Online Appendix for "Optimal Dynamic Allocation of Attention"*

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B.1 Omitted Proofs from Appendix A.

B.1.1 Proof of Proposition 7

Proof. Since the DM can stop immediately, we have $V^*(p) \geq U(p)$. For the second inequality, consider the problem of a decision maker who can choose $\alpha_t \in [0,1]$ and $\beta_t \in [0,1]$ without the constraint that $\alpha_t + \beta_t = 1$. Clearly the value of this problem exceeds $V^*(p)$ for all p. The value function of the unconstrained problem is $\max\left\{U(p), U^{FA}(p)\right\}$. To see this, note that it is optimal to choose $\alpha_t = \beta_t = 1$. Given this policy, the belief does not change over time if no breakthrough occurs. The optimal policy is therefore either to stop immediately or to wait without deadline until a breakthrough occurs. Hence the value of the unconstrained problem is $\max\left\{U(p), U^{FA}(p)\right\}$. Therefore $V^*(p) = U(p) = \max\left\{U(p), U^{FA}(p)\right\}$ if (EXP) is violated. □

B.1.2 Proof of Lemma 1

Proof. The result follows from straightforward algebra. □

B.1.3 Proof of Lemma 2

Proof. Suppose $V_0(p) = V_1(p) = V(p)$ for some $p \in (0,1)$. Solving (A7) and (A8) for $V_0'(p)$ and $V_1'(p)$ and some algebra yields

$$V_0'(p) - V_1'(p) = \frac{\lambda + 2\rho}{\lambda p(1-p)}\left(V(p) - U^S(p)\right).$$

Therefore $\operatorname{sgn}(V_0'(p) - V_1'(p)) = \operatorname{sgn}\left(V(p) - U^S(p)\right)$. □

B.1.4 Proof of Lemma 3

Proof. Consider first the case that $V_0(p)$ satisfies (A7). With $V = V_0(p)$, and substituting $V' = V_0'(p)$ from (A7), we have

$$\frac{\partial F_\alpha(p, V_0(p), V_0'(p))}{\partial \alpha} = \frac{2\rho + \lambda}{\lambda}\left(U^S(p) - V_0(p)\right).$$

This implies that $\alpha = 0$ is a maximizer if $V_0(p) \geq U^S(p)$, and the unique maximizer if the inequality is strict. This proves Part (a). The proof of Part (b) follows from a similar argument. □

B.1.5 Proof of Proposition 8

The following three lemmas establish properties of the function U^S, V^{FA}, V_{own} and V_{opp} that are used in the proof of Proposition 8. Some of these properties were already established in Appendix A and are repeated here for convenience.

Lemma 4 (Properties of $U^S(p)$ and $U^{FA}(p)$)**.** *(a) $U^S(p) < U^{FA}(p)$ for all $p \in [0,1]$.*

(b) $U^S(p)$ and $U^{FA}(p)$ are linear in p.

If $U^S(p) \geq U(p)$ for some $p \in [0,1]$, then $U'_\ell(p) < U^{S\prime}(p) < U'_r(p)$ for all $p \in [0,1]$.

If $U^{FA}(p) \geq U(p)$ for some $p \in [0,1]$, then $U'_\ell(p) < U^{FA\prime}(p) < U'_r(p)$ for all $p \in [0,1]$.

(c) $U^S(p), U^{FA}(p) < U(p)$ at $p \in \{0,1\}$.

For all $p \in [0,1]$, $U^S(p)$ and $U^{FA}(p)$ are strictly decreasing without bound in c.

Proof. (a) $U^S(p) < U^{FA}(p)$ is immediate from the expressions in (A2) and (A3).

(b) Linearity is obvious. Suppose $U^S(p) \geq U(p)$ for some $p \in [0,1]$. To show $U'_\ell(p) < U^{S\prime}(p)$ for all p, suppose by contradiction that $U^{S\prime}(p) \leq U'_\ell(p)$ for some p. Note that $U^S(0) = \frac{u_\ell^L \lambda - 2c}{\lambda + 2\rho} < u_\ell^L = U_\ell(0)$. Hence, $U^{S\prime}(p) \leq U'_\ell(p)$ and the linearity of these functions imply $U^S(p) < U_\ell(p) \leq U(p)$ for all p, which is a contradiction. The other inequalities are proven similarly.

Part (c) is obtained from straightforward algebra. □

The following lemma summarizes the properties of the own-biased strategy:

Lemma 5. *(a) $\underline{V}_{own}(p)$ and $\overline{V}_{own}(p)$ are continuously differentiable and convex on $(0,1)$;*

(b) $\underline{V}_{own}(p)$ is strictly convex and $\underline{V}_{own}(p) > U_\ell(p)$ on $(\underline{p}^*, 1]$, and $\overline{V}_{own}(p)$ is strictly convex and $\overline{V}_{own}(p) > U_r(p)$ on $[0, \overline{p}^*)$. $V_{own}(p) > U(p)$ for $p \in (\underline{p}^*, \overline{p}^*)$.

(c) If $\underline{p}^*, \overline{p}^* \in (0,1)$, they satisfy

$$U_\ell(\underline{p}^*) = U^{FA}(\underline{p}^*), \quad \textit{and} \quad U_r(\overline{p}^*) = U^{FA}(\overline{p}^*). \tag{B.1}$$

(d) Suppose (EXP) holds. Then, $0 < \underline{p}^* < \overline{p}^* < 1$, $\underline{V}_{own}(p) < U^{FA}(p)$ for $p \in (\underline{p}^*, 1)$, $\overline{V}_{own}(p) < U^{FA}(p)$ for $p \in (0, \overline{p}^*)$, and $V_{own}(p) = U(p) > U^{FA}(p)$ for $p \notin [\underline{p}^*, \overline{p}^*]$.

(e) If (EXP) is violated, then $V_{own}(p) = U(p)$ for all $p \in [0,1]$.

Proof. Parts (a)-(c) follow from straightforward algebra. For part (d), note that (EXP) together with part (c) and Lemma 4.(b) imply $0 < \underline{p}^* < \hat{p} < \overline{p}^* < 1$ and $U^{FA}(p) < U(p)$ for $p \notin [\underline{p}^*, \overline{p}^*]$. This implies $V_{own}(p) = U(p) > U^{FA}(p)$ for $p \notin [\underline{p}^*, \overline{p}^*]$. To show that $\underline{V}_{own}(p) < U^{FA}(p)$ for $p \in (\underline{p}^*, 1)$, note that $\underline{V}_{own}(\underline{p}^*) = U_\ell(\underline{p}^*) = U^{FA}(\underline{p}^*)$ from part (c), and $\underline{V}_{own}(1) = U^{FA}(1)$ from (A11). Since $U^{FA}(p)$ is linear by Lemma 4.(b) and $\underline{V}_{own}(p)$ is strictly convex $(\underline{p}^*, 1]$ by part (b), this implies implies that $\underline{V}_{own}(p) < U^{FA}(p)$ for $p \in (\underline{p}^*, 1)$. $\overline{V}_{own}(p) < U^{FA}(p)$ for $p \in (0, \overline{p}^*)$ is proven similarly.

Part (e) holds because by part (c), $\underline{p}^* > \overline{p}^*$ if (EXP) is violated. □

We next observe several properties of $V_{opp}(p)$.

Lemma 6. *(a) $V_{opp}(p)$ is continuously differentiable and strictly convex on $(0,1)$, and $V_{opp}(p) \geq U^S(p)$ for all $p \in [0,1]$ with strict inequality for $p \neq p^*$.*

(b) Then, $V_{opp}(p) \leq U^{FA}(p)$ for all $p \in [0,1]$, with equality if and only if $p \in \{0,1\}$.

Proof. Part (a) follows from straightforward algebra. For part (b), again by straightforward algebra we get $U^{FA}(0) = \underline{V}_{opp}(0) = V_{opp}(0)$ and $U^{FA}(1) = \overline{V}_{opp}(0) = V_{opp}(1)$. Since $U^{FA}(p)$ is linear and V_{opp} is strictly convex, this implies $V_{opp}(p) < U^{FA}(p)$ for all $p \in (0,1)$. □

We are now ready to prove Proposition 8. For the reader's convenience, we restate the proposition.

Proposition (Structure of V_{Env}). (a) If (EXP) holds and $V_{own}(p^*) \geq V_{opp}(p^*)$, then there exists a unique $\check{p} \in \left(\underline{p}^*, \overline{p}^*\right)$ such that $\underline{V}_{own}(\check{p}) = \overline{V}_{own}(\check{p})$ and

$$V_{Env}(p) = V_{own}(p) = \begin{cases} \underline{V}_{own}(p), & \text{if } p < \check{p}, \\ \overline{V}_{own}(p), & \text{if } p \geq \check{p}. \end{cases}$$

(b) If (EXP) holds and $V_{own}(p^*) < V_{opp}(p^*)$, then $p^* \in (\underline{p}^*, \overline{p}^*)$, and there exist a unique $\underline{p} \in (\underline{p}^*, p^*)$ such that $V_{own}(\underline{p}) = V_{opp}(\underline{p})$, and a unique $\overline{p} \in (p^*, \overline{p}^*)$ such that $V_{own}(\overline{p}) = V_{opp}(\overline{p})$ and

$$V_{Env}(p) = \begin{cases} \underline{V}_{own}(p), & \text{if } p < \overline{p}, \\ V_{opp}(p), & \text{if } p \in [\underline{p}, \overline{p}], \\ \overline{V}_{own}(p), & \text{if } p > \overline{p}. \end{cases}$$

Proof. **Part (a):** We first prove that $V_{own}(p) \geq V_{opp}(p)$ for all $p \in [0,1]$. Since $V_{own}(p) \geq U^{FA}(p) > V_{opp}(p)$ for $p \notin [\underline{p}^*, \overline{p}^*]$, it suffices to show $V_{own}(p) \geq V_{opp}(p)$ for $p \in [\underline{p}^*, \overline{p}^*]$. To this end, suppose first $p^* > \underline{p}^*$ and consider $p \in [\underline{p}^*, p^*]$ so that $V_{opp}(p) = \underline{V}_{opp}(p)$. Recall from Lemmas 5 and 6 that $\underline{V}_{own}(\underline{p}^*) = U^{FA}(\underline{p}^*) > V_{opp}(\underline{p}^*)$. Since $V_{opp}(\cdot) \geq U^S(\cdot)$, by the Crossing Lemma 2, $\underline{V}_{own}$ can cross $V_{opp} = \underline{V}_{opp}(p)$ only from above on $[\underline{p}^*, p^*)$. If $\underline{V}_{own}(p^*) \geq \underline{V}_{opp}(p^*)$, by the Crossing Lemma 2, $\underline{V}_{opp}(p) < \underline{V}_{own}(p) \leq V_{own}(p)$ for all $p \in [\underline{p}^*, p^*]$. If $\underline{V}_{own}(p^*) < \underline{V}_{opp}(p^*)$, then $\overline{V}_{own}(p^*) = V_{own}(p^*) \geq \underline{V}_{opp}(p^*)$. Since both $\overline{V}_{own}(p)$ and $\underline{V}_{opp}(p^*)$ satisfy (A7), we must have $\underline{V}_{opp}(p) \leq \overline{V}_{own}(p) \leq V_{own}(p)$ for all $p \in [\underline{p}^*, p^*]$. Either way, we have proven that $V_{opp}(p) = \underline{V}_{opp}(p) \leq V_{own}(p)$ for all $p \in [\underline{p}^*, p^*]$. A symmetric argument proves that $V_{opp}(p) \leq V_{own}(p)$ for all $p \in [p^*, \overline{p}^*]$ in case $p^* < \overline{p}^*$.

So far, we have shown that $V_{own}(p) \geq V_{opp}(p)$ for all $p \in [0,1]$. Recall from Lemma 5 that $\underline{V}_{own}(\underline{p}^*) = U^{FA}(\underline{p}^*) > \overline{V}_{own}(\underline{p}^*)$ and $\overline{V}_{own}(\overline{p}^*) = U^{FA}(\overline{p}^*) > \underline{V}_{own}(\overline{p}^*)$. By the intermediate value theorem, there exists $\check{p} \in \left(\underline{p}^*, \overline{p}^*\right)$ where $\underline{V}_{own}(\check{p}) = \overline{V}_{own}(\check{p})$. For any p we have $V_{own}(p) \geq V_{opp}(p)$ and $V_{opp}(p) \geq U^S(p)$ and hence $V_{own}(\check{p}) \geq U^S(\check{p})$. The Crossing Lemma 2 then implies that $\underline{V}_{own}$ cannot cross $\overline{V}_{own}$ from below at $\check{p}$.[1] This means that the intersection point $\check{p}$ is unique and the structure stated in part (a) obtains.

Part (b): We first prove that $p^* \in \left(\underline{p}^*, \overline{p}^*\right)$. By Lemma 5, $V_{own}(p) \geq U(p)$ for all $p \in [0, 1]$. This implies $V_{opp}(p^*) > U(p^*)$, and since $V_{opp}(p^*) = U^S(p^*) < U^{FA}(p^*)$, and since by Lemma 5.(d) $U^{FA}(p) \leq U(p)$ for $p \notin \left(\underline{p}^*, \overline{p}^*\right)$, we must have $p^* \in \left(\underline{p}^*, \overline{p}^*\right)$. Next, by Lemma 6.(b), $V_{opp}(\underline{p}^*) < U^{FA}(\underline{p}^*) = \underline{V}_{own}(\underline{p}^*)$. Therefore, $V_{opp}(p)$ and $\underline{V}_{own}(p)$ intersect at some $\underline{p} \in \left(\underline{p}^*, p^*\right)$ and by the Crossing Lemma 2, the intersection is unique since $V_{opp}(\underline{p}) > U^S(\underline{p})$ for $\underline{p} \in \left(\underline{p}^*, p^*\right)$ by Lemma 6.(a). Moreover, for $p < p^*$, we have $V_{opp}(p) > \overline{V}_{own}(p)$ since both satisfy (A7), and hence $\overline{V}_{own}(p) < \underline{V}_{own}(p)$ for all $p \in \left(\underline{p}^*, \underline{p}\right)$. This proves the result for $p \leq p^*$. For $p > p^*$ the arguments are symmetric. □

B.1.6 Proof of Proposition 9

Proof. If (EXP) is violated, $V_{Env}(p) = U(p)$ since $\underline{p}^* > \overline{p}^*$ by Proposition 7. Moreover Proposition 7 shows that $V^*(p) = U(p) = V_{Env}(p)$ in this case. Similarly, if (EXP) is satisfied, by Lemma 1 and Proposition 8, $V_{Env}(p) = U(p)$ for all $p \notin (\underline{p}^*, \overline{p}^*)$ and Proposition 7 shows that $V^*(p) = U(p) = V_{Env}(p)$ for $p \notin (\underline{p}^*, \overline{p}^*)$.

It remains to verify $V^*(p) = V_{Env}(p)$ for $p \in (\underline{p}^*, \overline{p}^*)$ when EXP is satisfied. In the remainder of this proof we write $V(p) = V_{Env}(p)$. Theorem III.4.11 in Bardi and Capuzzo-Dolcetta (1997) characterizes the value function of a dynamic programming problem with an optimal stopping decision as in (A1) as the (unique) viscosity solution of the HJB equation.[2] For all $p \in (0, 1)$ where $V(p)$ is differentiable, this requires that $V(p)$ satisfy (A4).

Consider points of differentiability $p \in \left(\underline{p}^*, \overline{p}^*\right)$. From (A11) and (A12), we obtain that $\underline{V}_{own}$ and $\overline{V}_{own}$ are strictly convex on $(\underline{p}^*, \overline{p}^*)$. Smooth pasting at $\underline{p}^*$ and $\overline{p}^*$, respectively, implies that $\underline{V}_{own}(p) > U_\ell(p)$ and $\overline{V}_{own}(p) > U_r(p)$, and therefore $V(p) \geq V_{own}(p) > U(p)$ for $p \in (\underline{p}^*, \overline{p}^*)$. This implies that (A4) is equivalent to (A6) for all $p \in \left(\underline{p}^*, \overline{p}^*\right)$. Since $V(p)$ satisfies (A7) or (A8) at points of differentiability, and $V(p) \geq V_{opp}(p) \geq U^S(p)$, the Unimprovability Lemma 3 implies that $V(p)$ satisfies (A6). Since V_{opp} is strictly convex by Lemma 6.(a), $V_{opp}(p) > U^S(p)$, and hence Lemma 3 implies that the optimal policy is unique at all points where $V(p)$ is differentiable except p^*. At p^*, the HJB equation is satisfied for any $\alpha \in [0, 1]$ but $\alpha = 1/2$ is the only maximizer that defines an admissible policy.

[1] $\underline{V}_{own}$ and $\overline{V}_{own}$ could be equal to U^S at $\check{p}$ which means that two branches are tangent. However, the convexity of both branches and the fact that $V_{own}(p) \geq U^S(p)$ for all p, means that $\underline{V}_{own}$ cannot cross $\overline{V}_{own}$ from below at any point of intersection. Therefore $\check{p}$ is unique.

[2] To formally apply their theorem, we have to use P_t as a second state-variable and define a value function $v(p, P) = PV(p)$. Since v is continuously differentiable in P, it is straightforward to apply the result directly to $V(p)$.

We have shown that $V(p)$ satisfies (A4) for all points of differentiability. For $V(p)$ to be a viscosity solution it remains to show that for all points of non-differentiability,

$$\max\{-c-\rho V(p)+F(p,V(p),z),U(p)-V(p)\}\leq 0, \tag{B.2}$$

for all $z\in\left[V'_-(p),V'_+(p)\right]$; and the opposite inequality holds for all $z\in\left[V'_+(p),V'_-(p)\right]$, where $V'_-(p)$ denotes the left derivative at p, and $V'_+(p)$ denotes the right derivative at p. By Proposition 8, non-differentiability occurs at $\check{p}$ if (EXP) holds and $V_{own}(p^*)\geq V_{opp}(p^*)$; and at $\underline{p}$ and $\overline{p}$ if (EXP) holds and $V_{own}(p^*)<V_{opp}(p^*)$. Since $V(p)\geq U^S(p)$, the Crossing Lemma 2 implies that $V(p)$ has convex kinks at all these points so that $V'_-(p)\leq V'_+(p)$. Therefore it suffices to check (B.2) for all $z\in\left[V'_-(p),V'_+(p)\right]$. F_α is linear in α (see (A5)), so it suffices to consider $\alpha\in\{0,1\}$. For $\alpha=1$ we have $F_1(p,V(p),z)\leq F_1(p,V(p),V'_-(p))$ and for $\alpha=0$ we have $F_0(p,V(p),z)\leq F_0(p,V(p),V'_+(p))$. Therefore if $U(p)\leq V(p)$, which holds for our candidate solution by construction, then

$$c+\rho V(p)\geq\max\left\{F_1(p,V(p),V'_-(p)),F_0(p,V(p),V'_+(p))\right\} \tag{B.3}$$

implies that (B.2) holds for all for $z\in\left[V'_-(p),V'_+(p)\right]$. We distinguish two cases.

