1, \quad r_f = 0, \quad R_b = \frac{r + \pi}{\pi} [(1 - \zeta)r + \delta].$$

Using (E2), we can show that

$$\begin{aligned} \frac{B}{K_b} &= \frac{\delta}{\pi} - \zeta Q_b - (1 - \alpha)\chi K_b^{\alpha-1} \\ &= \frac{\delta}{\pi} - \zeta Q_b - \chi \frac{(1 - \alpha)}{\alpha} R_b. \end{aligned}$$

The bubbly equilibrium requires $B > 0$. Using the preceding equations, we then obtain the necessary and sufficient conditions

$$0 \leq \chi < \frac{\alpha}{1 - \alpha} \frac{1}{r(1 - \zeta) + \delta} \left[\frac{\delta}{r + \pi} - \zeta \right].$$

This result shows that a stock price bubble can exist as long as the short-sale constraint for households is sufficiently tight. The analysis of Section V.E corresponds to the case of $\chi = 0$.

F. INTERTEMPORAL DEBT WITHOUT A MARKET FOR CAPITAL

In this appendix we show that the equilibrium system analyzed in Section V.E is equivalent to a setup where there is no market for capital goods. We replace intratemporal debt in the baseline model with intertemporal bonds with zero net supply. With intertemporal bonds, firms can raise new debt to payoff old debt. Let r_{ft} denote the interest rate on the bonds. Suppose that firms can invest and accumulate capital on their own. We allow the lender to seize both a fraction ζ of the defaulting firm's existing capital and a fraction η of its newly installed capital in the event of default.²³ The solution in Section V.E corresponds to the special case with $\eta = 0$.

ASSUMPTION 10: *Households cannot short intertemporal bonds. Firms do not own or trade each other's shares and do not issue new equity to finance investment. The only sources of finance are internal funds, savings, and intertemporal debt.*

We will derive equilibria in which investing firms borrow from non-investing firms and households do not hold any bonds. Let $L_t^h \geq 0$ denote the representative household's bond holdings. Let $L_t^j > (<)0$ denote firm j 's debt level (saving). The market-clearing condition for the bonds is $\int L_t^j dj = L_t^h$. Let $V_t(K_t^j, L_t^j)$ denote the ex ante equity value of firm j when its capital stock and debt level at time t are K_t^j and L_t^j , respectively, prior to the realization of the Poisson shock. We suppress the aggregate state variables in the argument. Then V_t satisfies the following Bellman equation in discrete time:

$$\begin{aligned} V_t(K_t^j, L_t^j) = & \max_{I_t^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} (1 - \pi \Delta) \left[D_{0t}^j \Delta + e^{-r\Delta} V_{t+\Delta}((1 - \delta\Delta)K_t^j, L_{t+\Delta}^j) \right] \\ & + \pi \Delta \left[D_{1t}^j + e^{-r\Delta} V_{t+\Delta}(K_{t+\Delta}^j, L_{1t+\Delta}^j) \right] \end{aligned}$$

subject to

$$(F1) \quad 0 \leq D_{0t}^j \Delta = R_t K_t^j \Delta + e^{-r_{ft}\Delta} L_{t+\Delta}^j - L_t^j,$$

²³If we introduce this assumption in Section V.E, then the resulting equilibrium system is equivalent to that studied in this appendix.

$$(F2) \quad 0 \leq D_{1t}^j = R_t K_t^j \Delta + e^{-r_{ft}\Delta} L_{1t+\Delta}^j - L_t^j - I_t^j,$$

$$(F3) \quad K_{t+\Delta}^j = (1 - \delta\Delta) K_t^j + I_t^j,$$

$$(F4) \quad V_{t+\Delta}(K_{t+\Delta}^j, L_{1t+\Delta}^j) \geq V_{t+\Delta}(K_{t+\Delta}^j, 0) - V_{t+\Delta}(\zeta(1 - \delta\Delta) K_t^j + \eta I_t^j, 0).$$

where $L_{1t+\Delta}^j$ ($L_{t+\Delta}^j$) represents the new debt level or saving when an investment opportunity arrives (no investment opportunity arrives). The price of the debt at time t that pays off one unit of consumption good at time $t + \Delta$ is $e^{-r_{ft}\Delta}$. By assumption, firm j cannot issue new equity to finance investment when an investment opportunity arrives so that $D_{1t}^j \geq 0$. Since there is no market for capital goods, the flow-of-funds constraints are different from those in the model of Section V.E. When firm j invests I_t^j with Poisson probability $\pi \Delta$, its capital stock jumps to $K_{t+\Delta}^j$ as shown in (F3).

