A Online Appendix A for "Selling to Advised Buyers" Andrey Malenko, Anton Tsoy: Omitted Proofs and Additional Results

A.1 Omitted Proofs

Omitted details for Theorem 3: Proof of (5)**.** We first show that for sufficiently large N, a strict inequality holds in (5) for all $k = 1, \ldots, K-1$. Denote

$$h(x|N) \equiv Nx^{N-1} - (N-1)x^{N}. \tag{8}$$

Let

$$H_k(v) \equiv \frac{h\left(F(v)|N\right) - h\left(F(\omega_{k-1})|N\right)}{h\left(F(\omega_k)|N\right) - h\left(F(\omega_{k-1})|N\right)}$$

be the c.d.f. of the distribution of the second-highest order statistic $v_{(2)}$ conditional on $v_{(2)} \in [\omega_{k-1}, \omega_k)$, and let

$$H_*(v) \equiv \frac{h\left(F(v)|N\right) - h\left(F(\omega_{K-1})|N\right)}{h\left(F(v^*)|N\right) - h\left(F(\omega_{K-1})|N\right)}$$

be the c.d.f. of the distribution of $v_{(2)}$ conditional on $v_{(2)} \in [\omega_{K-1}, v^*)$. We start with two auxiliary claims:

Claim 1. *Function $h(x|N)$ is strictly convex on $[0, 1 - \frac{1}{N-1})$. Further, for any $\tilde{x} \in [0,1)$ and $\varepsilon > 0$, it holds for all N sufficiently large that $h(\tilde{x}|N) \le h(x|N) + \varepsilon(\tilde{x} - x)$ for all $x \in [0, \tilde{x}]$.*

Proof. The first statement follows from

$$h''(x|N) = N(N-1)x^{N-3}((N-2) - (N-1)x)$$

is greater than 0 whenever $x \in [0, 1 - \frac{1}{N-1})$. To prove the second statement, note that since

$$h'(x|N) = N(N-1)(x^{N-2} - x^{N-1}) \le N(N-1)x^{N-2} \text{ for } x \in [0,1],$$

and $N\left(N-1\right)x^{N-2}$ decreases in N to zero for sufficiently large N, for all N sufficiently large it holds that $h'(x|N) \le \varepsilon$ for all $x \in [0, \tilde{x}]$. Hence, $h(\tilde{x}|N) \le h(x|N) + \varepsilon(\tilde{x} - x)$ for all $x \in [0, \tilde{x}]$. *q.e.d.* □

Claim 2. *For all sufficiently high N, $H_k(v)$ first-order stochastically dominates $F(v|v \in [\omega_{k-1}, \omega_k))$ for all $k = 1, \ldots, K-1$, and $H_*(v)$ first-order stochastically dominates $F(v|v \in [\omega_{K-1}, v^*))$.*

Proof. Fix $k = 1, \ldots, K-1$. It is sufficient to show that for all $v \in (\omega_{k-1}, \omega_k)$, $H_k(v) < F(v|v \in [\omega_{k-1}, \omega_k))$, or equivalently,

$$h(F(v)|N) < \frac{F(v) - F(\omega_{k-1})}{F(\omega_k) - F(\omega_{k-1})} h(F(\omega_k)|N) + \frac{F(\omega_k) - F(v)}{F(\omega_k) - F(\omega_{k-1})} h(F(\omega_{k-1})|N).$$

This inequality indeed holds for sufficiently large N, because $h(\cdot|N)$ is strictly convex on $[0, F^*(v))$ for sufficiently large N by Claim 1. The proof of the fact that $H_*(v)$ first-order stochastically dominates $F(v|v \in [\omega_{K-1}, v^*))$ is analogous. *q.e.d.* □

We proved earlier that $\omega_{K-1} < v^*$. Hence, by Claim 2, for all sufficiently large N it holds that

$$\begin{aligned}
\mathbb{E}[\min\{v_{(2)}, v^*\} + b|v_{(2)} \in [\omega_{k-1}, \omega_k)] &= \mathbb{E}[v_{(2)} + b|v_{(2)} \in [\omega_{k-1}, \omega_k)] \\
&> \mathbb{E}[v_{(2)}|v_{(2)} \in [\omega_{k-1}, \omega_k)] \\
&\geq \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] = m_k.
\end{aligned}$$

for all $k = 1, \ldots, K-1$. Thus, we have strict inequality in (5) for all $k = 1, \ldots, K-1$. We are left to show that $\mathbb{E}[\min\{v_{(2)}, v^*\} + b|v_{(2)} \geq \omega_{K-1}] \geq m_K$. We start with the following auxiliary claim.

Claim 3. For sufficiently large N, $\mathbb{P}(v \geq v^*|v \geq \omega_{K-1}) < \mathbb{P}\left(v_{(2)} \geq v^*|v_{(2)} \geq \omega_{K-1}\right)$.

Proof. Note that

$$\frac{1 - h(F(\omega_{K-1})|N)}{1 - F(\omega_{K-1})} \geq 1 - h(F(v^*)|N),$$

with $\lim_{N\to\infty} h(F(v^*)|N) = 0$, and so $(1 - h(F(\omega_{K-1})|N)) / (1 - F(\omega_{K-1})) > \frac{1}{2}$ for sufficiently high N. Applying Claim 1 with $\tilde{x} = F(v^*)$, $x = F(\omega_{K-1})$, and $\varepsilon = \frac{1}{2}$, we obtain that for sufficiently high N

$$\begin{aligned}
h(F(v^*)|N) &< h(F(\omega_{K-1})|N) + (F(v^*) - F(\omega_{K-1})) \frac{1}{2} \\
&< h(F(\omega_{K-1})|N) + (F(v^*) - F(\omega_{K-1})) \frac{1 - h(F(\omega_{K-1})|N)}{1 - F(\omega_{K-1})}.
\end{aligned}$$

The last inequality implies that

$$\frac{1 - F(v^*)}{1 - F(\omega_{K-1})} < \frac{1 - h(F(v^*)|N)}{1 - h(F(\omega_{K-1})|N)},$$

which proves the desired inequality. *q.e.d.* □

Combining Claims 2 and 3 implies that for sufficiently high N:

$$
\begin{aligned}
&\mathbb{E}\left[\min\{v_{(2)},v^*\}+b|v_{(2)}\geq\omega_{K-1}\right]\\
=&\mathbb{E}\left[v_{(2)}+b|v_{(2)}\in[\omega_{K-1},v^*)\right]\mathbb{P}\left(v_{(2)}\in[\omega_{K-1},v^*)|v_{(2)}\geq\omega_{K-1}\right)+(v^*+b)\mathbb{P}\left(v_{(2)}\geq v^*|v_{(2)}\geq\omega_{K-1}\right)\\
\geq&\mathbb{E}\left[v|v\in[\omega_{K-1},v^*)\right]\mathbb{P}\left(v_{(2)}\in[\omega_{K-1},v^*)|v_{(2)}\geq\omega_{K-1}\right)+\mathbb{E}\left[v|v\geq v^*\right]\mathbb{P}\left(v_{(2)}\geq v^*|v_{(2)}\geq\omega_{K-1}\right)\\
\geq&\mathbb{E}\left[v|v\in[\omega_{K-1},v^*)\right]\mathbb{P}\left(v\in[\omega_{K-1},v^*)|v\geq\omega_{K-1}\right)+\mathbb{E}\left[v|v\geq v^*\right]\mathbb{P}\left(v\geq v^*|v\geq\omega_{K-1}\right)=\mathbb{E}\left[v|v\geq\omega_{K-1}\right]
\end{aligned}
$$

The first inequality follows from Claim 2 and $b>0$; the second inequality follows from Claim 3. Since expected revenues in the second-price auction cannot exceed $\mathbb{E}\left[v|v\geq\omega_{K-1}\right]$, the capped delegation equilibrium in the English auction yields higher expected revenues than any equilibrium of the second-price auction if N is sufficiently high.