Case A: (EXP) *is satisfied and* $V_{own}(p^*)\geq V_{opp}(p^*)$. Consider $p=\check{p}$. (B.3) becomes

$$c+\rho\underline{V}_{own}(\check{p})=c+\rho\overline{V}_{own}(\check{p})\geq\max\left\{F_1(\check{p},\underline{V}_{own}(\check{p}),\underline{V}'_{own}(\check{p})),F_0(\check{p},\overline{V}_{own}(\check{p}),\overline{V}'_{own}(\check{p}))\right\}.$$

By the Unimprovability Lemma 3, this holds with equality since $\underline{V}_{own}(p)$ satisfies (A8) and $\overline{V}_{own}(p)$ satisfies (A7) at $\check{p}$. As we have argued earlier, $V_{own}(p)>U(p)$ for all $p\in\left(\underline{p}^*,\overline{p}^*\right)$ and hence $V(\check{p})>U(\check{p})$. (B.2) is thus satisfied at $\check{p}$.

Case B: (EXP) *is satisfied and* $V_{own}(p^*)<V_{opp}(p^*)$. The proof is similar to Case B.

We have thus shown that $V(p)$ is a viscosity solution of (A4) which is sufficient for $V(p)$ to be the value function of problem (A1). ☐

B.2 Proof of Proposition 1

Proof. We begin by deriving formulas for the waiting time. Suppose the DM has belief p and seeks L-evidence ($\alpha=0$) until her belief reaches $q>p$, then from (1) we obtain that the time it takes to reach q in the absence of an L-signal is

$$T(q,p;\alpha=0):=\frac{1}{\lambda}\log\left(\frac{q}{1-q}\frac{1-p}{p}\right).$$

This implies that $\tau(p)$ satisfies

$$\tau(p)=\tau(p;\alpha=0,q):=(1-p)\int_0^{T(q,p)}s\lambda e^{-\lambda s}ds+\left(p+(1-p)e^{-\lambda T(q,p)}\right)(T(q,p)+\tau(q))$$

$$= \frac{1}{\lambda}\left[\frac{q-p}{q} + p\log\left(\frac{q}{1-q}\frac{1-p}{p}\right)\right] + \frac{p}{q}\tau(q)$$

Similarly, if the DM has belief p and seeks R-evidence ($\alpha = 1$) until her belief reaches $q < p$, then the time it takes to reach q in the absence of an L-signal is

$$T(q, p; \alpha = 1) := \frac{1}{\lambda}\log\left(\frac{p}{1-p}\frac{1-q}{q}\right).$$

This implies that $\tau(p)$ satisfies

$$\tau(p) = \tau(p; \alpha = 1, q) := \frac{1}{\lambda}\left[\frac{p-q}{1-q} + (1-p)\log\left(\frac{p}{1-p}\frac{1-q}{q}\right)\right] + \frac{1-p}{1-q}\tau(q).$$

Direct computation shows concavity of the waiting time for $\alpha = 0$ or $\alpha = 1$, respectively:

$$\tau''(p; \alpha = 0, q) = -\frac{1}{\lambda(1-p)^2 p} < 0, \text{ and } \tau''(p; \alpha = 1, q) = -\frac{1}{\lambda(1-p)p^2} < 0.$$

(a) Equipped with these formulas, we prove part (a) of the proposition in several steps.

Step 1. *Suppose $c \in (0, \underline{c})$. Then, $\tau(p)$ is concave on $(\underline{p}, \overline{p})$.*

Proof. At p^*, the DM uses $\alpha = 1/2$. Hence the arrival rate of a signal is $\lambda/2$ and the expected delay is given by the expectation of the exponential distribution:

$$\tau(p^*) = \frac{2}{\lambda}.$$

Hence

$$\tau'(p) = \begin{cases} \frac{1}{\lambda}\left[\frac{1}{p} - \frac{1}{1-p^*} - \log\left(\frac{p}{1-p}\frac{1-p^*}{p^*}\right)\right] & \text{if } p > p^*, \\ \frac{1}{\lambda}\left[\frac{1}{p^*} - \frac{1}{1-p} + \log\left(\frac{p^*}{1-p^*}\frac{1-p}{p}\right)\right] & \text{if } p < p^*. \end{cases}$$

This implies $\tau(p^*_+) = \tau(p^*_-)$, i.e., $\tau(p)$ is differentiable on $(\underline{p}, \overline{p})$. Since we have shown that $\tau(p)$ is concave for $p > p^*$ and $p < p^*$, it is therefore concave on the whole interval $(\underline{p}, \overline{p})$. □

In the sequel, we let $\overline{p} = \underline{p} = \check{p}$ if $c \in [\underline{c}, \overline{c})$. Of course, $\overline{p}$ and $\underline{p}$ are well defined if $c \in (0, \underline{c})$.

Step 2. *Suppose $c \in (0, \overline{c})$. $\tau(\cdot)$ is concave on $[\overline{p}, \overline{p}^*)$ and on $(\underline{p}^*, \underline{p}]$. For $\rho > 0$ sufficiently small, $\tau(\cdot)$ is strictly decreasing on $[\max\{\overline{p}, p^*\}, \overline{p}^*)$ and strictly increasing on $(\underline{p}^*, \min\{\underline{p}, p^*\}]$.*

Proof. Given the symmetry, it suffices to prove the result for $p \in [\overline{p}, \overline{p}^*)$. For this region, the DM employs the own-biased strategy, i.e., she is seeking L-evidence. Hence, the expected delay is given by $\tau(p) = \tau(p; \alpha = 0, \overline{p}^*)$ which we have shown to be concave. Differentiating $\tau(p; \alpha = 0, \overline{p}^*)$ with respect to p we get

$$\tau'(p) = \frac{1}{\lambda}\left[-\frac{1}{\overline{p}^*(\rho)} + \log z(\rho) - \frac{1}{1-p}\right]. \tag{B.4}$$

where we define $z(\rho) := \left(\frac{\overline{p}^*(\rho)}{1-\overline{p}^*(\rho)}\right)\left(\frac{1-p}{p}\right)$ and write $\overline{p}^*(\rho)$ as a function ρ. We prove that there exists ρ_2 such that for all $\rho < \rho_2$, $\tau'(p) < 0$ for $p \in [\max\{\overline{p}, p^*\}, \overline{p}^*)$. To this end, consider $p = \max\{\overline{p}, p^*\}$. For ρ sufficiently small, $p > \underline{p}^*(0) + \varepsilon$ for some $\varepsilon > 0$ and hence

$$\tau'(p) < \frac{1}{\lambda}\left[-\chi + \log z\right],$$

where $\chi = -\frac{1}{\overline{p}^*(0)} - \frac{1}{1-\underline{p}^*(0)} < 2$, where we write $\underline{p}^*(\rho)$ as a function of ρ (evaluated at 0). Therefore it suffices to show that $\lim_{\rho\to 0}\log z(\rho) \leq 2$.

Note that $\overline{V}_{own}(p) \geq \overline{V}_{opp}(p)$ at that p. We use this condition to derive an upper bound for z. We can use our closed form solutions for $\overline{V}_{own}(p)$ and $\overline{V}_{opp}(p)$ to rewrite this condition:

$$-kp + \frac{u_\ell^L y - k}{y+1}(1-p) + \frac{y(k+u_r^R)}{y+1}p(z(\rho))^{-\frac{1}{y}} \geq -k(1-p) + \frac{u_r^R y - k}{y+1}p + \frac{y}{2+y}\frac{y(k+u_\ell^L)}{y+1}\left(\frac{1-p}{p}\right)^{\frac{1}{y}}(1-p),$$

where $k := c/\rho$, $y := \lambda/\rho$. Simplifying, we get

$$(z(\rho))^{-\frac{1}{y}} \geq \frac{(1-\phi)k + u_r^R - \phi u_\ell^L}{k+u_r^R} + \frac{y}{2+y}\frac{k+u_\ell^L}{k+u_r^R}\phi^{\frac{1+y}{y}},$$

where $\phi = (1-p)/p$. We define $\phi^* := (1-p^*)/p^* = \frac{k+u_r^R}{k+u_\ell^L}$, and rewrite the condition:

$$(z(\rho))^{-\frac{1}{y}} \geq 1 - \frac{\phi}{\phi^*} + \frac{y}{2+y}\frac{\phi}{\phi^*}\phi^{\frac{1}{y}} =: g(\phi).$$

$g(\phi)$ is convex and is minimized at $\phi^{**} = \left(\frac{2+y}{1+y}\right)^y$. As $\rho \to 0$, we have $\phi^* \to 1$ and $\phi^{**} \to e$. Therefore, for $\rho > 0$ sufficiently small, $g'(\phi) \leq 0$, for any $\phi \leq \phi^*$. This implies that

$$\begin{aligned}(z(\rho))^{-\frac{1}{y}} &\geq g(\phi) \geq g(\phi^*) = \left(\frac{y}{2+y}\right)(\phi^*)^{\frac{1}{y}},\\ \iff z(\rho) &\leq \left(1+\frac{2}{y}\right)^y \frac{1}{\phi^*}.\end{aligned}$$

Since $\phi^* \to 1$, the bound for $(z(\rho))$ converges to e^2 and hence $\lim_{\rho\to 0}\log z(\rho) \leq 2$ as needed for the proof. Therefore $\tau'(\max\{\overline{p}, p^*\}) < 0$ and since $\tau(p)$ is concave for $p \in [\max\{\overline{p}, p^*\}, \overline{p}^*)$, we have $\tau'(p) < 0$ for $p \in [\max\{\overline{p}, p^*\}, \overline{p}^*)$. □

Step 3. *Suppose $c \in (0, \underline{c})$. Then, for ρ sufficiently small, $\tau(\overline{p}_-) \geq \tau(\overline{p}_+)$ and $\tau(\underline{p}_-) \leq$*

$\tau(\underline{p}_+)$.

Proof. As before, we prove only the first statement (with the second obtained by a symmetric argument). Suppose $\rho = 0$. If at $\overline{p}$, the DM follows the own-biased strategy, she enjoys the payoff of

$$\left[\overline{p}u_r^R + (1-\overline{p})u_\ell^L\right] - \overline{p}\left[u_r^R - u_\ell^R\right] - c\int_0^{\overline{T}^*(\overline{p})}(1-H(t))dt. \tag{B.5}$$

where H is the distribution of the time at which the DM makes a decision.

Suppose instead that the DM follows the opposite-biased strategy. In this case her expected payoff (for $r = 0$) is given by

$$\left[\overline{p}u_r^R + (1-\overline{p})u_\ell^L\right] - c\int_0^{\infty}(1-G(t))dt. \tag{B.6}$$

where G is the distribution of time at which the DM makes a decision.

Since at $\overline{p}$, the DM is indifferent between both strategies we must have

$$\int_0^{\overline{T}^*(\overline{p})}(1-H(t))dt < \int_0^{\infty}(1-G(t))dt,$$

i.e., the DM will take a strictly longer time for decision if she chooses a opposite-biased strategy instead. This proves that $\tau(\overline{p}_+) < \tau(\overline{p}_-)$ for $\rho =$. By continuity the result extends to ρ in a neighborhood of zero. □

Suppose $c \in (0, \underline{c})$. Then, $\underline{p} < p^* < \overline{p}$. Hence, combining Steps 1-3 implies that, $\tau(p)$ is quasi-concave in p and attains its maximum in $[\underline{p}, \overline{p}]$.

The case $c \in [\underline{c}, \overline{c})$ requires a different proof depending on whether $\Delta^L = \Delta^R$ or $\Delta^L \neq \Delta^R$, where $\Delta^L = u_\ell^L - u_r^L$ and $\Delta^R = u_r^R - u_\ell^R$. We begin with the case $\Delta^L \neq \Delta^R$.

Step 4. *Fix u_x^ω, λ, and $c > 0$ such that $c \in [\underline{c}, \overline{c})$ for all ρ in a neighborhood of zero. If $\Delta^L \neq \Delta^R$, then there exists $\rho_4 > 0$ such that for all $\rho < \rho_4$, $\check{p} \neq 1/2$ and $\tau(\check{p}_-) < \tau(\check{p}_+)$ if $\check{p} < 1/2$, and $\tau(\check{p}_-) > \tau(\check{p}_+)$ if $\check{p} > 1/2$.*

Proof. We first show several results for the case that $\rho = 0$, and then prove Step 4 by continuity.

At $\check{p}$ we have $\overline{V}_{own}(\check{p}) = \underline{V}_{own}(\check{p})$. The explicit expression of the value functions for $c = 0$ are:

$$\overline{V}_{own}(p) = pu_r^R + (1-p)u_\ell^L + \frac{c}{\lambda}\left(\log\left(\frac{p}{1-p}\frac{1-\overline{p}^*}{\overline{p}^*}\right)p - 1\right),$$
$$\underline{V}_{own}(p) = pu_r^R + (1-p)u_\ell^L + \frac{c}{\lambda}\left(\log\left(\frac{1-p}{p}\frac{\underline{p}^*}{1-\underline{p}^*}\right)(1-p) - 1\right).$$

Equating these expressions we get

$$\log\left(\frac{\overline{p}^*}{1-\overline{p}^*}\frac{1-\check{p}}{\check{p}}\right)\check{p} = \log\left(\frac{\check{p}}{1-\check{p}}\frac{1-\underline{p}^*}{\underline{p}^*}\right)(1-\check{p}) \tag{B.7}$$

Claim 1. *Let $\rho = 0$ and fix u_x^ω, λ, and $c > 0$ such that $c \in [\underline{c}, \overline{c})$. Then*

$$\begin{cases} \check{p} < 1/2 & \text{if } \Delta^R > \Delta^L, \\ \check{p} = 1/2 & \text{if } \Delta^R = \Delta^L, \\ \check{p} > 1/2 & \text{if } \Delta^R < \Delta^L. \end{cases}$$

Proof of Claim 1. Substituting $\overline{p}^*$ and $\underline{p}^*$ in (B.7) we obtain the following condition for $\check{p}$:

$$\check{p}\log\left(\frac{\Delta^L\lambda - c}{c}\frac{1-\check{p}}{\check{p}}\right) = (1-\check{p})\log\left(\frac{\check{p}}{1-\check{p}}\frac{\Delta^R\lambda - c}{c}\right). \tag{B.8}$$

Proposition 8.(a) shows that this condition has a unique solution if $c \in [\underline{c}, \overline{c}]$. (B.8) implies that $\check{p} = 1/2$ if and only if $\Delta^L = \Delta^R$. Next consider $\Delta^R > \Delta^L$. The argument for $\Delta^L > \Delta^R$ is symmetric. From Proposition 3 we have that

$$\frac{\partial \check{p}}{\partial \Delta^L} > 0 \quad \text{and} \quad \frac{\partial \check{p}}{\partial \Delta^R} < 0.$$

This implies that $\check{p} < 1/2$ if $\Delta^R > \Delta^L$. To see this consider two separate cases. In the first case $c \leq \overline{c}(\Delta^L, \Delta^R = \Delta^L)$, in this case for any $\hat{\Delta}^R \in [\Delta^L, \Delta^R]$, we have $c \in [\underline{c}(\Delta^L, \hat{\Delta}^R), \overline{c}(\Delta^L, \hat{\Delta}^R)]$. Since $\check{p} = 1/2$ if $\hat{\Delta}^R = \Delta^L$, $\frac{\partial \check{p}}{\partial \Delta^R} < 0$ implies that $\check{p} < 1/2$ if $\Delta^R > \Delta^L$. In the second case $c > \overline{c}(\Delta^L, \Delta^R = \Delta^L)$. Then we set $\hat{\Delta}^R$ so that $c = \overline{c}(\Delta^L, \hat{\Delta}^R)$. If $c = \overline{c}(\Delta^L, \hat{\Delta}^R)$, then $\check{p} = \underline{p}^* = \overline{p}^* = \frac{c}{\hat{\Delta}^R\lambda} < 1/2$, where the inequality is obtained as follows:

$$c = \overline{c} \iff c = \frac{\lambda\hat{\Delta}^R\Delta^L}{\hat{\Delta}^R + \Delta^L} \iff \frac{c}{\hat{\Delta}^R\lambda} = \frac{\Delta^L}{\hat{\Delta}^R + \Delta^L} < 1/2.$$

Hence, as before $\frac{\partial \check{p}}{\partial \Delta^R} < 0$ implies that $\check{p} < 1/2$ for the original values $\Delta^R > \Delta^L$. This completes the proof of the claim. □

Now we prove Step 4 for $\rho = 0$.

Lemma 7. *Suppose $\rho = 0$ and fix u_x^ω, λ, and $c > 0$, such that $c \in [\underline{c}, \overline{c})$. Then $\tau(\check{p}_-) < \tau(\check{p}_+)$ if $\check{p} < 1/2$; $\tau(\check{p}_-) = \tau(\check{p}_+)$ if $\check{p} = 1/2$; and $\tau(\check{p}_-) > \tau(\check{p}_+)$ if $\check{p} > 1/2$.*

Proof. We want to determine the sign of $\tau(\check{p}_+) - \tau(\check{p}_-)$. Since $\tau(\overline{p}^*) = \tau(\underline{p}^*) = 0$, we have

$$\tau(p) = \tau(p; \alpha = 0, \overline{p}^*) = \frac{1}{\lambda}\left[1 - \frac{p}{\overline{p}^*} + p\log\left(\frac{\overline{p}^*}{1-\overline{p}^*}\frac{1-p}{p}\right)\right], \text{ if } p > \check{p}$$

$$\tau(p) = \tau(p;\alpha=1,\underline{p}^*) = \frac{1}{\lambda}\left[1 - \frac{1-p}{1-\underline{p}^*} + (1-p)\log\left(\frac{p}{1-p}\frac{1-\underline{p}^*}{\underline{p}^*}\right)\right], \text{ if } p < \check{p}$$

Substituting (B.7) in $\tau(\check{p}_+)$ we get

$$\begin{aligned} &\tau(\check{p}_+) - \tau(\check{p}_-) \geq 0, \\ \iff &\frac{1-\check{p}}{\check{p}}\frac{\overline{p}^*}{1-\underline{p}^*} \geq 1, \end{aligned} \tag{B.9}$$

and the second line holds with strict inequality if and only if τ has a jump at $\check{p}$. We have

$$\underline{p}^* = \frac{c}{\Delta^R\lambda}, \qquad \text{and} \qquad \overline{p}^* = \frac{\Delta^L\lambda - c}{\Delta^L\lambda}.$$

Hence (B.9) can be written as (we omit the arguments λ and c of $\xi(\cdot)$):

$$\xi(\Delta^L, \Delta^R) := \frac{1-\check{p}}{\check{p}}\frac{\Delta^L\Delta^R\lambda - \Delta^R c}{\Delta^L\Delta^R\lambda - \Delta^L c} \geq 1 \tag{B.10}$$

The proof of the lemma amounts to showing that $\xi(\Delta^L, \Delta^R) > 1$ whenever $\check{p} < 1/2$ and $\xi(\Delta^L, \Delta^R) < 1$ whenever $\check{p} > 1/2$. Claim 1 implies that to show the Lemma, it suffices to show that $\xi(\Delta^L, \Delta^R) > 1$ if $\Delta^R > \Delta^L$; $\xi(\Delta^L, \Delta^R) = 1$ if $\Delta^R = \Delta^L$; and $\xi(\Delta^L, \Delta^R) < 1$ if $\Delta^R < \Delta^L$.

