# Online Appendix to "Optimal Regulation of Financial Intermediaries"

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**Abstract**

Appendix A provides omitted proofs, the setting with heterogenous assets, and the phase diagram of the competitive equilibrium. Appendix B develops the contractual environment in detail, provides verification theorems and shows how to solve the competitive equilibrium and the planner's problem as a system of PDEs. Appendix C provides a discrete-time version of the setting.

# Appendix A

In the first part of this Appendix I provide proofs for Propositions 1 and 2, which correspond to the baseline model. In the second part I extend the baseline model to include heterogenous asset classes and intermediaries, as in Section 6, and I provide the proof of Proposition 3. For simplicity I don't consider retirement, as in the baseline model in the paper. Proofs can be easily extended to incorporate retirement. Finally, Figure 1 shows the stationary distribution of the competitive equilibrium and the phase diagram.

## 1 Omitted proofs in the baseline model

### Proof of Proposition 1

Consider an optimal plan $\mathcal{P}$ and the associated value function $S$ and policy functions $\hat{c}$, $g$, $\sigma_x$ and the law of motion of the endogenous state, $\mu_X$ and $\sigma_X$, all functions of $(X, Y)$. We can build a recursive equilibrium using the same law of motion $\mu_X$ and $\sigma_X$. From the FOC for growth we get

$$q = \iota'(g) \tag{1}$$

From the FOC for households' consumption we get $\zeta = \left(\frac{a-\iota(g)-\hat{c}X}{S}\right)^{-1/\psi} = \hat{c}_h^{-1/\psi}$ and from intermediaries' FOC for $\hat{c}$ we get $\xi = \left(\hat{c}^{-1/\psi} + \frac{\gamma}{\psi}(\frac{\phi q \nu}{X})^2 \hat{c}^{-2/\psi-1}\right)^{-1}$, where we have used $\hat{k} = X^{-1}$. Notice that we get by construction $\xi\zeta = \Lambda = -S_X$, from the planner's optimality condition (37). From $S = \zeta(q + T - \xi X)$ we get $T = \frac{S}{\zeta} - q + \xi X$.

We can use the representative households' HJB to pin down $r$, and from the FOC for $\sigma_w$ we pin down $\pi$, as follows. Define $w = (q + T - \xi X)k$, and obtain an expression for $\sigma_w$. We then set $\pi = \gamma\sigma_w - (1-\gamma)\sigma_\zeta$, and $r$ so that households' HJB is satisfied. We are in effect choosing $r$ and $\pi$ so that $(a - \iota(g) - \hat{c}X)k = c_h$ is the optimal choice of consumption for the household, and their wealth $w = (q + T - \xi X)k$. Intermediaries' FOC for $\sigma_x$ will be satisfied automatically, because the planner's optimality condition for aggregate risk sharing (36) coincides with the competitive equilibrium's (22). Since the FOC for $\hat{c}$ is satisfied by construction, we just need to set $\tau^k$ so that the FOC for $\hat{k}$ is satisfied:

$$\frac{a - \iota(g)}{q} + \mu_q + g + \sigma'\sigma_q - (r + \tau^k) - \pi(\sigma + \sigma_q) = \gamma\xi(\hat{c}^{-1/\psi}\phi\nu)^2 \frac{q}{X} \tag{2}$$

Now we want to prove that $\xi$ will satisfy intermediaries' HJB equation. For this we will use the planner's HJB equation (32). Multiply by $S$ on both sides, take the derivative with respect to $X$ using the envelop theorem, and divide throughout by $S_X$ to obtain

$$\begin{aligned}
\frac{\rho}{1-\frac{1}{\psi}} = \frac{(a-\iota(g)-\hat{c}X)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} \frac{1}{\psi} S^{\frac{1}{\psi}-1} - \frac{(a-\iota(g)-\hat{c}X)^{-\frac{1}{\psi}} S^{\frac{1}{\psi}}}{S_X}\hat{c} + (g - \frac{\gamma}{2}\sigma^2) \\
+ \frac{S_{XY}}{S_X}\mu_Y + \frac{S_{XX}}{S_X}\mu_X X \\
+ \mu_X - \gamma(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota'(g)}{X}\nu)^2 + \frac{1}{2}\frac{S_{XYY}}{S_X}\sigma_Y^2 + \frac{1}{2}\frac{S_{XXX}}{S_X}(\sigma_X X)^2 + \frac{S_{XX}}{S_X}\sigma_X^2 X + \frac{S_{XXY}}{S_X}\sigma_X X\sigma_Y \\
+ \frac{S_{XY}}{S_X}\sigma_X\sigma_Y + (1-\gamma)\sigma\left(\frac{S_{XX}}{S_X}\sigma_X X + \frac{S_{XY}}{S_X}\sigma_Y\right) + (1-\gamma)\sigma\sigma_X \\
-\gamma\left(\frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y\right)\left(\frac{S_{XX}}{S_X}\sigma_X X + \frac{S_{XY}}{S_X}\sigma_Y + \sigma_X\right) + \frac{\gamma}{2}\left(\frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y\right)^2
\end{aligned} \tag{3}$$

Now use $-\xi\zeta = S_X$ to obtain

$$\frac{S_{XX}}{S_X} = \frac{\xi_X}{\xi} + \frac{\zeta_X}{\zeta}, \quad \frac{S_{XY}}{S_X} = \frac{\xi_Y}{\xi} + \frac{\zeta_Y}{\zeta}, \quad \frac{S_{XXX}}{S_X} = \frac{\zeta_{XX}}{\zeta} + 2\frac{\xi_X}{\xi}\frac{\zeta_X}{\zeta} + \frac{\xi_{XX}}{\xi},$$

$$\frac{S_{XYY}}{S_X} = \frac{\zeta_{YY}}{\zeta} + 2\frac{\zeta_Y}{\zeta}\frac{\xi_Y}{\xi} + \frac{\xi_{YY}}{\xi}, \quad \frac{S_{XXY}}{S_X} = \frac{\zeta_{XY}}{\zeta} + \frac{\xi_Y}{\xi}\frac{\zeta_X}{\zeta} + \frac{\xi_X}{\xi}\frac{\zeta_Y}{\zeta} + \frac{\xi_{XY}}{\xi}$$

Now plug this into (3), use the definition of $\mu_X$ and the FOC for $\hat{c}$ in the private contract (which we already know holds), and simplify to obtain

$$0=\frac{\frac{1}{\psi}}{1-\frac{1}{\psi}}\tilde{c}_h^{1-\frac{1}{\psi}}\zeta^{\frac{1}{\psi}-1}+\mu_\xi+\mu_\zeta-\frac{1}{\psi}\frac{\hat{c}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}+\frac{\gamma}{2}\sigma_X^2-\gamma(\frac{1}{2}-1/\psi)(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota'(g)}{X}\nu)^2$$

$$+\sigma_\xi\sigma_\zeta+\sigma_X\left(\sigma_\xi+\sigma_\zeta\right)+(1-\gamma)\sigma\left(\sigma_\xi+\sigma_\zeta\right)$$

$$-\gamma\sigma_S\left(\sigma_\xi+\sigma_\zeta+\sigma_X\right)+\frac{\gamma}{2}\sigma_S^2$$

Now from household's HJB and using $\sigma_w=\sigma_S-\sigma_\zeta+\sigma$ we get

$$\frac{\rho}{1-\frac{1}{\psi}}=\frac{1}{\psi}\frac{\tilde{c}_h^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\zeta^{\frac{1}{\psi}-1}+r+\frac{\gamma}{2}\left(\sigma_S-\sigma_\zeta+\sigma\right)^2+\mu_\zeta-\frac{\gamma}{2}\sigma_\zeta^2$$

which we plug into our expression. After some algebra using $\sigma_S=\sigma_X+\frac{1}{\gamma}(\sigma_\xi+\sigma_\zeta)$ and $\sigma_x=\sigma_X+\sigma$, as well as the FOC for $\hat{c}$ and $\sigma_x$, we get

$$\gamma\xi(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota'(g)}{X}\nu)^2-\hat{c}$$

$$+\xi\left\{r+\frac{\hat{c}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}-\frac{\rho}{1-\frac{1}{\psi}}-\frac{\gamma}{2}(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota'(g)}{X}\nu)^2-\mu_\xi+\sigma_x\pi+\sigma_\xi\pi-\frac{\gamma}{2}\sigma_x^2-\sigma_\xi\sigma_x\right\}=0$$

Because $\tau^k$ is chosen so that the pricing equation for capital holds we get

$$\gamma\xi(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota'(g)}{X}\nu)^2=\frac{q}{X}\left(\frac{a-\iota(g)}{q}+\mu_q+g+\sigma'\sigma_q-(r+\tau^k)-\pi(\sigma+\sigma_q)\right)$$

and plugging this in, we obtain experts' HJB.

Finally, we just need to check that the pricing equation for taxes is satisfied. First, use the planner's HJB and households' HJB to obtain a version of the dynamic budget constraint of the household. Write $S=\zeta(q+T-\xi X)$, $w=(q+T-\xi X)k$, and then

$$q\mu_q+T\mu_T-\xi X(\mu_\xi+\mu_X+\sigma_\xi\sigma_X)+g(q+T-\xi X)+\sigma(q\sigma_q+T\sigma_T-\xi X(\sigma_\xi+\sigma_X))$$

$$=r(q+T-\xi X)+\pi(q\sigma_q+T\sigma_T-\xi X(\sigma_\xi+\sigma_X)+\sigma(q+T-\xi X))-\tilde{c}_h(q+T-\xi X)$$

Multiply experts' HJB by $X$ to obtain

$$a-\iota(g)-\hat{c}X+q(g+\mu_q+\sigma\sigma_q-(r+\tau^k)-(\sigma+\sigma_q)\pi)$$

$$+\xi X\left\{r+\frac{1}{1-\frac{1}{\psi}}(\hat{c}^{1-\frac{1}{\psi}}-\rho)-\frac{\gamma}{2}\sigma_x^2-\frac{\gamma}{2}(\hat{c}^{-\frac{1}{\psi}}\phi\frac{\iota(g)}{X}v)^2+\sigma_x(\pi-\sigma_\xi)-\mu_\xi+\sigma_\xi\pi\right\}=0$$

Combining these two expressions, and using the definition of $\mu_X$, $\sigma_X$, and $\tilde{c}_h=\frac{a-\iota(g)-\hat{c}X}{q+T-\xi X}$ we get the pricing equation for taxes. Verifying that we indeed have a competitive equilibrium can then be done as described in part B of this Appendix, and depends on the particular application. Finally, comparing the planner's optimality condition for $g$ (33) with the equilibrium condition (35) we obtain $T_t/q_t = \eta_t$.