Details of the Proof of Proposition 3. In this proof we assume that $y(\theta)$ is strictly increasing in the range $[\underline{v}+b,\overline{v}]$ and show that contract $\theta_i(\omega_i)=b+\min\{\omega_i,v^*\}$ strictly dominates any other contract $\theta_i(\cdot)$. In contrast, if $y(\theta)$ is only weakly increasing in the range $[\underline{v}+b,\overline{v}]$, then the arguments in this proof show that bidder i does at least as well with contract $\theta_i(\omega_i)$ as with any other contract $\theta_i(\cdot)$, but need not do strictly better. Since $y(\cdot)$ may have mass points, we need to account for the possibility of winning at ties. Let $y(\theta_-)$ and $y(\theta_+)$ denote the left and right limits of $y(\cdot)$ at θ, respectively, and let ρ_θ denote the probability of winning conditional on bidding θ and tying. Let $x(\theta)\equiv y(\theta_-)+(y(\theta_+)-y(\theta_-))\rho_\theta$ denote the probability of winning if bidder i submits bid θ. Let $t(\theta)\equiv\int_0^{\theta-}c\,dy(c)+(y(\theta_+)-y(\theta_-))\rho_\theta\theta$ denote the expected transfer of bidder i to the seller in this case. Note that we can re-write it as $t(\theta)=\int_0^\theta c\,dx(c)$. Then, bidder i's problem is to maximize $\mathbb{E}[x(\theta(v))v-t(\theta(v))]$ over all contracts $\theta(v)$ that satisfy the incentive compatibility condition for the advisor:

$$x(\theta(v))(v+b)-t(\theta(v))\geq x(\theta(w))(v+b)-t(\theta(w)),\ \forall v,w\in[\underline{v},\overline{v}].$$

We prove the proposition in three steps.

Step 1: Deriving Expected Payoffs of the Bidder and the Advisor Let $u(\theta,v)\equiv x(\theta)v-t(\theta)$ denote the expected utility of a bidder with valuation v who bids θ in the second-price auction (follows the strategy of bidding up to price θ in the English auction). Then, $u(\theta,v+b)$ is the expected payoff of the advisor of type v if he induces the bidder to bid θ.

The next lemma shows that the advisor's expected payoff is inverted U-shaped in bid θ with the maximum at $\theta=v+b$:

Lemma 1. *$u(\theta,v+b)$ is weakly increasing in θ for $\theta<v+b$ (strictly for $\theta\in[\underline{v}+b,v+b)$), and weakly decreasing in θ for $\theta>v+b$ (strictly for $\theta\in(v+b,\overline{v}]$). For any $v\in(\underline{v},\overline{v}-b)$, $u(\theta,v+b)$*

reaches a unique maximum at $\theta = v + b$.

Proof. Using the expression for $t(\theta)$, we have $u(\theta, v+b) = x(\theta)(v+b) - \int_0^\theta c dx(c)$. Thus, for any θ_2 and $\theta_1 > \theta_2$,

$$u(\theta_1, v+b) - u(\theta_2, v+b) = (x(\theta_1) - x(\theta_2))(v + b - \mathbb{E}[c | c \in [\theta_2, \theta_1]]),$$

where c is distributed according to $x(\cdot)$. Since $x(\theta)$ is weakly increasing, it follows that $u(\theta, v+b)$ is weakly increasing in the range $\theta < v + b$ and weakly decreasing in the range $\theta > v + b$. Both relationships are strict wherever $x(\theta)$ is strictly increasing, i.e., for all $\theta \in [\underline{v} + b, \overline{v}]$. Thus, for any $v \in (\underline{v}, \overline{v} + b)$, $u(\theta, v+b)$ reaches a unique maximum in θ at $\theta = v + b$. $\square$

Since contract $\theta(v)$ is incentive compatible for the advisor, $u(\theta(v), v+b) = \max_{\hat{v}} x(\theta(\hat{v}))(v+b) - t(\theta(\hat{v}))$ and by the generalized envelope theorem (Milgrom and Segal (2002)) the payoff of the advisor of type v can be written as

$$u(\theta(v), v+b) = \int_{\underline{v}}^{v} x(\theta(u)) du + u(\theta(\underline{v}), \underline{v} + b), \tag{9}$$

where the expected payoff of the lowest type of the advisor is $u(\theta(\underline{v}), \underline{v} + b) = x(\theta(\underline{v}))(\underline{v} + b) - t(\theta(\underline{v}))$.

Conditional on the advisor's type v, the bidder's expected payoff equals $u(\theta(v), v) = u(\theta(v), v+b) - bx(\theta(v))$. Using (9) and integrating over the distribution of v by parts, we obtain the bidder's expected payoff from contract $\theta(v)$:

$$\begin{aligned} &\int_{\underline{v}}^{\overline{v}} (u(\theta(v), v+b) - bx(\theta(v))) dF(v) \\ &= \int_{\underline{v}}^{\overline{v}} (1 - F(v) - bf(v)) x(\theta(v)) dv + u(\theta(\underline{v}), \underline{v} + b). \end{aligned} \tag{10}$$

Step 2: Shape of Incentive Compatible Contracts For any $\tilde{v} \in [\underline{v}, \overline{v}]$, define $\theta^+(\tilde{v}) \equiv \lim_{v \to \tilde{v}+} \theta(v)$ and $\theta^-(\tilde{v}) \equiv \lim_{v \to \tilde{v}-} \theta(v)$. Since the bidder's valuation cannot exceed $\overline{v}$, the optimal contract cannot have $\theta(v) > \overline{v}$, so we restrict attention to contracts with $\theta(v) \leq \overline{v}$. We next show that any incentive-compatible contract must have a rather specific form. This result is a counter-part of Proposition 1 in Melumad and Shibano (1991).

Lemma 2 (**Melumad and Shibano, 1991**). *An incentive-compatible* $\theta(v)$ *must satisfy the following:*

1. $\theta(\cdot)$ *is weakly increasing;*

2. *If* $\theta(\cdot)$ *is strictly increasing and continuous on an open interval* (v_1, v_2), *then* $\theta(v) = v + b$ *on it;*

3. If $\theta(\cdot)$ is discontinuous at $\tilde{v}$, the discontinuity must be a jump discontinuity that satisfies

$$x\left(\theta^{-}(\tilde{v})\right)(\tilde{v}+b)-t\left(\theta^{-}(\tilde{v})\right)=x\left(\theta^{+}(\tilde{v})\right)(\tilde{v}+b)-t\left(\theta^{+}(\tilde{v})\right); \tag{11}$$

$$\begin{aligned}\theta(v)&=\theta^{-}(\tilde{v}),\forall v\in[\theta^{-}(\tilde{v})-b,\tilde{v});\\ \theta(v)&=\theta^{+}(\tilde{v}),\forall v\in(\tilde{v},\theta^{+}(\tilde{v})-b].\end{aligned} \tag{12}$$

Proof. The proof follows the argument in Melumad and Shibano (1991) with the adjustment to the fact that we do not assume differentiability or strict concavity of the advisor's utility function. Weak monotonicity of $\theta(\cdot)$ follows by the same argument as the weak monotonicity of $m(\cdot)$ in the proof of Part 2 of Proposition 1. Consider the second statement. By contradiction, suppose that $\theta(v)<v+b$ for some $v\in(v_1,v_2)$ (case $\theta(v)>v+b$ is analogous). Since $\theta(\cdot)$ is continuous and strictly increasing, there exists $\varepsilon>0$: $\theta(v)<\theta(v+\varepsilon)<v+b$. Then, by Lemma 1, $u(\theta(v+\varepsilon),v+b)>u(\theta(v),v+b)$, i.e., type v is better off deviating to reporting $v+\varepsilon$ rather than ε, which is a contradiction. Finally, consider the last statement. Equality (11) follows from the incentive compatibility. We next show the first line in (12). The proof of the second line is symmetric. Consider $v\in[\theta^{-}(\tilde{v})-b,\tilde{v})$. By construction, $\theta^{-}(\tilde{v})\leq v+b$. By the first statement of the lemma, $\theta(v)\leq\theta^{-}(\tilde{v})$. If $\theta(v)<\theta^{-}(\tilde{v})\leq v+b$, then by Lemma 1 $u(\theta(\tilde{v}),v+b)>u(\theta(v),v+b)$, implying that type v would be better off reporting $\tilde{v}$ than v, which contradicts incentive compatibility of $\theta(v)$. Thus, $\theta(v)=\theta^{-}(\tilde{v})$. $\square$