Note first that inserting $\Delta^L = \Delta^R$ and $\check{p} = 1/2$ in (B.10) yields $\xi(\Delta^L, \Delta^L) = 1$. To establish the result for $\Delta^R > \Delta^L$, we write $\xi(\Delta^L, \Delta^R)$ as follows

$$\xi(\Delta^L, \Delta^R) = \xi(\Delta^L, \hat{\Delta}^R) + \int_{\hat{\Delta}^R}^{\Delta^R} \frac{\partial\xi(\Delta^L, z)}{\partial\Delta^R} dz, \tag{B.11}$$

and show the following claim (a symmetric argument holds for $\Delta^L > \Delta^R$):

Claim 2. *Suppose $\Delta^R > \Delta^L$ and $c < \overline{c}(\Delta^L, \Delta^R)$.*

(a) Then there exists $\hat{\Delta}^R \in [\Delta^L, \Delta^R]$ such that $\xi(\Delta^L, \hat{\Delta}^R) = 1$ and $c \leq \overline{c}(\Delta^L, z)$ for all $z \in [\hat{\Delta}^R, \Delta^R]$,

(b) and $\frac{\partial\xi(\Delta^L, z)}{\partial\Delta^R} > 0$ for all $z \in (\hat{\Delta}^R, \Delta^R]$.

Claim 2, together with (B.11), implies that $\xi(\Delta^L, \Delta^R) > 1$ if $\Delta^R > \Delta^L$ which completes the proof of the Lemma.

Proof of Claim 2. For part (a) note that:

$$c \leq \overline{c}(\Delta^L, z) \iff c \leq \frac{\lambda z \Delta^L}{z + \Delta^L}$$

We distinguish two cases. First, suppose that $\Delta^L\lambda/2 \geq c$, in this case $c \leq \overline{c}(\Delta^L, z)$ for all $z \in [\Delta^L, \Delta^R]$ and $\xi(\Delta^L, \Delta^L) = 1$. Hence we can set $\hat{\Delta}^R = \Delta^L$.

Second suppose that $\Delta^L\lambda/2 < c$. In this case we set $\hat{\Delta}^R = c\Delta^L/(\lambda\Delta^L - c)$, which is equivalent to $c = \overline{c}(\Delta^L, \hat{\Delta}^R)$. This implies $c \leq \overline{c}(\Delta^L, z)$ for all $z \in [\Delta^L, \hat{\Delta}^R]$. Moreover, $\check{p}(\Delta^L, \hat{\Delta}^R) = \underline{p}^*(\Delta^L, \hat{\Delta}^R) = \overline{p}^*(\Delta^L, \hat{\Delta}^R) = \frac{c}{\hat{\Delta}^R\lambda}$ and we have

$$\xi(\Delta^L, \hat{\Delta}^R) = \frac{\hat{\Delta}^R\lambda - c}{c} \frac{\Delta^L\hat{\Delta}^R\lambda - \hat{\Delta}^R c}{\Delta^L\hat{\Delta}^R\lambda - \Delta^L c} = \hat{\Delta}^R \frac{\Delta^L\lambda - c}{c\Delta^L} = 1$$

This completes the proof of part (a) of the claim.

For part (b) we have (where we sometimes write $\check{p}(\Delta^L, \Delta^R)$ to indicate the dependence on the parameters):

$$\begin{aligned}
&\frac{\partial\xi(\Delta^L, z)}{\partial\Delta^R} > 0 \\
\iff\ & -\frac{1}{(\check{p})^2}\frac{\partial\check{p}(\Delta^L, z)}{\partial\Delta^R}\frac{\Delta^L z\lambda - zc}{z\Delta^L\lambda - \Delta^L c} \\
+\frac{1-\check{p}}{\check{p}} & \frac{\left(\Delta^L\lambda - c\right)\left(\Delta^L z\lambda - \Delta^L c\right) - z\left(\Delta^L\lambda - c\right)\Delta^L\lambda}{\left(z\Delta^L\lambda - \Delta^L c\right)^2} > 0 \\
\iff\ & \frac{\partial\check{p}(\Delta^L, z)}{\partial\Delta^R} < -\frac{\check{p}\left(1-\check{p}\right)c}{z\left(z\lambda - c\right)}
\end{aligned} \tag{B.12}$$

In this derivation we have used that $\Delta^L\lambda - c > 0$ and $z\lambda - c > 0$ if $z \in [\hat{\Delta}^R, \Delta^R]$. To see this, note that $z \in [\hat{\Delta}^R, \Delta^R]$ implies

$$c \leq \overline{c}(\Delta^L, z) \iff \lambda z - c \geq \frac{zc}{\Delta^L} \geq c. \tag{B.13}$$

Similarly, we obtain $\Delta^L\lambda - c > 0$.

To show that the last line in (B.12) holds, we compute the partial derivative of $\check{p}$ using the implicit function theorem. (B.8) can be rearranged to:

$$\log\left(\frac{\check{p}}{1-\check{p}}\right) + \check{p}\log\left(\frac{c}{\Delta^L\lambda - c}\right) - (1-\check{p})\log\left(\frac{c}{\Delta^R\lambda - c}\right) = 0.$$

Therefore we have:

$$\frac{\partial\check{p}(\Delta^L, z)}{\partial\Delta^R} = -\frac{(1-\check{p})\frac{\lambda}{(z\lambda - c)}}{\frac{1}{\check{p}(1-\check{p})} + \log\left(\frac{c}{\Delta^L\lambda - c}\frac{c}{z\lambda - c}\right)} < 0$$

The inequality (which follows from Proposition 3) implies that the denominator is positive. Substituting the derivative in (B.12) we have

$$\frac{1}{(1-\check{p})} + \check{p}\log\left(\frac{c}{\Delta^L\lambda - c}\frac{c}{z\lambda - c}\right) < \frac{\lambda z}{c} \tag{B.14}$$

Where we have used that $z\lambda - c > 0$ if $c \leq \overline{c}$, and that the denominator of $\partial\check{p}(\Delta^L, z)/\partial\Delta^R$

is positive as noted before.

In (B.14) we have $1/(1-\check{p}) \leq 2$ since $\check{p}(\Delta^L, z) \leq 1/2$ if $z \in [\hat{\Delta}^R, \Delta^R]$, and the LHS is greater or equal 2, which is seen as follows:

$$c < \bar{c} \iff c < \frac{\lambda z \Delta^L}{z + \Delta^L} \overset{(z \geq \Delta^L)}{\Longrightarrow} c < \frac{\lambda z z}{z + z} = \frac{\lambda z}{2} \iff \frac{\lambda z}{c} > 2.$$

Hence, a sufficient condition for (B.14) is

$$\log \left(\frac{c}{\Delta^L \lambda - c} \frac{c}{z\lambda - c} \right) \leq 0.$$

Remember that he have noted in (B.13) that

$$z\lambda - c \geq c \frac{z}{\Delta^L} > 0$$

and similarly we obtain

$$\Delta^L \lambda - c \geq c \frac{\Delta^L}{z} > 0$$

with strict inequalities if $z > \hat{\Delta}^R$ (which implies $c < \bar{c}(\Delta^L, z)$). Multiplying these two inequalities we get

$$(\Delta^L \lambda - c)(z\lambda - c) \geq c^2 \iff \log \left(\frac{c}{\Delta^L \lambda - c} \frac{c}{\Delta^R \lambda - c} \right) \leq 0$$

with strict inequalities if $z > \hat{\Delta}^R$. Remember that Condition (B.14) is equivalent to $\frac{\partial \xi(\Delta^L, z)}{\partial \Delta^R} > 0$ if $z \in [\hat{\Delta}^R, \Delta^R]$ and we have now shown that a sufficient condition for (B.14) holds if $z > \hat{\Delta}^R$. This completes to proof of part (b) of the claim. $\square$

This completes the proof of the Lemma. $\square$

To finish the proof of Step 4, we argue by continuity. Suppose $\Delta^R > \Delta^L$. We have assumed that for ρ sufficiently small $c \in [\underline{c}, \bar{c})$. Moreover, note that if $c > 0$, $\underline{p}^* > 0$ and $\overline{p}^* < 1$. The waiting time for the R-biased strategy depends on ρ only through $\overline{p}^*(\rho)$. We have for $p \in (0, \overline{p}^*(\rho))$:

$$\begin{aligned}
&|\tau(p; \alpha = 0, \overline{p}^*(\rho)) - \tau(p; \alpha = 0, \overline{p}^*(0))| \\
=&\frac{1}{\lambda} \left| \frac{\overline{p}^*(\rho) - \overline{p}^*(0)}{\overline{p}^*(\rho)\overline{p}^*(0)} p + p \log \left(\frac{\overline{p}^*(\rho)}{1 - \overline{p}^*(\rho)} \frac{1 - \overline{p}^*(0)}{\overline{p}^*(0)} \right) \right| \\
\leq&\frac{1}{\lambda} \left[\left| \frac{\overline{p}^*(\rho) - \overline{p}^*(0)}{\overline{p}^*(\rho)\overline{p}^*(0)} \right| + \left| \log \left(\frac{\overline{p}^*(\rho)}{1 - \overline{p}^*(\rho)} \frac{1 - \overline{p}^*(0)}{\overline{p}^*(0)} \right) \right| \right] \longrightarrow 0 \text{ (as } \rho \to 0).
\end{aligned}$$

Since the last line is independent of p, we have uniform convergence (uniform in p) of $\tau(\cdot; \alpha = 0, \overline{p}^*(\rho))$ as $\rho \to 0$. Similarly, we obtain uniform converges of $\tau(\cdot; \alpha = 1, \underline{p}^*(\rho))$

as $\rho \to 0$. Moreover $\check{p}(\rho) \to \check{p}(0)$. Hence, there exists $\varepsilon > 0$ such that $\check{p}(0) + \varepsilon < 1/2$, and $\rho_4 > 0$ such that for any $\rho < \rho_4$, $|\check{p}(\rho) - \check{p}(0)| < 0$, and $\tau(p; \alpha = 0, \overline{p}^*(\rho)) > \tau(p; \alpha = 1, \overline{p}^*(\rho))$ for all $p \in B_\varepsilon(\check{p}(0))$ (by uniform convergence of the two functions, and the fact that for $\rho = 0$ there is a jump at $\check{p}(0)$). This shows that for $\rho < \rho_4$, $\tau(\check{p}_+) > \tau(\check{p}_-)$. A symmetric argument shows Step 4 for $\Delta^L > \Delta^R$. $\square$

Suppose $c \in (\underline{c}, \overline{c})$. Let $\check{p}(0) := \lim_{\rho \to 0} \check{p}(\rho)$. There are two cases. Suppose first $\check{p}(0) \neq 1/2$, which is equivalent ot $\Delta^L \neq \Delta^R$. We consider the case where $\check{p}(0) < 1/2$ (which is equivalent to $\Delta^L < \Delta^R$). Since $p^* \to 1/2$ as $\rho \to 0$, for ρ sufficiently small, we have $\check{p}(\rho) < p^*$. Then, Step 2 implies that for $\rho < \rho_2$, $\tau(\cdot)$ is strictly increasing on $(\underline{p}^*, \check{p}(\rho))$, and that $\tau(\cdot)$ is concave on $(\check{p}(\rho), \overline{p}^*)$. Meanwhile, Step 4 implies that $\tau(\check{p}(\rho)_-) < \tau(\check{p}(\rho)_+)$. Hence, we have the quasi-concavity of τ. The case of $\check{p}(0) > 1/2$ is symmetric.

Next, suppose that $\check{p}(0) = 1/2$ which is equivalent to $\Delta^L = \Delta^R$. In this case Step 2 implies that

$$\frac{\partial \tau(1/2, \alpha = 1, \underline{p}^*(0))}{\partial p} > 0, \quad \text{and} \quad \frac{\partial \tau(1/2, \alpha = 0, \overline{p}^*(0))}{\partial p} < 0.$$

We can show that $\partial \tau(p, \alpha = 1, \underline{p}^*(\rho))/\partial p$ and $\partial \tau(p, \alpha = 0, \overline{p}^*(\rho))/\partial p$ are uniformly continuous in ρ.[3] Since $\check{p}(\rho) \to 1/2$, this implies that

$$\frac{\partial \tau(\check{p}(\rho), \alpha = 1, \underline{p}^*(\rho))}{\partial p} > 0, \quad \text{and} \quad \frac{\partial \tau(\check{p}(\rho), \alpha = 0, \overline{p}^*(\rho))}{\partial p} < 0.$$

for ρ sufficienctly close to zero. Together with concavity for $p < \check{p}(\rho)$ and $p > \check{p}(\rho)$ this implies that $\tau(p)$ is concave on $[\underline{p}^*, \overline{p}^*]$. This completes the proof of part (a).

(b) Consider $p > \check{p}$ if $c \in [\underline{c}, \overline{c})$, so that the DM uses $\alpha = 0$ according to the opposite-biased strategy. Time it takes to reach $\overline{p}^*$ in the absence of a breakthrough is

$$\overline{T}^* = T(\overline{p}^*, p; \alpha = 0) = \frac{1}{\lambda} \log\left(\frac{\overline{p}^*}{\overline{p}^* - \overline{p}^*} \frac{1-p}{p}\right).$$

The probability of a mistake is therefore

$$(1-p)\left(1 - e^{-\lambda \overline{T}^*}\right) = \frac{(1-\overline{p}^*)(\overline{p}^* - p)}{\overline{p}^*(1-p)}.$$

[3] To see this note that

$$\left|\frac{\partial \tau(p, \alpha = 0, \overline{p}^*(\rho))}{\partial p} - \frac{\partial \tau(p, \alpha = 0, \overline{p}^*(0))}{\partial p}\right| = \left|\frac{1}{\lambda}\left[\frac{\overline{p}^*(\rho) - \overline{p}^*(0)}{\overline{p}^*(\rho)\overline{p}^*(0)} + \log\left(\frac{\overline{p}^*(\rho)}{1 - \overline{p}^*(\rho)} \frac{1 - \overline{p}^*(0)}{\overline{p}^*(0)}\right)\right]\right|$$

which is independent of p and converges to zero as $\rho \to 0$.

Differentiating this with respect to p, we get

$$-\frac{\overline{p}^* - p}{\overline{p}^*(1-p)} < 0.$$

This proves that the probability of a mistake decreases in the distance to $\overline{p}^*$ for $p > \check{p}$. THe proofs for the remaining cases are similar. $\square$

B.3 Proof of Proposition 2

Proof. (a) By (B.1), $\underline{p}^*$ and $\overline{p}^*$ are given by the intersections of $U(p)$ and $U^{FA}(p)$. Since $U(p)$ is independent of ρ and c, and $U^{FA}(p)$ is strictly decreasing in both parameters, the experimentation region expands as ρ or c fall. As $(\rho, c) \to (0, 0)$, we have $U^{FA}(p) \to U(p)$ for $p \in \{0, 1\}$, hence the experimentation region converges to $(0, 1)$.

(b) The dependence of $\underline{p}^*$ and $\overline{p}^*$ on u_ℓ^R and u_r^L is straightforward from the expressions for the cutoffs in (A9) and (A10).

(c) By (B.1), $\underline{p}^*$ is the intersection between $U_\ell(p)$ and $U^{FA}(p)$. The former is independent of u_r^R and the latter is increasing in u_r^R. Hence $\partial \underline{p}^* / \partial u_r^R < 0$. Also by (B.1), $\overline{p}^*$ is the intersection between $U_r(p)$ and $U^{FA}(p)$. We have

$$\frac{\partial U_r(p)}{\partial u_r^R} = p > \frac{\lambda}{r+\lambda} p = \frac{\partial U^{FA}(p)}{\partial u_r^R}.$$

This implies that $\partial \overline{p}^* / \partial u_r^R < 0$. The comparative statics with respect to u_ℓ^L is derived similarly. $\square$

B.4 Proof of Proposition 3

Proof. (a) We prove $\partial \check{p} / \partial u_\ell^R > 0$; the other case follows from a symmetric argument. Consider $\overline{V}_{own}(p)$. Since the right branch of the own-biased value function is obtained from a strategy that takes action ℓ only if a signal has been received, its value is independent of u_ℓ^R, as can be seen from (A12). On the other hand we have $\partial \underline{V}_{own}(p) / \partial u_\ell^R > 0$ from (A11). Therefore the point of intersection of $\underline{V}_{own}$ and $\overline{V}_{own}$ is increasing in u_ℓ^R.

(b) It is clear from (A18) that $\underline{c}$ is decreasing in u_ℓ^R and u_r^L. Therefore, it suffices to consider the case that $c < \underline{c}$. We prove that $\underline{p} \to 0$ monotonically as $u_\ell^R \to -\infty$. If a opposite-biased region exists, $\underline{p} \in (\underline{p}^*, p^*)$ is defined as the unique intersection between $\underline{V}_{opp}(p)$ and $\underline{V}_{own}(p)$. Note that $\underline{V}_{opp}(p)$ is independent of u_ℓ^R since the opposite-biased strategy never leads to a mistake. As in (a), we have $\partial \underline{V}_{own}(p) / \partial u_\ell^R > 0$. Moreover, Lemma 2 shows that $\underline{V}_{own}(p)$ crosses $\underline{V}_{opp}(p)$ from above at $\underline{p}$. Since V_{opp} is independent of u_ℓ^R this implies that of $\underline{p}$ is monotonically increasing in u_ℓ^R.

Since $\underline{p}$ is bounded from below, there exists $q = \lim_{u_\ell^R \to -\infty} \underline{p} < p^*$. Suppose by contradiction that $q > 0$. Notice that, for each $p \in [q, p^*]$, as $u_\ell^R \to -\infty$.

$$\underline{V}_{own}(p) \to \frac{\lambda u_r^R p\rho - \lambda c(1-p) - c\rho}{(\rho+\lambda)\,\rho} =: \underline{V}^\circ_{own}(p),$$

where we used the fact that $\underline{p}^*/(1-\underline{p}^*) \to 0$ as $u_\ell^R \to -\infty$.

Note that the convergence is uniform on $[q, p^*]$ since $q > 0$.[4] Simple algebra yields $\underline{V}^\circ_{own}(p^*) \le U^S(p^*)$, and $\underline{V}^{\circ\prime}_{own}(p) > U^{S\prime}(p)$. Since $\underline{V}^\circ_{own}(p)$ is linear in p, this implies that $\underline{V}_{opp}(q) \ge U^S(q) > \underline{V}^\circ_{own}(q)$ which is a contradiction and we must have $q = 0$. The proof for $\overline{p}$ is essentially the same. □

B.5 Proof of Proposition 4

Proof. (a) We have

$$\frac{\partial \overline{c}}{\partial \rho} = -\frac{u_r^R u_\ell^L - u_\ell^R u_r^L}{(u_r^R + u_\ell^L) - (u_\ell^R + u_r^L)},$$

and hence $\operatorname{sgn}(\partial\overline{c}/\partial\rho) = \operatorname{sgn}\left(u_\ell^R u_r^L - u_r^R u_\ell^L\right)$. It is straightforward to verify that $U(\hat{p}) > 0$ if and only if $u_r^R u_\ell^L - u_\ell^R u_r^L > 0$.