### Proof of Proposition 2

Use expression (34) for $\eta$

$$\eta_t = \frac{\Lambda_t X_t}{\hat{c}_{h,t}^{-1/\psi}} \gamma(\hat{c}_t^{-1/\psi} \phi_t \frac{\nu_t}{X_t})^2 \iota_t''(g)$$

Use $\zeta_t = \hat{c}_{h,t}^{-1/\psi}$ and $\Lambda_t = \xi_t \zeta_t$ to get

$$\eta_t = \xi_t X_t \gamma(\hat{c}_t^{-1/\psi} \phi_t \frac{\nu_t}{X_t})^2 \iota_t''(g)$$

$$\eta_t = \frac{1}{\xi_t X_t} \gamma(\xi_t \hat{c}_t^{-1/\psi} \phi_t \nu_t)^2 \iota_t''(g)$$

Now use $\tilde{\phi}_t = \xi_t \hat{c}_t^{-1/\psi} \phi_t$, and multiply and divide by $\iota_t'(g_t)$ to get

$$\eta_t = \frac{\iota_t'(g_t)}{\xi_t X_t} \gamma(\tilde{\phi}_t \nu_t)^2 \frac{\iota_t''(g)}{\iota_t'(g_t)}$$

Now recall $q_t = \iota_t'(g_t)$, and $X_t = \int_{\mathbb{I}} x_{i,t} di$, so $\xi_t X_t = n_t/k_t$. Plug this in to get

$$\eta_t = \frac{q_t k_t}{n_t} \gamma(\tilde{\phi}_t \nu_t)^2 \frac{\iota_t''(g)}{\iota_t'(g_t)}$$

Finally, recall $\alpha_t = \frac{q_t k_t}{n_t} \gamma(\tilde{\phi}_t \nu_t)^2$ and the definition $\epsilon_t = \frac{\iota_t''(g)}{\iota_t'(g_t)}$ to get

$$\eta_t = \alpha_t \epsilon_t$$

This completes the proof.

## 2 Heterogenous asset classes and intermediaries

Each intermediary has continuation utility

$$dU_{i,t} = -f(c_{i,t}, U_{i,t})dt + \sigma_{U,i,t} dZ_t + \sum_j \tilde{\sigma}_{U,i,j,t} dW_{j,t} \tag{4}$$

The incentive compatibility constraint requires that we expose the intermediary to idiosyncratic risk in each of his asset classes

$$\tilde{\sigma}_{U,i,j,t} \geq \partial_c f(c_{i,t}, U_{i,t})\phi_{i,t}q_{j,t}k_{i,j,t}\nu_{i,j,t} = \frac{c_{i,t}^{-1/\psi}}{((1-\gamma)U_{i,t})^{\frac{\gamma-1/\psi}{1-\gamma}}}\phi_{i,j,t}q_{j,t}k_{i,j,t}\nu_{i,j,t} \geq 0 \tag{5}$$

The process $\xi_i$ depends on intermediary $i$'s type (we get one process $\xi$ for each intermediary type). The HJB equation must then be adjusted slightly

$$\begin{aligned} r_t\xi_{i,t} \quad = \quad & \min_{\hat{c},\hat{k},\sigma_x} \hat{c} - \sum_j q_{j,t}\hat{k}_j\alpha_{i,j,t} + \xi_{i,t}\Big\{\frac{1}{1-\frac{1}{\psi}}(\rho - \hat{c}^{1-1/\psi}) - \sigma_x\pi_t \\ & +\mu_{\xi,i,t} - \sigma_{\xi,i,t}\pi_t + \frac{1}{2}\gamma\sigma_x^2 + \sum_j \frac{1}{2}\gamma\left(\hat{c}^{-1/\psi}\phi_{i,j,t}q_{j,t}\hat{k}_j\nu_{i,j,t}\right)^2 + \sigma_{\xi,i,t}\sigma_x\Big\} \end{aligned} \tag{6}$$

The FOC for each $\hat{k}_j$ gives us the asset pricing equation for capital for all $(i,j)$ such that $\hat{k}_{i,j,t} > 0$,

$$\underbrace{\frac{a_{i,j,t} - \iota_{j,t}(g_{j,t})}{q_{j,t}} + g_{j,t} + \mu_{q,j,t} + \sigma_{j,t}\sigma'_{q,j,t} - (r_t + \tau^k_{j,t}) - (\sigma_{j,t} + \sigma_{q,j,t})\pi_t}_{\text{risk-adjusted excess return } \equiv\, \alpha_{i,j,t}} = \underbrace{\gamma\frac{q_{j,t}k_{i,j,t}}{n_{i,t}}(\tilde{\phi}_{i,j,t}\nu_{i,j,t})^2}_{\text{id. risk premium}} \tag{7}$$

where $\tilde{\phi}_{i,j,t} = \xi_{i,t}\hat{c}_{i,t}^{-1/\psi}\phi_{i,j,t}$. Note that now the intermediary might have to keep different equity stakes for each asset class. If $\phi_{i,j,t} = \phi_{i,t}$ for all $j$, the equity stake is common across asset classes, and we can implement the optimal contract with an equity constraint. Otherwise, we need an incentive scheme that treats the returns on different asset classes differently.

The FOC for investment in each asset class is

$$\iota'_{j,t}(g_{j,t}) = q_{j,t}$$

$$\implies \hat{c}_{h,t}^{-1/\psi}\iota'_{j,t}(g_{j,t})(1 + T_{j,t}/q_{j,t}) = \hat{c}_{h,t}^{-1/\psi}(q_{j,t} + T_{j,t}) \tag{8}$$

Since front-loading consumption can relax the equity constraint across asset classes, the FOC for $\hat{c}$ is now

$$\xi_{i,t}\hat{c}_{i,t}^{-1/\psi} + \xi_{i,t}\sum_j \frac{\gamma}{\psi}(\phi_{i,j,t}q_{j,t}\hat{k}_{i,j,t}\nu_{i,j,t})^2\hat{c}_{i,t}^{-2/\psi-1} = 1 \tag{9}$$

Since households' FOC for consumption is unchanged, we get the MRS $\Lambda_{i,t} = \xi_{i,t}\zeta_t$

$$\Lambda_{i,t} = \frac{\hat{c}_{h,t}^{-1/\psi}}{\hat{c}_{i,t}^{-1/\psi} + \sum_j \frac{\gamma}{\psi}(\phi_{i,j,t}\iota_{j,t}(g_{j,t})\hat{k}_{i,j,t}\nu_{i,j,t})^2\hat{c}_{i,t}^{-2/\psi-1}} \tag{10}$$

and the FOC for aggregate risk sharing yields

$$\sigma_{x,i,t} - \sigma_h = -\frac{1}{\gamma}\sigma_{\Lambda_{i,t}} \tag{11}$$

The logic is the same as in the baseline model; optimal contracts give more utility to an intermediary when the cost of his utility $\Lambda_{i,t}$ is low.

It is still the case that every intermediary of each type gets the same policy function $\hat{c}_f$, $\hat{k}_{f,j}$ and $\sigma_{x,f}$ for each $f = 1...F$. We now have more endogenous state variables: for each intermediary type $f = 1...F$ we have $X_{f,t} = \frac{\int_{\mathbb{I}_f} x_{i,t}di}{k_t}$; and for each asset class $j = 1...J$ we have $\theta_{j,t} = \frac{k_{j,t}}{k_t}$, where $k_t = \sum_j k_{j,t}$ is the total capital stock:

$$\frac{dk_t}{k_t} = \underbrace{\sum_j g_{j,t}\theta_{j,t}}_{g_t} dt + \underbrace{\sum_j \sigma_{j,t}\theta_{j,t}}_{\sigma_t} dZ_t$$

Each intermediary type has aggregate wealth $n_{f,t} = \xi_{f,t}X_{f,t}k_t$, and the representative households' wealth is $w_t = (\sum_j(q_{j,t} + T_{j,t})\theta_{j,t} - \sum_f \xi_{f,t}X_{f,t})k_t$. Total output from each asset class is $a_{j,t}k_{j,t} = \sum_f \left(a_{f,j,t}\hat{k}_{f,j,t}X_{f,t}\right) k_t$. Market clearing conditions must be adjusted

$$\tilde{c}_{h,t}w_t + \sum_f \hat{c}_{f,t}X_{f,t} = \sum_j \left((a_{j,t} - \iota_{j,t}(g_{j,t}))\theta_{j,t}\right) \tag{12}$$

$$\sum_f \hat{k}_{f,j,t}X_{f,t} = \theta_{j,t} \tag{13}$$

The law of motion of each $X_f$ must be adjusted:

$$\mu_{X,f,,t} = \frac{\rho}{1-1/\psi} - \frac{\hat{c}_{f,t}^{1-1/\psi}}{1-1/\psi} + \frac{\gamma}{2}\sigma_{x,f,t}^2 + \sum_j \frac{\gamma}{2}(\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\hat{k}_{f,j,t}\iota'_{j,t}(g_{j,t})\nu_{f,j,t})^2 - g_t - \sigma_t\sigma_{x,f,t} + \sigma_t^2$$

$$\sigma_{X,f,t} = \sigma_{x,f,t} - \sigma_t$$

and the law of motion of each $\theta_j$ is

$$d\theta_{j,t} = \theta_{j,t}\left(g_{j,t} - g_t + \sigma_t(\sigma_{j,t} - \sigma_t)\right)dt + \theta_{j,t}\left(\sigma_{j,t} - \sigma_t\right)dZ_t$$

The social planner has the same laws of motion. His value function takes the form $\frac{(S(X,\theta,Y)k)^{1-\gamma}}{1-\gamma}$. We must adjust his HJB equation, and also allow him to choose how to allocation assets to intermediaries:

$$\frac{\rho}{1-1/\psi} = \max_{g_j,\hat{c}_f,\hat{k}_{f,j},\sigma_{x,f}} \frac{\left(\sum_j \left((a_{j,t}-\iota_{j,t}(g_{j,t}))\theta_{j,t}\right)-\sum_f \hat{c}_f X_f\right)^{1-1/\psi}}{1-1/\psi} S^{1/\psi-1} \tag{14}$$

$$+\mu_S + g - \frac{\gamma}{2}\sigma_S^2 - \frac{\gamma}{2}\sigma^2 + (1-\gamma)\sigma_S\sigma \tag{15}$$

where $\mu_S$ and $\sigma_S$ are obtained from Ito's lemma on $S(X,\theta,Y)$.

The FOC for $\hat{c}_f$ and $\sigma_{f,t}$ are the same as in the competitive equilibrium. We get equations (10) and (11), where $\Lambda_{f,t} = -S'_{X_f}$. The FOC for $g_j$ is

$$\underbrace{\hat{c}_{h,t}^{-1/\psi}}_{\partial_{c_h}(Sk)} \iota_{j,t}(g_{j,t})(1+\eta_{j,t}) = \underbrace{S_t + \sum_f \Lambda_{f,t}X_{f,t} + \left(S'_{\theta_j} - \sum_m \theta_{m,t}S'_{\theta_m}\right)}_{\partial_{k_j}(Sk)} \tag{16}$$

$$\eta_{jt} = \sum_f \frac{\Lambda_{f,t}X_{f,t}}{\hat{c}_h^{-1/\psi}\theta_{j,t}}\gamma(\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\hat{k}_{f,j,t}\nu_{f,j,t})^2\iota''_{j,t}(g_{j,t}) \tag{17}$$

Comparing this FOC with (8) we see that $\eta_{j,t}$ captures the externality related to asset class $j$, analogous to the case with homogeneous capital and intermediaries. Here when the planner raises the marginal cost of capital, it affects idiosyncratic risk sharing for all intermediaries that hold that asset.