Step 3: Optimality of the Capped Delegation We first rule out discontinuous jumps in the optimal $\theta(\cdot)$. By contradiction, suppose that there is a discontinuous jump in $\theta(\cdot)$ at some $\tilde{v}\in(\underline{v},\overline{v})$. Consider new contract $\tilde{\theta}(v)$, identical to $\theta(v)$ for all types but $v\in[\theta^{-}(\tilde{v})-b,\theta^{+}(\tilde{v})-b]$, where $\tilde{\theta}(v)=v+b$. Let $v^{j}\equiv\theta^{j}(\tilde{v})-b$ and $x^{j}\equiv x\left(\theta^{j}(\tilde{v})\right)$ for $j\in\{+,-\}$. Since contract $\theta(\cdot)$ is incentive-compatible, the advisor with type $\tilde{v}$ is indifferent between inducing bids $\theta^{-}(\tilde{v})$ and $\theta^{+}(\tilde{v})$. Using $t(\theta)=\int_0^{\theta}c\,dx(c)$, (11) implies

$$\int_{v^{-}}^{\tilde{v}}\left(x(v+b)-x^{-}\right)dv=\int_{\tilde{v}}^{v^{+}}\left(x^{+}-x(v+b)\right)dv. \tag{13}$$

Subtracting the bidder's expected payoff (10) under contract $\theta(v)$ from the expected payoff under contract $\tilde{\theta}(v)$:

$$\int_{v^{-}}^{\tilde{v}}(1-F(v)-bf(v))\left(x(v+b)-x^{-}\right)dv-\int_{\tilde{v}}^{v^{+}}(1-F(v)-bf(v))\left(x^{+}-x(v+b)\right)dv>0,$$

from (13) and the fact that $F(v)+bf(v)$ is strictly increasing. Therefore, contract $\theta(\cdot)$ is dominated by contract $\tilde{\theta}(\cdot)$, which contradicts optimality of $\theta(\cdot)$.

By Lemma 2 and continuity of the optimal contract, it is sufficient to consider contracts $\theta(\cdot)$ that are comprised of, at most, one upward sloping part $\theta(v)=v+b$ for $v\in[v_1,v_2]$ and two flat parts, $\theta(v) = v_1 + b$ for $v \in [\underline{v}, v_1]$ and $\theta(v) = v_2 + b$ for $v \in [v_2, \overline{v}]$. Any other contract satisfying Lemma 2 exhibits a discontinuous jump in $\theta(v)$ at some v, which is inconsistent with the optimality of $\theta(\cdot)$ by the previous paragraph. For any pair of $v_1 \geq \underline{v}$ and $v_2 \in [v_1, \overline{v}]$, letting $x_1 \equiv x(v_1 + b)$ and $x_2 \equiv x(v_2 + b)$ and using (10), the expected payoff of the bidder can be written as:

$$\begin{aligned} & x_1 \int_{\underline{v}}^{v_1} (1 - F(v) - bf(v))\, dv + \int_{v_1}^{v_2} (1 - F(v) - bf(v))\, x(v+b)\, dv \\ & + x_2 \int_{v_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv + x_1 (\underline{v} + b) - \int_0^{v_1+b} v dx(v) \end{aligned} \tag{14}$$

Integrating by parts, $\int_0^{v_1+b} v dx(v) = x_1(v_1 + b) - \int_0^{v_1+b} x(v)\, dv$. Taking the difference of (14) for $v_1' \in (v_1, v_2)$ and v_1 and simplifying the expression, we obtain:

$$-\left(x_1' - x_1\right) \int_{\underline{v}}^{v_1} (F(v) + bf(v))\, dv - \int_{v_1}^{v_1'} \left(x_1' - x(v+b)\right) (F(v) + bf(v))\, dv,$$

where we denote $x_1' \equiv x(v_1' + b)$. This expression is strictly negative, because $x(\cdot)$ is strictly increasing on $[\underline{v} + b, \overline{v}]$. herefore, for any $v_1 > \underline{v}$, contract $\theta(v)$ is dominated by contract $\tilde{\theta}(v)$ that coincides with $\theta(v)$ but lowers v_1 by an infinitesimal amount. Hence, the optimal contract has no flat region at the bottom: $v_1 = \underline{v}$.

Step 4: Optimal Pooling Region at the Top The previous steps imply that the optimal contract $\theta(v)$ take form of capped delegation: $\theta(v) = b + \min\{v, v_2\}$ for some v_2. We next show that $v_2 = v^*$. Taking the difference of the bidder's expected payoff (14) for $v_2 + \varepsilon$ and v_2, we obtain:

$$\int_{v_2}^{v_2+\varepsilon} (1 - F(v) - bf(v))\, (x(v+b) - x_2)\, dv + \left(x_2' - x_2\right) \int_{v_2+\varepsilon}^{\overline{v}} (1 - F(v) - bf(v))\, dv,$$

where we denote $x_2' \equiv x(v_2 + b + \varepsilon)$. Taking the limit as $\varepsilon \to 0$ and using the fact that $x(\cdot)$ is strictly increasing, we obtain that the difference has the same sign as $\int_{v_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv$. Since $F(v) + bf(v)$ is strictly increasing in v, the bidder's expected payoff (14) is inverted U-shaped in v_2 with the maximum at $v_2 = \hat{v}_2$, which is the unique solution to $\int_{\hat{v}_2}^{\overline{v}} (1 - F(v) - bf(v))\, dv = 0$. Note that this equation can be re-written as $MRL(\hat{v}_2) = b$, and thus $\hat{v}_2 = v^*$. Therefore, the unique optimal cap is $v^* + b$.

A.2 Additional Results for Subsection 4.1

Here, we derive several additional results. First, Proposition 4 provides explicit formulas for the cut-off advisor types in any equilibrium communication partition.

Proposition 4. *In any standard auction with continuous payments, in any equilibrium with K partition intervals, thresholds $(\omega_k)_{k=0}^K$ satisfy $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and*

$$G(\omega_{k-1},\omega_k)(1-\Lambda_k)(\omega_k+b-m_k) = -G(\omega_k,\omega_{k+1})\Lambda_{k+1}(\omega_k+b-m_{k+1}), \tag{15}$$

where

$$m_k \equiv \mathbb{E}\left[v|v\in[\omega_{k-1},\omega_k)\right], \tag{16}$$

and

$$\Lambda_k \equiv \frac{1}{G(\omega_{k-1},\omega_k)}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\frac{F(\omega_{k-1},\omega_k)^n F(\omega_{k-1})^{N-1-n}}{n+1}\right) \tag{17}$$

is the probability of winning conditional on a tie at bid m_k. Further if $(\omega_k)_{k=0}^K$ satisfy $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and (15)*, then they are part of equilibrium communication partition.*

Proof. By Proposition 1 it is without loss to restrict attention to communication in equilibria of the second-price auction. By the interval partition form of communication, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k given by (16).

As an auxiliary step, let us derive the probability that a bidder with bid $m_k = \mathbb{E}[v|v\in[\omega_{k-1},\omega_k)]$ wins a tie, conditional on the tie taking place at bid $m_k = \mathbb{E}[v|v\in[\omega_{k-1},\omega_k)]$. Without loss of generality, we refer to this bidder as bidder N and to her rivals as bidders $i\in\{1,2,...,N-1\}$. Denote such a probability by Λ_k. Since ties are broken randomly,

$$\Lambda_k = \mathbb{E}\left[\frac{1}{\tilde{n}_k+1}|\hat{v}\in[\omega_{k-1},\omega_k)\right],$$

where $\tilde{n}_k = \sum_{i=1}^{N-1} 1\left\{v_i\in[\omega_{k-1},\omega_k)\right\}$ is a random variable, denoting the number of rival bidders with the same bid m_k. Re-writing,

$$\begin{aligned}\Lambda_k &= \sum_{n=1}^{N-1}\frac{1}{n+1}\frac{\Pr\left[\tilde{n}_k=n,\hat{v}\in[\omega_{k-1},\omega_k)\right]}{G\left(\omega_{k-1},\omega_k\right)}\\ &= \sum_{n=1}^{N-1}\frac{1}{n+1}\frac{\Pr\left[\tilde{n}_k=n,\sum_{i=1}^{N-1}1\left\{v_i<\omega_{k-1}\right\}=N-1-n\right]}{G\left(\omega_{k-1},\omega_k\right)}\\ &= \sum_{n=1}^{N-1}\frac{1}{n+1}\frac{\binom{N-1}{n}F\left(\omega_{k-1},\omega_k\right)^n F\left(\omega_{k-1}\right)^{N-1-n}}{G\left(\omega_{k-1},\omega_k\right)},\end{aligned}$$

which coincides with expression (17).