Debt is subject to the credit constraint (F4). Firm j borrows $L_{1t+\Delta}^j$ at time t when an investment opportunity arrives. It may default on debt $L_{1t+\Delta}^j$ at time $t + \Delta$. If it does not default, it obtains continuation value $V_{t+\Delta}(K_{t+\Delta}^j, L_{1t+\Delta}^j)$. If it defaults, debt is renegotiated and the repayment $L_{1t+\Delta}^j$ is relieved. The lender can seize a fraction ζ of depreciated capital $(1 - \delta\Delta) K_t^j$ and a fraction η of newly installed capital I_t^j .²⁴ The lender keeps the firm running with these assets by reorganizing the firm. Thus the threat value to the lender is $V_t(\zeta(1 - \delta\Delta) K_t^j + \eta I_t^j, 0)$. Assume that firm j has a full bargaining power so that the renegotiated repayment is given by the threat value to the lender. The expression on the right-hand side of (F4) is the value to the firm if it chooses to default. We then have the incentive constraint given in (F4).

Conjecture that

$$V_t(K_t^j, L_t^j) = a_t K_t^j - a_t^L L_t^j + b_t.$$

Define $Q_t = e^{-r\Delta} a_{t+\Delta}$. Here Q_t is Tobin's marginal Q or the shadow price of capital, instead of the market price of capital. Substituting this conjecture and equations (F1), (F2), and (F3) into the Bellman equation yields

$$\begin{aligned} & a_t K_t^j - a_t^L L_t^j + b_t \\ = & \max_{I_t^j, K_{t+\Delta}^j, K_{1t+\Delta}^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} R_t K_t^j \Delta - L_t^j + e^{-r\Delta} b_{t+\Delta} \\ & + (1 - \pi \Delta) \left[e^{-r_{ft}\Delta} L_{t+\Delta}^j + Q_t (1 - \delta\Delta) K_t^j - e^{-r\Delta} a_{t+\Delta}^L L_{t+\Delta}^j \right] \\ & + \pi \Delta \left[e^{-r_{ft}\Delta} L_{1t+\Delta}^j + Q_t (1 - \delta\Delta) K_t^j - e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j \right] \\ & + \pi \Delta (Q_t - 1) I_t^j \end{aligned}$$

²⁴If $\zeta = \eta$, the lender effectively seizes the firm's future capital $\zeta K_{t+\Delta}^j$.

subject to

$$(F5) \quad I_t^j \leq R_t K_t^j \Delta + e^{-r_{ft}\Delta} L_{1t+\Delta}^j - L_t^j,$$

$$(F6) \quad a_{t+\Delta}^L L_{1t+\Delta}^j \leq b_{t+\Delta} + a_{t+\Delta} \zeta (1 - \delta \Delta) K_t^j + a_{t+\Delta} \eta I_t^j,$$

where (F5) follows from $D_{1t}^j \geq 0$ and says that investment is financed by internal funds, savings, and debt only. Credit constraint (F6) follows from (F4).

By the linear property of the Bellman function, the first-order condition for $L_{t+\Delta}^j$ yields

$$(F7) \quad e^{-r_{ft}\Delta} = e^{-r\Delta} a_{t+\Delta}^L,$$

and hence $L_{t+\Delta}^j$ is indeterminate. This implies that firm j is indifferent between saving and borrowing when no investment opportunity arrives. Multiplying the two sides of inequality (F6) by $e^{-r\Delta}$ and using (F7), we obtain

$$(F8) \quad e^{-r_{ft}\Delta} L_{1t+\Delta}^j = e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j \leq B_t + Q_t \zeta (1 - \delta \Delta) K_t^j + Q_t \eta I_t^j,$$

where we have used $Q_t = e^{-r\Delta} a_{t+\Delta}$ and the definition

$$(F9) \quad B_t \equiv e^{-r\Delta} b_{t+\Delta}.$$

When $1 < Q_t < 1/\eta$, the financing constraint (F5) and the credit constraint (F8) bind so that optimal investment is given by

$$I_t^j = \frac{1}{1 - \eta Q_t} \left[R_t K_t^j \Delta + B_t + Q_t \zeta (1 - \delta \Delta) K_t^j - L_t^j \right],$$

where the multiplier $1/(1 - \eta Q_t)$ reflects the leverage effect.

