Training and Effort Dynamics in Apprenticeship: Online Appendix

Drew Fudenberg and Luis Rayo

1 Proofs of Lemmas A1 and A2

1.1 Proof of Lemma A1

Define $y(X, a, b) := f(X, a) + g(b)$. The conclusion of the lemma will follow from a series of claims.

Claim 1 *The principal obtains a strictly positive profit by contracting with the agent.*

Proof. Fix $X' \in (\underline{X}, \overline{X})$, a', and b' s.t. $y(X', a', b') > \underline{v}$ (which is feasible because $v(\overline{X}) > \underline{v}$) and then pick $T' > 0$ s.t. $e^{-rT'}v(\overline{X}) - (1 - e^{-rT'})c(a' + b') > v(X')$. Now consider the contract where the principal pays 0 wages, brings the agent's knowledge up to X' at time 0, and asks them to maintain efforts (a', b') until time T', at which point the principal brings the agent's knowledge up to $\overline{X}$. This contract satisfies the agent's participation and liquidity constraints, and gives the principal a positive payoff. ■

This proves part 1 of the lemma.

Claim 2 *Any contract where $W_\infty > 0$ is strictly dominated by some finite-duration contract where $W_\infty = 0$.*

Proof. If contract $\mathcal{C}$ with potentially infinite graduation date T prescribes $W_\infty > 0$ and is not strictly dominated, by the previous claim it must have $\Pi_0(\mathcal{C}) > 0$, so $U_0(\mathcal{C}) < \frac{1}{r} v(X_\infty)$. Now let $T' \in (0, T)$ satisfy

$$\left(e^{-rT'} - e^{-rT}\right) \frac{1}{r} v(X_\infty) + \int_{T'}^{T} e^{-rt} c(a_t + b_t)\, dt = W_\infty,$$

and consider a new contract $\mathcal{C}'$ where the agent earns zero wages, graduates at date T' with knowledge X_∞, and for $t < T'$,

$$X'_t, a'_t, b'_t = X_t, a_t, b_t.$$

By construction,

$$\begin{aligned} U_0(\mathcal{C}') &= e^{-rT'}\frac{1}{r}v(X_\infty) - \int_0^{T'} e^{-rt}c(a_t+b_t)\,dt \\ &= e^{-rT}\frac{1}{r}v(X_\infty) + W_\infty - \int_0^{T} e^{-rt}c(a_t+b_t)\,dt = U_0(\mathcal{C}). \end{aligned}$$

In addition, for $t < T'$,

$$\begin{aligned} U_t(\mathcal{C}') - U_t(\mathcal{C}) &= \left(e^{-r(T'-t)} - e^{-r(T-t)}\right)\frac{1}{r}v(X_\infty) \\ &\quad + \int_{T'}^{T} e^{-r(\tau-t)}c(a_\tau+b_\tau)\,d\tau - \int_t^T e^{-r(\tau-t)}w_\tau d\tau \\ &= e^{rt}\left[W_\infty - \int_t^T e^{-r\tau}w_\tau d\tau\right] \geq 0. \end{aligned}$$

As a result, since the original contract satisfied the participation and liquidity constraints, the new contract satisfies them as well.

Finally, we have

$$\begin{aligned} &\Pi_0(\mathcal{C}') + U_0(\mathcal{C}') - [\Pi_0(\mathcal{C}) + U_0(\mathcal{C})] \\ &\geq \int_{T'}^{T} e^{-rt}\left[v(X_\infty) - v(X_t)\right]dt > 0, \end{aligned}$$

where the strict inequality follows from the facts that v is strictly increasing and that $X_t < X_\infty$ for all $t \in (T', T)$. Since $U_0(\mathcal{C}') = U_0(\mathcal{C})$, it follows that $\Pi_0(\mathcal{C}') > \Pi_0(\mathcal{C})$, so $\mathcal{C}'$ strictly dominates $\mathcal{C}$. ■

This proves the first clause in part 2 of the lemma.