Now, we can show (15). Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bidder. From submitting a message m_k, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k),$$

where the expected utility from bidding m_k when the other bidders submit bids below m_k and when some bidders tie with the bidder is captured by the first and second terms, respectively. Analogously, from submitting a message m_{k+1}, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}).$$

Type ω_k should be indifferent between the two which gives the eq. (15). Thus, any PBE communication strategy can be described by a solution $(\omega_k)_{k=0}^K$ to recursion (15) where $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$.

The last statement of the proposition is immediate from the specification of m_k in (16) and the fact that cut-off types ω_k are indifferent between the two adjacent equilibrium bids m_k and m_{k+1}, and the single-crossing property of advisor's payoffs. □

The next proposition verifies that there indeed exist equilibria that satisfy NITS.

Proposition 5. *In any static auction, there is an equilibrium that satisfies the NITS condition.*

We first show the following auxiliary lemma:

Lemma 3. *If $\omega_{k+1} = \omega_k$, then $k = 0$. Furthermore, there exists $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $k = 1, \ldots, K$.*

Proof. To prove the first statement, suppose by contradiction that $\omega_{k+1} = \omega_k$ for some $0 < k \leq K$. This implies that $G(\omega_k, \omega_{k+1}) = 0$ and so, from (15), $G(\omega_{k-1}, \omega_k)(1 - \Lambda_{k-1})(\omega_k + b - m_k) = 0$ which in turn implies that $\omega_k + b = m_k$ or $\omega_{k-1} = \omega_k$. If $\omega_{k-1} < \omega_k$, then $m_k < \omega_k < \omega_k + b$ which is a contradiction. If $\omega_{k-1} = \omega_k$, then choose j so that $\omega_{k-j-1} < \omega_{k-j} = \cdots = \omega_{k-1} = \omega_k = \omega_{k+1}$ and the argument proceeds as in the case $\omega_{k-1} < \omega_k$.

To prove the second statement, suppose $\omega_{k+1} - \omega_k > 0$ for some k. By the first statement of the lemma, $k > 0$. Since $\omega_k + b - m_k > \omega_k + b - m_{k+1}$, it follows from (15) and $\omega_{k+1} - \omega_k > 0$ that

$$\omega_k + b \leq \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{18}$$

If to contradiction for any $\varepsilon > 0$, there were an equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq \omega_k + \varepsilon < \omega_k + b$ which would contradict (18) for $\varepsilon < b$. Thus, there is $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k \leq K$.

□

We can now proceed to the proof of Proposition 5.

Proof of Proposition 5. We consider separately cases $\overline{v} < \infty$ and $\overline{v} = \infty$.

Case 1: $\overline{v} < \infty$

Lemma 3 implies that there is an upper bound $\bar{K}$ on the number of intervals in the communication strategy. To prove there is an equilibrium that satisfies NITS, we adapt the proof of Proposition 1 from Chen et al. (2008) for our problem. It is useful to introduce the following notations:

$$\begin{aligned}
\Psi(\omega_{k-1}, \omega_k) = \quad G(\omega_{k-1}, \omega_k)(1 - \Lambda_k) = \quad & \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1}, \\
\Phi(\omega_k, \omega_{k+1}) = \quad G(\omega_k, \omega_{k+1})\Lambda_{k+1} = \quad & \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_k, \omega_{k+1})^n F(\omega_k)^{N-1-n} \frac{1}{n+1}.
\end{aligned}$$

Denote $m(\omega_{k-1}, \omega_k) = \mathbb{E}[v | v \in (\omega_{k-1}, \omega_k)]$ and

$$H(\omega_{k-1}, \omega_k, \omega_{k+1}) \equiv \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) + \Phi(\omega_k, \omega_{k+1})(\omega_k + b - m(\omega_k, \omega_{k+1})). \quad (19)$$

Note that an equilibrium with K intervals is given by recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$. To prove the statement, we will show that if an equilibrium with K intervals $\omega = (\omega_0, \omega_1, ..., \omega_K)$, violates the NITS condition, then for all $k = 1, ..., K$, there exists a solution to recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $k+1$ intervals, ω^k, that satisfies $\omega_0^k = \underline{v}$, $\omega_k^k > \omega_{k-1}$, and $\omega_{k+1}^k = \omega_k$. After this result is established, the statement of the proposition follows from the following argument. By contradiction, suppose that the most informative equilibrium (i.e., one with $\bar{K}$ intervals) violates the NITS condition. Applying the result above for $k = \bar{K}$, there must be a solution to $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$ with $\bar{K}+1$ intervals satisfying boundary conditions $\omega_0^{\bar{K}} = \underline{v}$ and $\omega_{\bar{K}+1}^{\bar{K}} = \omega_{\bar{K}} = \bar{v}$. By Propositions 1 and 4, this is an equilibrium, which contradicts the statement that $\bar{K}$ is the highest number of equilibrium intervals.

We show the result by induction on k. As an induction base, consider $k = 1$. If the equilibrium with K intervals $\omega = (\omega_0, \omega_1, ..., \omega_K)$ violates the NITS, it must be that $\underline{v} + b < m(\underline{v}, \omega_1)$, and hence, $H(\underline{v}, \underline{v}, \omega_1) < 0$. At the same time, $H(\underline{v}, \omega_1, \omega_1) > 0$, since $\omega_1 > m(\underline{v}, \omega_1)$ and $b > 0$. By continuity, there exists $x \in (\underline{v}, \omega_1)$ at which $H(\underline{v}, x, \omega_1) = 0$. Hence, the claim holds for $k = 1$: $\omega^1 = \left(\omega_0^1, \omega_1^1, \omega_2^1\right) = (\underline{v}, x, \omega_1)$ solves $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$ with $\omega_0^1 = \underline{v}$, $\omega_1^1 > \omega_0 = \underline{v}$, and $\omega_2^1 = \omega_1$. Consider the difference $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) - H(\omega_{k-1}, \omega_k, \omega_{k+1})$:

$$\begin{aligned}
&\Psi\left(\omega_k^k, \omega_k\right)\left(\omega_k + b - m\left(\omega_k^k, \omega_k\right)\right) - \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) \\
=&F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F\left(\omega_k^k, \omega_k\right)}{F(\omega_k)}\right)^n\left(\frac{F\left(\omega_k^k\right)}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)\left(\omega_k + b - m\left(\omega_k^k, \omega_k\right)\right) \\
&- F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F(\omega_{k-1}, \omega_k)}{F(\omega_k)}\right)^n\left(\frac{F(\omega_{k-1})}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)(\omega_k + b - m(\omega_{k-1}, \omega_k)).
\end{aligned}$$

Since $\omega_k^k > \omega_{k-1}$, we have two implications. First, $m\left(\omega_k^k, \omega_k\right) > m\left(\omega_{k-1}, \omega_k\right)$, implying $\omega_k + b - m\left(\omega_k^k, \omega_k\right) < \omega_k + b - m\left(\omega_{k-1}, \omega_k\right)$. Second, binomial distribution with success probability $\frac{F(\omega_{k-1},\omega_k)}{F(\omega_k)}$ dominates binomial distribution with success probability $\frac{F\left(\omega_k^k,\omega_k\right)}{F(\omega_k)}$ in the sense of first-order stochastic dominance, implying $\Psi\left(\omega_k^k, \omega_k\right) < \Psi\left(\omega_{k-1}, \omega_k\right)$. Therefore, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) - H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) < 0$. Since $H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) = 0$, we conclude that $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$.