Substituting the investment rule back into the Bellman equation and matching coefficients, we derive

$$a_t = R_t \Delta + Q_t (1 - \delta \Delta) + \pi \Delta (Q_t - 1) \frac{R_t \Delta + Q_t \zeta (1 - \delta \Delta)}{1 - \eta Q_t},$$

$$a_t^L = 1 + \pi \Delta \frac{Q_t - 1}{1 - \eta Q_t},$$

$$b_t = e^{-r\Delta} b_{t+\Delta} + \pi \Delta \frac{(Q_t - 1) B_t}{1 - \eta Q_t}.$$

Using (F7) and (F9) and the preceding three equations, we can derive

$$Q_t = e^{-r\Delta} \left[R_{t+\Delta} \Delta + Q_{t+\Delta} (1 - \delta \Delta) + \pi \Delta (Q_{t+\Delta} - 1) \frac{R_{t+\Delta} \Delta + Q_{t+\Delta} \zeta (1 - \delta \Delta)}{1 - \eta Q_t} \right],$$

$$e^{-r_{ft}\Delta} = e^{-r\Delta} \left[1 + \pi \Delta \frac{Q_{t+\Delta} - 1}{1 - \eta Q_t} \right],$$

$$B_t = e^{-r\Delta} \left[1 + \pi \Delta \frac{Q_{t+\Delta} - 1}{1 - \eta Q_t} \right] B_{t+\Delta}.$$

Taking the continuous-time limit as $\Delta \rightarrow 0$ yields

$$(F10) \quad \dot{Q}_t = (r + \delta) Q_t - R_t - \frac{\pi (Q_t - 1) Q_t \zeta}{1 - \eta Q_t},$$

$$(F11) \quad \dot{B}_t = r B_t - \frac{\pi (Q_t - 1)}{1 - \eta Q_t} B_t,$$

$$(F12) \quad r_{ft} = r - \frac{\pi (Q_t - 1)}{1 - \eta Q_t} < r.$$

We now show that this solution is the same as that in the continuous-time setup. We derive the continuous-time limit of the dynamic programming problem as

$$(F13) \quad r V_t(K_t^j, L_t^j) = \max_{D_{0t}^j, D_{1t}^j, I_t^j, L_{1t}^j} D_{0t}^j + \dot{V}_t(K_t^j, L_t^j) \\ + \pi \left[D_{1t}^j + V_t(K_t^j + I_t^j, L_{1t}^j) - V_t(K_t^j, L_t^j) \right]$$

subject to

$$(F14) \quad \dot{L}_t^j = r_{ft} L_t^j + D_{0t}^j - R_t K_t^j,$$

$$(F15) \quad D_{1t}^j = L_{1t}^j - L_t^j - I_t^j,$$

$$(F16) \quad I_t^j \leq L_{1t}^j - L_t^j,$$

$$(F17) \quad V_t(K_t^j + I_t^j, L_{1t}^j) \geq V_t(K_t^j + I_t^j, 0) - V_t(\zeta K_t^j + \eta I_t^j, 0).$$

When no investment opportunity arrives, capital simply depreciates so that $\dot{K}_t^j = -\delta K_t^j$. Whenever an investment opportunity arrives, capital jumps to $K_t^j + I_t^j$.

Conjecture the value function takes the form

$$(F18) \quad V_t(K_t^j, L_t^j) = Q_t K_t^j - L_t^j + B_t,$$

and hence the credit constraint (F17) becomes

$$(F19) \quad L_{1t}^j \leq Q_t \zeta K_t^j + \eta Q_t I_t^j + B_t,$$

where $B_t \geq 0$ is the bubble component of equity value.

Substituting the conjectured value function into the Bellman equation yields

$$r \left(Q_t K_t^j - L_t^j + B_t \right) = \max \left[\dot{L}_t^j - r_{f_t} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j - \dot{L}_t^j + \dot{B}_t \right. \\ \left. + \pi \left[\left(L_{1t}^j - L_t^j - I_t^j \right) + \left(Q_t I_t^j - L_{1t}^j + L_t^j \right) \right] \right].$$

Thus \dot{L}_t^j cancels itself out so that firm j is indifferent between saving and borrowing when no investment opportunity arrives. Simplifying yields

$$(F20) \quad r \left(Q_t K_t^j - L_t^j + B_t \right) = \max \left[-r_{f_t} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j \right. \\ \left. + \dot{B}_t + \pi (Q_t - 1) I_t^j \right].$$

Using the credit constraint (F19) and the financing constraint (F16), we obtain

$$I_t^j \leq L_{1t}^j - L_t^j \leq \zeta Q_t K_t^j + \eta Q_t I_t^j + B_t - L_t^j.$$

If $1 < Q_t < 1/\eta$, it is profitable for firm j to invest as much as possible and both constraints bind. In this case firm j borrows by selling bonds. We then have

$$I_t^j = \frac{\zeta Q_t K_t^j + B_t - L_t^j}{1 - \eta Q_t}.$$

Substituting this investment rule back into the Bellman equation (F13) and matching coefficients, we derive the equations for Q_t , B_t , and r_{f_t} given above.