(b) Denoting $Z(x) := (x+1)/\left(1 + (2x+1)^{\frac{1}{x}}\right)$, we have

$$\frac{\partial \underline{c}}{\partial \rho} = \begin{cases} Z'(\rho/\lambda)\left(u_r^R - u_\ell^R\right) - u_r^R & \text{if } (\lambda Z(\rho/\lambda) - \rho)\left(u_r^R - u_\ell^L\right) - \lambda Z(\rho/\lambda)\left(u_\ell^R - u_r^L\right) < 0, \\ Z'(\rho/\lambda)\left(u_\ell^L - u_r^L\right) - u_\ell^L & \text{if } (\lambda Z(r/\lambda) - \rho)\left(u_r^R - u_\ell^L\right) - \lambda Z(\rho/\lambda)\left(u_\ell^R - u_r^L\right) > 0. \end{cases}$$

Consider the first case. Since $Z'(x) \in \left[\left(1+3e^2\right)/\left(1+e^2\right)^2, 1/2\right]$,

$$Z'(\rho/\lambda)\left(u_r^R - u_\ell^R\right) - u_r^R < \frac{1}{2}\left(u_r^R - u_\ell^R\right) - u_r^R = -\frac{1}{2}\left(u_r^R + u_\ell^R\right),$$

which is negative if $u_r^R > \left|u_\ell^R\right|$. Conversely, if u_ℓ^R is sufficiently negative $Z'(\rho/\lambda)\left(u_r^R - u_\ell^R\right) - u_r^R > 0$, which is equivalent to $u_\ell^R/u_r^R < 1 - 1/Z'(\rho/\lambda)$. A sufficient condition is $u_\ell^R/u_r^R < 1 - 1/Z'(0) \approx -2.04$. The argument for the second case is similar. □

B.6 Proof of Proposition 5

Proof. Let $F_t(p)$ be the distribution function of beliefs in the whole population at time t. Denote the density, whenever it exists by $f_t(p)$. Denote by $\delta_t(p) = F_t(p) - F_t^-(p)$ the mass at p if there is a mass point.

(a) To simplify notation, we set $\underline{p} = \overline{p} = \check{p}$ if $c \ge \underline{c}$. For part (a) we consider the subpopulation of voters with prior beliefs in $\mathcal{P}_{own} = [\underline{p}^*, \underline{p}] \cup [\overline{p}, \underline{p}^*]$. Initially, these

[4] Recall from (A11) that $\underline{V}_{own}(p) \to \infty$ as $p \to 0$, hence the condition $q > 0$ is necessary here.

voters consume own-biased news. If we consider the same subpopulation at later points $t > 0$, then their beliefs either remain inside $\mathcal{P}_{own}$, or they jump to $p_t = 0$ after an L-breakthrough. Therefore, for $t > 0$ we consider the subpopulation of voters with beliefs in $\mathcal{P}^0_{own} = \mathcal{P}_{own} \cup \{0\}$. Within $\mathcal{P}^0_{own}$ we consider the median belief for voters with $p_t > 1/2$, denoted m^r_t and the median belief for voters with $p_t < 1/2$, denoted by m^ℓ_t.

We first consider m^r_t which is given by

$$m^r_t = F_t^{-1}\left(F_t(\overline{p}) + \frac{F_t(\overline{p}^*) - F_t(\overline{p})}{2}\right)$$

We show that this is increasing in t whenever $m^r_t < \overline{p}^*$. All individuals in$\mathcal{P}^0_{own} \cap (1/2, 1]$ consume R-biased news. This leads to two possible changes in their beliefs that effects the median. First, for voters who receive breakthrough news the belief becomes 0 so that they leave the set $\mathcal{P}^0_{own} \cap (1/2, 1]$. Note that conditional on the state being L all individuals who acquire information receive L-breakthroughs at rate λ. If $m^r_t < \overline{p}^*$, all voters in $\mathcal{P}^0_{own} \cap (1/2, 1]$ below the median still acquire information but some voters above the median have already stopped. Therefore, more voters below the median receive breakthrough than above the median. This increases the median m^r_t.

Second, absent a breakthrough the belief of a voter in $\mathcal{P}^0_{own} \cap (1/2, 1]$ drifts upwards. The upward drift also increases the median. Hence, if $m^r_t < \overline{p}^*$, m^r_t is increasing over time. If $m^r_t = \overline{p}^*$, it remains constant for all $t' > t$.

Next consider the subpopulation of individuals with beliefs in $\mathcal{P}^0_{own} \cap [0, 1/2)$. This subpopulation is composed of (i) the voters who initially consume own-biased news and have a prior $p_0 < 1/2$, and (ii) voters who initially consume own-biased news and have a prior of $p > 1/2$, but received breakthrough news at some time $t' \le t$. The median belief at time t of individuals with beliefs below 1/2 in this subset is given by

$$m^\ell_t = \begin{cases} 0, & \text{if } F_t(0) \ge \delta_t(\underline{p}^*) + F_t(\underline{p}) - F_t(\underline{p}^*) \\ F_t^{-1}\left(F_t(\underline{p}) - \frac{\delta_t(0) + F_t(\underline{p}) - F_t^-(\underline{p}*)}{2}\right), & \text{otherwise.} \end{cases}$$

m^ℓ_t is moved by two forces. First, individuals with $p > 1/2$ who receive breakthroughs enter the population with $p < 1/2$, and since they have a belief $p = 0$ after the breakthrough this reduces the median. Second, individuals with beliefs $p < 1/2$ who consume own-biased news never receive breakthroughs if the true state is L. Therefore their beliefs drift downwards which further decreases m^ℓ_t.

In summary we have shown that $m^r_t - m^\ell_t$ is increasing which concludes the proof of part (a).

Part (b) follows from similar arguments since all voters who consume any news choose own-biased news by assumption. Therefore their belief dynamics as in case (a). The remaining voters do not consume any news so that their beliefs remain constant and leave the median in the subpopulations above and below 1/2 unaffected.

The proof of part (c) is immediate from the definition of the opposite-biased strategy.

□

C Extensions

C.1 Discrete Time Foundation

Proof of Proposition 6. If the DM chooses an experiment with parameters a and $b = 1 + \lambda dt - a$, then the posteriors are $q^R := p\,(\lambda dt + 1 - a)\,/\,(p\lambda dt + (1-a))$ when the R-signal is received, and $q^L := p\,(a - \lambda dt)\,/\,(a - p\lambda dt)$ when an L-signal is received. The unconditional probabilities of the signals are $\text{Prob}\,[R\text{-signal}] = p\lambda dt + (1-a)$, and $\text{Prob}\,[L\text{-signal}] = a - p\lambda dt$. Hence the DM maximizes

$$\max_{a \in [\lambda dt, 1]} (p\lambda dt + (1-a))\,\widetilde{V}(q^R) + (a - p\lambda dt)\,\widetilde{V}(q^L) \tag{C.1}$$

where $\widetilde{V}(q) = \max\left\{U(q), e^{-\rho dt}V(q) - c\right\}$ and $V(p)$ is the optimal value function. We note that $V(p)$ is weakly convex.[5] Therefore the *continuation value* $\widetilde{V}(p)$ is also weakly convex.

In the following, we fix an arbitrary weakly convex continuation value $\widetilde{V}$ and belief $p \in (0,1)$. We show that (C.1) is maximized by $\alpha = \lambda dt$ or $\alpha = 1$. To do this, we rewrite the objective in (C.1) for an arbitrary choice $\hat{a} \in [\lambda dt, 1]$ in a way that can be bounded by the value for $\alpha = \lambda dt$ or $\alpha = 1$.

So we fix any $\hat{a} \in [\lambda dt, 1]$ and denote the implied posteriors by $\hat{q}^R$ and $\hat{q}^L$. To rewrite the objective in (C.1), we construct alternative payoff parameters $\hat{u}^\omega_x$ so that the resulting stopping payoffs satisfy $\widehat{U}_\ell(\hat{q}^L) = \widetilde{V}(\hat{q}^L)$ and $\widehat{U}'_\ell(\hat{q}^L) = \widetilde{V}'(\hat{q}^L)$, as well as $\widehat{U}_\rho(\hat{q}^R) = \widetilde{V}(\hat{q}^R)$ and $\widehat{U}'_r(\hat{q}^R) = \widetilde{V}'(\hat{q}^R)$.[6] Theses conditions yields:

$$\begin{aligned}
\hat{u}^R_r &:= \widetilde{V}(\hat{q}^R) + (1-\hat{q}^R)\widetilde{V}'(\hat{q}^R), \qquad & \hat{u}^R_\ell &:= \widetilde{V}(\hat{q}^R) - \hat{q}^R\widetilde{V}'(\hat{q}^R),\\
\hat{u}^L_r &:= \widetilde{V}(\hat{q}^L) + (1-\hat{q}^L)\widetilde{V}'(\hat{q}^L), \qquad & \hat{u}^R_\ell &:= \widetilde{V}(\hat{q}^L) - \hat{q}^L\widetilde{V}'(\hat{q}^L).
\end{aligned}$$

By definition, $\widehat{U}(p)$ is tangent to $\tilde{V}(p)$ at $p = \hat{q}^L$ and at $p = \hat{q}^R$, and is everywhere weakly below $\widetilde{V}(p)$, given the convexity of $\widetilde{V}(p)$.

[5] To see this, note that the expected value of a fixed strategy (i.e. a mapping that specifies the attention choice and action for each history) is linear in the prior belief. The value function is therefore the upper envelope of a family of linear functions, which implies convexity.

[6] We use the notation $\widehat{U}_\ell(p) := p\hat{u}^R_\ell + (1-p)\hat{u}^R_\ell$, $\widehat{U}_r(p) := p\hat{u}^R_r + (1-p)\hat{u}^R_r$, and $\widehat{U}(p) := \max\{\widehat{U}_\ell(p), \widehat{U}_r(p)\}$.

The objective in (C.1) for $\hat{a}$ can be rearranged and bounded as follows:

$$\begin{aligned}
&(p\lambda dt + (1-\hat{a}))\,\widetilde{V}(\hat{q}^R) + (\hat{a} - p\lambda dt)\,\widetilde{V}(\hat{q}^L)\\
=&(p\lambda dt + (1-\hat{a}))\,\widehat{U}(\hat{q}^R) + (\hat{a} - p\lambda dt)\,\widehat{U}(\hat{q}^L)\\
=&p(1+\lambda dt-\hat{a})\hat{u}_r^R + (1-p)(1-\hat{a})\hat{u}_r^L + p(a-\lambda dt)\hat{u}_\ell^R + (1-p)\hat{a}\hat{u}_\ell^L\\
\le& \max_{a\in\{\lambda dt,1\}} p(1+\lambda dt-a)\hat{u}_r^R + (1-p)(1-a)\hat{u}_r^L + p(a-\lambda dt)\hat{u}_\ell^R + (1-p)a\hat{u}_\ell^L\\
=&(p\lambda dt + (1-\hat{a}^*))\,\widehat{U}(\hat{q}^{R*}) + (\hat{a}^* - p\lambda dt)\,\widehat{U}(\hat{q}^{L^*})\\
\le&(p\lambda dt + (1-\hat{a}^*))\,\widetilde{V}(\hat{q}^{R*}) + (\hat{a}^* - p\lambda dt)\,\widetilde{V}(\hat{q}^{L^*}).
\end{aligned}$$

In the second line, we have replaced $\widetilde{V}$ by $\widehat{U}$. Writing this out in the third line, we see that the expression is linear in a. Therefore, maximizing over $a \in \{\lambda dt, 1\}$, we get a weakly higher value. In the fifth line $\hat{a}^*$ denotes a maximizer from the forth line and $\hat{q}^{\omega *}$ denotes the corresponding posterior beliefs. The last inequality follows from the fact that $\widetilde{V}$ is weakly above $\widehat{U}$. This shows that the optimal a can be found in $\{\lambda dt, 1\}$. □

C.2 Non-Exclusivity of Attention

The proofs of our main results only require minor modifications. One important change is that the full attention strategy has to be defined using $\alpha = \beta = \overline{\alpha}$. Without this modification, Lemmas 1 and 4–6 are no longer valid. We also have to replace V_0 and V_1 by solutions to $c + \rho V(p) = F_\alpha(p, V(p), V'(p))$ for $\alpha = \underline{\alpha}$ and $\alpha = \overline{\alpha}$, respectively. The value of the stationary strategy $U^S(p)$ is unchanged as discussed in the main text. The crucial Lemmas 2 and 3 continue to hold without modification.

Explicit expressions for the boundaries of the experimentation region and the absorbing point p^* are now given by

$$\begin{aligned}
\underline{p}^* &= \frac{u_\ell^L\rho + c}{\rho\,(u_\ell^L - u_\ell^R) + (u_r^R - u_\ell^R)\,\lambda\overline{\alpha}},\\
\overline{p}^* &= \frac{\left(u_\ell^L - u_r^L\right)\lambda\overline{\alpha} - u_r^L\rho - c}{\rho\,(u_r^R - u_r^L) + (u_\ell^L - u_r^L)\,\lambda\overline{\alpha}},\\
p^* &= \frac{\left(u_\ell^L\rho + c\right)}{(u_r^R\rho + c) + (u_\ell^L\rho + c)}.
\end{aligned}$$

One can see from the first two expressions that $\underline{p}^*$ increases and $\overline{p}^*$ decreases if we decrease the upper bound $\overline{\alpha}$. This confirms the claim that the experimentation region shrinks if the constraint on α is tightened.

The cutoffs $\overline{c}$, $\underline{c}$ are given by:

$$\overline{c} := 0 \vee \frac{\lambda\overline{\alpha}\left(u_r^R - u_\ell^R\right)\left(u_\ell^L - u_r^L\right) - \rho\left(u_r^R u_\ell^L - u_\ell^R u_r^L\right)}{(u_r^R - u_\ell^R) + (u_\ell^L - u_r^L)}, \tag{C.2}$$

$$\underline{c} := 0 \vee \begin{cases} \overline{c} \wedge \frac{\lambda}{1+e^2} \min\left\{\left(u_r^R - u_\ell^R\right), \left(u_\ell^L - u_r^L\right)\right\} & \text{if } \rho = 1 - \overline{\alpha} = 0, \\ \overline{c} \wedge \min\left\{ \frac{(\rho+\lambda\overline{\alpha})\left(u_r^R - u_\ell^R\right)}{1+\left(\frac{2\rho+\lambda}{(2\overline{\alpha}-1)\lambda}\right)^{\frac{2\overline{\alpha}-1}{(1-\overline{\alpha})+\frac{\rho}{\lambda}}}} - \rho u_r^R, \frac{(\rho+\lambda\overline{\alpha})\left(u_\ell^L - u_r^L\right)}{1+\left(\frac{2\rho+\lambda}{(2\overline{\alpha}-1)\lambda}\right)^{\frac{2\overline{\alpha}-1}{(1-\overline{\alpha})+\frac{\rho}{\lambda}}}} - \rho u_\ell^L \right\} & \text{otherwise.} \end{cases} \tag{C.3}$$

From the first expression it is immediately clear that $\overline{c}$ decreases if we reduce the upper bound $\overline{\alpha}$. It is less obvious that $\underline{c}$ increases at the same time. To see this, remember from the proof of Theorem 1 that $c > \underline{c}$ is equivalent to

$$\max\left\{\underline{V}_{own}(p^*), \overline{V}_{own}(p^*)\right\} > U^S(p^*).$$

The right-hand side of this inequality is independent of $\overline{\alpha}$. The left-hand however, is decreasing in $\overline{\alpha}$.

C.3 Asymmetric Returns to Attention

In this section we revisit three crucial results that are used to prove Theorem 1, and outline how they are changed if $\overline{\lambda}^R \neq \overline{\lambda}^L$. Throughout we assume that $\overline{\lambda}^R \geq \overline{\lambda}^L$. Up to relabeling this is without loss of generality. The three crucial results are:

(a) The Crossing Lemma 2 and the Unimprovability Lemma 3. In Appendix A, we considered solutions V_0 and V_1 to the HJB equation where we set $\alpha = 0$ or $\alpha = 1$, respectively. If we generalize the HJB equation to allow for $\overline{\lambda}^R > \overline{\lambda}^L$, we can obtain similar solutions V_0 and V_1. Lemma 2 also uses the value of the stationary strategy as a benchmark. The definition of the stationary strategy has to be modified if $\overline{\lambda}^R > \overline{\lambda}^L$. The Bayesian updating formula in the absence of a signal is now given by:

$$\dot{p}_t = -\left(\overline{\lambda}^R \alpha_t - \overline{\lambda}^L \beta_t\right) p_t (1 - p_t), \tag{C.4}$$

Hence the stationary attention strategy is given by

$$\alpha^S = \frac{\overline{\lambda}^L}{\overline{\lambda}^R + \overline{\lambda}^L}.$$

Note that this coincides with the definition of our main model where $\alpha^S = 1/2$ if $\overline{\lambda}^R = \overline{\lambda}^L$. The value of the stationary strategy is now

$$U^S(p) := p\frac{\alpha^S \overline{\lambda}^R u_r^R - c}{\rho + \alpha^S \overline{\lambda}^R} + (1-p)\frac{\beta^S \overline{\lambda}^L u_\ell^L - c}{\rho + \beta^S \overline{\lambda}^L}.$$

With this definition, Lemmas 2 and 3 continue to hold.