The FOC for $\hat{k}_{f,j}$ is

$$\hat{c}_{h,t}^{-1/\psi}a_{f,j,t} - \Lambda_{f,t}\gamma(\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\iota'_{j,t}(g_{j,t})\nu_{f,j,t})^2\hat{k}_{f,j,t} = \lambda_{j,t}$$

where $\lambda_{j,t}$ is the Lagrange multiplier on the constraint (13). Using $\Lambda_{f,t} = \xi_{f,t}\zeta_t = \xi_{f,t}\hat{c}_{h,t}^{-1/\psi}$, the definition of $\tilde{\phi}_{f,j,t}$ and $q_{j,t} = \iota_{j,t}(g_{j,t})$ in the competitive equilibrium, we get that all intermediaries who hold asset of class $j$ must have the same

$$a_{i,j,t} - \gamma\frac{q_{j,t}k_{i,j,t}}{n_{i,t}}(\tilde{\phi}_{i,j,t}\nu_{i,j,t})^2 q_{j,t}$$

which is exactly what the competitive equilibrium does according to the pricing equation (7). So we see that $\eta_{j,t}$ is still the only source of inefficiency in the economy. All other optimality conditions for the planner are satisfied.

## Proof of Proposition 3

The proof follows the same lines as the proof of Proposition 1. To establish that each intermediary's HJB holds, we now take derivatives on the planner's HJB equation with respect to each $X_f$ (instead of a single $X$ as before). Notice that $S_t = S(\{X_f\},\{\theta_j\},Y)$, so the law of motion of the $\theta's$ must be taken into account when computing $\mu_S$ and $\sigma_S$. Also, now we don't have $\hat{k} = X^{-1}$. Instead, $\hat{k}_{f,j}$ are controls for the planner, so the envelope theorem allows us to ignore it when taking derivatives with respect to $X_f$; but $X_f$ appears in the formula for $a_{j,t}\theta_{j,t} = \sum_f \left( a_{f,j,t}\hat{k}_{f,j,t}X_{f,t} \right)$ and each of the (13) constraints (for each $j$). Once this is taken into account, we obtain essentially the same formulas, extended to account for heterogenous asset classes and intermediaries.

To establish the validity of the sufficient statistic (43) for each asset class, we follow the proof of Proposition 2 but using expression (17):

$$\eta_{j,t} = \sum_f \frac{\Lambda_{f,t}X_{f,t}}{\hat{c}_h^{-1/\psi}\theta_{j,t}} \gamma(\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\hat{k}_{f,j,t}\nu_{f,j,t})^2 \iota''_{j,t}(g_{j,t})$$

Use $\zeta_t = \hat{c}_{h,t}^{-1/\psi}$ and $\Lambda_{f,t} = \xi_{f,t}\zeta_t$ to get

$$\eta_{j,t} = \sum_f \frac{\xi_{f,t}X_{f,t}}{\theta_{j,t}} \gamma(\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\hat{k}_{f,j,t}\nu_{f,j,t})^2 \iota''_{j,t}(g_{j,t})$$

$$\eta_{j,t} = \sum_f \frac{X_{f,t}\hat{k}_{f,j,t}}{\xi_{f,t}\theta_{j,t}} \gamma(\xi_{f,t}\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\nu_{f,j,t})^2 \hat{k}_{f,j,t}\iota'_{j,t}(g_{j,t}) \frac{\iota''_{j,t}(g_{j,t})}{\iota'_{j,t}(g_{j,t})}$$

$$\eta_{j,t} = \sum_f \frac{X_{f,t}}{\theta_{j,t}} \hat{k}_{f,j,t}\gamma(\xi_{f,t}\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}\nu_{f,j,t})^2 \frac{\hat{k}_{f,j,t}\iota'_{j,t}(g_{j,t})}{\xi_{f,t}} \frac{\iota''_{j,t}(g_{j,t})}{\iota'_{j,t}(g_{j,t})}$$

Now use $\tilde{\phi}_{f,j,t} = \xi_{f,t}\hat{c}_{f,t}^{-1/\psi}\phi_{f,j,t}$,

$$\eta_{j,t} = \sum_f \frac{k_{f,j,t}}{k_{j,t}} \gamma(\tilde{\phi}_{f,j,t}\nu_{f,j,t})^2 \frac{q_{j,t}k_{f,j,t}}{n_{f,t}} \frac{\iota''_{j,t}(g_{j,t})}{\iota'_{j,t}(g_{j,t})}$$

Notice that for any $f$ such that $\hat{k}_{f,j,t} > 0$ we have $\alpha_{f,j,t} = \gamma(\tilde{\phi}_{f,j,t}\nu_{f,j,t})^2 \frac{q_{j,t}k_{f,j,t}}{n_{f,t}}$. Multiply and divide by $q_{j,t} = \iota'_{j,t}(g_{j,t})$ to get:

$$\eta_{j,t} = \sum_f \frac{q_{j,t}k_{f,j,t}}{q_{j,t}k_{j,t}} \alpha_{f,j,t}\epsilon_{j,t} = \alpha_{j,t}\epsilon_{j,t}$$

where $\alpha_{j,t} = \sum_f \frac{q_{j,t}k_{f,j,t}}{q_{j,t}k_{j,t}}\alpha_{f,j,t}$ is the value-weighted risk-adjusted expected excess return that intermediaries obtain on assets of class $j$.

**Stationary distribution under competitive equilibrium**

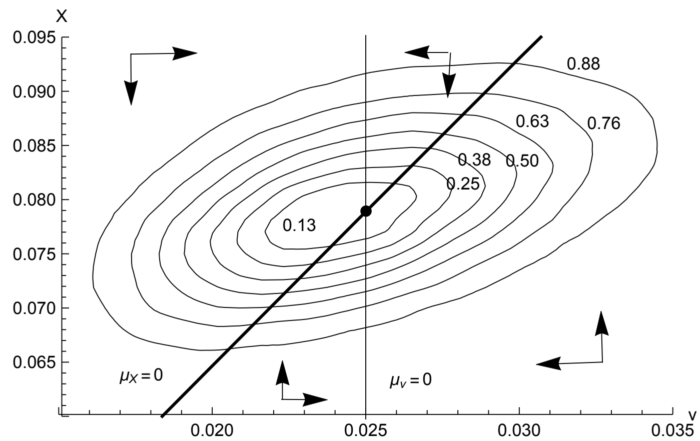


Figure 1: Stationary distribution of state variables $\nu$ and $X$ in the competitive equilibrium. The lines indicate the loci of $\mu_X = 0$ and $\mu_\nu = 0$, and the arrows the direction in which the economy moves. The intersection represents the steady state of the economy, but uncertainty shocks continually move the economy away from it. The contour lines indicate the probability contained within.

# Appendix B

In Section 1 I develop the contract environment in detail. I provide a verification theorem for the HJB equation, show how the optimal contract can be implemented with a constrained portfolio problem, and show that it is renegotiation-proof. In Section 2 I show how the competitive equilibrium and the planner's problem can be characterized with a system of second order PDEs that can be easily solved numerically.

## 3 Optimal Contracts

This section develops the contractual environment in detail. It is useful for the applications and numerical solutions to include an exogenous retirement time $\tau_i$ which arrives with independent Poisson intensity $\theta$ (this will yield a stationary distribution). After retirement the agent cannot manage capital any longer, and the contract just delivers a terminal utility by giving the agent consumption (he becomes a household). The results in the main paper can be obtained by letting $\theta \to 0$.

### 3.1 Setting

Let $(\Omega, P, \mathcal{F})$ be a complete probability space. Throughout this appendix, all stochastic processes are adapted to the filtration $\mathbb{F}_i$ generated by the aggregate brownian motion $Z$, the idiosyncratic brownian motion $W_i$, and an idiosyncratic Poisson process $N_i$ with arrival rate $\theta$ associated with retirement of intermediary $i$. I will use a weak formulation of the problem, which is equivalent to the strong one in the paper. Prices and coefficients depend only on the history of aggregate shocks, e.g. $r$, $\pi$, $\sigma$, $\nu$, $\phi$, $q$, $g$, $\iota(g)$, $\tau^k$, $\zeta$. In what follows I will drop the reference $i$ to the intermediary in order to simplify notation.

There is a complete financial market, with risk free interest rate $r$ and price of aggregate risk $\pi$ (and associated equivalent martingale measure $Q$). The agent can continuously trade capital $k$ at price $q$, and obtains a return per dollar invested in capital

$$dR_t = \left(\frac{a - \iota_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t\sigma'_{q,t} - \tau^k_t\right) dt + (\sigma_t + \sigma_{q,t})dZ_t + \nu_t dW_t$$

The agent starts with net worth $n_0$ and signs a contract $\mathcal{C} = \left(c, \bar{U}, k\right)$ with full commitment. The contract specifies consumption $c = \{c_t \geq 0; t \leq \tau\}$, a terminal utility $\bar{U} = \{\bar{U}_t; t \leq \tau\}$, and capital under management $k = \{k_t \geq 0; t \leq \tau\}$. After retirement the agent cannot manage capital any longer, and the principal delivers utility $\bar{U}_\tau$.

After signing the contract the intermediary can choose a hidden action $s = \{s_t; t \leq \tau\}$ which we interpret as a stealing process. Stealing changes the distribution of observed outcomes from $P$ to $P^s$ so that the return can be written

$$dR_t = \left(\frac{a - \iota_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t\sigma'_{q,t} - \tau^k_t - s_t\right) dt + (\sigma_t + \sigma_{q,t})dZ_t + \nu_t dW^s_t$$

where $W^s_t = W_t + \int_0^t \frac{s_u}{\nu_u} du$ is a brownian motion under $P^s$. For each dollar stolen, the intermediary keeps a fraction $\phi_t \in (0, 1)$ which he adds to his consumption: $\tilde{c} = c + \phi qks$ (the intermediary doesn't have access to hidden savings). As a result he gets utility $U^s(\tilde{c}) = U^s_0$, where the utility process $U^s = \{U^s_t; t \leq \tau\}$ is given by

$$U^s_t = \mathbb{E}^s_t\left[\int_t^\tau f(\tilde{c}_t, U^s_t)ds + \bar{U}_\tau\right] \tag{18}$$

In this environment it is always optimal to implement no stealing $s = 0$, for the same reasons as in DeMarzo and Sannikov (2006) for example.[1] The principals' objective is to minimize the cost of delivering utility $u_0$ to the intermediary $F(\mathcal{C}) = F_0$ where the continuation cost of the contract $F = \{F_t; t \leq \tau\}$ is

$$F_t = \mathbb{E}^Q_t\left[\int_t^\tau e^{-\int_t^u r_m dm}\left(c_u - q_u k_u \alpha_u\right) du + e^{-\int_t^\tau r_m dm}\bar{F}_\tau(\bar{U}_\tau)\right]$$

and $\alpha_t \equiv \frac{a - \iota_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t\sigma'_{q,t} - r_t - \tau^k_t - (\sigma_t + \sigma_{q,t})\pi_t$. Here $\bar{F}_t(\bar{U})$ is the cost of delivering utility $\bar{U}$ to the intermediary who has retired and cannot manage capital any longer. The intermediary has in fact become a household, so the cost of delivering utility

[1] If a contract implements stealing $s$ in equilibrium, then the principal can do better by giving $c' = c + \phi qks$ to the agent as legitimate consumption, and implementing no stealing $s' = 0$.

takes the form

$$\bar{F}_t(\bar{U}) = \zeta_t^{-1}\left((1-\gamma)\bar{U}\right)^{\frac{1}{1-\gamma}}$$

and we assume $\zeta_t^{-1} > 0$ is bounded.