We now compute aggregate investment

$$I_t = \int I_t^j dj = \frac{\zeta Q_t K_t + B_t - \int L_t^j dj}{1 - \eta Q_t}.$$

Since $r_{f_t} < r$, households' short-sale constraints bind so that $L_t^h = 0$ and the bond market-clearing condition becomes $\int L_t^j dj = 0$. Thus

$$(F21) \quad I_t = \frac{\zeta Q_t K_t + B_t}{1 - \eta Q_t}.$$

We can then derive the law of motion for aggregate capital

$$\int K_{t+\Delta}^j dj = \int (1 - \delta \Delta) K_t^j dj + \pi \Delta \int I_t^j dj.$$

Taking the limit as $\Delta \rightarrow 0$ yields

$$\dot{K}_t = -\delta K_t + \pi I_t.$$

Substituting (F21) into the above equation yields the equation for K_t

$$(F22) \quad \dot{K}_t = -\delta K_t + \pi \frac{\xi Q_t K_t + B_t}{1 - \eta Q_t},$$

Finally, we can use the same procedure in the proof of Proposition 2 to derive $R_t = \alpha K_t^{\alpha-1}$. The equilibrium system for (Q_t, B_t, r_{ft}, K_t) consists of (F10), (F11), (F12), and (F22) when $1 < Q_t < 1/\eta$. The usual transversality conditions must be satisfied. We can see that the equilibrium system presented in Proposition 12 is the special case with $\eta = 0$.

We can also prove the following result.

PROPOSITION 14: *For the model in this subsection with intertemporal bonds, if*

$$(F23) \quad 0 < \xi < \frac{\delta(1 - \eta)}{r + \pi},$$

then the bubbly and bubbleless steady states with $1 < Q < 1/\eta$ coexist. Moreover, the interest rates in the bubbleless and bubbly steady states are given by $r_f^ = r + \pi - \delta(1 - \eta)/\xi < 0$ and $r_f = 0$, respectively.*

PROOF:

We first derive the bubbly steady state in which $B > 0$. Using the equilibrium system derived above, we can show that

$$(F24) \quad Q_b = \frac{r + \pi}{\eta r + \pi}, \quad r_f = 0,$$

$$(F25) \quad R_b = \alpha K_b^{\alpha-1} = \frac{r + \pi}{\eta r + \pi} [(1 - \xi)r + \delta],$$

$$(F26) \quad \frac{B}{K_b} = \frac{\delta}{\pi} - \frac{\xi(r + \pi)}{\pi(1 - \eta)}.$$

Since $\eta \in (0, 1)$, we have $1 < Q_b < 1/\eta$. Given condition (F23), we have $B > 0$ and hence a bubbly steady state exists.

We next derive the bubbleless steady state in which $B = 0$. Using the equilibrium system derived above, we can show that

$$Q^* = \frac{\delta}{\pi \xi + \eta \delta},$$

$$R^* = \alpha K^{*\alpha-1} = \frac{\delta r}{\pi \xi + \eta \delta} + \delta,$$

$$r_f^* = r + \pi - \delta(1 - \eta)/\xi.$$

Under condition (F23), we have $1 < Q^* < 1/\eta$. Thus a bubbleless steady state exists.

G. CROSS-HOLDINGS

In this appendix we assume that households hold a fraction $1 - H$ shares of a market portfolio of all firm stocks and firms hold $H \in (0, 1)$ shares of the market portfolio in the model of Section V.E. For technical convenience we consider the continuous-time setup. Assume that firms do not use the market portfolio to finance investment for the reasons discussed in Section V.F.

Let $V_t(K_t^j, L_t^j, H_t^j)$ denote the ex ante market value of firm j , where H_t^j denotes firm j 's holdings of the market portfolio prior to the investment opportunity shock. Then V_t satisfies the continuous-time Bellman equation

$$(G1) \quad rV_t(K_t^j, L_t^j, H_t^j) = \max D_{0t}^j + \dot{V}_t(K_t^j, L_t^j, H_t^j) + \pi \left[D_{1t}^j + V_t(K_{1t}^j, L_{1t}^j, H_{1t}^j) - V_t(K_t^j, L_t^j, H_t^j) \right]$$

subject to the flow-of-funds constraints

$$(G2) \quad \dot{L}_t^j = r_{ft}L_t^j + D_{0t}^j - R_tK_t^j + Q_t(\dot{K}_t^j + \delta K_t^j) + P_t\dot{H}_t^j - X_tH_t^j,$$