(b) Properties of the own-biased strategy in Lemma 5:[7] In Appendix A we have constructed the own-biased strategy by first obtaining the boundary points $\underline{p}^*$ and $\overline{p}^*$ from value matching and smooth pasting. Following the same steps while allowing for $\overline{\lambda}^R > \overline{\lambda}^L$ we get

$$\underline{p}^* = \frac{u_\ell^L \rho + c}{\rho \left(u_\ell^L - u_\ell^R\right) + \left(u_r^R - u_\ell^R\right) \overline{\lambda}^R}, \tag{C.5}$$

$$\overline{p}^* = \frac{\left(u_\ell^L - u_r^L\right) \overline{\lambda}^L - u_r^L \rho - c}{\rho \left(u_r^R - u_r^L\right) + \left(u_\ell^L - u_r^L\right) \overline{\lambda}^L}. \tag{C.6}$$

The branches of the own-biased solution are then given by particular solutions V_0 and V_1 that satisfy the boundary conditions $V_0(\overline{p}^*) = U_r(\overline{p}^*)$ and $V_1(\underline{p}^*) = U_\ell(\underline{p}^*)$. Lemma 5.(a)-(b) hold unchanged if $\overline{\lambda}^R > \overline{\lambda}^L$. For the other results in Lemma 5, we need to modify the definition of the full-attention strategy. We compute separately the value of full attention if the DM can obtain both types of evidence at rate $\overline{\lambda}^R$:

$$U_R^{FA}(p) := p \frac{\overline{\lambda}^R u_r^R - c}{\rho + \overline{\lambda}^R} + (1-p) \frac{\overline{\lambda}^R u_\ell^L - c}{\rho + \overline{\lambda}^R} = \frac{\overline{\lambda}^R \left(p u_r^R + (1-p) u_\ell^L\right) - c}{\rho + \overline{\lambda}^R},$$

and at rate $\overline{\lambda}^L$:

$$U_L^{FA}(p) := p \frac{\overline{\lambda}^L u_r^R - c}{\rho + \overline{\lambda}^L} + (1-p) \frac{\overline{\lambda}^L u_\ell^L - c}{\rho + \overline{\lambda}^L} = \frac{\overline{\lambda}^L \left(p u_r^R + (1-p) u_\ell^L\right) - c}{\rho + \overline{\lambda}^L}.$$

Generalizing Lemma 5.(c) we obtain now obtain:

$$U_\ell(\underline{p}^*) = U_R^{FA}(\underline{p}^*), \qquad \text{and} \qquad U_r(\overline{p}^*) = U_r^{FA}(\overline{p}^*).$$

Lemma 5.(d) refers to the condition (EXP). If $\overline{\lambda}^R > \overline{\lambda}^L$, we need to define two separate conditions

$$U_R^{FA}(\hat{p}) > U(\hat{p}), \tag{EXP$_R$}$$

$$U_L^{FA}(\hat{p}) > U(\hat{p}). \tag{EXP$_L$}$$

With this Lemma 5.(d) generalizes as follows: If (EXP$_R$) holds, $0 < \underline{p}^* < \hat{p}$ and $\underline{V}_{own}(p) < U_R^{FA}(p)$ for all $p \in (\underline{p}^*, 1)$, $\underline{V}_{own}(p) > U_R^{FA}(p)$ for $p < \underline{p}^*$ and $\underline{V}_{own}(p) = U_R^{FA}(p)$ if $p \in \left\{\underline{p}^*, 1\right\}$. If (EXP$_L$) holds, $0 < \underline{p}^* < \overline{p}^* < \hat{p}$ and $\overline{V}_{own}(p) < U_L^{FA}(p)$ for all $p \in (0, \overline{p}^*)$, $\overline{V}_{own}(p) > U_L^{FA}(p)$ for $p > \overline{p}^*$ and $\overline{V}_{own}(p) = U_L^{FA}(p)$ if $p \in \{0, \overline{p}^*\}$. Lemma (5).(e) generalizes as follows: If (EXP$_L$) is violated, then $\overline{V}_{own} = U(p)$ for all $p \in [\hat{p}, 1]$. If (EXP$_R$) is violated, then $V_{own} = U(p)$ for all $p \in [0, \hat{p}]$.

[7] Lemma 5 repeats the statements of Lemma 1 so we do not discuss Lemma 1 separately.

(c) Properties of the opposite-biased solution in Lemma 6: As in the main model, we can use smooth pasting and value matching with U^S to obtain p^* as follows:

$$p^* = \frac{\left(u_\ell^L\rho + c\right)\overline{\lambda}^L}{\left(u_r^R\rho + c\right)\overline{\lambda}^R + \left(u_\ell^L\rho + c\right)\overline{\lambda}^L}. \tag{C.7}$$

As before we obtain the branches of the opposite-biased strategy as particular solutions V_0 and V_1 with the boundary condition $V_0(p^*), V_1(p^*) = U^S(p^*)$, and set

$$V_{own}(p) := \begin{cases} \underline{V}_{own}(p), & \text{if } p < p^*, \\ \overline{V}_{own}(p), & \text{if } p \geq p^*. \end{cases}$$

With this definition Lemma 6.(a) holds unchanged. Part (b) of the Lemma has to be modified: $V_{opp}(p) = \underline{V}_{opp}(p) \leq U_L^{FA}(p)$ for all $p \in [0, p^*]$ with strict inequality for $p \neq p^*$, and $V_{opp}(p) = \overline{V}_{own}(p) \leq U_R^{FA}(p)$ for all $p \in [p^*, 1]$ with strict inequality for $p \neq p^*$.

The fact that the important Lemmas 2 and 3 continue to hold and we still have $V_{opp}(p) > U^S(p)$ for all $p \neq p^*$ from Lemma 6.(a), together imply that many of the structural properties of $V_{Env}(p) = \max\{V_{own}(p), V_{opp}(p)\}$ are preserved and the structure of the optimal policy is similar to the main model with $\overline{\lambda}^R = \overline{\lambda}^L$.

However, there is one crucial difference. It is now possible that $\overline{V}_{own}(p)$ is dominated by $\underline{V}_{own}(p)$ or by $V_{opp}(p)$ for all $p < \overline{p}^*$. We only consider the case that $V_{own}(p^*) > U^S(p^*)$. In this case we can use similar steps as in the proof of Proposition 8.(a) to show that $V_{Env}(p) = V_{own}(p)$ for all $p \in [0, 1]$, i.e., opposite-biased learning is never optimal. However, it is no longer guaranteed that there exists a point of intersection $\check{p} \in (\underline{p}^*, \overline{p}^*)$ between $\underline{V}_{own}(p)$ and $\overline{V}_{own}(p)$. This is most easily seen by considering the generalization of Lemma 5.(c) outlined above. It is easy to see that $U_R^{FA}(p) > U_L^{FA}(p)$ for all $p \in [0, 1]$ since $\overline{\lambda}^R > \overline{\lambda}^L$. Since both functions are strictly decreasing in c, we can find levels of c for which $U_L^{FA}(p) < U(p)$ for all $p \in [0, 1]$ but $U_R^{FA}(p) > U(p)$ for some p. In this case

$$V_{own}(p) = \max\left\{\underline{V}_{own}(p), \overline{V}_{own}(p)\right\} = \max\left\{\underline{V}_{own}(p), U_r(p)\right\}, \quad \forall p \in [0, 1]. \tag{C.8}$$

More generally, (C.8) may also hold if $U_L^{FA}(p) > U(p)$ for some $p \in [0, 1]$. Before we could rule out this case since for $\overline{\lambda}^R = \overline{\lambda}^L$, $\underline{V}_{own}(p) < U^{FA}(p)$ for all $p \in (\underline{p}^*, 1)$ and $\overline{V}_{own}(\overline{p}^*) = U^{FA}(\overline{p}^*)$. This is not longer true if $\overline{\lambda}^R > \overline{\lambda}^L$. The example in Panel (a) of Figure 7 depicts such a case. Moreover we can argue that, as claimed in Section VI.C, this case only arises if $\overline{\lambda}^R - \overline{\lambda}^L$ is sufficiently large. To see this fix $\overline{\lambda}^R$ such that $\underline{V}_{own}(p) > U(p)$ for some p. Note that $\underline{V}_{own}(p)$ is independent of $\overline{\lambda}^L$, since $\underline{p}^*$ does not depend on $\overline{\lambda}^L$ and $\underline{V}_{own}(p)$ is the value of seeking R-evidence. Moreover, we have $\underline{V}_{own}(\underline{p}^*) > U_r(\underline{p}^*)$ and one can easily verify that $\underline{V}_{own}(1) = U_R^{FA}(1) < U_r(p)$. Since $U_r(p)$ is linear in p and $\underline{V}_{own}(p)$

can be verified to be strictly convex, there exists a unique $\overline{q}$ such that $\overline{V}_{own}(\underline{q}) = U_r(\overline{q})$. If $\overline{q} \geq \overline{p}^*$, the Crossing Lemma 2 implies that there exists no intersection of $\underline{V}_{own}(p)$ and $\overline{V}_{own}(p)$ between $\underline{p}^*$ and $\overline{p}^*$. In this case (C.8) holds. Conversely, if $\overline{q} < \overline{p}^*$ an intersection point $\check{p} \in (\underline{p}^*, \overline{p}^*)$ exists and $V_{own}(p)$ has the same structure as in our main model. It remains to argue that $\overline{q} \geq \overline{p}^*$ only if $\overline{\lambda}^R - \overline{\lambda}^L$ is sufficiently large. For $\overline{\lambda}^R - \overline{\lambda}^L = 0$, $\underline{V}_{own}(\overline{p}^*) < U_R^{FA}(\overline{p}^*) = U_L^{FA}(\overline{p}^*) = \overline{V}_{own}(\overline{p}^*) = U_r(\overline{p}^*)$. Therefore, $\overline{q} < \overline{p}^*$. Decreasing λ^L while holding λ^R fixed does not change $\overline{q}$ but decreases $\overline{p}^*$. Hence, there exists a cutoff for λ^L below which (for given λ^R), $\overline{q} \geq \overline{p}^*$.

C.4 Diminishing Returns to Attention

As an extension to the main model in Section II, we show that the general structure of the solution is preserved if the arrival rate of breakthroughs from a given news source does not increase linearly in the amount of attention allocated to the source. For the proofs we adopt a different notation than in Section VI.D. Note that each choice of attention gives rise to a pair of arrival rates (λ^R, λ^L). For given $g(x)$ a pair (λ^R, λ^L) is feasible if there exists $\alpha \in [0, 1]$ such that $\lambda^R \leq \lambda g(\alpha)$ and $\lambda^L \leq \lambda g(1 - \alpha)$. Instead of working with the function $g(x)$ we introduce a function $\Gamma(\lambda^R)$ that characterizes the upper bound of the set of feasible pairs (λ^R, λ^L) as follows:[8]

$$\left\{(\lambda^R, \lambda^L) \in \mathbb{R}_+ \,\middle|\, \lambda^L \leq \Gamma(\lambda^R)\right\}.$$

Remember from Section VI.D that $\lambda^R = \lambda g(\alpha)$ and $\lambda^L = \lambda g(1 - \alpha)$. If we normalize $\lambda = 1$, we can derive $\Gamma(\lambda)$ from the function $g(x)$:[9]

$$\Gamma(\lambda^R) = g(1 - g^{-1}(\lambda^R)).$$

Clearly, the DM will only chose pairs of arrival rates on the upper bound, i.e., $(\lambda^R, \Gamma(\lambda^R))$, so we can describe her choice by λ^R. To simplify the notation we omit the superscript and write λ instead of λ^R. Moreover we assume that $\lambda \in [0, 1]$. We

[8] The feasible set of arrival rates can also be derived from a model with many news sources but constant returns to attention. In this model, a news source is now characterized by two parameters (λ^R, λ^L). If an amount of attention α_i is directed to a news-source given by $(\lambda_i^R, \lambda_i^L)$, the DM will receive a signal from that source that confirms state R with Poisson arrival rate $\lambda_i^R \alpha_i$ if the state is indeed R and she will receive a signal that confirms state L with Poisson arrival rate $\lambda_i^L \alpha_i$ if the state is L. Hence, when allocating her attention over two news sources with parameters $(\lambda_i^R, \lambda_i^L)$ and $(\lambda_j^R, \lambda_j^L)$ with attention levels α_i and $\alpha_j = 1 - \alpha_i$, the DM will receive a signal that confirms R with Poisson rate $\lambda^R = \alpha_i \lambda_i^R + (1 - \alpha_j)\lambda_j^R$, and a signal that confirms L with Poisson rate $\lambda^L = \alpha_i \lambda_i^L + (1 - \alpha_i)\lambda_j^L$. The set of feasible arrival rates (λ^R, λ^L) is thus a weakly convex subset of $\mathbb{R}_+$. We denote the upper bound of this set $\Gamma(\lambda^R)$ and note that weak convexity of the set implies weak concavity $\Gamma(\lambda^R)$. In the main model studied before we had $\Gamma(\lambda^R) = 1 - \lambda^R$, which is the linear boundary that is spanned by the two primitive news sources given by $(1, 0)$ and $(0, 1)$.

[9] The normalization of the upper bound is without loss of generality since only the ratios ρ/λ and c/λ matter.

maintain the following assumptions about the function Γ.

Assumption 1. $\Gamma : [0,1] \to [0,1]$ *is twice continuously differentiable, strictly decreasing, strictly convex, and satisfies* $\Gamma(0) = 1$, $\Gamma(1) = 0$ *and* $\Gamma'(\gamma) = -1$, *where* γ *is the unique fixed point of* Γ.

Note that $\Gamma'(\gamma) = -1$ is always fulfilled if Γ is derived from a differentiable function g since $\Gamma(\Gamma(x)) = x$ in this case which implies that the graph of Γ is symmetric with respect to the 45-degree line.

Example 1. A parametric example is obtained by setting $g(x) = \sqrt{1 + 4x + x2} - 2$. The inverse is $g^{-1}(x) = 2\sqrt{4 - 2x - x^2}$ and we obtain

$$\Gamma(\lambda^R) = g(1 - g^{-1}(\lambda^R)) = \sqrt{6\sqrt{4 - 2\lambda^R - (\lambda^R)^2} + \lambda^R(2 + \lambda^R) - 8} - 1.$$

This is the example used in Figure 8 in Section VI.D.

C.4.1 The Decision Maker's Problem

The DM's posterior evolves according to

$$\dot{p}_t = -p_t(1 - p_t)\left(\lambda_t - \Gamma(\lambda_t)\right), \tag{C.9}$$

The objective is given by

$$J\left((\lambda_t)_{t\geq 0}, T; p_0\right) := \begin{Bmatrix} \int_0^T e^{-\rho t} P_t(p_0, (\lambda_\tau))\left(p_t \lambda_t u_r^R + (1 - p_t)\Gamma(\lambda_t) u_\ell^L\right) dt \\ + e^{-\rho T} P_T(p_0, (\lambda_\tau)) U(p_T) \end{Bmatrix},$$
$$\text{where}\quad P_t(p_0, (\lambda_\tau)) := p_0 e^{-\int_0^t \lambda_s ds} + (1 - p_0) e^{-\int_0^t \Gamma(\lambda_s) ds}.$$

The DM solves the problem ($\mathcal{P}^\Gamma$) given by:

$$V(p_0) := \sup_{\left((\lambda_t)_{t\geq 0}, T\right)} J\left((\lambda_t)_{t\geq 0}, T; p_0\right) \quad \text{s.t. (C.9), and } \lambda_t \in [0,1]. \tag{$\mathcal{P}^\Gamma$}$$

We define

$$H(p, V(p), V'(p), \lambda) := \begin{Bmatrix} \lambda p\left(u_r^R - V(p)\right) + \Gamma(\lambda)(1 - p)\left(u_\ell^L - V(p)\right) \\ -p(1 - p)(\lambda - \Gamma(\lambda)) V'(p) \end{Bmatrix}.$$

The HJB equation for ($\mathcal{P}^\Gamma$) is

$$\max\left\{-c - \rho V(p) + \max_{\lambda\in[0,1]} H(p, V(p), V'(p), \lambda), U(p) - V(p)\right\} = 0. \tag{C.10}$$

If $V(p) > U(p)$ this simplifies to

$$c + \rho V(p) = \max_{\lambda \in [0,1]} H(p, V(p), V'(p), \lambda). \tag{C.11}$$

The first-order condition is given by

$$\frac{\partial H(p, V(p), V'(p), \lambda)}{\partial \lambda} = \left\{ \begin{matrix} p\left(u_r^R - V(p)\right) + \Gamma'(\lambda)(1-p)\left(u_\ell^L - V(p)\right) \\ -p(1-p)(1-\Gamma'(\lambda))V'(p) \end{matrix} \right\} = 0. \tag{C.12}$$

For a given policy $\lambda(p)$, we obtain the differential equation

$$\begin{aligned} c + \rho V(p) &= H(p, V(p), V'(p), \lambda(p)) \qquad \text{(C.13)} \\ \iff c + \rho V(p) &= \left\{ \begin{matrix} \lambda(p)p\left(u_r^R - V(p)\right) + \Gamma(\lambda(p))(1-p)\left(u_\ell^L - V(p)\right) \\ -p(1-p)(\lambda(p) - \Gamma(\lambda(p)))V'(p) \end{matrix} \right\}. \end{aligned}$$

As in our original model, we will define two candidate value functions. For this purpose, we state the HJB equation for problems in which the DM is either restricted to choose $\lambda \geq \gamma$,

$$c + \rho V_+(p) \quad = \max\nolimits_{\lambda \in [\gamma,1]} H(p, V_+(p), V_+'(p), \lambda), \tag{C.14}$$

or $\lambda \leq \gamma$:

$$c + \rho V_-(p) \quad = \max\nolimits_{\lambda \in [0,\gamma]} H(p, V_-(p), V_-'(p), \lambda). \tag{C.15}$$

we denote policies corresponding to solution to (C.14) and (C.15) by $\lambda_+(p)$ and $\lambda_-(p)$, respectively.

C.4.2 Preliminary results

We first revisit some definitions made for the original model. The stationary strategy is now given by choosing $\lambda = \gamma$ until a signal arrives and then taking an optimal action according to the signal. The value of this strategy is now given by

$$U^S(p) = \frac{\gamma}{\rho + \gamma} U^*(p) - \frac{c}{\rho + \gamma},$$

where

$$U^*(p) = p u_r^R + (1-p) u_\ell^L$$

is the first best value that is achieved if the DM can learn the state without any delay.

As in the original model, we obtain a crossing condition for functions that satisfy (C.14) and (C.15) and a condition under which solutions to (C.14) and (C.15) satisfy

(C.11).

Lemma 8 (Crossing Lemma)**.** *Suppose $V_+(p)$ is $\mathcal{C}^1$ at p and satisfies (C.14) and $V_-(p)$ is $\mathcal{C}^1$ at p and satisfies (C.15). If $V_+(p) = V_-(p) \geq U^S(p)$, then $V_+'(p) \leq V_-'(p)$. If $V_+(p) = V_-(p) > U^S(p)$, then $V_+'(p) < V_-'(p)$.*

Proof of Lemma 8. Suppose $V(p) := V_+(p) = V_-(p) \geq U^S(p)$ at p and denote the maximizers in (C.14) and (C.15) by $\lambda_+(p)$ and $\lambda_-(p)$ respectively.

From (C.14) and (C.15), we obtain

$$\begin{aligned}
&p(1-p)(\Gamma(\lambda_-(p)) - \lambda_-(p))(\lambda_+(p) - \Gamma(\lambda_+(p)))(V_-'(p) - V_+'(p)) \\
&= (\delta(p)\rho + \Delta(p)) \left[V(p) - \frac{\frac{\Delta(p)}{\delta(p)}}{\frac{\Delta(p)}{\delta(p)} + \rho} U^*(p) + \frac{1}{\frac{\Delta(p)}{\delta(p)} + \rho} c \right] \\
&\geq (\delta(p)\rho + \Delta(p)) \left[V(p) - \frac{\gamma}{\gamma + \rho} U^*(p) + \frac{1}{\gamma + \rho} c \right] \\
&= (\delta(p)\rho + \Delta(p)) \left[V(p) - U^S(p) \right],
\end{aligned}$$

where

$$\begin{aligned}
\delta(p) :=& \Gamma(\lambda_-(p)) - \lambda_-(p) + \lambda_+(p) - \Gamma(\lambda_+(p)) > 0, \\
\Delta(p) :=& \lambda_+(p)\Gamma(\lambda_-(p)) - \lambda_-(p)\Gamma(\lambda_+(p)) > 0,
\end{aligned}$$

since $\lambda_+(p) > \gamma > \lambda_-(p)$. The inequality can be seen as follows. First, one can verify that $(\Delta(p)/\delta(p), \Delta(p)/\delta(p))$ is the point of intersection between the forty-five degrees line and the line segment between two points, $(\lambda_-(p), \Gamma(\lambda_-(p)))$ and $(\lambda_+(p), \Gamma(\lambda_+(p)))$. Since Γ is concave, we must have $\Delta(p)/\delta(p) < \gamma$. Since $\delta(p), \Delta(p) \geq 0$, if $V(p) \geq U^S(p)$, the last expression is non-negative, and if $V(p) > U^S(p)$, it is strictly positive. $\square$

Lemma 9 (Unimprovability)**.** *(a) Suppose $V_+(p)$ is $\mathcal{C}^1$ at p and satisfies (C.14). If $V_+(p) \geq \max\{U^S(p), U(p)\}$, then $V_+(p)$ satisfies (C.11) at p.*

(b) Suppose $V_-(p)$ is $\mathcal{C}^1$ at p and satisfies (C.15). If $V_-(p) \geq \max\{U^S(p), U(p)\}$, then $V_-(p)$ satisfies (C.11) at p.