Claim 3 *Any infinite-duration contract is strictly dominated by some finite-duration contract with* $W_\infty = 0$.

Proof. In any infinite-duration contract, the initial participation constraint requires $W_\infty \geq \frac{1}{r}\max\{\underline{v}, v(\underline{X})\} > 0$, so from by the previous claim the contract is strictly dominated. ■

Claim 4 *Any finite-duration contract with* $W_\infty = 0$ *and* $X_T < \overline{X}$ *is strictly dominated by some finite-duration contract with* $W_\infty = 0$ *and* $X_T = \overline{X}$.

Proof. If a finite-duration contract with $W_\infty = 0$ has $X_T < \overline{X}$, then there is a time interval Δ and effort levels a', b' such that $y(X', a', b') > \underline{v}$ and $e^{-r\Delta} v(\overline{X}) - (1 - e^{-r\Delta}) c(a' + b') > v(X_T)$, and so the principal could obtain strictly higher profits by extending the agent's contract to $T' = T + \Delta$ paying no additional wages, setting $X_t = X_T$ and $(a_t, b_t) = (a', b')$ for $t \in [T, T')$, and setting $X_{T'} = \overline{X}$. ■

Claims 3 and 4 prove the second clause in part 2 of the lemma.

Claim 5 *Any contract is weakly dominated by some finite-duration contract with* $X_T = \overline{X}$ *and zero wages.*

Proof. From Claims 3 and 4, we can restrict to finite-duration contracts such that $X_T = \overline{X}$ and $W_\infty = 0$. Let S be one such contract, and consider an alternative contract S' that is identical to S except for the fact that wages are 0 at all times. The two contracts deliver identical profits. In addition, for all t,

$$\begin{aligned} e^{-rt} \left[U_t(\mathcal{C}) - U_t(\mathcal{C}') \right] &= \int_t^T e^{-r\tau} w_\tau d\tau \\ &= W_\infty - W_t \leq 0, \end{aligned}$$

where the inequality follows from the fact that $W_t \geq 0$ and $W_\infty = 0$. As a result, $U_t(\mathcal{C}') \geq U_t(\mathcal{C})$ and therefore $\mathcal{C}'$ satisfies the participation constraint. ■

This proves the third clause in part 2 of the lemma and so completes its proof.

1.2 Proof of Lemma A2

We will show each clause of the lemma in turn.

Claim 6 *Any contract is weakly dominated by a contract that sets the agent's participation constraints to hold with equality.*

Proof. In a contract with zero wages, $U_t = e^{-r(T-t)} \frac{1}{r} v\left(\overline{X}\right) - \int_t^T e^{-r(\tau - t)} c\left(a_\tau + b_\tau\right) d\tau$, which is strictly increasing (because $v(\overline{X}) > 0$) and continuous. Thus if $U_t > \frac{1}{r} v(X_t)$ for some times t, the contract with the same effort path and terminal date, and $X'_t = \max\{X_t, v^{-1}(rU_t)\}$ at all times will satisfy the participation constraints and give the principal a weakly higher payoff at each date. Moreover, if the times where $U_t > \frac{1}{r} v(X_t)$ had positive measure, the new contract would give the principal a strictly higher payoff overall. ■

Claim 7 *Any contract is weakly dominated by a contract where at each t total effort $a_t + b_t$ is allocated across tasks to maximize output.*

Proof. Given any contract where at some times $y_t \neq y^*(X_t, (a_t + b_t))$, consider the alternative contract where the time paths of knowledge and total effort are the same but effort is allocated to maximize output at each time. Since the agent's knowledge stock and effort cost are the same, the participation constraints are still satisfied, and the principal does at least as well, and strictly better if the times where $y_t \neq y^*(X_t, (a_t + b_t))$ had positive measure. ■

This completes the proof of the lemma.

2 Training certificates

Suppose the principal has the ability to grant the agent a certificate worth $\Delta > 0$ in flow terms. The agent's outside option is $v(X_t)$ with the certificate, and $v(X_t) - \Delta$ without it, and so the certificate is worth Δ/r in present value. The agent's productivity inside the relationship is independent of the agent being certified. The principal has the option of granting the certificate at any time, but it is without loss to assume that she grants it at the end of the contract, as doing so relaxes the agent's dynamic participation constraint.