We say a contract $\mathcal{C} = (c, \bar{U}, k)$ is *admissible* if 1) there is a solution $U^0$ to (18), with

$$\mathbb{E}\Big[\Big(\int_0^t f(c_u, U_u^0)du\Big)^2 + (U_t^0)^2\Big] < \infty \tag{19}$$

for all $t$, and 2)

$$\mathbb{E}^Q\left[\int_0^\tau e^{-\int_0^t r_m dm}|c_t - q_t k_t \alpha_t|\,dt + \sup_{u\le\tau} e^{-\int_0^u r_m dm}\bar{F}_u(U_u)\right] < \infty \tag{20}$$

Given a contract $\mathcal{C}$, we say that a stealing process s is *valid* if 1) there is a $U^s$ solution to (18), and 2) $\frac{s}{\nu} \ge 0$ is bounded and there is a constant $T \in \mathbb{R}_+$ such that $s_t = 0\,\forall t \ge T$.[2] Let $\mathbb{S}(\mathcal{C})$ be the set of valid stealing plans given contract $\mathcal{C}$. We say an admissible contract $\mathcal{C}$ is *incentive compatible* if[3]

$$0 \in \arg\max_{s\in\mathbb{S}(\mathcal{C})} U^s(c + \phi qks)$$

Let $\mathbb{IC}$ be the set of incentive compatible contracts. For an initial utility $u_0$ for the agent, an incentive compatible contract is *optimal* if it minimizes the cost of delivering initial utility $u_0$ to the agent, that is

$$J(u_0) = \min_{\mathcal{C}\in\mathbb{IC}} F(\mathcal{C})$$

$$st: \quad U^0(\mathcal{C}) \ge u_0$$

By changing $u_0$ we can trace the Pareto frontier for this problem. In particular, at time $t = 0$ the intermediary has net worth $n_0$ which he gives to the principal in exchange for the contract. We set $u_0$ so that the principal breaks even

$$n_0 + J(u_0) = 0$$

Let $J_t = F(\mathcal{C}^*)$ be the continuation cost of an optimal contract $\mathcal{C}^*$. This is how much it would cost the agent to "buy into" the contract at that time, and therefore this is the net worth of the intermediary at time $t$.

[2]Note the intermediary can choose $T$ as large as desired, as well as the bound on $\frac{s_t}{\nu_t}$. These regularity conditions can be relaxed with some work, but are economically innocuous.

[3]Notice $\mathbb{A}(\mathcal{C}) \neq \emptyset$ because $0 \in \mathbb{S}(\mathcal{C})$ for an admissible contract.

### 3.2 Recursive formulation

We can use the continuation utility of the intermediary as a state variable in order to provide incentives for not stealing. First we obtain a law of motion for continuation utility.

**Lemma 1.** *For any admissible contract* $\mathcal{C} = (c, \bar{U}, k)$*, the intermediary's continuation utility* $U^0$ *satisfies*

$$dU_t^0 = -f(c_t, U_t^0)dt + \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t - \lambda_t\left(dN_t - \theta dt\right) \tag{21}$$

*for some* $\sigma_U$ *and* $\tilde{\sigma}_U$ *in* $\mathcal{L}^2$*, and* $\lambda_t = U_{t^-}^0 - \bar{U}_t$*.*[4]

*Proof.* Consider the process

$$Y_t = \mathbb{E}_t\left[\int_0^\tau f(c_u, U_u^0)du + \bar{U}_\tau\right] = \int_0^t f(c_u, U_u^0)du + U_t^0$$

on $\{t \leq \tau\}$. Since $Y$ is an $\mathbb{F}$-adapted $P$-martingale, and $\mathbb{F}$ is generated by $Z$, $W$ and $N$, we can apply a martingale representation theorem to obtain

$$dY_t = f(c_t, U_t^0)dt + dU_t^0 = \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t - \lambda_t\left(dN_t - \theta dt\right)$$

Rearranging we get (21). Since $U_\tau = \bar{U}_\tau$ it must be that $\lambda_t = U_{t^-}^0 - \bar{U}_t$, and from admissibility of $\mathcal{C}$ we get $\mathbb{E}\left[Y_t^2\right] < \infty$ for all $t$, so therefore $\sigma_U$ and $\tilde{\sigma}_U$ are in $\mathcal{L}^2$.[5] $\square$

Notice that because retirement is contractible, the agent's utility can in principle "jump" when the agent retires. However, if $U_t^0$ jumps down when the agent retires, for example, then while he doesn't retire it must drift up to compensate the agent. To obtain equation (6) in the main text, we just drop the jump term.

Faced with contract $\mathcal{C}$, the intermediary can consider a valid stealing process $s \in \mathbb{S}(\mathcal{C})$, getting consumption $\tilde{c} = c + \phi qks$ under probability $P^s$. The following lemma gives necessary and sufficient conditions for an admissible contract to be incentive compatible, for the parameter configuration that is of interest to us.

**Lemma 2.** *If EIS* $\psi > 1$ *and the risk aversion* $\gamma > 1$*, an admissible contract* $\mathcal{C} = (c, \bar{U}, k)$ *is incentive compatible if and only if*

$$0 \in \arg\max_{s \geq 0} f(c_t + \phi_t q_t k_t s, U_t^0) - \tilde{\sigma}_{U,t}\frac{s_t}{\nu_t} - f(c_t, U_t^0) \tag{22}$$

*Remark.* The result of this lemma can be extended to other combinations of $\psi$ and $\gamma$.

[4] In this context, $\mathcal{L}^2$ is the set of $\mathbb{F}$-adapted processes $x$ such that $\mathbb{E}\left[\int_0^t x_u^2 du\right] < \infty$ for any $t$.

[5] Notice that for $t \geq \tau$, $Y_t$ is constant, so $\sigma_{U,t}$, $\tilde{\sigma}_{U,t}$, and $\lambda_t$ are zero, but this is not relevant for our purposes.

*Proof.* Suppose the agent picks a valid stealing plan $s$. His utility is $U^s$, defined by

$$U_t^s = \mathbb{E}_t^s \left[ \int_t^\tau f(c_u + \phi_t q_u k_u s_u, U_u^s) du + \bar{U}_\tau \right]$$

We would like to compare this with the utility from good behavior $U^0$. To do this, it's useful to first express $U^0$ as an expectation under $P^s$. Using (21) and $dW_t = dW_t^s - \frac{s_t}{\nu_t} dt$, we obtain

$$dU_t^0 = \left( -f(c_t, U_t^0) - \frac{s_t}{\nu_t} \right) dt + \sigma_{U,t} dZ_t + \tilde{\sigma}_{U,t} dW_t^s - \lambda_t \left( dN_t - \theta dt \right)$$

Now we can integrate, bearing in mind that since $\frac{s}{\nu}$ is bounded $\sigma_U$ and $\tilde{\sigma}_U$ are both in $\mathcal{L}^2(P^s)$ as well. We get:

$$U_t^0 = \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \left( f(c_u, U_u^0) + \frac{s_u}{\nu_u} \tilde{\sigma}_{U,u} \right) du + U_{T \wedge \tau}^0 \right]$$

where $s_t = 0$ for all $t \geq T$. Notice $U_{T \wedge \tau}^s = U_{T \wedge \tau}^0$. Now we can obtain

$$U_t^s - U_t^0 = \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \left( f(c_u + \phi_t q_u k_u s_u, U_u^s) - f(c_u, U_u^0) - \frac{s_u}{\nu_u} \tilde{\sigma}_{U,u} \right) du \right] \tag{23}$$

on $\{t \leq T \wedge \tau\}$.

To prove necessity, suppose (22) fails. Pick a bounded stealing strategy such that

$$f(c_t + \phi_t q_t k_t s, U_t^0) - \tilde{\sigma}_{U,t} \frac{s_t}{\nu_t} - f(c_t, U_t^0) > 0$$

on a set of positive measure $A$, and zero outside (we can pick $T$ as large as desired). Look at the integrand in (23), and write

$$\begin{aligned} f(c_u + \phi_t q_u k_u s_u, U_u^s) - f(c_u, U_u^0) - \tilde{\sigma}_{U,u} \frac{s_u}{\nu_u} &= f(c_u + \phi_t q_u k_u s_u, U_u^s) - f(c_u + \phi_t q_u k_u s_u, U_u^0) \\ &\quad + f(c_u + \phi_t q_u k_u s_u, U_u^0) - f(c_u, U_u^0) - \tilde{\sigma}_{U,u} \frac{s_u}{\nu_u} \\ &\geq f(c_u + \phi_t q_u k_u s_u, U_u^s) - f(c_u + \phi_t q_u k_u s_u, U_u^0) \end{aligned}$$

with strict inequality on $A$. Now we use an interesting fact about the EZ aggregator $f(c, U)$: if $\gamma > 1$ and $\psi > 1$, then there is a constant $\kappa > 0$ such that $f(c, y) - f(c, x) \leq \kappa(y - x)$ for $y \geq x$, and any $c$.[6] We can then write

$$\underbrace{f(c_u + \phi_t q_u k_u s_u, U_u^s) - f(c_u, U_u^0) - \tilde{\sigma}_{U,u} \frac{s_u}{\nu_u}}_{H_u} \geq \kappa \underbrace{(U_u^s - U_u^0)}_{M_u} \quad \text{when } \underbrace{U_u^s - U_u^0}_{M_u} \leq 0$$

[6] see Proposition 3.2 in Kraft et al. (2011)

and the inequality is strict on $A$. Now define the process $M_t = U_t^s - U_t^0$ and rewrite the previous condition as

$$M_t = U_t^s - U_t^0 = \mathbb{E}_t^s \left[ \int_t^{T\wedge\tau} H_u du \right] \quad \text{with } H_t \geq \kappa M_t \text{ whenever } M_t \leq 0$$

We can now use a generalized version of Skiadas' Lemma[7] to obtain that $M_t = U_t^s - U_t^0 \geq 0$ as follows. Let $\tau_0 = \inf\{t \geq 0 : U_t^0 \leq U_t^s\}$ and write

$$M_t \mathbb{1}_{\{\tau_0 > t\}} \geq \mathbb{E}_t^s \left[ \int_t^{\tau_0 \wedge T \wedge \tau} \kappa M_u \mathbb{1}_{\{\tau_0 > t\}} du + M_{\tau_0 \wedge T \wedge \tau} \mathbb{1}_{\{\tau_0 > t\}} \right]$$

$$M_t \mathbb{1}_{\{\tau_0 > t\}} \geq \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \kappa M_u \mathbb{1}_{\{\tau_0 > u\}} du + M_{\tau_0} \mathbb{1}_{\{\tau_0 > T \wedge \tau\}} \right] = \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \kappa M_u \mathbb{1}_{\{\tau_0 > u\}} du \right]$$

Applying the stochastic Gronwall-Bellman inequality,[8] we get that $M_t \mathbb{1}_{\{\tau_0 > t\}} \geq 0$ for $0 \leq t \leq T \wedge \tau$. Since $M_0 \mathbb{1}_{\{\tau_0 = 0\}} \geq 0$ , we conclude $M_0 \geq 0$. We can apply a similar argument for any $u$ (redefining the stopping time $\tau_u = \inf\{t \geq u : U_t^0 \leq U_t^s\}$) and get $M_u \geq 0$ for all $0 < u < T \wedge \tau$. For $u \geq T \wedge \tau$ we already know that $M_{u \wedge \tau} = 0$.