Proof of Lemma 9. We prove the first statement; the second follows symmetrically. Suppose the optimal policy satisfies $\lambda_+(p) > \gamma$. By the condition, it is not improvable by an immediate action or by any $\lambda \geq \gamma$. Hence, it suffices to show that it is not improvable by any $\lambda_- < \gamma$.

Substituting $V_+'(p)$ from (C.14) and rearranging we get

$$H(p, V_+(p), V_+'(p), \lambda_+(p)) - H(p, V_+(p), V_+'(p), \lambda_-)$$

$$
\begin{aligned}
=&\frac{\hat{\delta}(p)\rho+\hat{\Delta}(p)}{\lambda_+(p))-\Gamma(\lambda_+(p)))}\left[V_+(p)-\frac{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)}}{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)}+\rho}U^S+\frac{1}{\frac{\hat{\Delta}(p)}{\hat{\delta}(p)}+\rho}c\right]\\
\geq&\frac{\hat{\delta}(p)\rho+\hat{\Delta}(p)}{\lambda_+(p))-\Gamma(\lambda_+(p)))}\left[V_+(p)-U^S(p)\right],
\end{aligned}
$$

where

$$
\hat{\delta}(p):=\Gamma(\lambda_-)-\lambda_-+\lambda_+(p)-\Gamma(\lambda_+(p)) \text{ and } \hat{\Delta}(p):=\lambda_+(p)\Gamma(\lambda_-)-\lambda_-\Gamma(\lambda_+(p)).
$$

The inequality follows from the same observation as in the proof of Lemma (8). $\square$

Before constructing the value function for ($\mathcal{P}^\Gamma$), we make one general observation about the boundaries of the experimentation region and the value opposite-biased signals at the boundaries.

For this purpose we consider a model in which the DM has full attention. In this case we have $\lambda^R=1=\lambda^L$ and the DM only chooses when to stop. Note that Assumption 1 precludes the DM from choosing $\lambda^R=1=\lambda^L$ so the full attention model only serves as a hypothetical benchmark.

The value of this stopping problem is given by

$$
\widehat{V}(p):=\max\left\{U(p),U^{FA}(p)\right\},
$$

where

$$
U^{FA}(p)=\frac{1}{\rho+1}U^*(p)-\frac{c}{\rho+1}.
$$

Moreover, we note that by Assumption 1, $(\lambda,\Gamma(\lambda))\leq(1,1)$ for all $\lambda\in(0,1)$. Therefore, $\widehat{V}(p)$ is an upper bound for the value function of the problem ($\mathcal{P}^\Gamma$).

Remember that in our original model, the boundaries of the experimentation region are given by the points of intersection between $U^{FA}(p)$ and $U(p)$:

$$
U^{FA}(\overline{p}^*)=U_r(\overline{p}^*). \tag{C.16}
$$

$$
U^{FA}(\underline{p}^*)=U_\ell(\underline{p}^*). \tag{C.17}
$$

If (EXP) is satisfied, we have $\underline{p}^*<\overline{p}^*$. We now show that the value of ($\mathcal{P}^\Gamma$) is equal to $\widehat{V}$ at these boundaries. This immediately shows that $\underline{p}^*$ and $\overline{p}^*$ are the boundaries of the experimentation region in ($\mathcal{P}^\Gamma$). Moreover, we show that under Assumption 1, at these boundaries, the DM does not benefit from interior choices $\lambda\in(0,1)$.

Proposition 10. *Suppose (EXP) is satisfied. Then $\underline{p}^*$ and $\overline{p}^*$ given by (C.16) and (C.17) are the boundaries of the the experimentation region for the optimal solution to ($\mathcal{P}^\Gamma$). At $\underline{p}^*$ and $\overline{p}^*$, the value of ($\mathcal{P}^\Gamma$) coincides with the value of our original model and equals*

$U^{FA}(p)$ The loss of restricting the DM to chose $\lambda \in \{0,1\}$ vanishes as $p \downarrow \underline{p}^*$ and $p \uparrow \overline{p}^*$.

Proof. If the DM is restricted to chose $\lambda \in \{0,1\}$, her optimal strategy coincides with the optimal strategy in our original model. The value in our original model is a lower bound for the value of ($\mathcal{P}^\Gamma$). Since at $\underline{p}^*$ and $\overline{p}^*$ the value in our original model coincides with the upper bound $U^{FA}(p)$, it must also coincide with the value of ($\mathcal{P}^\Gamma$). □

Note that while Assumption 1 requires $\Gamma(\lambda) < 1$ for $\lambda > 0$, it does not rule out an Inada condition like $\lim_{\lambda\to 0} \Gamma'(\lambda) = 0$. This shows that at the boundaries of the experimentation region, the value of a opposite-biased signal is zero even if it is cost-less to obtain. We will see below when we characterize the value function that without an Inada condition, there exist neighborhoods of $\underline{p}^*$ and $\overline{p}^*$ such that the DM does not suffer any loss if in these neighborhoods she uses $\lambda = 1$ and $\lambda = 0$, respectively.

C.4.3 Construction of Solutions to the HJB equation

For the remainder of this section, we will focus on the cases that the payoffs are symmetric. This simplifies the derivations and is sufficient to understand the main features of the optimal solution in the extension. Formally we impose:

Assumption 2. *$u_r^R = u_\ell^L = U^S$ and $u_\ell^R = u_r^L = \underline{u}$ for some $\overline{u} > \underline{u} > 0$.*

In contrast to our original model, it may now be optimal to choose $\lambda \in (0,1)$ for beliefs $p \in \left(\underline{p}^*, \overline{p}^*\right)$, i.e., in the interior of the experimentation region. For an interval where this is the case, we will obtain a differential equation for $\lambda(p)$ and furthermore an equation that expresses $V(p)$ as a function of $\lambda(p)$. We begin with the latter. To state the result in concise form we define

$$A(\lambda) := \frac{\Gamma(\lambda) - \Gamma'(\lambda)\,\lambda}{\Gamma(\lambda) - \Gamma'(\lambda)\,\lambda + \rho\,(1 - \Gamma'(\lambda))}, \quad \text{and} \quad B(\lambda) := \frac{1 - \Gamma'(\lambda)}{\Gamma(\lambda) - \Gamma'(\lambda)\,\lambda + \rho\,(1 - \Gamma'(\lambda))}.$$

A basic observation that we will use at several points is that these two functions are (inverse) U-shaped with (maximum) minimum at $\lambda = \gamma$.

Lemma 10. *If Assumption 1 is satisfied,*

$$A'(\lambda) > (<) 0 \iff B'(\lambda) < (>) 0 \iff \lambda > (<) \gamma.$$

Proof. The Lemma follows from straightforward algebra which we omit here. □

Lemma 11. *Suppose Assumptions 1 and 2 are satisfied. If $p \in (0,1)$, $V(p)$ is continuously differentiable at p and satisfies (C.11) with maximizer $\lambda(p) \neq \gamma$, then*

$$V(p) \geq A(\lambda(p))\overline{u} - B(\lambda(p))\,c \geq U^S(p) \tag{C.18}$$

If λ satisfies (C.12) at p, then the first inequality binds. The statement continues to hold if we replace V, λ, and (C.11), by V_+, λ_+ and (C.14), or V_-, λ_- and (C.15).

Proof of Lemma 11. We define the LHS of (C.12) as

$$X := (p + (1-p)\Gamma'(\lambda))\,(\overline{u} - V(p)) - p(1-p)(1-\Gamma'(\lambda))V'(p). \tag{C.19}$$

Eliminating $V'(p)$ from (C.13) and (C.19) we obtain an expression for $V(p)$ in terms of $\lambda(p)$ and X:

$$V(p) = A(\lambda(p))\overline{u} - B(\lambda(p))\,c + \frac{X\,(\lambda - \Gamma(\lambda(p)))}{\Gamma(\lambda) - \Gamma'(\lambda)\,\lambda + \rho\,(1-\Gamma'(\lambda))}.$$

If $\lambda(p)$ is a maximizer in (C.11), we must have

$$X \begin{cases} \geq 0 & \text{if } \lambda = 1, \\ = 0 & \text{if } \lambda \in (0,1), \\ \leq 0 & \text{if } \lambda = 0. \end{cases}$$

Since $\lambda = 1$ implies $\lambda - \Gamma(\lambda(p)) > 0$ and $\lambda = 0$ implies $\lambda - \Gamma(\lambda(p)) < 0$ we have

$$V(p) \geq A(\lambda(p))\overline{u} - B(\lambda(p))\,c,$$

and the inequality holds with equality if $X = 0$ which is equivalent to λ satisfying (C.12). This proves the first inequality and the first statement.

The second inequality follows from Lemma 10 and $A(\gamma)\overline{u} - B(\gamma)\,c = U^S(p)$, which is obtained from straightforward algebra. It is straightforward to adapt the proofs to V_+ and V_-. $\square$

Using Lemma 11 we can obtain an ODE for λ that holds whenever the optimal policy is interior, i.e., it satisfies (C.12).

Lemma 12. *Suppose Assumptions 1 and 2 are satisfied. If $p \in (0,1)$, V is continuously differentiable at p and satisfies (C.13) and the maximizer is $\lambda(p) \neq \gamma$ and satisfies (C.12) at p, then*

$$\lambda'(p) = \frac{[p + (1-p)\Gamma'(\lambda(p))]\,[\Gamma(\lambda(p)) - \Gamma'(\lambda(p))\,\lambda(p) + \rho\,(1-\Gamma'(\lambda(p)))]}{p(1-p)\,(\Gamma(\lambda(p)) - \lambda(p))\,\Gamma''(\lambda(p))}. \tag{C.20}$$

The statement continues to hold if we replace V and λ, by V_+ and λ_+, or V_- and λ_-.

Proof of Lemma 12. If $\lambda(p) \neq \gamma$ satisfies (C.12), then by Lemma 11

$$V(p) = A(\lambda(p))\overline{u} - B(\lambda(p))\,c,$$

$$\text{and} \quad V'(p) = A'(\lambda(p))\lambda'(p)\overline{u} - B'(\lambda(p))\lambda'(p)\, c.$$

Inserting these two equations in (C.13) and solving for $\lambda'(p)$ we get (C.20) $\square$

Next, we state a Lemma that identifies conditions under which the solution to (C.20) remains bounded away from $\lambda = 0$ or $\lambda = 1$.

Lemma 13. *Suppose Assumptions 1 and 2 are satisfied. Then there exists function $p^1(x) > 1/2$ for $x > \gamma$ and $p^0(x) < 1/2$ for $x < \gamma$ such that*

$$\begin{aligned}
\lambda(p) = \lambda_+ > \gamma \quad &\Rightarrow \quad \left\{\lambda'(p) < 0 \iff p < p^1(\lambda_+)\right\}, \\
\lambda(p) = \lambda_- < \gamma \quad &\Rightarrow \quad \left\{\lambda'(p) > 0 \iff p > p^0(\lambda_-)\right\}.
\end{aligned}$$

Proof. Inserting $\lambda(p) = \lambda_+ > \gamma$ in (C.20) yields

$$\begin{aligned}
&\lambda'(p) < 0 \\
\iff [p + (1-p)\Gamma'(\lambda_+)]\, &\frac{\Gamma(\lambda_+) - \Gamma'(\lambda_+)\,\lambda_+ + \rho\,(1 - \Gamma'(\lambda_+))}{p(1-p)\,(\Gamma(\lambda(p)) - \lambda(p))\,\Gamma''(\lambda(p))} < 0 \\
&\iff p + (1-p)\Gamma'(\lambda_+) < 0 \\
&\iff p < p^1(\lambda_+) = \frac{|\Gamma'(\lambda_+)|}{1 + |\Gamma'(\lambda_+)|}
\end{aligned}$$

Since $|\Gamma'(\lambda_+)| > 1$ $p^1 > 1/2$. The proof for $\lambda(p) = \lambda_- < \gamma$ is similar. $\square$

Next, we show the following property that relates sufficiency of the FOC (C.12) to convexity of the value function.

Lemma 14. *Suppose Assumptions 1 and 2 are satisfied.*

(a) Let $W : [0,1] \to \mathbb{R}$ be weakly convex and satisfy $W(p) = U(p)$ in neighborhoods of 0 and 1. Then $H(p, W(p), W'(p), \lambda)$ is weakly concave in λ for all p and strictly concave whenever $W(p) > U(p)$.

(b) Let $\lambda(p)$ be a solution to (C.20) such that $\lambda(p) \in (0,1)$ at some p. Let

$$\pi(\ell) = \frac{(\rho + \ell)\,\Gamma'(\ell)}{(\rho + \ell)\,\Gamma'(\ell) - (\rho + \Gamma(\ell))}.$$

Then

$$\frac{\partial^2\left[A(\lambda(p))\overline{u} - B(\lambda(p))c\right]}{\partial p^2} \geq 0 \quad \text{if} \quad \begin{cases} & \lambda(p) > \gamma \text{ and } p \leq \pi(\lambda(p)), \\ \text{or} & \lambda(p) < \gamma \text{ and } p \geq \pi(\lambda(p)). \end{cases}$$

$\pi(\ell) > 1/2$ if $\ell > \gamma$, and $\pi(\ell) < 1/2$ if $\ell < \gamma$.

Proof. (a) Some algebra yields

$$\frac{\partial^2 H(p, W(p), W'(p), \lambda)}{\partial \lambda^2} \leq 0 \qquad \iff W(p) - pW'(p) \leq U^S.$$

The latter inequality is satisfied under the assumptions on W and both are strict if $W(p) > U(p)$.

(b) Differentiating $A(\lambda(p))\overline{u} - B(\lambda(p))c$ with respect to p, substituting $\lambda'(p)$ from (C.20) and differentiating again yields (after some algebra):

$$\begin{aligned} & \frac{\partial^2 \left[A(\lambda(p))\overline{u} - B(\lambda(p))c\right]}{\partial p^2} < 0 \\ \iff & -\frac{\left(p^2 - (1-p)^2\Gamma'(\lambda(p))\right)\left(\rho + \Gamma(\lambda(p)) - (\rho + \lambda(p))\Gamma'(\lambda(p))\right)}{p(1-p)\left(\rho + p\lambda(p) + (1-p)\Gamma(\lambda(p))\right)\Gamma''(\lambda(p))} > \lambda'(p). \end{aligned}$$

Substituting $\lambda'(p)$ from (C.20) in the last line and rearranging we get

$$\left(\lambda(p) - \Gamma(\lambda(p))\right)\left(p\left[\rho + \Gamma(\lambda(p))\right] + (1-p)\left[\rho + \lambda(p)\right]\Gamma'(\lambda(p))\right) < 0.$$

Solving for p this yields an upper bound if $\lambda(p) > \gamma$ so that the first term is positive and a lower bound if $\lambda(p) < \gamma$. The bound is $\pi(\lambda(p))$ in both cases. If $\ell > \gamma > \Gamma(\ell)$ we have

$$\begin{aligned} \pi(\ell) &= \frac{(\rho + \ell)\left|\Gamma'(\ell)\right|}{(\rho + \ell)\left|\Gamma'(\ell)\right| + (\rho + \Gamma(\ell))} \\ &> \frac{(\rho + \ell)\left|\Gamma'(\ell)\right|}{(\rho + \ell)\left|\Gamma'(\ell)\right| + (\rho + \ell)} \\ &= \frac{\left|\Gamma'(\ell)\right|}{\left|\Gamma'(\ell)\right| + 1} \\ &> 1/2. \end{aligned}$$

where the last step follows because Assumption 1 implies that $|\Gamma'(\ell)| > 1$ if $\ell > \gamma$. Similarly we obtain $\pi(\ell) < 1/2$ if $\ell < \gamma$. □

C.4.4 Solution Candidates

Own-Biased Learning The first candidate is obtained by assuming that the DM uses an own-biased attention strategy. In contrast to our original model, where we choose $\lambda \in \{0, 1\}$, we will now also use interior values for λ. In an own-biased strategy, the DM may now receive breakthrough news for both states but with a higher likelihood in the state that she find relatively unlikely. For instance, for low posterior beliefs p, the own-biased strategy involves $\lambda > \gamma$. At the same time, the belief moves in the same direction as the initial bias if now breakthrough arrives: $\dot{p}_t < 0$ if $\lambda > \gamma$. We have already identified the boundaries of the experimentation region.

Lemma 15. *Suppose* (EXP) *is satisfied. Then* $\underline{p}^*$ *and* $\overline{p}^*$ *satisfy*

$$\underline{p}^* = \inf\left\{ p \in [0, \hat{p}) \,\middle|\, c + \rho U_\ell(p) \leq \max_{\lambda \in [\gamma, 1]} \left\{ \begin{array}{l} (\lambda p + \Gamma(\lambda)(1-p))\,(\overline{u} - U_\ell(p)) \\ -p(1-p)(\lambda - \Gamma(\lambda))U'_\ell(p) \end{array} \right\} \right\}, \tag{C.21}$$

$$\overline{p}^* = \sup\left\{ p \in (\hat{p}, 1] \,\middle|\, c + \rho U_r(p) \leq \max_{\lambda \in [0, \gamma]} \left\{ \begin{array}{l} (\lambda p + \Gamma(\lambda)(1-p))\,(\overline{u} - U_r(p)) \\ -p(1-p)(\lambda - \Gamma(\lambda))U'_r(p) \end{array} \right\} \right\}, \tag{C.22}$$

and the maximizers on the right-hand side are given by $\lambda = 1$ *and* $\lambda = 0$*, respectively. Moreover,*

$$\begin{aligned} & U_\ell(\underline{p}^*) \geq A(1)\overline{u} - B(1)c, \\ \textit{and} \quad & U_r(\overline{p}^*) \geq A(1)\overline{u} - B(1)c. \end{aligned}$$

The first inequality is strict if and only if $\Gamma'(1)$ *is finite. The second is strict if and only if* $\Gamma'(0) < 0$*.*

Proof of Lemma 15. We only give the proof for $\underline{p}^*$, the other case is symmetric. Consider the maximization problem in (C.21). The derivative of the objective function simplifies to $p\,(\overline{u} - \underline{u})$. Therefore we can set $\lambda = 1$ and (C.21) reduces to the definition via smooth pasting and value matching as in our original model.