On the other hand, since $\omega_k > m\left(\omega_k^k, \omega_k\right)$ and $b > 0$, we have $\omega_k + b > m\left(\omega_k^k, \omega_k\right)$, implying $H\left(\omega_k^k, \omega_k, \omega_k\right) > 0$. This, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$, and continuity imply that there exists $x \in (\omega_k, \omega_{k+1})$ at which $H\left(\omega_k^k, \omega_k, x\right) = 0$. Since $\omega_{k+1}^k = \omega_k$, the same x satisfies $H\left(\omega_k^k, \omega_{k+1}^k, x\right) = 0$. That is, there exists a solution in which the $(k+1)$st intervals ends at $\omega_{k+1}^k = \omega_k$ and the $(k+2)$nd interval ends at $x < \omega_{k+1}$. By continuity, there exists a solution to the recursion in which the $(k+2)$nd interval ends at any $\omega \in \left(\omega_{k+1}^k, \omega_{k+1}\right)$. By continuity, for one such ω, denoted ω_{k+1}^{k+1}, the $(k+2)$nd interval ends exactly at ω_{k+1}, i.e., $\omega_{k+2}^{k+1} = \omega_{k+1}$. Hence, there exists a solution to recursion $H\left(\omega_{k-1}, \omega_k, \omega_{k+1}\right) = 0$, with $k+2$ intervals, ω^{k+1}, that satisfies $\omega_0^{k+1} = \underline{v}$, $\omega_{k+1}^{k+1} > \omega_k$, and $\omega_{k+2}^{k+1} = \omega_{k+1}$. This completes the proof of the inductive step.

Case 2: $\overline{v} = \infty$

Consider a sequence of $\overline{v}_j = j, j = 1, 2, \ldots$. By case 1, there exists an equilibrium partition $(\omega_k(j))_{k=0}^{K_j}$ in the game with distribution of types truncated from above at $\overline{v}_j$ that satisfies the NITS condition, i.e.,

$$\underline{v} + b \geq \mathbb{E}[v | v \leq \omega_1(j)]. \tag{20}$$

Let us construct partition $(\omega_k^*)_{k=0}^K$ as follows. Consider a sequence $(\omega_1(j))_{j=1,2,\ldots}$. If there is a divergent subsequence in $(\omega_1(j))_{j=1,2,\ldots}$, then set $K = 1$ and $\omega_1^* = \infty$. If there is no divergent subsequence in $(\omega_1(j))_{j=1,2,\ldots}$, then consider an arbitrary convergent subsequence $(\omega_1(j(i)))_{i=1,2,\ldots}$ of $(\omega_1(j))_{j=1,2,\ldots}$ and set $\omega_1^* = \lim_{i\to\infty} \omega_1\left(j(i)\right)$. We next move to $k = 2$. Again, if there is a divergent subsequence in $(\omega_1(j(i)))_{i=1,2,\ldots}$, then set $K = 2$ and $\omega_2^* = \infty$. Otherwise, consider a convergent subsequence of $(\omega_1(j(i)))_{i=1,2,\ldots}$ and set ω_2^* to its limits. We proceed this way to construct $(\omega_k^*)_{k=0}^K$.

For any k, $\omega_k(j)$ satisfies

$$H\left(\omega_{k-1}(j), \omega_k(j), \omega_{k+1}(j) | \overline{v}_j\right) = 0, \tag{21}$$

where function $H(\cdot | \overline{v}_j)$ is defined as in (19) with F replaced by $F(\cdot | v \leq \overline{v}_j)$. By taking the limit of (21) over the subsequence used to obtain ω_{k+1}^*, we get that each ω_k^* satisfies (19) or equivalently (15). By Proposition 4, $(\omega_k^*)_{k=0}^K$ are part of the equilibrium communication partition.

Finally, by construction, $\omega_1^* = \lim_{i\to\infty} \omega_1\left(j(i)\right)$ and it holds (20). By taking the limit (20) as $i \to \infty$ we get that $\underline{v} + b \geq \mathbb{E}[v | v \leq \omega_1^*]$. Thus, the constructed partition satisfies NITS, which completes the proof. $\square$

A.3 Additional Result for Section 4.2

In the somewhat unnatural case when neither Assumption A nor B are satisfied, the English auction can have multiple capped delegation equilibria. However, for the case of two bidders, we can generalize our comparison results:

Proposition 6. *Suppose that $b > 0$ and $N = 2$. Then, the pooling region (if it is not empty) in any capped delegation equilibrium in the English auction is finer than the top interval in any equilibrium of the second-price auction: $v^* \geq \omega_{K-1}$. If, in addition, $\varphi(\cdot)$ is strictly increasing, then any capped delegation equilibrium in the English auction brings higher expected revenues than any equilibrium satisfying NITS in the second-price auction. Both comparisons are strict if $v^* > \underline{v}$.*

Proof. We prove the first statement by contradiction. Suppose there exists a capped delegation equilibrium in the English auction and an equilibrium in the second-price auction satisfying $v^* < \omega_{K-1}$. Then, in the set of intervals in equilibrium of the second-price auction, there exists an interval (ω_{k-1}, ω_k) satisfying $\omega_{k-1} \leq v^* < \omega_k$. Consider the indifference condition of type ω_k in the second-price auction. When $N = 2$, $G(\omega_k, \omega_{k+1}) = F(\omega_k, \omega_{k+1})$ and $\Lambda_k = \Lambda_{k+1} = \frac{1}{2}$. Therefore, (15) can be simplified to

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} m_{k+1} = \mathbb{E}\left[v | v \in (\omega_{k-1}, \omega_{k+1})\right]. \tag{22}$$

Since $\omega_{k+1} \leq \bar{v}$ and $\omega_{k-1} \leq v^*$, the right-hand side of (22) satisfies:

$$\mathbb{E}\left[v | v \in (\omega_{k-1}, \omega_{k+1})\right] \leq \mathbb{E}\left[v | v \geq \omega_{k-1}\right] \leq \mathbb{E}\left[v | v \geq v^*\right].$$

On the other hand, since $v^* < \omega_k$, the left-hand side of (22) satisfies $\omega_k + b > v^* + b$. Hence, $v^* + b < \mathbb{E}[v|v \geq v^*]$. On the other hand, by the argument in Theorem 3, whenever $v^* < \overline{v}$ it is necessary that $\mathbb{E}[v|v \geq v^*] = v^* + b$, which gives us the contradiction. Therefore, $v^* \geq \omega_{K-1}$. If $v^* > \underline{v}$, then $v^* > \omega_{K-1}$, since $v^* = \omega_{K-1}$ cannot be by contradiction. Indeed, in this case, equation (22) implies $v^* + b = \mathbb{E}[v|v \geq \omega_{K-2}] < \mathbb{E}[v|v \geq v^*]$, which again contradicts $\mathbb{E}[v|v \geq v^*] = v^* + b$. The second statement of the theorem follows from the same argument as Theorem 1, since it relies only on higher efficiency of an equilibrium of the English auction than an equilibrium of the second-price auction and on the NITS condition.

□

B Online Appendix B for "Selling to Advised Buyers" Andrey Malenko, Anton Tsoy: Selection of the Capped Delegation Equilibria

In this appendix, we introduce the dynamic version of NITS and show that it selects capped delegation equilibria.

B.1 Dynamic Version of NITS

We require that the NITS condition holds in every round of the game. For any history h, let $\mu(h)$ be the bidder's posterior after history h in the beginning of the current round (before the advisor sends a message in the current round). Let

$$v_w(h) \equiv \min\{v | v \in supp(\mu(h))\} \tag{23}$$

be the weakest remaining (according to the bidder's beliefs) type of the advisor after history h. Similarly to Chen et al. (2008), an equilibrium violates the dynamic version of NITS condition if after some history h, the advisor of type $v_w(h)$ is better off claiming that he is the weakest remaining type than playing his equilibrium strategy. To capture this condition, we require that any unexpected message is interpreted as a signal of the weakest type (then the advisor's sequential rationality implies that after any history, the equilibrium strategy is weakly preferred to signaling that you are the weakest type). Formally, the dynamic version of NITS that we impose is stated as follows:

Definition 3. *An equilibrium of the English auction $(m, a, \tilde{\mu})$ satisfies the NITS condition if the following holds. Consider any p-round history h, in which the advisor deviates in round p' for the first time and sends $\tilde{m} \notin \bigcup_{v \in supp(\mu(h'))} m(v, p', \mu(h'))$, where h' is a truncation at round p' of history h. Then $\mu(h)$ assigns probability one to $v_w(h')$.*[17]

Several observations are in order. First, Definition 3 states that after the first unexpected message, the bidder assigns probability one to the weakest type in the round when the deviation happened and never updates her belief after that. Second, in dynamic auctions the weakest type can (and will) change as the auction progresses. Third, the condition in Definition 3 requires that any unsent message is perceived as a signal of the weakest type which is slightly stronger than assuming that the lowest type of advisor does not want to reveal itself in equilibrium.