Now to make the inequality strict, if $M_t = 0$ a.e. on $[0, \tau] \times \Omega$, then $H_t \geq \kappa M_t = 0$, and the inequality is strict on positive measure subset $A$, and therefore $M_0 > 0$. If $M_t > 0$ for at least some $(\omega, t)$ with $t < \tau$, with positive probability, we do the following. For some small $\epsilon > 0$, let $\tau^\epsilon = \inf\{t : M_t \geq \epsilon\}$. If we take $\epsilon$ small enough, the probability that we get to such a point before $\tau$ is positive: $P^s(\{\tau^\epsilon \wedge \tau < \tau\}) > 0$ (for any $P^s$ since they are equivalent). It must be that there is some stealing going on after this, since otherwise $M_{\tau^\epsilon}$ would be zero. Now consider the alternative stealing plan $s'$ that steals only until $\tau^\epsilon$ and then stops, that is $s'_t = s_t$ for $t < \tau^\epsilon$ and $s' = 0$ after this. By a similar argument as before, $U_0^0 \leq U_0^{s'}$. Utility under this plan satisfies $U_{\tau^\epsilon \wedge \tau}^{s'} = U_{\tau^\epsilon \wedge \tau}^0 < U_{\tau^\epsilon \wedge \tau}^s$ if $\tau^\epsilon \wedge \tau < \tau$, and equal otherwise. Now if we compare $s$ and $s'$, both plans induce the same probability measure until $\tau^\epsilon \wedge \tau$, and the same consumption stream, but the payoff at $\tau^\epsilon \wedge T$ is larger for $s$ (strictly so with positive probability):

$$U_t^{s'} = \mathbb{E}_t^s \left[ \int_t^{\tau^\epsilon \wedge \tau} f(c_u + \phi_t q_u k_u s_u, U_u^{s'}) + U_{\tau^\epsilon \wedge \tau}^0 \right]$$

$$U_t^s = \mathbb{E}_t^s \left[ \int_t^{\tau^\epsilon \wedge \tau} f(c_u + \phi_t q_u k_u s_u, U_u^s) + U_{\tau^\epsilon \wedge \tau}^s \right]$$

By strict monotonicity of EZ preferences with respect to terminal value, we get $U_0^s > U_0^{s'} \geq U_0^0$. This proves stealing is attractive if (22) fails.

[7]The strategy is similar to Theorem A.2 in Kraft et al. (2011).

[8]See Duffie and Epstein (1992)

For sufficiency, suppose (22) holds, so that for any valid stealing plan we have

$$f(c_t + \phi_t q_t k_t s, U_t^0) - \tilde{\sigma}_{U,t}\frac{s_t}{\nu_t} - f(c_t, U_t^0) \le 0$$

Using the same properties of the EZ aggregator as before but with the opposite inequalities, we get

$$M_t = U_t^s - U_t^0 = \mathbb{E}_t^s\left[\int_t^{T\wedge\tau} H_u du\right] \quad \text{with } H_t \le \kappa M_t \text{ whenever } M_t \ge 0$$

The same reasoning as before now yields $M_0 = U_0^s - U_0^0 \le 0$, so the contract is indeed incentive compatible. □

The FOC for (22) are

$$\partial_c f(c_t, U_t^0)\phi_t q_t k_t = \tilde{\sigma}_{U,t}\frac{1}{\nu_t}$$

which yields the "skin in the game" condition (**??**) in the main text

$$\tilde{\sigma}_{U,t} = \partial_c f(c_t, U_t^0)\phi_t q_t k_t \nu_t \tag{24}$$

### 3.3 HJB equation

Because of homothetic preferences, the value function for the principal's cost minimization problem takes the form $J_t = \xi_t x_t$. Here $x_t = \left((1-\gamma)U_t^0\right)^{\frac{1}{1-\gamma}} > 0$ is a monotone (and concave) transformation of the intermediary's continuation utility. As a result, we can also interpret it as the intermediary's continuation utility. The stochastic process $\xi = \{\xi_t; t \le \tau\}$ captures the investment opportunity set, and has a law of motion

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t + \left((\xi_t\zeta_t)^{-1} - 1\right)dN_t \tag{25}$$

It depends only on the aggregate shocks $Z$ that affect market prices, and on whether the intermediary has retired. The last term ensures that $\xi_\tau = \zeta_\tau^{-1}$.

Use the following normalization: $k_t = \hat{k}_t x_t$, $c_t = \hat{c}_t x_t$, $\sigma_{U,t} = \sigma_{x,t}(1-\gamma)U_t^0$, $\tilde{\sigma}_{U,t} = \hat{c}_t^{-\frac{1}{\psi}}\phi q_t \hat{k}_t \nu_t (1-\gamma)U_t^0$, and $\lambda_t = \hat{\lambda}_t(1-\gamma)U_t^0$. Then we can write

$$\begin{aligned}\frac{dx_t}{x_t} &= \left(\frac{1}{1-\frac{1}{\psi}}\left(\rho - \hat{c}_t^{1-\frac{1}{\psi}}\right) + \frac{1}{2}\gamma\sigma_{x,t}^2 + \frac{1}{2}\gamma\left(\hat{c}_t^{-\frac{1}{\psi}}\phi_t q_t \hat{k}_t \nu_t\right)^2 + \theta\hat{\lambda}_t\right)dt \\ &\quad + \sigma_{x,t}dZ_t + \left(\hat{c}_t^{-\frac{1}{\psi}}\phi_t q_t \hat{k}_t \nu_t\right)dW_t + \left(\left(1 - \hat{\lambda}(1-\gamma)\right)^{\frac{1}{1-\gamma}} - 1\right)dN_t\end{aligned} \tag{26}$$

The principal's cost minimization problem is now a standard optimal control problem. The associated HJB BSDE is

$$r_t J_t dt = \min_{c,k,\sigma_x,\lambda} (c - q_t k \alpha_t) dt + \mathbb{E}_t^Q [dJ_t] \tag{27}$$

subject to (25), as well as $c \geq 0$ and $k \geq 0$. Using the normalization of the controls, the form of the value function, and the fact that $Z_t = Z_t^Q - \int_0^t \pi_u du$, where $Z^Q$ is a brownian motion under $Q$, we can rewrite the HJB equation

$$\begin{aligned} r_t \xi_t \quad = \quad & \min_{\hat{c},\hat{k},\sigma_x,\hat{\lambda}} \hat{c} - q_t \hat{k} \alpha_t + \xi_t \Bigg\{ \frac{1}{1-\frac{1}{\psi}} \left(\rho - \hat{c}_t^{1-\frac{1}{\psi}}\right) - \sigma_{x,t}\pi_t + \mu_{\xi,t} - \sigma_{\xi,t}\pi_t \\ & + \frac{1}{2}\gamma\sigma_{x,t}^2 + \frac{1}{2}\gamma \left(\hat{c}^{-\frac{1}{\psi}} \phi_t q_t \hat{k} \nu_t\right)^2 + \sigma_{\xi,t}\sigma_{x,t} + \theta\Big(\left(1 - \hat{\lambda}(1-\gamma)\right)^{\frac{1}{1-\gamma}} \frac{1}{\zeta_t \xi_t} - 1 + \hat{\lambda}\Big)\Bigg\} \end{aligned} \tag{28}$$

This should be interpreted together with (25).[9] The expression on the right hand side is convex if $\psi > 2$, and the FOC are sufficient. If $\psi \leq 2$ then the optimal contract does not exist, unless capital pays zero excess return, as explained in the paper. Focusing on the $\psi > 2$ case, we get the following FOC

$$\xi_t \hat{c}_t^{-\frac{1}{\psi}} + \xi_t \frac{\gamma}{\psi} \left(\phi_t q_t \hat{k}_t \nu_t\right)^2 \hat{c}_t^{-\frac{2}{\psi}-1} = 1 \tag{29}$$

$$\frac{a - \iota_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t \sigma'_{q,t} - (r_t + \tau_t^k) = \underbrace{(\sigma_t + \sigma_{q,t})\pi_t}_{\text{agg. risk premium}} + \underbrace{\gamma \xi_t \left(\hat{c}_t^{-\frac{1}{\psi}} \phi_t \nu_t\right)^2 q_t \hat{k}_t}_{\text{id. risk premium}} \tag{30}$$

$$\sigma_{x,t} = \frac{\pi_t}{\gamma} - \frac{\sigma_{\xi,t}}{\gamma} \tag{31}$$

$$\left(1 - \hat{\lambda}(1-\gamma)\right)^{\frac{\gamma}{1-\gamma}} \frac{1}{\zeta_t \xi_t} = 1 \tag{32}$$

The FOC for $\hat{c}$ has the cost of delivering consumption to the intermediary on the right hand side, and the benefit of a lower promised utility on the left hand side. This is the standard tradeoff we would expect. In addition, however, there is another benefit to giving consumption to the agent: it relaxes the "skin in the game constraint". By front loading consumption, the principal can reduce the marginal private benefit of stealing and consuming, and therefore improve idiosyncratic risk sharing. As a result, there is a tradeoff between distortions in intertemporal consumption and idiosyncratic risk sharing.

The FOC for $\hat{k}$ gives us a pricing equation for capital. As usual, capital pays an excess return because it is exposed to aggregate risk with a market price of $\pi_t$. But in addition, capital must also pay an excess return for its exposure to idiosyncratic risk, even though

[9] That is, a solution to (28) is a process $\xi$ that satisfies (25) for some $\mu_\xi$ and $\sigma_\xi$, and ($\xi$, $\mu_\xi$, $\sigma_\xi$) satisfy (28).

this risk is not priced by the financial market. The reason for this is that the principal knows that if he gives more capital to the intermediary, he will have to expose him to risk for incentive reasons, and this is costly because the intermediary is risk averse.