The first inequality is equivalent to

$$\frac{1}{(1+\rho)\Gamma'(1) - \rho} \leq 0,$$

which holds under Assumption 1. The inequality is strict if and only if $\Gamma'(1)$ is finite. The second inequality is equivalent to

$$\frac{\Gamma'(0)}{1 + \rho - \rho\Gamma'(0)} \leq 0,$$

which is strict if and only if $\Gamma'(0) < 0$. □

We are now ready to define the opposite-biased strategy. Given that we impose Assumption 1, we only describe the construction for the left branch which is used for $p \leq 1/2$. There are up to four intervals where the opposite-biased strategy takes a different form. First, for $p \leq \underline{p}^*$, the DM takes immediate action. Then there is an interval $(\underline{p}^*, \underline{q}^b]$ where the DM uses the own-biased strategy from our original model. $\underline{q}^b$ is given by

$$\frac{\partial H(\underline{q}^b, \underline{V}_{own}(\underline{q}^b), \underline{V}'_{own}(\underline{q}^b), 1)}{\partial \lambda} = 0.$$

Rearranging this we get

$$\frac{(1+\rho)\Gamma'(\underline{q}^b)}{\rho-(1+\rho)\Gamma'(\underline{q}^b)}+\underline{q}^b+(1-\underline{q}^b)\left(\frac{1-\underline{q}^b}{\underline{q}^b}\frac{\underline{p}^*}{1-\underline{p}^*}\right)^{\rho}=0,$$

which is equivalent to

$$\underline{V}_{opp}(\underline{q}^b)=A(1)\overline{u}-B(1)c.$$

By Lemma 15, $\underline{q}^b=\underline{p}^*$ if $\Gamma'(1)$ is infinite and otherwise $\underline{q}^b>\underline{p}^*$. If $\underline{q}^b\geq 1/2$ we define the own-biased strategy as in our original model. If $\underline{q}^b<1/2$, Lemma 13 implies that $\lambda'(\underline{q}^b)<0$ if we impose the boundary condition $\lambda(\underline{q}^b)=1$. Denote the unique solution for $p\geq\underline{q}^b$ to (C.20) with $\lambda(\underline{q}^b)=1$ by $\lambda(p;\underline{q}^b,1)$. Since by Lemma 13, $\lambda'(p;p,1)<0$ for all $p\leq 1/2$, we have $\lambda(p;\underline{q}^b,1)<1$ for $p\in(\underline{q}^b,1/2)$. Finally we need to take care of the possibility that there exists $\underline{q}^s\in(\underline{q}^b,1/2]$ such that $\lambda(p;\underline{q}^b,1)=\gamma$. If no such $\underline{q}^s$ exists we set $\underline{q}^s=1/2$. If Assumption 2 is satisfied, a symmetric construction can be used for the right branch with cutoffs $\overline{q}^b=1-\underline{q}^b$ and $\overline{q}^s=1-\underline{q}^s$.

We thus define the opposite biased strategy as follows. For $p\notin\left(\underline{p}^*,\overline{p}^*\right)$: take the optimal immediate action. For $p\in\left(\underline{p}^*,\overline{p}^*\right)$, experiment according to the following attention strategy:

$$\lambda^{\Gamma}_{own}(p)=\begin{cases}1, & \text{if } p\in(\underline{p}^*,\underline{q}^b],\\ \lambda(p;\underline{q}^b,1), & \text{if } p\in(\underline{q}^b,\underline{q}^s],\\ \gamma, & \text{if } p\in(\underline{q}^s,\overline{q}^s),\\ \lambda(p;\overline{q}^b,0), & \text{if } p\in[\overline{q}^s,q^b),\\ 0, & \text{if } p\in[q^b,\overline{p}^*),\end{cases}$$

and take an action corresponding to the signal if one is received.[10] Note that by Lemma 13, $\lambda^{\Gamma}_{own}(p)$ is strictly decreasing if $p\in(\underline{q}^b,\underline{q}^s]\cup[\overline{q}^s,q^b)$. The value of this strategy is given by

$$V^{\Gamma}_{own}(p)=\begin{cases}V_{own}(p), & \text{if } p\leq\underline{q}^b,\\ A\left(\lambda(p;\underline{q}^b,1)\right)\overline{u}-B\left(\lambda(p;\underline{q}^b,1)\right)c, & \text{if } p\in(\underline{q}^b,\underline{q}^s],\\ U^S(p), & \text{if } p\in(\underline{q}^s,\overline{q}^s),\\ A\left(\lambda(p;\overline{q}^b,0)\right)\overline{u}-B\left(\lambda(p;\overline{q}^b,0)\right)c, & \text{if } p\in[\overline{q}^s,q^b),\\ V_{own}(p), & \text{if } p\geq q^b,\end{cases}$$

where $V_{own}(p)$ denotes the value of the opposite-biased strategy from our original model. Note that since we focus attention on the symmetric case (Assumption 2), the belief that separates the "left branch" and the "right branch" of the opposite-biased solution is given

[10] If $\underline{q}^s=\overline{q}^s$, $\lambda^{\Gamma}_{own}(\overline{q}^s)\in\left\{\lambda(\underline{q}^s;\underline{q}^b,1),\lambda(\underline{q}^s;\overline{q}^b,0)\right\}$ with an arbitrary tie-breaking rule.

by $\check{p}$. Note also, that in contrast to our original model, we defined the own-biased strategy in a way that it is always weakly greater than $U^S(p)$.

The implied dynamics of the posterior as well as the attention strategy are summarized by the following diagram:

$$\Big| \underbrace{\qquad\qquad\qquad}_{\text{immediate action } b} \underline{p}^* \overbrace{\underbrace{\leftarrow\!\!-\!\!\leftarrow\!\!-}_{\lambda=1} \underline{q}^b \underbrace{\leftarrow\!\!-\!\!\leftarrow\!\!-}_{\lambda\in(\gamma,1)} \underline{q}^s \underbrace{-\check{p}-}_{\lambda=\gamma} \overline{q}^s \underbrace{-\!\!\to\!\!-\!\!\to}_{\lambda\in(0,\gamma)} \overline{q}^b \underbrace{-\!\!\to\!\!-\!\!\to}_{\lambda=0}}^{\text{information acquisition}} \overline{p}^* \underbrace{\qquad\qquad\qquad}_{\text{immediate action } a} \Big|$$

Lemma 16. *Suppose Assumptions 1 and 2 are satisfied. Then V_{own}^{Γ} is continuously differentiable and convex on $[0, \underline{q}^s)$ and on $(\overline{q}^s, 1]$, respectively, and satisfies (C.11) on $[\underline{p}^*, \underline{q}^s)$ and on $(\overline{q}^s, \overline{p}^*]$, respectively.*

Proof. We show the Lemma for $p \le 1/2$. The remaining results follow from a symmetric argument.

We need to show that V_{own}^{Γ} is continuously differentiable at $\underline{q}^b$. For $r > 0$, some algebra yields for $p \le 1/2$[11]

$$\begin{aligned} & V_{own}^{\Gamma} = A(1)\overline{u} - B(1)c \\ \iff & \left(\frac{\underline{p}^*}{1-\underline{p}^*}\frac{1-p}{p}\right)^{\rho} = 1 - \frac{r}{(1-p)\left(\rho - (1+\rho)\Gamma'[1]\right)}. \end{aligned}$$

Substituting this expression in $V_{own}^{\Gamma\prime}(p)$ yields

$$\begin{aligned} V_{own}^{\Gamma\prime}(p)\big|_{V_{own}(p)=A(1)\overline{u}-B(1)c} &= \frac{\left(c+\rho U^S\right)\left(p+(1-p)\Gamma'[1]\right)}{(1-p)\,p\left(\rho-(1+\rho)\Gamma'[1]\right)} \\ &= \left.\frac{\partial\left[A(\lambda(p))\overline{u} - B(\lambda(p))c\right]}{\partial\lambda}\right|_{\lambda(p)=1}. \end{aligned}$$

Convexity on $[\underline{p}^*, \underline{q}^s]$ follows from strict convexity of V_{own} (Lemma 5) and strict convexity of $A(\lambda(p))\overline{u} - B(\lambda(p))c$ (Lemma 14.(b)) and continuous differentiability.

Note that by Lemma 9, it suffices to show that V_{own}^{Γ} satisfies (C.14) for all $[\underline{p}^*, \underline{q}^s)$ since $V_{own}^{\Gamma}(p) > U^S(p)$ for $p < \underline{q}^s$. We have derived V_{own}^{Γ} from the first order-condition (C.12) and the respective Kuhn-Tucker condition of $p < \underline{q}^b$. Therefore it suffices to show that the maximization problem in the HJB equation is concave. By Lemma 14.(a), this is the case since we have shown that V_{own}^{Γ} is weakly convex. $\square$

Opposite-Biased Learning The second candidate for the value function is obtained by assuming that the DM uses an opposite-biased attention strategy. Specifically, we define a "reference belief" p^* such that the DM chooses $\lambda < \gamma$ for lower beliefs $p < p^*$ and $\lambda > \gamma$ for higher beliefs $p > p^*$. The implied dynamics of the posterior as well as the

[11]The derivation for $r = 0$ is similar.

attention strategy are summarized by the following diagram:

$$|\underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\lambda\in[0,\gamma)} p^* \underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\lambda\in(\gamma,1]}|$$

The reference belief is absorbing and we assume that once p^* is reached, the DM adopts the stationary attention strategy $\lambda = \gamma$. Under Assumption 2, we have $p^* = 1/2$. This can also be derived from value matching

$$V(p^*) = U^S(p^*)(= U^S), \tag{C.23}$$

and the tangency condition

$$V'(p^*) = U^{S\prime}(p^*)(= 0). \tag{C.24}$$

Substituting these two conditions together with $\lambda = \gamma$ in (C.12) yields $p^* = 1/2$.[12]

We would now like to construct the opposite-biased strategy in a similar fashion as the own-biased solution, that is, we will identify two types of regions. If $\lambda \in \{0, 1\}$, we will use solutions to (A7) or (A8) (with $\overline{\lambda} = 1$, $\underline{\lambda} = 1$ and α replace by λ.) On the other hand, if $\lambda \in (0, 1)$ we will use solutions to (C.20) with a suitable boundary condition. A problem arises since we want to impose the boundary condition $\lambda(p^*) = \gamma$. Note that this implies $\lambda'(p^*) = 0/0$. We therefore begin by identifying a solution to (C.20) that satisfies $\lambda(p^*) = \gamma$ as well as $\lambda'(p^*) > 0$.

Lemma 17. *Suppose Assumptions 1 and 2 are satisfied. Then there exists a unique continuously differentiable function $\hat{\lambda}_{opp}(p)$ which satisfies (C.20) for all p in a neighborhood of $p^* = 1/2$, such that $\lambda(p^*) = \gamma$ and $\lambda'(p^*) > 0$. The derivative at p^* is given by*

$$\hat{\lambda}'_{opp}(p^*) = -(\rho+\gamma) + \sqrt{(\rho+\gamma)^2 - \frac{8(\rho+\gamma)}{\Gamma''(\gamma)}}.$$

Proof of Lemma 17. The ODE (C.20) can be written as

$$\lambda'(p) = \frac{P(p, \lambda(p))}{Q(p, \lambda(p))},$$

where

$$\begin{aligned} P(p, \lambda) &= \left[\Gamma(\lambda) - \Gamma'(\lambda)\,\lambda + \rho\,(1 - \Gamma'(\lambda))\right] \times \left[p + \Gamma'(\lambda)\,(1-p)\right], \\ Q(p, \lambda) &= p(1-p)\Gamma''(\lambda)\left[\Gamma(\lambda) - \lambda\right]. \end{aligned}$$

Since P and Q are both continuous and have continuous partial derivatives, the behavior

[12] Note that in contrast to the linear model, we cannot use the HJB equation because for $\lambda = \gamma$, $V'(p)$ vanishes so that substituting (C.24) has no bite.

of solutions that go through points in a neighborhood of (p^*, γ) is, under some conditions (see below), the same as for[13]

$$\lambda'(p) = \frac{a\,(p - p^*) + b\,(\lambda(p) - \gamma)}{c\,(p - p^*) + d\,(\lambda(p) - \gamma)}, \tag{C.25}$$

where

$$\begin{aligned}
a &= \partial_p P(p^*, \gamma) = 4\,(\rho + \gamma) > 0, \\
b &= \partial_\lambda P(p^*, \gamma) = (\rho + \gamma)\,\Gamma''(\gamma) < 0, \\
c &= \partial_p Q(p^*, \gamma) = 0, \\
d &= \partial_\lambda Q(p^*, \gamma) = -\frac{1}{2}\Gamma''(\gamma) > 0.
\end{aligned}$$

The characteristic equation is

$$x^2 - bx - ad = 0.$$

Since $ad > 0$, the characteristic equation has two reals roots of opposite sign. This implies that (p^*, γ) is a saddle point and there are two continuously differentiable solutions $\lambda(p)$ that pass through (p^*, γ). In the case of a saddle point, the behavior of the solutions of (C.25) in a neighborhood of (p^*, γ) corresponds to the behavior of the solutions to (C.20). Hence there exist two continuously differentiable solutions $\lambda(p)$ that satisfy the boundary condition $\lambda(p^*) = \gamma$.

Next we want to obtain $\lambda'(p^*)$ for these solutions, and show that only one of them has a positive derivative. We have

$$\begin{aligned}
\lambda'(p^*) = \lim_{p \to p^*} \lambda'(p) &= \lim_{p \to p^*} \frac{P(p, \lambda(p))}{Q(p, \lambda(p))} \\
&= \lim_{p \to p^*} \frac{\partial_p P(p, \lambda(p)) + \partial_\lambda P(p, \lambda(p))\lambda'(p)}{\partial_p Q(p, \lambda(p)) + \partial_\lambda Q(p, \lambda(p))\lambda'(p)} \\
&= \frac{a + b\lambda'(p^*)}{d\lambda'(p^*)}.
\end{aligned}$$

Hence $\lambda'(p^*)$ solves

$$\begin{aligned}
x^2 - \frac{b}{d}x - \frac{a}{d} = 0, \\
\lambda'(p^*) = \frac{b}{2d} \pm \sqrt{\left(\frac{b}{2d}\right)^2 + \frac{a}{d}}.
\end{aligned}$$

Since $a/d > 0$, there is one positive and one negative solution. For the opposite-biased

[13] See e.g. Bronshtein, Semendyayev, Musiol, and Muehlig (2007).

solution, we are interested in a solution that satisfies $\lambda'(p^*) > 0$. Hence we have

$$\lambda'(p^*) = \frac{b}{2d} + \sqrt{\left(\frac{b}{2d}\right)^2 + \frac{4(\rho+\gamma)}{d}}$$
$$= -(\rho+\gamma) + \sqrt{(\rho+\gamma)^2 - \frac{8(\rho+\gamma)}{\Gamma''(\gamma)}}.$$

□

Lemma 17 provides the solution $\hat{\lambda}_{opp}$ which together with $V(p) = A(\hat{\lambda}_{opp}(p))\overline{u}$ defines V_{opp} in a neighborhood of p^*. To extend this definition to $(0,1)$ we first extend $\hat{\lambda}_{opp}$ to the maximal interval $(\underline{q}, \overline{q})$ where $\hat{\lambda}_{opp}(p) \in (0,1) \setminus \{\gamma\}$ unless $p = p^*$.

Lemma 18. *Suppose Assumptions 1 and 2 are satisfied. There exist two points $0 \le \underline{q} < p^* < \overline{q} \le 1$ such that*

(a) $\hat{\lambda}_{opp}(p)$ is well defined as the unique $\mathcal{C}^1$-solution to (C.20) that satisfies the properties in Lemma 17

(b) $\hat{\lambda}_{opp}(p) > \gamma$ if $p > p^*$ and $\hat{\lambda}_{opp}(p) < \gamma$ if $p < p^*$.

(c) Either $\underline{q} = 0$ or $\hat{\lambda}_{opp}(\underline{q}) = 0$.

(d) Either $\overline{q} = 1$ or $\hat{\lambda}_{opp}(\overline{q}) = 1$.

Note that Properties (c) and (d) mean that the interval $(\underline{q}, \overline{q})$ is the maximal interval where $\hat{\lambda}_{opp}(p) \in (0,1)$.

Proof of Lemma 18. Consider the interval $(\underline{q}, p^*)$. $\hat{\lambda}_{opp}(p) \in (0,\gamma)$ in a neighborhood of p^*. Moreover, (C.20) satisfies local Lipschitz continuity if $p \in (0, p^*)$ and $\lambda \neq \gamma$. Hence, if there exists a $\mathcal{C}^1$ solution to (C.20) with initial condition $\hat{\lambda}_{opp}(p^* - \varepsilon) \in (0,\gamma)$ that satisfies $\hat{\lambda}_{opp}(p) \in (0,\gamma)$ for all $p \in (\underline{q}, p^*)$, then it is the unique such solution. We first show that by extending the interval from a neighborhood of p^* to $(\underline{q}, p^*)$, we do not violate $\hat{\lambda}_{opp}(p) < \gamma$. Suppose by contradiction that there exists $p' < p^*$ such that $\lim_{p \searrow p'} \hat{\lambda}_{opp}(p) \nearrow \gamma$. Note that

$$p' + \Gamma'(\gamma)(1-p') < p^* + \Gamma'(\gamma)(1-p^*) = 0.$$

Hence, since $\Gamma'' < 0$,$\lim_{p \searrow p'} \hat{\lambda}'_{opp}(p) \to \infty$ which contradicts $\lim_{p \searrow p'} \hat{\lambda}_{opp}(p) \nearrow \gamma$. Therefore we can extend the domain of $\hat{\lambda}_{opp}(p)$ to the left until either $p = 0$ or $\hat{\lambda}_{opp}(p) = 0$. This completes the proof for $p < p^*$ and the argument for $p > p^*$ is similar. □

If $\underline{q} > 0$ and $\overline{q} < 1$, respectively, then we further extend $\lambda_{opp}(p)$ to $(0,1)$ by setting $\lambda = 0$ for $p < \underline{q}$ and $\lambda = 1$ for $p > \overline{q}$. We define

$$\lambda^{\Gamma}{}_{opp}(p) := \begin{cases} 0, & \text{if } p \le \underline{q}, \\ \hat{\lambda}_{opp}(p), & \text{if } p \in (\underline{q}, \overline{q}), \\ 1, & \text{if } p \ge \overline{q}. \end{cases}$$

The value of this strategy is given by

$$V^{\Gamma}_{opp}(p) := \begin{cases} V_0\left(p; \underline{q}, A(0)\overline{u} - B(0)c\right) & \text{if } p \leq \underline{q}, \\ A(\lambda_{opp}(p))\overline{u} & \text{if } p \in (\underline{q}, \overline{q}), \\ V_1\left(p; \overline{q}, A(1)\overline{u} - B(1)c\right) & \text{if } p \geq \overline{q}. \end{cases}$$

Lemma 19. *Suppose Assumptions 1 and 2 are satisfied. Then $V^{\Gamma}_{opp}(p)$ is a $\mathcal{C}^1$ solution to (C.11) and $V^{\Gamma}_{opp}(p)$ is strictly convex on $\left(\underline{q}, \overline{q}\right)$.*

Proof. The proof has several steps. We give arguments for $p \geq 1/2$. The Lemma then follows by symmetry (Assumption 2) and the fact that $V^{\Gamma}_{opp}(p)$ is constructed in a way that is continuously differentiable at p^* (see (C.24)). Suppose in the following that $p > 1/2$.