[17] We implicitly assume that the set of messages is rich enough so that there is always an "unused" message in any equilibrium.

B.2 Online Strategies

We first show that it is sufficient to look for equilibria of the English auction in which the advisor gives a real-time recommendation of the action ("quit" or "stay") to the bidder, both advisors' and bidders' strategies are of the threshold form, and bidders follow the recommendations of their advisors on equilibrium path. We refer to these equilibria as equilibria in online threshold strategies.

Definition 4. *An equilibrium in the English auction is* in online threshold strategies *if the strategies of each advisor and bidder satisfy:*

$$m(v,p,\mu) = \begin{cases} 1, & \text{if } p \geq \hat{p}(v,\mu), \\ 0, & \text{if } p < \hat{p}(v,\mu), \end{cases} \quad a(p,\tilde{\mu}) = \begin{cases} 1, & \text{if } p \geq \bar{p}(\tilde{\mu}), \\ 0, & \text{if } p < \bar{p}(\tilde{\mu}), \end{cases} \tag{24}$$

for some $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$, where $\tilde{\mu}$ denotes the posterior belief of the bidder at price p, having observed her advisor's message in this round. Functions $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$ are such that on equilibrium path the bidder exits the auction the first time her advisor sends message $\tilde{m} = 1$.

Intuitively, at any price p, the advisor sends a binary message to his bidder recommending to quit the auction immediately or stay in it, and on equilibrium path, the bidder follows the advisor's recommendation. The next lemma shows that the restriction to equilibria in online threshold strategies is without loss of generality:

Lemma 4. *For any equilibrium there is also an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path. For any equilibrium that satisfies NITS there is an equilibrium in online threshold strategies that satisfies NITS and results in the same bidding behavior on equilibrium path.*

The first statement is that any equilibrium with a general communication strategy has an equivalent in online threshold strategies. The proof is the manifestation of the sure-thing principle (Savage (1972)), stating that if an action is optimal for a decision-maker in every state, then it must be optimal if she does not know the state. Intuitively, since the advisor's information is only relevant for determining the price level at which the bidder quits the auction, any equilibrium quitting strategy can be achieved by the advisor delaying communication as much as possible, which occurs when she sends a recommendation to quit immediately when the price hits the level at which the bidder is supposed to quit. The second statement implies that the NITS condition is not stronger for equilibria in online threshold strategies than in general: If an equilibrium with a general communication strategy satisfies NITS, then an equivalent in online threshold strategies also satisfies NITS.

Proof of Lemma 4. The proof of the first statement follows the argument of the proof of Lemma IA.2 in Grenadier et al. (2016). Specifically, for any pure-strategy PBEM we construct an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path.

Consider any pure-strategy equilibrium E with some strategies $\bar{m}(v,p,\mu)$ and $\bar{a}(p,\tilde{\mu})$. It implies an equilibrium exit price $\bar{\tau}(v)$, which is the price at which the bidder exits the auction, if the valuation is v, provided that the bidder and her advisor play the equilibrium strategies $\bar{m}(\cdot)$ and $\bar{a}(\cdot)$. Note that $\bar{\tau}(v)$ must be weakly increasing in v. To see this, suppose by contradiction that $\bar{\tau}(v_1) > \bar{\tau}(v_2)$ for some $v_1 \in [\underline{v}, \bar{v})$ and $v_2 \in (v_1, \bar{v}]$. Since the advisor's payoff from acquiring the asset at any price p is higher for type v_2 than for type v_1 ($v_2 + b - p > v_1 + b - p$), the advisor's continuation value from not exiting the auction at any price p cannot be lower for type v_2 than for type v_1. The payoff from exiting the auction at any current price p does not depend on the type and equals zero. Thus, $\bar{\tau}(v_2) \geq \bar{\tau}(v_1)$. Let $\varrho \equiv \{p : \exists v \in [\underline{v}, \bar{v}] \text{ such that } \bar{\tau}(v) = p\}$ be the set of prices at which the bidder exits the auction for some realization of v. It will be convenient to define $v_l(p) \equiv \inf\{v : \bar{\tau}(v) = p\}$ and $v_h(p) \equiv \sup\{v : \bar{\tau}(v) = p\}$ for any $p \in \varrho$. We extend the definition of $v_l(p)$ for any $p \notin \varrho$ by setting $v_l(p) \equiv \inf\{v : \bar{\tau}(v) \geq p\}$.

Consider an online threshold strategy of the advisor, $m(v,p,\mu)$, with $\hat{p}(v,\mu) = \bar{\tau}(v)$ and the following belief updating rule of the bidder. For any belief μ, price p, and message $\tilde{m}$ such that there is $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}$, belief μ is updated via the Bayes rule. Any other message $\tilde{m}$ (i.e., a message for which there is no $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}$) is treated as some message $\tilde{m}'$ for which there is some $v \in supp(\mu(h))$ with $m(v,p,\mu) = \tilde{m}'$, and belief μ is updated following message $\tilde{m}$ in the same way as following message $\tilde{m}'$.[18] Given this, the posterior belief of the bidder for any history h is as follows. A sequence of messages $m = 0$ for all prices $p' \leq p$ up to price p implies that the bidder's posterior belief is given by the prior distribution of valuations truncated from below at $v_l(p)$. A sequence of messages $m = 0$ for all prices $p' < p'' \in \varrho$ and message $m = 1$ at price $p'' \in \varrho$ and any history of messages after that results in the bidder's posterior belief given by the prior distribution of valuations truncated at $v_l(p'')$ from below and at $v_h(p'')$ from above. Any history involving off-equilibrium messages leads to the posterior belief equivalent to one of these two posterior beliefs by construction of the updating rule. Given this, consider an online threshold strategy of the bidder, $a(p,\tilde{\mu})$, with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \geq v_l(p)]$ for the posterior belief $\tilde{\mu}$ in the history of the first kind (i.e., when the advisor never recommended quitting at one of prices $p \in \varrho$ in the past), and with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \in [v_l(p''), v_h(p'')]]$ for the posterior belief $\tilde{\mu}$ in the history of the second type (i.e., when the advisor recommended to quit the auction at price $p'' \in \varrho$). Let E' denote a combination of these online threshold strategies of the advisor and the bidder and the belief updating rule. Below we show that E' is indeed an equilibrium and that it results in the same equilibrium exit price $\bar{\tau}(v)$ as equilibrium E.

For the collection of strategies and beliefs E' to be an equilibrium, we need to verify the incentive compatibility (IC) conditions of the advisor and the bidder.

[18] Intuitively, according to this updating rule, the bidder effectively ignores unexpected messages. As a consequence, it is sufficient to consider only deviations to on-path (expected) messages. Because no deviation to an off-path message can be beneficial, we do not lose any equilibria by focusing on this belief updating rule.

1 - IC of the advisor. First, we verify that the advisor is not better off deviating from (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$. Because of the above definition of the off-path beliefs, it is sufficient to consider only deviations to $m \in \{0,1\}$ at $p \in \varrho$. First, consider a deviation of type v to $m = 1$ at $p \in \varrho$ at which $p < \bar{\tau}(v)$. This deviation is equivalent to mimicking the communication strategy of type $v' : \bar{\tau}(v') = p$. Since mimicking the communication strategy of type v' is not profitable for type v in equilibrium E (otherwise, it would not be an equilibrium), it is also not profitable here. Second, consider a deviation of type v to $m = 0$ at $p = \bar{\tau}(v)$. Depending on her communication strategy at later prices, this deviation will result in exit at price $\bar{\tau}(v')$ for some $v' \geq v_h(\bar{\tau}(v))$. Hence, any such deviation is equivalent to mimicking the communication strategy of type v'. Since it is not profitable for type v in equilibrium E, it is also not profitable here.