The FOC for $\sigma_x$ has the following interpretation. Delivering utility to the intermediary is costly for the principal. He would therefore prefer to promise him more utility in states of the world where it is relatively cheaper. This can happen because the value of a unit of consumption in that state is lower (captured by the $\pi_t$ term) or because the cost of delivering utility in that state is lower (captured by the $\sigma_{\xi,t}$ term). A similar logic explains the FOC for $\hat{\lambda}$. After retirement the intermediary cannot manage capital any longer, so delivering utility to him is more costly. This is captured by $\zeta_t^{-1} \geq \xi_t$. As a result, the optimal contract has $\hat{\lambda} \geq 0$. The principal prefers to promise the intermediary less utility after he retires (when it is more costly to deliver utility to him) even if it means promising him more utility while he doesn't retire.

We can plug in the FOC into the HJB equation (28). If we find a solution $\xi$ to the HJB equation, we can use it to build the optimal contract using the policy functions $\hat{c}$, $\hat{k}$, $\sigma_x$, and $\hat{\lambda}$ (so the HJB holds with equality) and the law of motion of $x$, (26) with initial condition $x_0 = ((1-\gamma)u_0)^{\frac{1}{1-\gamma}}$. Let $\mathcal{C}^* = (c^*, \bar{U}^*, k^*)$ be the candidate optimal contract thus constructed, with associated state $x^*$. We have the following verification theorem.

**Theorem 1.** *Let $\xi$ be a strictly positive solution to the HJB equation (28) bounded above by $\zeta^{-1}$. Then,*

*1) For any incentive compatible contract $\mathcal{C}$ that delivers at least utility $u_0$ to the agent, we have $\xi_0 \left((1-\gamma)u_0\right)^{\frac{1}{1-\gamma}} \leq F_0(\mathcal{C})$.*

*2) Let $\mathcal{C}^*$ be a candidate contract constructed as described above. If $\mathcal{C}^*$ is admissible and delivers utility $u_0$ to the agent, then it is optimal, with cost $F_0(\mathcal{C}^*) = J_0(u_0) = \xi_0 \left((1-\gamma)u_0\right)^{\frac{1}{1-\gamma}}$.*

*Proof.* For the first part, consider an incentive compatible contract $\mathcal{C} = (c, \bar{U}, k)$ that delivers utility $u_0$ to the agent, and has an associated state variable $x$. Use the HJB equation to obtain

$$e^{-\int_0^{\tau^n\wedge\tau} r_u du}\xi_{\tau^n\wedge\tau}x_{\tau^n\wedge\tau} \geq \xi_0 x_0 - \int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\left(c_t - q_t k_t \alpha_t\right)dt$$

$$+\int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\xi_t x_t\left(\sigma_{\xi,t}+\sigma_{x,t}\right)dZ_t^Q + \int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\xi_t x_t\big(\hat{c}_t^{-\frac{1}{\psi}}\phi q_t \hat{k}_t \nu_t\big)dW_t$$

$$+\int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\xi_t x_t \theta\Big(\left(1-\hat{\lambda}(1-\gamma)\right)^{\frac{1}{1-\gamma}}\frac{1}{\zeta_t\xi_t}-1\Big)\Big)\left(dN_t - \theta dt\right)$$

Here we are using the localizing sequence $\{\tau^n\}_{n\in\mathbb{N}}$:

$$\tau^n = \inf\left\{T \geq 0 : \int_0^T \left|e^{-\int_0^t r_u du}\xi_t x_t \left(\sigma_{\xi,t} + \sigma_{x,t}\right)\right|^2 dt + \int_0^T \left|e^{-\int_0^t r_u du}\xi_t x_t \big(\hat{c}_t^{-\frac{1}{\psi}}\phi q_t \hat{k}_t \nu_t\big)\right|^2 dt \geq n\right\}$$

The stochastic integrals are therefore martingales, so we can take expectations under $Q$ to obtain

$$\mathbb{E}_0^Q\left[e^{-\int_0^{\tau^n\wedge\tau} r_u du}\xi_{\tau^n\wedge\tau}x_{\tau^n\wedge\tau}\right] \geq \xi_0 x_0 - \mathbb{E}_0^Q\left[\int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\left(c_t - q_t k_t \alpha_t\right)dt\right]$$

Now we will use the dominated convergence theorem to take the limit $n \to \infty$. First,

$$\begin{aligned}\left|\int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\left(c_t - q_t k_t\alpha_t\right)dt\right| &\leq \int_0^{\tau^n\wedge\tau} e^{-\int_0^t r_u du}\left|c_t - q_t k_t\alpha_t\right| dt \\ &\leq \int_0^{\tau} e^{-\int_0^t r_u du}\left|c_t - q_t k_t\alpha_t\right| dt\end{aligned}$$

which is integrable because $\mathcal{C}$ is admissible. Second,

$$e^{-\int_0^{\tau^n\wedge\tau} r_u du}\xi_{\tau^n\wedge\tau}x_{\tau^n\wedge\tau} \leq \sup_{t\leq\tau} e^{-\int_0^t r_u du}\xi_t x_t \leq \sup_{t\leq\tau} e^{-\int_0^t r_u du}\zeta_t^{-1}x_t$$

which is also integrable because $\mathcal{C}$ is admissible. So letting $n \to \infty$, we get $\tau^n \wedge \tau \to \tau$ a.s. and therefore using $\xi_\tau = \zeta_\tau^{-1}$:

$$\mathbb{E}_0^Q\left[e^{-\int_0^\tau r_u du}\zeta_\tau^{-1}x_\tau\right] \geq \xi_0 x_0 - \mathbb{E}_0^Q\left[\int_0^\tau e^{-\int_0^t r_u du}\left(c_t - q_t k_t\alpha_t\right)dt\right]$$

Rearranging we get the desired result.

For the second part, let $\mathcal{C}^*$ be the candidate optimal contract with associated utility process $x^*$. By construction, the HJB holds with equality, so the same argument shows that $\mathcal{C}^*$ in fact has a cost $F_0(\mathcal{C}^*) = \xi_0\left((1-\gamma)u_0\right)^{\frac{1}{1-\gamma}}$. We know $\mathcal{C}^*$ delivers utility $u_0$ by assumption, and Lemma 2 ensures it is incentive compatible. □

### 3.4 Implementation

Finally, from (27) we can easily see that the net worth of the agent $n_t = J_t = \xi_t x_t$ follows the law of motion

$$dn_t = (r_t n_t + q_t k_t\alpha_t - c_t + \sigma_{n,t}n_t\pi_t)dt + \sigma_{n,t}n_t dZ_t + \tilde{\phi}_t q_t k_t \nu_t dW_t + \lambda_{n,t}n_t(dN_t - \theta dt)$$

where c, and $\lambda_{n,t} = \hat{\lambda}_t(\gamma - 1)$. With $\theta = 0$ we obtain equation (5) in the main body.

We can implement the optimal contract $(c, k)$ with a constrained portfolio problem as follows. Set a required retained equity stake $\tilde{\phi}_t = \xi_t\hat{c}^{-1/\psi}\phi$, and the consumption rate

$\tilde{c}_t = c_t/(\xi_t x_t)$ as functions of the history of returns $R$ and aggregate shocks $Z$. This is a lower bound on the fraction of retained equity, but since the intermediary wants to keep as little idiosyncratic risk as possible, it is always binding (he can get aggregate risk in other ways). We then let the intermediary choose how much to invest in capital $\tilde{k}_t = k_t/n_t$ and his exposure to aggregate risk $\sigma_{n,t}$ to maximize his utility subject to the budget constraint. Formally, the intermediary solves

$$\max_{\tilde{k},\sigma_n} U(\tilde{c}n)$$

$$st: \quad \frac{dn_t}{n_t} = (r_t + q_t\tilde{k}_t\alpha_t - \tilde{c}_t + \sigma_{n,t}\pi_t)dt + \sigma_{n,t}dZ_t + \tilde{\phi}_t q_t \tilde{k}_t \nu_t dW_t$$

and $n_t \geq 0$. Let $(\tilde{k}, \sigma_n)$ be the solution to this constrained portfolio problem $(\tilde{c}, \tilde{\phi})$, with associated process for net worth $n$. We say the constrained portfolio problem $(\tilde{c}, \tilde{\phi})$ *implements* the optimal contract $(c, k)$ if $c = \tilde{c}n$ and $k = \tilde{k}n$.

**Lemma 3.** *Let $\mathcal{C} = (c, k)$ be an optimal contract, with associated processes $\xi$, $\hat{c}$, and $x$. The constrained portfolio problem with retained equity stake $\tilde{\phi}_t = \xi_t \hat{c}^{-1/\psi}\phi$ and consumption rate $\tilde{c}_t = c_t/(\xi_t x_t)$ implements the optimal contract.*

*Proof.* Let the value function of the portfolio problem be $U_t(n) = \frac{(\tilde{\xi}_t n_t)^{1-\gamma}}{1-\gamma}$ for some stochastic process $\tilde{\xi}$ with law of motion

$$\frac{d\tilde{\xi}_t}{\tilde{\xi}_t} = \mu_{\tilde{\xi},t}dt + \sigma_{\tilde{\xi},t}dZ_t$$

The HJB equation for the portfolio problem is

$$\frac{\rho}{1-1/\psi} = \max_{\tilde{k},\sigma_n} \frac{\tilde{c}^{1-1/\psi}}{1-1/\psi}\tilde{\xi}_t^{1/\psi-1} + r_t + q_t\tilde{k}_t\alpha_t - \tilde{c}_t + \sigma_{n,t}\pi_t + \mu_{\tilde{\xi},t} - \frac{\gamma}{2}\sigma_{\tilde{\xi},t}^2 - \frac{\gamma}{2}\sigma_{n,t}^2 + (1-\gamma)\sigma_{\tilde{\xi},t}\sigma_{n,t} - \frac{\gamma}{2}(\tilde{\phi}_t q_t \tilde{k}_t \nu_t)^2 \tag{33}$$

with FOC

$$\alpha_t = \gamma(\tilde{\phi}_t\nu_t)^2 q_t \tilde{k}_t$$

$$\pi_t = \gamma\sigma_{n,t} - (1-\gamma)\sigma_{\tilde{\xi},t}$$

Now guess and verify that the solution is $\tilde{\xi} = \xi^{-1}$, $\tilde{k} = \hat{k}/\xi_t$, and $\sigma_n = \sigma_\xi + \sigma_x$. We can check that the FOCs hold using the FOC of the HJB of the optimal contract. If we plug this into the HJB (33), and use also the definition of $\tilde{\phi}$ and $\tilde{c}$, we will obtain the HJB of the optimal contract:

$$\frac{\rho}{1-1/\psi} = \frac{\hat{c}^{1-1/\psi}}{1-1/\psi} + r_t + \frac{1}{\xi_t}\left(q_t\hat{k}_t\alpha_t - \hat{c}_t\right) + (\sigma_{\xi,t}+\sigma_{x,t})\pi_t - \mu_{\xi,t} + \sigma_{\xi,t}^2 - \frac{\gamma}{2}\sigma_{\xi,t}^2 - \frac{\gamma}{2}(\sigma_{\xi,t}+\sigma_{x,t})^2$$