First we note that $V^{\Gamma}_{opp}(p)$ is continuously differentiable. This holds by construction for $p \neq \overline{q}$ and at $\overline{q}$ it follows by the same argument as in the proof of Lemma 16.

Second, we shows that $V^{\Gamma}_{opp}(p)$ is strictly convex. For $p > 1/2$, $\lambda^{\Gamma}_{opp}(p) > \gamma$. Therefore, by Lemma 14, strict convexity on $(p^*, \overline{q})$ follows if $p < \pi(\lambda^{\Gamma}_{opp}(p))$ for all $p \in (p^*, \overline{q})$. Note that $\pi(\lambda^{\Gamma}_{opp}(p^*)) = \pi(\gamma) = 1/2$. We show that whenever $p = \pi(\lambda^{\Gamma}_{opp}(p))$, then $\pi'(\lambda^{\Gamma}_{opp}(p))\lambda^{\Gamma\prime}_{opp}(p) > 1$. This implies that $p < \pi(\lambda^{\Gamma}_{opp}(p))$ for all $p \in (p^*, \overline{q})$. We have

$$\begin{aligned} & \pi'(\lambda^{\Gamma}_{opp}(p^*))\lambda^{\Gamma\prime}_{opp}(p^*) > 1 \\ \iff \frac{2 - (\rho+\gamma)\Gamma''(\gamma)}{4(\rho+\gamma)} \left(\sqrt{(r+\gamma)^2 - \frac{8\,(\rho+\gamma)}{\Gamma''(\gamma)}} - (\rho+\gamma) \right) & > 1 \\ & \iff \Gamma''(\gamma) < 0. \end{aligned}$$

for $p > p^*$, we substitute $p = \pi(\lambda^{\Gamma}_{opp}(p))$ in (C.20), which yields (after some algebra)

$$\pi'(\lambda^{\Gamma}_{opp}(p^*))\lambda^{\Gamma\prime}_{opp}(p^*) = 1 + \frac{\Gamma'(\lambda^{\Gamma}_{opp}(p))\left(\rho + \Gamma(\lambda^{\Gamma}_{opp}(p)) - \left(\rho + \lambda^{\Gamma}_{opp}(p)\right)\Gamma'(\lambda^{\Gamma}_{opp}(p))\right)}{\left(\rho + \lambda^{\Gamma}_{opp}(p)\right)\left(\rho + \Gamma(\lambda^{\Gamma}_{opp}(p))\right)\Gamma''(\gamma)} > 1.$$

This completes the proof of convexity on $(p^*, \overline{q})$. For $p > \overline{q}$, convexity has been shown in Lemma 6. Since $V^{\Gamma}_{opp}(p)$ is continuously differentiable at $p = \overline{q}$, $V^{\Gamma}_{opp}(p)$ is strictly convex on $[0, 1]$.

Third, by Lemma 14.(a), convexity implies that the maximization problem in (C.14) is concave so that the first-order condition is sufficient. Therefore, $V^{\Gamma}_{opp}(p)$ satisfies (C.14) or for $p > p^*$.

Finally, convexity, together with (C.23) and (C.24) implies that $V^{\Gamma}_{opp}(p) \geq U^S(p)$ for $p \geq p^*$. Lemma 9 then implies that $V^{\Gamma}_{opp}(p)$ satisfies (C.11). □

Finally we show that $\lambda^{\Gamma}_{opp}(p)$ is strictly increasing.

Lemma 20. *Suppose Assumptions 1 and 2 are satisfied and let $\underline{q}, \overline{q}$ be given as in Lemma*

18. Then $\lambda^{\Gamma}_{opp}(p)$ is strictly increasing on $\left(\underline{q},\overline{q}\right)$.

Proof. For $p \in (\underline{q},\overline{q})$, $V^{\Gamma}_{opp}(p) = A(\lambda_{opp}(p))\overline{u}$. Differentiating with respect to p we get

$$V^{\Gamma\prime}_{opp}(p) = A'(\lambda_{opp}(p))\lambda'_{opp}(p)\overline{u}.$$

Hence if $\lambda'(p) = 0$ for $p \neq 1/2$, we must have $V^{\Gamma\prime}_{opp}(p) = 0$. Since $V^{\Gamma\prime}_{opp}(1/2) = 0$, this violates strict convexity of $V^{\Gamma}_{opp}(p)$. Therefore $\lambda'(p) \neq 0$ for all $p \in (\underline{q},\overline{q})$. Since $\lambda'(1/2) > 0$, this implies that $\lambda'(p) > 0$ if $p \in (\underline{q},\overline{q})$. □

C.4.5 Optimal Solution

As in our original model we show that the value function V^{Γ} is the upper envelope of the two solution candidates. In contrast to our original model, the optimal policy is not a bang-bang solution. We show that inside the own-biased region, $\alpha(p) = g^{-1}(\lambda(p))$ is decreasing whenever it is not a corner-solution. This means that more extreme beliefs lead to a more own-biased news-diet. In the opposite-biased region, $\alpha(p)$ is strictly increasing. This implies that more moderate beliefs lead to a more balanced news-diet.

Theorem 2. *Suppose Assumptions 1 and 2 are satisfied.*

(a) If (EXP) is violated then $V^{\Gamma}(p) = U(p)$ for all $p \in [0,1]$.

(b) If (EXP) is satisfied and $V^{\Gamma}_{own}(p) > U^{S}(p)$ for all $p \neq 1/2$, then $V^{\Gamma}(p) = V^{\Gamma}_{own}(p)$ for all $p \in [0,1]$, and $\alpha(p) = g^{-1}(\lambda(p))$ is strictly decreasing if $V^{\Gamma}(p) > U(p)$ and $\alpha(p) = g^{-1}(\lambda(p)) \in (0,1)$.

(c) If (EXP) is satisfied and $V^{\Gamma}_{own}(p) = U^{S}(p)$ for some $p \neq 1/2$, then $V^{\Gamma}(p) = \max\left\{V^{\Gamma}_{own}(p), V^{\Gamma}_{opp}(p)\right\}$, and $\alpha(p) = g^{-1}(\lambda(p))$ is strictly decreasing if $V^{\Gamma}(p) = V^{\Gamma}_{own}(p) > U(p)$ and $\lambda(p) \in (0,1)$, and strictly increasing if $V^{\Gamma}(p) = V^{\Gamma}_{opp}(p)$.

Proof of Theorem 2. Follows from the same arguments as the proof of Theorem 1. □

C.5 Multiple Actions

In this Appendix, we extend the model in Section II to include a third action $x = m$ which yields u^{R}_{m} and u^{L}_{m} in states R and L. Up to relabeling of the actions it is without loss to assume that $u^{R}_{m} \in \left(u^{R}_{\ell}, u^{R}_{r}\right)$. Further we assume $u^{L}_{m} < u^{L}_{\ell}$ which guarantees that action m does not dominate action ℓ for all beliefs.

The optimal policy will be affected by the availability of action m if it is optimal to take this action for some beliefs. To identify when this is the case, we define a strategy that specifies a stopping region $[\underline{p}_m, \overline{p}_m]$ in which action m is taken immediately. For $p > \overline{p}_m$, the strategy prescribes attention to the L-biased news source ($\alpha = 1$) and for $p < \underline{p}_m$, the strategy prescribes attention to the R-biased news source ($\alpha = 0$). We call this strategy the "m-strategy." It has the following structure:

$$\underset{p=0}{|}\underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}\underline{p}_m\underbrace{\rule{6cm}{0.4pt}}_{\text{immediate action } m}\overline{p}_m\underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}\underset{1}{|}$$

If this strategy is part of the optimal solution (for some range of belief), the boundary points $\underline{p}_m$ and $\overline{p}_m$ must satisfy value-matching and smooth-pasting conditions that resemble those used to define $\underline{p}^*$ and $\overline{p}^*$. We will define $\overline{p}_m$ by imposing smooth pasting and value matching with $U_m(p)$ in (A8):

$$c + \rho U_m(p) = \lambda p\left(u_r^R - U_m(p)\right) - \lambda p(1-p)U_m'(p). \tag{C.26}$$

Similarly we will define $\underline{p}_m$ by imposing smooth pasting and value matching with $U_m(p)$ in (A7):

$$c + \rho U_m(p) = \lambda(1-p)\left(u_\ell^L - U_m(p)\right) + \lambda p(1-p)U_m'(p). \tag{C.27}$$

The following lemma identifies when solutions to (C.26) and (C.27) exist, and when these solutions can be used to define the cutoffs $\underline{p}_m$ and $\overline{p}_m$ in a way the m-strategy only prescribes information acquisition if it is not dominated by immediate action m or by the stationary strategy.

Lemma 21.

(a) Let $u_m^R \geq U^{FA}(1)$ or $c + \rho u_m^L \leq 0$. If $q_1 \in (0,1)$ is a solution to (C.26), then $V_1(p; q_1, U_m(q_1)) \leq U_m(p)$ for all $p \in [q_1, 1]$.

(b) If $u_m^R < U^{FA}(1)$ and $c + \rho u_m^L > 0$, then there exists a unique solution $q_1 \in (0,1)$ to (C.26) given by

$$q_1 = \frac{u_m^L \rho + c}{\rho\left(u_m^L - u_m^R\right) + \left(u_r^R - u_m^R\right)\lambda}. \tag{C.28}$$

and $V_1(p; q_1, U_m(q_1))$ is strictly convex on $[q_1, 1]$.

(c) Let $u_m^L \geq U^{FA}(0)$ or $c + \rho u_m^R \leq 0$. If $q_2 \in (0,1)$ is a solution to (C.27), then $V_0(p; q_2, U_m(q_2)) \leq U_m(p)$ for all $p \in [0, q_2]$.

(d) If $u_m^L < U^{FA}(0)$ and $c + \rho u_m^R > 0$, then there exists a unique solution $q_2 \in (0,1)$ to (C.27) given by

$$q_2 = \frac{\left(u_\ell^L - u_m^L\right)\lambda - u_m^L \rho - c}{\rho\left(u_m^R - u_m^L\right) + \left(u_\ell^L - u_m^L\right)\lambda} \tag{C.29}$$

and $V_0(p; q_2, U_m(q_2))$ is strictly convex on $[0, q_2]$.

(e) Suppose $u_m^R < U^{FA}(1)$ and $c + \rho u_m^L > 0$, and $u_m^L < U^{FA}(0)$ and $c + \rho u_m^R > 0$. If $U_m(q_1) \geq U^S(q_1)$ and $U_m(q_2) \geq U^S(q_2)$, then $q_1 \geq q_2$.

Proof. For (a) and (b) we note that the general solution to (A8) is given by

$$V_1(p) = \underbrace{\frac{p\rho u_r^R \lambda - c\,(\rho + (1-p)\lambda)}{\rho(\rho+\lambda)}}_{=:z(p)} + \left(\frac{1-p}{p}\right)^{\frac{\rho}{\lambda}} (1-p)\,C,$$

there C is the constant of integration. Clearly, the sign of C determines whether the solution is convex or concave since

$$\frac{d^2}{dp^2}\left(\left(\frac{1-p}{p}\right)^{\frac{\rho}{\lambda}} (1-p)\right) > 0$$

Moreover, we note the $V_1(1) = U^{FA}(1)$ regardless of the value of the constant C.

For the proof of (a) we distinguish several cases: Case 1: If $u_m^R = U^{FA}(1)$ and $c+\rho u_m^L = 0$. In this case, $U_m(p) = z(p)$ for all p. Hence any $q_1 \in (0,1)$ satisfies (C.26) and smooth pasting but $V(p; q_1, U_m(q_1)) = U_m(p)$ for all $p \in [0,1]$.

Case 2: $u_m^R > U^{FA}(1)$. We first show that if (C.26) and smooth pasting is satisfied for $p' \in (0,1)$, then $U_m(p') < z(p)$. Suppose by contradiction that $U_m(p') \geq z(p')$. If (C.26) is satisfies at p', then $U_m(p)$ is tangent to $V_1(p; p', U_m(p')$at p' and since $V_1(p; p', U_m(p')) \geq z(p)$, $V_1(p; p', U_m(p'))$ is weakly convex as a function of p. But this implies that $V_1(1; p', U_m(p')) \geq U_m(1) = u_m^R > U^{FA}(1)$. This is a contradiction since we argued above that any solution to (A8) satisfies $V_1(1) = U^{FA}(1)$. Hence (C.26) or smooth pasting is violated at p' if $U_m(p') \geq z(p')$. If $U_m(p') \leq z(p')$, (C.26), and smooth pasting is satisfied for $p' \in (0,1)$, then $V_1(p; p', U_m(p'))$ is strictly concave as a function of p and tangent to $U_m(p)$ at p'. Hence $V_1(p; p', U_m(p')) < U_m(p)$ for all $p > p'$.

Case 3: $u_m^R < U^{FA}(1)$. If $c + \rho u_m^L \leq 0$, then $U_m(0) < z(0)$ and since $z(1) = U^{FA}(1)$ we have $U_m(p) < z(p)$ for all p. As in case 2, if $p' \in (0,1)$ satisfies (C.26) and smooth pasting, then $V_1(p; p', U_m(p')) < U_m(p)$ for all $p > p'$ which contradicts $V_1(1; p', U_m(p')) = U^{FA}(1)$. Hence there is no solution to (C.26) that satisfies smooth pasting. This concludes the proof of (a).

For (b), note that if $u_m^R < U^{FA}(1)$ and $c + \rho u_m^L > 0$, then $U_m(p)$ crosses $z(p)$ from above. As in case 3 in the proof for part (a), $z(p') > U_m(p)$ implies that (C.26) and smooth pasting cannot be both satisfied. Next we identify a solution q_1 to (C.26) for which $V'(q_1; q_1, U_m(q_1)) = U_m'(q_1)$. If $U_m(q_1) = z(q_1)$ then $V'(q_1; q_1, U_m(q_1)) = z'(q_1) = U_m'(q_1)$. On the hand $\lim_{q_1 \to 0} V'(q_1; q_1, U_m(q_1)) = -\infty$. Therefore, the intermediate value theorem implies that there exists $q_1 \in (0,1)$ such that $V'(q_1; q_1, U_m(q_1)) = U_m'(q_1)$ and simple algebra shows that is is given by (C.28).

The proofs of (c) and (d) follow from a similar argument. For part (e) suppose by contradiction that $q_1 < q2$. Since both $V_1(p; q_1, U_m(q_1))$ and $V_0(p; q_0, U_m(q_0))$ are strictly convex on $[q_1, q_2]$ and coincide with $U_m(p)$ at q_1 and q_2, respectively, there exists $p' \in$

(q_1, q_2) such that $V_1(p'; q_1, U_m(q_1)) = V_0(p'; q_0, U_m(q_0)) > U_m(p')$ and $V_1'(p'; q_1, U_m(q_1)) > V_0'(p'; q_0, U_m(q_0))$. Since $U_m(p) \geq U^S(p)$ for $p \in q_1, q_2$ and both U_m and U^S are linear, we have $V_1(p'; q_1, U_m(q_1)) = V_0(p'; q_0, U_m(q_0)) > U_S(p')$. By Lemma 2 this implies $V_1'(p'; q_1, U_m(q_1)) < V_0'(p'; q_0, U_m(q_0))$ which is a contradiction. Therefore we must have $q_1 \geq q_2$. $\square$

Based on the results of this lemma, we define $\underline{p}_m$ and $\overline{p}_m$ as follows:

$$\overline{p}_m = \begin{cases} q_1, & \text{if } u_m^R < U^{FA}(1),\ c + \rho u_m^L > 0, \text{ and } U_m(q_1) \geq U^S(q_1) \\ 1, & \text{otherwise.} \end{cases}$$

$$\underline{p}_m = \begin{cases} q_2, & \text{if } u_m^L < U^{FA}(0),\ c + \rho u_m^R > 0, \text{ and } U_m(q_2) \geq U^S(q_2) \\ 0, & \text{otherwise.} \end{cases}$$

Consider $\overline{p}_m$. By Lemma 21.(a)-(b) $u_m^R < U^{FA}(1)$ together with $c + \rho u_m^L > 0$ is a necessary and sufficient condition for the existence of a solution in $(0, 1)$ to (C.26) that satisfies smooth pasting and is not dominated by immediate action m. Hence if the necessary and sufficient condition is violated we set $\overline{p}_m = 1$. Similarly, Lemma 21.(c)-(d) motivates the definition of $\underline{p}_m = 0$ if $u_m^B \geq U^{FA}(0)$ and $c + \rho u_m^R \leq 0$.

The requirements that $U_m(q_1) \geq U^S(q_1)$ in the definition of $\overline{p}_m$ and $U_m(q_2) \geq U^S(q_2)$ in the definition of $\underline{p}_m$, guarantee, respectively, that the m-strategy always has the structure depicted in the diagram above because it avoids defining $\overline{p}_m = q_1$ and $\underline{p}_m = q_2$ when $q_2 > q_1$.

The value of the m-strategy is

$$V_m(p) := \begin{cases} V_0(p; \underline{p}_m, U_m(\underline{p}_m)), & \text{for } p < \underline{p}_m, \\ U_m(p), & \text{for } p \in \left[\underline{p}_m, \overline{p}_m\right], \\ V_1(p; \overline{p}_m, U_m(\overline{p}_m)), & \text{for } p > \overline{p}_m. \end{cases}$$

The Lemmas leading to the upper envelope characterization of the value function in Proposition 9 depend on the properties of branches defined by particular solutions to (A7) and (A8). Therefore the same steps can be applied in this extension and we obtain that the value function of the extended problem is given by:

$$V(p) = \max \{V_{own}(p), V_{opp}(p), V_m(p)\}.$$

It is straightforward to extend this to more than three actions. Suppose we have actions ℓ, r as well as additional actions $m_1, m_2, \ldots$, where for all $i = 1, 2, \ldots$, $(u_{m_i}^R, u_{m_i}^A)$ satisfy the conditions formulated for action m at the beginning of this section. In this case we define an m_i-strategy for each of the actions in the same way as above. Denote the value of strategy m_i by $V_{m_i}(p)$. The value function of the DM's problem is then given by

$$V(p) = \max \{V_{own}(p), V_{opp}(p), V_{m_1}(p), V_{m_2}(p), \ldots\}.$$

References

Bardi, M., and I. Capuzzo-Dolcetta (1997): *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birhäuser.

Bronshtein, I. N., K. A. Semendyayev, G. Musiol, and H. Muehlig (2007): *Handbook of Mathematics*. Springer, 5th edn.