2 - IC of the bidder after observing $m = 1$ at $p \in \varrho$ and $m = 0$ before. We argue that $\bar{p}(\tilde{\mu}) \leq p$ in this case, so the bidder's best response is to quit the auction immediately. Given this history, the bidder's posterior belief is that $v \in [v_l(p), v_h(p)]$. Because the bidder expects the advisor to follow (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$, she expects the advisor to send $m = 1$ at any later price. Since the bidder expects to not learn anything new about v, her optimal exit strategy is given by the expected valuation, i.e., $\mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. It follows that the bidder exits immediately if $p \geq \mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. Next, we show that $\bar{\tau}(p)$ in equilibrium E must satisfy this condition at any $p \in \varrho$. Since exiting at price p is optimal for the bidder for any realization $v \in [v_l(p), v_h(p)]$ of the valuation, it must be that $p \geq \mathbb{E}\left[v|\mathcal{H}_p^E\right]$ for any history $\mathcal{H}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_l(p), v_h(p)]$. It follows that $p \geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right]$, where $\mathbb{H}_p^E$ denotes the set of such histories. Using the law of iterated expectations and fact that the maximum of a random variable cannot be below its mean,

$$
\begin{aligned}
p &\geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right] \geq \mathbb{E}\left[\mathbb{E}\left[v|\mathcal{H}_p^E\right]|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] \\
&= \mathbb{E}\left[v|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] = \mathbb{E}\left[v|v \in [v_l(p), v_h(p)]\right].
\end{aligned}
$$

Therefore, when the bidder observes message $m = 1$ at $p \in \varrho$ for the first time, she finds it optimal to quit the auction immediately.

3 - IC of the bidder after observing a sequence of messages $m = 0$ up to price $p < \bar{\tau}(\bar{v})$. We argue that $\bar{p}(\tilde{\mu}) > p$ for any such history, i.e., it is optimal for the bidder to wait. Given this history, the bidder's posterior is that $v \in [v_h(p'), \bar{v}]$ for highest $p' \in \varrho$ satisfying $p' < p$. Consider equilibrium E and any history $\tilde{\mathcal{H}}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_h(p'), \bar{v}]$. Denote the set of such histories by $\tilde{\mathbb{H}}_p^E$. Since the bidder finds it optimal to wait, the payoff from waiting is weakly above the payoff from quitting the auction immediately (i.e., zero) for any such history $\tilde{\mathcal{H}}_p^E$. In strategy profile E' the bidder learns no less between price p and the exit price than in strategy profile E. Hence, the fact that waiting is optimal for any history $\tilde{\mathcal{H}}_p^E \in \tilde{\mathbb{H}}_p^E$ implies that waiting is also optimal when the bidder expects the advisor to follow (24) with $\hat{p}(v,\mu) = \bar{\tau}(v)$.

Therefore, the collection of strategies and beliefs E' is an equilibrium. Furthermore, on equilibrium path, the advisor with type v recommends to quit the auction when the price reaches $\bar{\tau}(v)$, and the bidder exits the auction immediately. Therefore, E' results in the same bidding behavior as E.

The second statement of the lemma can be proved by contradiction. Consider equilibrium E that satisfies NITS, and suppose that an equilibrium in online threshold strategies with the same bidding behavior violates NITS. Hence, there exists price p such that the advisor with type $v_l(p)$ is better off credibly revealing itself at price p than getting the expected (as of information at price p) payoff in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself then exceeds the time-0 expected payoff of the advisor of type $v_l(p)$ in equilibrium E'. Now, consider equilibrium E, and the strategy of the advisor of type $v_l(p)$ to send equilibrium message $\bar{m}(v,p',\mu)$ for all $p' < p$ and to credibly reveal itself at price p (by definition of $v_l(p)$, type $v_l(p) = \inf\{v|v \in supp(\mu(h))\}$ for any history induced by this message strategy up to price p). In equilibrium E, bidding behavior of other bidders is the same and the bidder's reaction to the advisor credibly revealing itself at price p is the same as in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from this strategy is the same as the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself at price p in equilibrium E', which is strictly higher than the time-0 equilibrium expected payoff of the advisor of type $v_l(p)$. Hence, equilibrium E also violates the NITS condition, which is a contradiction. $\square$

B.3 Selection Result

Lemma 4 implies that any equilibrium of the second-price auction has an equivalent equilibrium in online threshold strategies in the English auction. Indeed, suppose that there is a single round of informative communication in the English auction, followed by babbling at any other round, where bidders believe that any message has no information content. Since there is no informative communication in later rounds, bidding in the English auction is equivalent to bidding in the second-price auction for the same distribution of bidder types induced by communication at the initial round. Thus, any equilibrium from Proposition 1 has an equivalent equilibrium in the English auction. By Lemma 4, the English auction has also an equivalent equilibrium in online threshold strategies.[19] Thus, the set of equilibria in the English auction is potentially very large. Nevertheless, it turns out that all equilibria

[19]Specifically, consider any equilibrium in the second-price auction with partition $\{\omega_k\}_{k=0}^{K}$. An equivalent equilibrium in online threshold strategies in the English auction has the following form. Type $v \in (\omega_{k-1}, \omega_k)$ plays the strategy of sending message $m = 0$, if $p < \mathbb{E}[v|v \in (\omega_{k-1}, \omega_k)]$, and message $m = 1$, otherwise. Let $\mathcal{P} \equiv \{p : \mathbb{E}[v|v \in (\omega_{k-1}, \omega_k)] = p \text{ for some } p\}$ denote the set of informative prices – prices at which some type switches from $m = 0$ to $m = 1$ with some probability. Let the bidder's belief updating rule for any $p \notin \mathcal{P}$ be such that any message is interpreted the same way (i.e., belief is unchanged for any message sent at such price). Finally, the bidder's strategy is to exit at the first informative price $p \in \mathcal{P}$, at which she gets message $m = 1$.

of the English auction satisfying NITS take a simple form of capped delegation strategies:

Proposition 7. *Any equilibrium in the English auction that satisfies the NITS condition is in capped delegation strategies with cutoff v^* satisfying:*

$$\begin{aligned} &\text{if } v^* \in (\underline{v}, \overline{v}), \text{ then } b = \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \underline{v}, \text{ then } b \geq \mathbb{E}[v|v \geq v^*] - v^*; \\ &\text{if } v^* = \overline{v}, \text{ then } \overline{v} = \infty \text{ and } b \leq \lim_{s\to\infty} \mathbb{E}[v|v \geq s] - s. \end{aligned} \tag{25}$$

Proof of Proposition 7. By Lemma 4, it is without loss of generality to focus on equilibria in online threshold strategies. In the proof of Lemma 4, we introduced function $\bar{\tau}(v)$, which denotes the equilibrium exit price of the bidder if the advisor's type is v. In an equilibrium in online threshold strategies, $\bar{\tau}(v)$ is also the first price at which the advisor with type v sends message "quit" to the bidder.

Any equilibrium generates partition Π of $[\underline{v}, \overline{v}]$ satisfying $\bar{\tau}(v) = \bar{\tau}(v')$ for any $v, v' \in \pi$ for any element $\pi \in \Pi$. As shown in Lemma 4, $\bar{\tau}(v)$ is weakly increasing, so any $\pi \in \Pi$ is an interval (possibly consisting of one element). We say that types in $\pi \in \Pi$ *pool* if $\bar{\tau}(v)$ is constant on $v \in \pi$, i.e., these types start sending message "quit" at the same price. We say that types in $[v', v'']$ *separate*, if $\bar{\tau}(v)$ is strictly increasing on $[v', v'']$, i.e., these types start sending message "quit" at different prices. Let Π^P and $\Pi^S \equiv [\underline{v}, \overline{v}]\backslash\Pi^P$ be the sets of all types that pool with some other type and that separate, respectively. Denote by $\partial\Pi^P$ the boundary of Π^P.

Babbling ($\bar{\tau}(v) = \mathbb{E}[v]\ \forall v$) is an equilibrium of the English auction, and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. This proves case $v^* = \underline{v}$ of the proposition. Hence, we can consider the case in which there is a non-trivial information transmission in equilibrium.