$$(G3) \quad D_{1t}^j = Q_tI_t^j + L_{1t}^j - L_t^j - I_t^j + Q_tK_t^j - Q_tK_{1t}^j + P_t(H_t^j - H_{1t}^j),$$

the financing constraint

$$(G4) \quad I_t^j \leq L_{1t}^j - L_t^j,$$

and the credit constraint

$$(G5) \quad V_t(K_{1t}^j, L_{1t}^j, H_{1t}^j) \geq V_t(K_{1t}^j, 0, H_{1t}^j) - V_t(\xi K_t^j, 0, 0),$$

where H_{1t}^j denotes firm j 's holdings of the market portfolio when an investment opportunity arrives. Here P_t denotes the value of the market portfolio,

$$P_t = \int V_t(K_t^j, L_t^j, H_t^j) dj,$$

and X_t denotes the total dividends of the portfolio

$$X_t = \int D_t^j dj = \int D_{0t}^j dj + \pi \int D_{1t}^j dj.$$

Note that the value of the market portfolio does not jump even if the value of an individual

firm can jump when an investment opportunity arrives. This is because

$$P_{t+\Delta} = (1 - \pi \Delta) \int V_{t+\Delta} \left(K_{t+\Delta}^j, L_{t+\Delta}^j, H_{t+\Delta}^j \right) dj \\ + \pi \Delta \int V_{t+\Delta} \left(K_{1t+\Delta}^j, L_{1t+\Delta}^j, H_{1t+\Delta}^j \right) dj$$

so that $P_{t+\Delta} \rightarrow P_t$ as $\Delta \rightarrow 0$.

The financing constraint (G4) means that firm j only uses debt and savings to finance investment. The interpretation of the credit constraint (G5) is similar to that in Section V.E. In particular, the lender can only recover a fraction ζ of capital and take over the firm in the event of default.

Conjecture that the value function takes the form

$$(G6) \quad V_t \left(K_t^j, L_t^j, H_t^j \right) = Q_t K_t^j + B_t - L_t^j + P_t H_t^j.$$

Substituting this conjecture and the flow-of-funds constraints into the preceding Bellman equation yields

$$r \left(Q_t K_t^j - L_t^j + B_t + P_t H_t^j \right) \\ = \max \left[\dot{L}_t^j - r_{ft} L_t^j + R_t K_t^j - Q_t \dot{K}_t^j - Q_t \delta K_t^j - P_t \dot{H}_t^j + X_t H_t^j \right. \\ \left. + Q_t \dot{K}_t^j + \dot{Q}_t K_t^j - \dot{L}_t^j + \dot{B}_t + \dot{P}_t H_t^j + P_t \dot{H}_t^j \right. \\ \left. + \pi \left[(Q_t - 1) I_t^j + L_{1t}^j - L_t^j + Q_t K_t^j - Q_t K_{1t}^j + P_t H_t^j - P_t H_{1t}^j \right] \right. \\ \left. + \pi \left[Q_t K_{1t}^j - L_{1t}^j + B_t + P_t H_{1t}^j - \left(Q_t K_t^j - L_t^j + B_t \right) - P_t H_t^j \right] \right].$$

Given the conjectured value function, the credit constraint becomes

$$L_{1t}^j \leq Q_t \zeta K_t^j + B_t.$$

If $Q_t > 1$, the financing constraint and the credit constraint bind so that optimal investment is given by

$$I_t^j = Q_t \zeta K_t^j + B_t - L_t^j.$$

Substituting this investment rule back into the Bellman equation and matching coefficients, we obtain (20), (21), (63), and

$$r P_t = X_t + \dot{P}_t.$$

Thus the rate of return on the market portfolio is equal to r . Aggregation yields the law of motion for aggregate capital (28). Thus the equilibrium system for (Q_t, K_t, B_t, r_{ft}) is the same as that in Section V.E and online Appendix B.5 and hence Proposition 13 still holds. The only difference lies in the valuation of the firm.

Since $\int H_t^j = H$, aggregation of (G6) yields

$$P_t = \frac{Q_t K_t + B_t}{1 - H}.$$

As discussed in Fedenia, Hodder, and Triantis (1994) and Elliott, Golub, and Jackson (2014), the equation above and equation (G6) show that cross-holdings inflate the market capitalization. Since households hold $1 - H$ shares of all firms, the portfolio value to the households is $Q_t K_t + B_t$. Thus cross-holdings do not have any effects on welfare and real allocation as long as cross-holdings do not help finance investment.