$$-(1-\gamma)\sigma_{\xi,t}(\sigma_{\xi,t}+\sigma_{x,t}) - \frac{\gamma}{2}(\hat{c}^{-1/\psi}\phi q_t \hat{k}_t \nu_t)^2$$

multiply throughout by $\xi_t$

$$0 = q_t\hat{k}_t\alpha_t - \hat{c}_t + \xi\left\{\frac{\hat{c}^{1-1/\psi}-\rho}{1-1/\psi} + r_t + (\sigma_{\xi,t}+\sigma_{x,t})\pi_t - \mu_{\xi,t} + \sigma_{\xi,t}^2 - \frac{\gamma}{2}\sigma_{\xi,t}^2 - \frac{\gamma}{2}(\sigma_{\xi,t}+\sigma_{x,t})^2\right.$$
$$\left. -(1-\gamma)\sigma_{\xi,t}(\sigma_{\xi,t}+\sigma_{x,t}) - \frac{\gamma}{2}(\hat{c}^{-1/\psi}\phi q_t\hat{k}_t\nu_t)^2\right\}$$

and expand the squares and simplify to obtain:

$$0 = q_t\hat{k}_t\alpha_t - \hat{c}_t + \xi\left\{\frac{\hat{c}^{1-1/\psi}-\rho}{1-1/\psi} + r_t + (\sigma_{\xi,t}+\sigma_{x,t})\pi_t - \mu_{\xi,t} - \frac{\gamma}{2}\sigma_{x,t}^2 - \sigma_{\xi,t}\sigma_{x,t} - \frac{\gamma}{2}(\hat{c}^{-1/\psi}\phi q_t\hat{k}_t\nu_t)^2\right\}$$

We can do the process in reverse. Since we know that the HJB of the optimal contract holds, the HJB of the portfolio problem holds as well, which verifies our guess. Finally, if we let $n = \xi x$, and use $\tilde{k} = \hat{k}/\xi_t$, $\sigma_n = \sigma_\xi + \sigma_x$, along with $\tilde{\phi}_t = \xi_t\hat{c}^{-1/\psi}\phi$ and $\tilde{c}_t = c_t/(\xi_t x_t)$ we get the budget constraint. This means that $n = \xi x$ is the process for net worth associated with the solution to the portfolio problem. So we get that $k = \hat{k}x = \tilde{k}\xi x = \tilde{k}n$ and $c = \hat{c}x = \tilde{c}\xi x = \tilde{c}n$. This shows that the portfolio problem implements the optimal contract. $\square$

### 3.5 Renegotiation

Suppose at any time the principal and the intermediary can write a new continuation contract that is also incentive compatible looking forward and delivers at least the same utility to both the principal and the intermediary. An incentive compatible contract is *renegotiation-proof* if the best such contract they can write at any time is the continuation contract.

**Lemma 4.** *The optimal long-term contract is renegotiation proof.*

*Proof.* We can see that the optimal contract is renegotiation-proof by noticing that the continuation contract after any history is incentive compatible (or the optimal contract wouldn't be IC), and it must minimize the continuation cost to the principal subject to delivering the promised utility to the intermediary at that point, within the class of IC contracts. Otherwise we could replace this continuation with a better IC continuation contract and obtain an IC contract that dominates the optimal contract. We can patch IC contracts this way because for incentive purposes only the promised utility of the continuation contract matters. So the only way in which they could improve on the optimal contract at this point is if giving more utility to the intermediary reduced the cost to the principal: $\xi_t < 0$. But we know that $\xi_t > 0$, or else the optimal contract would not exist. This proves that the optimal contract is indeed renegotiation-proof. $\square$

## 4 Numerical solution

In this section I show how the competitive equilibrium and the planner's allocation can be solved as a system of PDEs. In Section 3 I illustrate the method with a concrete example where the economy is hit by uncertainty shocks. It is useful to introduce retirement among intermediaries as explained in Section 3, in order to obtain a non-degenerate stationary distribution for the economy. Retirement arrives with Poison arrival rate $\theta$, at which point the intermediary becomes a household. If we set $\theta \to 0$ we obtain the setting in the paper.

### 4.1 Competitive equilibrium

The HJB equation for intermediaries with retirement is given by (28), with FOCs (29), (30), (31), and (32). These are the same as in the paper, except for the FOC for $\hat{\lambda}_t$. The representative household's HJB and FOC are unchanged, as are the market clearing conditions. The only equilibrium condition that needs to be tweaked is the drift of the endogenous state variable $X_t = \frac{\int_{\mathbb{I}_t} x_{i,t} di}{k_t}$

$$\mu_X = \frac{\rho}{1-1/\psi} - \frac{\hat{c}^{1-1/\psi}}{1-1/\psi} + \frac{\gamma}{2}\sigma_x^2 + \frac{\gamma}{2}(\hat{c}^{-1/\psi}\frac{\phi\iota'(g)\nu}{X})^2 - g - \sigma\sigma_x + \sigma^2 + \theta\left(\hat{\lambda} - 1\right) \tag{34}$$

Notice how with $\theta = 0$ we obtain equation (18)) in the paper. With $\theta > 0$, because each intermediary's utility will jump down on retirement, it must drift up while they don't retire to compensate them. As a result, $X$ gains a positive drift $\theta\hat{\lambda}$. On the other hand, when intermediaries retire their continuation utility is not counted in $X$ any longer, since they are now households, so we get the term $-\theta$.

The strategy to solve the competitive equilibrium is to use Ito's lemma to transform the problem into a system of PDEs for $q$, $\xi$, and $\zeta$. Suppose we are given these functions. We can build $S = \zeta(q - \xi X)$ and $\Lambda = \xi\zeta$, and use Ito's lemma to compute the drift and volatility of all these objects, in terms of $\mu_X$ and $\sigma_X$, which we still don't know, and $\mu_Y$ and $\sigma_Y$, which are exogenously given. From the equilibrium condition for the allocation of aggregate risk, (??), we obtain

$$\sigma_X - \sigma_S = -\frac{1}{\gamma}\sigma_\Lambda$$

From Ito's lemma we get

$$\sigma_\Lambda = \frac{\Lambda_X}{\Lambda}\sigma_X X + \frac{\Lambda_Y}{\Lambda}\sigma_Y \qquad \sigma_S = \frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y$$

There is a two-way feedback loop: $\sigma_X$ depends on how the MRT $\Lambda$, responds to aggregate shocks, $\sigma_\Lambda$; but $\sigma_\Lambda$ is an endogenous object that depends, among other things, on how $X$

responds to aggregate shocks, $\sigma_X$. We can solve for a fixed point for $\sigma_X$ to obtain

$$\sigma_X = \frac{\frac{S_Y}{S} - \frac{1}{\gamma}\frac{\Lambda_Y}{\Lambda}}{1 - \left(\frac{S_X}{S} - \frac{1}{\gamma}\frac{\Lambda_X}{\Lambda}\right) X}\sigma_Y \tag{35}$$

At this point we have $\sigma_X$ and therefore the volatility of $q$, $\xi$, $\zeta$ (and therefore $S$ and $\Lambda$) in terms of their first and second derivatives. Now we can use the definition of $\sigma_X$ to write $\sigma_x = \sigma_X + \sigma$, and use the FOC for $\sigma_x$ to write $\pi = \gamma\sigma_x + \sigma_\xi$. Then using the FOC for $\sigma_w$, in households' HJB, we get $\sigma_w = \frac{\pi}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_\zeta$. Now we need to compute the drifts. First, use the FOCs to obtain $\hat{c}$ and $\hat{c_h}$, and we can use the definition of $\mu_X$ to compute it. With this we have the drift of $q$, $\xi$, $\zeta$ (and therefore $S$ and $\Lambda$).

Finally, use households' HJB to compute $r$. We end up with intermediaries' HJB (with $\hat{k} = X^{-1}$ from market clearing for capital), the FOC for capital, and the market clearing condition for consumption goods. This is a system of two second order PDEs and an algebraic constraint (the market clearing for consumption goods) for $q$, $\xi$, and $\zeta$.

**Boundary conditions.** We don't need to impose conditions at the boundary of the domain. Rather, we have global conditions. We are looking for a solution with $p$, $\xi$, and $\zeta$ strictly positive, as well as $q - \xi X$ (households' wealth). $\xi$ and $\zeta$ should be bounded away from zero (so $\xi^{1-\gamma}$ and $\zeta^{1-\gamma}$ are bounded above), and we want the resulting process for $X$ and $S$ to remain positive, and intermediaries' and households' plans to be admissible and deliver utility $\frac{(Sk)^{1-\gamma}}{1-\gamma}$ and $\frac{(Xk)^{1-\gamma}}{1-\gamma}$. If we find a solution with these properties, then we have a competitive equilibrium. We know the HJB, their FOCs, and the market clearing conditions hold by construction. We only need to make sure these plans are truly optimal: with $\xi^{1-\gamma}$ and $\zeta^{1-\gamma}$ bounded above this is guaranteed (Theorem 1 for intermediaries and standard arguments for households).

**Numerical algorithm.** The system of equations can be solved by adding a fictitious finite time horizon $T$, with some terminal values for these functions. A time derivative must be added to the computation of all drifts, and we can then solve backwards in time. In this respect we have a system of first order ODEs with respect to time, which can be solved with a standard integrator, such as Runge-Kutta 4 for example. If the time derivatives vanish as we solve backwards, we have a solution to the system of PDEs we were interested in (infinite horizon). Terminal conditions are not important as long as the time derivatives vanish in the limit. Since the market clearing condition for consumption is an algebraic constraint, it is easier to differentiate it with respect to time to obtain a differential equation. We just need to make sure that terminal conditions are consistent with market clearing for consumption goods, and the algorithm will preserve this as we solve backwards. We can also verify ex-post that this condition is satisfied by the solution.

There are two complications. The first is that the FOC for $\hat{c}$ cannot be solved analytically, and solving it numerically at each step would make the algorithm much slower. What we can do is add $\hat{c}$ as a function to be solved for, and differentiate the FOC for $\hat{c}$ with respect to time, like we did for market clearing for consumption. We get an extra unknown but also an extra differential equation, and terminal conditions must be chosen so that the FOC for $\hat{c}$ is satisfied. This can also be verified ex-post (the benefit is we only solve the FOC for $\hat{c}$ once at the beginning).

The second complication is that the domain of the system $(X, Y) \in D \subset \mathbb{R}_+^2$ is unknown. Basically, for a given $Y$ we know that $X \in (0, \bar{X}(Y))$, but we don't know what is the maximum utility that can be delivered to intermediaries for each exogenous state $Y$. To deal with this we can do a change of variables, such as $\tilde{X} = \frac{X}{X+\zeta(q-\xi X)} \in (0,1)$, and solve the resulting system.