Claim 4. *For any $\pi, \pi' \in \Pi^P$, π and π' are not adjacent.*

Proof: By contradiction, suppose that there are two adjacent intervals of types, π and π', such that $\bar{\tau}(v) = p\ \forall v \in \pi$ and $\bar{\tau}(v) = p'\ \forall v \in \pi'$. Without loss of generality, $p' > p$. Consider the advisor with type $\tilde{v}$ on the boundary of π and π'. By continuity, the advisor with type $\tilde{v}$ is indifferent between his bidder quitting the auction at prices p and p'. The benefit of the latter is winning against types in π, while the cost is risking to win against types in π' and paying p'. The indifference of type $\tilde{v}$ implies that $p' > \tilde{v} + b$. Consider running price $\frac{p'+p}{2}$. Type $\tilde{v}$ is the weakest remaining type at this price. Since $p' > \tilde{v} + b$, following his equilibrium strategy of waiting to send recommendation $m = 1$ until price p' generates negative expected payoff to the advisor at this point. In contrast, claiming that he is the weakest remaining type at the current price of $\frac{p'+p}{2}$ will lead to the bidder quitting immediately, yielding the payoff of zero to the advisor. This contradicts the NITS condition. *q.e.d.*

We next show that whenever types within an interval separate, they start recommending to quit the auction at their most preferred time.

Claim 5. *If $\bar{\tau}(v)$ is strictly increasing on (v', v''), then $\bar{\tau}(v) = v + b$ for any $v \in (v', v'')$.*

Proof: By contradiction, suppose there is $v \in (v', v'')$ with $\bar{\tau}(v) \neq v + b$. Then, either $\bar{\tau}(v) > v + b$ or $\bar{\tau}(v) < v + b$. First, consider the former case. Since v is interior, there exists a subset of (v', v'') of types $v + \varepsilon > v$ with positive measure with $\bar{\tau}(v) > v + \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v + \varepsilon) > \bar{\tau}(v) > v + \varepsilon + b$. Therefore, any such type $v + \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v + \varepsilon)$: by doing this, the advisor ensures that the bidder does not win when the valuation of the strongest rival is in $(v, v + \varepsilon)$, in which case the bidder overpays relative to the advisor's maximum willingness to pay of $v + \varepsilon + b$. Hence, it cannot be that $\bar{\tau}(v) > v + b$. Second, consider the case $\bar{\tau}(v) < v + b$. Now, there exists a subset of (v', v'') of types $v - \varepsilon < v$ with positive measure with $\bar{\tau}(v) < v - \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v - \varepsilon) < \bar{\tau}(v) < v - \varepsilon + b$. Therefore, any such type $v - \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v - \varepsilon)$: by doing this, the advisor ensures that the bidder wins when the valuation of the strongest rival is in $(v - \varepsilon, v)$, in which case the advisor gets a positive payoff, since the bidder pays below the advisor's maximum willingness to pay. Therefore, it cannot be that $\bar{\tau}(v) < v + b$. We conclude that $\bar{\tau}(v) = v + b$. *q.e.d.*

Claim 6. *If* $\Pi^P \neq \emptyset$ *and* $\Pi^S \neq \emptyset$, *then* Π^P *contains a single interval* $\pi^P = [v^*, \overline{v}]$, *where* $v^* > \underline{v}$.

Proof: By contradiction, suppose that Π^P contains more than one interval or that it contains one interval that is to the left of Π^S. In the former case, Claim 4 implies that the intervals are not adjacent. Therefore, there is an interval $\pi \in \Pi^P$ that lies to the left of an interval in Π^S. Let $v \in \pi$ be the highest type in this interval. Since it must be indifferent between separation and pooling and $\bar{\tau}(v) = v + b$ in the separation region by Claim 2, we have $v + b = \bar{\tau}(w)$ for any $w \in \pi$. Therefore, $\bar{\tau}(w) > w + b$ for any $w \in \pi$, $w \neq v$. In particular, it holds for the lowest type in the interval $w' = \min_{w \in \pi} w$. However, this violates the NITS condition. Indeed, consider running price $p = \frac{\bar{\tau}(w) + w' + b}{2}$. Type w' is the weakest remaining type of the advisor at this price. Since $p > w' + b$, following his equilibrium strategy of waiting until price $\bar{\tau}(w)$, $w \in \pi$ to send recommendation $m = 1$ generates negative expected payoff to the advisor at the current point. In contrast, claiming that he is the weakest remaining type at the current price will lead to the bidder quitting immediately, yielding the payoff of zero. Therefore, Π^P contains only one interval that lies to the right of Π^S, i.e., the interval is of the form $\pi^P = [v^*, \overline{v}]$ for some $v^* > \underline{v}$. *q.e.d.*

Claim 7. *Cut-off* v^* *satisfies* (25).

Proof: Case $v^* = \underline{v}$ (the babbling equilibrium) was covered above (before Claim 4). Consider case $v^* = \overline{v}$, i.e., $\Pi^P = \emptyset$. By Claim 5, $\bar{\tau}(v) = v + b$ for any $v \in (\underline{v}, \bar{v})$. If $\bar{v} < \infty$, the upper bound on the bidder's utility in round $p = v + b$ is $\overline{v} - (v + b) \overset{v \to \overline{v}}{\to} -b$, which contradicts the optimality of the bidder to follow the advisor's recommendation. Hence, it must be that $\bar{v} = \infty$. Next, by contradiction, suppose that $b > \lim_{s \to \infty} \mathbb{E}[v | v \geq s] - s$. By continuity, there is $\overline{s} < \infty$ such that $b > \mathbb{E}[v | v \geq s] - s$ for any $s > \overline{s}$. If the bidder wins in any round $p \geq s + b$, then her expected utility equals $\mathbb{E}[v | v \geq s] - s - b < 0$ and so, the value of following the advisor's recommendations is negative, which is a contradiction.

Therefore, it must be that $b \leq \lim_{s \to \infty} \mathbb{E}[v|v \geq s] - s$.

Finally, consider case $v^* \in (\underline{v}, \bar{v})$. By contradiction, suppose that $v^* + b \neq \mathbb{E}[v|v \geq v^*]$. By indifference of type v^*, it must be that $\bar{\tau}(v) = v^* + b$ for any $v \in [v^*, \bar{v}]$. If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v)$, $v > v^*$ violates the incentive compatibility condition of the bidder. To see this, consider running price just below $v^* + b$. The equilibrium behavior prescribes the bidder to exit the auction in the next instant, which is below her maximum willingness to pay of $\mathbb{E}[v|v \geq v^*]$. By waiting a little beyond price $\bar{\tau}(v) = v^* + b$, the bidder ensures that she wins the auction with probability one and pays below her estimated valuation of $\mathbb{E}[v|v \geq v^*]$. Since this strategy results in a discontinuous upward jump in the expected utility of the bidder, she is better off deviating. Hence, it cannot be that $v^* + b < \mathbb{E}[v|v \geq v^*]$. If $v^* + b > \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v) = v^* + b$, $v \geq v^*$ violates the incentive compatibility condition of the bidder, because she would prefer to exit the auction slightly earlier. Consider the running price $p = v^* + b - \varepsilon$ for an infinitesimal positive ε and suppose that the bidder has got a sequence of recommendations $\tilde{m} = 0$. Her posterior belief is that the valuation is in the range $(v^* - \varepsilon, \overline{v}]$. Suppose that the bidder follows her equilibrium play. If $v \in (v^* - \varepsilon, v^*)$ and the bidder wins, she pays $v + b$ above her valuation v. If $v \in (v^*, \bar{v}]$ and the bidder wins, she pays $\bar{\tau}(v) = v^* + b$, which is, on average, above her valuation v ($\mathbb{E}[v|v \geq v^*]$). Since the bidder wins with positive probability, her expected payoff from following the equilibrium play is negative. In contrast, immediate exit yields zero expected payoff. Hence, the bidder is better off deviating and exiting the auction immediately. Therefore, $v^* + b = \mathbb{E}[v|v \geq v^*]$. *q.e.d.*

The proof of Proposition 7 follows from Claims 5, 6, and 7.

$\square$