## 4.2 Planner's problem

Retirement requires modifying the planner's HJB equation and the law of motion of $X$. First, $X$ now captures the continuation utility of currently remaining intermediaries. Likewise, $S$ is the continuation utility of current households, including previously retired intermediaries. The HJB becomes

$$\frac{\rho}{1-\frac{1}{\psi}} = \max_{g,\hat{c},\sigma_x,\hat{\lambda}} \frac{(a-\iota(g)-\hat{c}X)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} S^{\frac{1}{\psi}-1} + \mu_S + g - \frac{\gamma}{2}\sigma_S^2 - \frac{\gamma}{2}\sigma^2 + (1-\gamma)\sigma_S\sigma \tag{36}$$

$$- \theta\frac{X}{S}\left(1-\hat{\lambda}(1-\gamma)\right)^{\frac{1}{1-\gamma}} \tag{37}$$

The $\theta$ term captures the fact that current households only get a part of future continuation utility of households in the future, because future households include current intermediaries that will retire in the meantime. Because retirement for each intermediary is observable and contractible, their continuation utility can jump down on retirement. The $\hat{\lambda}$ term captures this. Likewise, $\mu_X$ is the same as in the competitive equilibrium, (34). We use Ito's lemma to obtain $\mu_S$ and $\sigma_S$, and plug it into the HJB equation:

$$\mu_S = \frac{S_Y}{S}\mu_Y + \frac{S_X}{S}\mu_X X + \frac{1}{2}\frac{S_{YY}}{S}\sigma_Y^2 + \frac{1}{2}\frac{S_{XX}}{S}(\sigma_X X)^2 + \frac{S_{XY}}{S}\sigma_X X\sigma_Y \tag{38}$$

$$\sigma_S = \frac{S_X}{S}\sigma_X X + \frac{S_Y}{S}\sigma_Y \tag{39}$$

Now we have an extra FOC for $\hat{\lambda}$:

$$\frac{S_X'}{S}X + \frac{X}{S}\left(1-\hat{\lambda}(1-\gamma)\right)^{\frac{1}{1-\gamma}-1} = 0$$

$$\implies \hat{\lambda} = \frac{1 - \Lambda^{\frac{1-\gamma}{\gamma}}}{1-\gamma}$$

All the other FOC are unchanged. Notice that the planner's FOC for $\hat{\lambda}$ coincides with the private FOC (32), using $\Lambda = \xi\zeta$. Asa result, it is still the case that the only inefficiency is in the FOC for $g$, because of the hidden trade. Retirement does not introduce any source of inefficiency.

**Numerical solution.** The planner's HJB is a PDE for $S(X, Y)$. As in the competitive equilibrium, we can solve it by adding a fictitious finite horizon $T$. This requires us to add a time derivative when computing $\mu_S$. We can then solve backward for arbitrary terminal conditions. If the time derivative vanishes as we solve back, we found the original PDE.

Just like in the competitive equilibrium case, we need to deal with two complications. The first is that now the FOC for both $\hat{c}$ and $g$ are difficult to solve analytically, so we add both as functions of $(X, Y)$ and differentiate the FOCs with respect to time to obtain two more PDEs. We just need to ensure that terminal conditions satisfy the FOCs (the benefit, as before, is that we only solve them numerically once). We can check at the end that the FOC are satisfied. The second problem is that as before we don't know the domain, so we need to do a change of variables as in the competitive equilibrium, such as $\tilde{X} = \frac{X}{X+S} \in (0, 1)$, and solve the resulting system.

# Appendix C

I provide a discrete-time version of the setting in the paper.

### Setting

There is an aggregate shock $z = \{z_t\}$ and an intermediary-specific idiosyncratic shock $w_i = \{w_{it}\}$ for each intermediary, all independent. Both take values in $\{-\Delta, \Delta\}$ for some small $\Delta > 0$, with equal probability each period. Denote $z^t$ and $w_i^t$ the history of shocks up to time $t$. For simplicity, time is finite, $t = 0, 2...T$.

The aggregate state of the economy depends on the history of aggregate shocks; that is, $\sigma_t(z^t)$, $\nu_t(z^t)$, $\iota_t(g; z^t)$. The price of capital $q_t(z^t)$ also depends on the history of aggregate shocks, as well as aggregate investment $g_t(z^t) = (\iota_t')^{-1}(q_t(z^t); z^t)$ and the tax on capital $\tau_t^k(z^t)$. There is a complete financial market, where the price of a consumption good after aggregate history $z^t$ is denoted $\eta_t(z^t)$. Idiosyncratic risk is spanned by the market, but it's priced fairly because idiosyncratic risk can be eliminated in the aggregate.[10]

[10]So the price of a consumption good after aggregate history $z^t$ and idiosyncratic history $w_i^t$ for agent $i$ is simply $\eta_t(z^t)Prob(w_i^t)$.

The representative household's problem is to pick a consumption path $c_h = \{c_t(z^t)\}$ to maximize $U(c_h)$ subject to the budget constraint

$$\sum_{t=0}^{T} \sum_{z^t} \eta_t(z^t) c_{ht}(z^t) \leq w_0$$

**Intermediaries' contracts.** Each intermediary signs a contract $\mathcal{C} = (c, k)$ specifying his consumption $c = \{c_t(z^t, \tilde{w}^t) \geq 0\}$ and capital under management $k = \{k_t(z^t, \tilde{w}^t) \geq 0\}$ after every history $(z^t, \tilde{w}^t)$ of observable aggregate shocks $z^t$ and reported idiosyncratic shock $\tilde{w}^t$ (dropping the $i$ to avoid clutter).

The moral hazard problem arises because the true idiosyncratic shock $w_t$ is not contractible. After observing the current period's shocks $z_t$ and $w_t$, the agent can choose to steal $s_t$ such that he reports $\tilde{w}_t = w_t - s_t \in \{-\Delta, \Delta\}$. This means that if his true idiosyncratic shock is good, $w_t = \Delta$ he can pick $s_t \in \{0, 2\Delta\}$, but if his true shock is $w_t = -\Delta$ he can only choose $s_t = 0$. When he steals he diverts capital $k_{t-1}\nu_{t-1}s_t$ keeps a fraction $\phi$ and sells it at price $q_t$, and immediately consumes the proceeds, so he adds $\phi q_t k_{t-1}\nu_{t-1}s_t$ to his current consumption $c_t$. After signing the contract the agent chooses a stealing strategy $s = \{s_t(z^t, w^t)\}$. We say a contract $\mathcal{C} = (c, k)$ is incentive compatible if it is optimal for the intermediary to not steal:

$$0 \in \arg \max_{\{s_t(z^t, w^t)\}} U(\{c_t(z^t, \tilde{w}^t) + \phi q_t(z^t) k_{t-1}(z^{t-1}, \tilde{w}^{t-1}) \nu_{t-1}(z^{t-1}) s_t(z^t, w^t)\}) \tag{40}$$

The principal's objective function is to minimize the cost of delivering utility to the intermediary:[11]

$$\begin{aligned} J_0(u_0) = \min_{(c,k)} \sum_{t=0}^{T} \sum_{z^t} \sum_{w^t} \eta_t(z^t) Prob(w^t) \big( & c_t(z^t, w^t) - (a - \iota_t(g_t(z^t); z^t)) k_t(z^t, w^t) \\ & + q_t(z^t) k_t(z^t, w^t)(1 + \tau^k(z^t)) \\ & - q_t(z^t) k_{t-1}(z^{t-1}, w^{t-1})(1 + g_{t-1}(z^{t-1})(1 + \sigma_{t-1}(z^{t-1}) z_t)(1 + \nu_{t-1}(z^{t-1}) w_t) \big) \end{aligned}$$

subject to:

$$U(c) \geq u_0$$

$$(c, k) \text{ is incentive compatible}$$

The intermediary's initial utility is pinned down by free-entry on the principal's side: $J_0(u_0) = n_0$.

[11] with the convention that $k_{t-1} = 0$ as well as $g_{t-1}$.

**Resource constraints.** The aggregate capital stock $k = \{k_t(z^t)\}$ follows the law of motion

$$k_t(z^t) = k_{t-1}(z^{t-1})(1 + g_{t-1}(z^{t-1}))(1 + \sigma_{t-1}(z^{t-1})z_t) \tag{41}$$

where the idiosyncratic shocks $w_{it}$ are aggregated away.

The resource constraints in the economy are

$$\int_0^1 c_{it}(z^t, w_i^t)di + c_{ht}(z^t) = k_t(z^t)(a - \iota_t(g_t(z^t); z^t) \tag{42}$$

$$\int_0^1 k_{it}(z^t, w_i^t)di = k_t(z^t) \tag{43}$$

**Competitive Equilibrium.** Given an initial wealth distribution $(\{\theta_i\}, \theta_h)$ and initial capital stock $k_0$ a competitive equilibrium is a sequence of aggregate stochastic processes $q$, $T$, $\eta$, $g$, $k$; a contract $\mathcal{C}_i = (c_i, k_i)$ for each intermediary; and a consumption plan $c_h$ for the representative household, such that:

i. The representative households' plan and each intermediary's contract is optimal, given prices and initial wealth $n_{i0} = \theta_i(q_0 + T_0)k_0$ and $w_0 = \theta_h(q_0 + T_0)k_0$.

ii. Investment is optimal: $\iota_t'(g_t(z^t)) = q_t(z^t)$

iii. The value of government transfers is

$$T_t(z^t) = \sum_{s=t}^{T} \frac{\eta_{t+s}(z^{t+s})}{\eta_t(z^t)} \tau_{t+s}^k(z^{t+s})k_{t+s}(z^{t+s})$$

iv. Resource constraints (42) and (43) hold

v. The aggregate capital stock satisfies the law of motion (41) with initial $k_0$.

**Planner problem.** The planner faces the same environment as private agents, with moral hazard and hidden trade. The hidden price of capital is $\tilde{q} = \{\tilde{q}_t(z^t)\}$ with $\tilde{q}_t(z^t) = \iota_t'(g_t(z^t); z^t)$. The incentive compatibility constraint is therefore, for every intermediary:

$$0 \in \arg\max_{\{s_t(z^t, w^t)\}} U(\{c_t(z^t, \tilde{w}^t) + \phi\iota_t'(g_t(z^t); z^t)k_{t-1}(z^{t-1}, \tilde{w}^{t-1})\nu_{t-1}(z^{t-1})s_t(z^t, w^t)\}) \tag{44}$$

The planner therefore chooses an allocation $(c_h, g, k, \{c_i, k_i\})$ to maximize the utility of the representative household $U(c_h)$ subject to delivering utility $\{u_{i0}\}$ to each intermediary, and satisfying the resource constraints (42) and (43), and law of motion for capital (41) with initial $k_0$, and incentive compatibility (44) for each intermediary.

## References

DeMarzo, P. and Y. Sannikov (2006): "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *The Journal of Finance*, 61, pp. 2681–2724.

Duffie, D. and L. Epstein (1992): "Stochastic Differential Utility," *Econometrica*, 60, pp. 353–394.

Kraft, H., F. Seifried, and M. Steffensen (2011): "Consumption-Portfolio Optimization with Recursive Utility in Incomplete Markets," .