Lemma S1 *Every contract is strictly dominated by a contract where:*

1. *The agent earns zero overall wages, that is $W_\infty = 0$.*
2. *The agent receives all knowledge in finite time.*
3. *At all times where $X_t < \overline{X}$, the agent's participation constraint holds with equality.*
4. *At (almost all) times, total effort is allocated to maximize total output.*

Proof. The proof is a straightforward extension of the proof of Lemmas A1 and A2. ■

Lemma S2 *Every optimal contract has two phases. Phase 1: over time interval $[0, T]$, with $T \geq 0$, the agent's participation constraints hold with equality, knowledge is strictly increasing, and $X_T = \overline{X}$. Phase 2: over time interval $(T, T']$, with $T' > T$, the agent's participation constraints are slack and the certificate is granted at time T'.*

Proof. Let T' denote the terminal time when the agent receives the certificate. Lemma S1 implies that the participation constraint for time $t \leq T'$ is

$$\frac{1}{r}\left[v\left(X_t\right)-\Delta\right] \leq e^{-r(T'-t)}\frac{1}{r}v\left(\overline{X}\right)-\int_t^{T'} e^{-r(\tau-t)}c\left(q_\tau\right)d\tau,$$

with equality whenever $X_t < \overline{X}$. Since the right-hand side is increasing and continuous in t, there is a time $T \leq T'$ such that $X_T = \overline{X}$ and $X_t < \overline{X}$ for all $t < T$. Therefore, between times 0 and T the participation constraint holds with equality and knowledge is strictly increasing. Moreover, the participation constraint for time $t \geq T$ is

$$\frac{1}{r}\left[v\left(\overline{X}\right)-\Delta\right] \leq e^{-r(T'-t)}\frac{1}{r}v\left(\overline{X}\right)-\int_t^{T'} e^{-r(\tau-t)}c\left(q_\tau\right)d\tau.$$

Since this constraint holds with equality for $t = T$, and $\Delta > 0$, it follows that $T' > T$. And since the right-hand side is strictly increasing in t, it follows that the participation constraint is slack for all $t > T$. ■

The principal's problem is

$$\max_{u_0,T,T',(q_t)_{t=0}^{T'}} \underbrace{\int_0^T e^{-rt}\ y^*\left(\phi\left(u_t\right),q_t\right)dt}_{\text{phase 1 profits}} + \underbrace{\int_T^{T'} e^{-rt}\ y^*\left(\overline{X},q_t\right)dt}_{\text{phase 2 profits}}$$

subject to constraints

$$\begin{aligned} u_0 &\in \left[\max\left\{v\left(\underline{X}\right),\underline{v}+\Delta\right\},v\left(\overline{X}\right)\right],\ u_T = v\left(\overline{X}\right), \\ \dot{u}_t &= r\left[u_t-\Delta+c\left(q_t\right)\right] \text{ for } 0<t<T, \end{aligned} \tag{1}$$

where $u_t := v\left(X_t\right)$, and subject to the time T participation constraint

$$\frac{1}{r}\left[v\left(\overline{X}\right)-\Delta\right] = e^{-r(T'-T)}\frac{1}{r}v\left(\overline{X}\right)-\int_T^{T'} e^{-r(t-T)}c\left(q_t\right)dt. \tag{2}$$

Since the choice of T', $(q_t)_{t=T}^{T'}$ does not affect phase 1 profits or constraints (1), for any given T the optimal such choice solves

$$\max_{T',(q_t)_{t=T}^{T'}} \underbrace{e^{-rT}\int_T^{T'} e^{-r(t-T)}\ y^*\left(\overline{X},q_t\right)dt}_{\text{phase 2 profits}}$$

subject to (2).

Lemma S3 *For any given T, the optimal T' and effort path $(q_t)_{t=T}^{T'}$ uniquely satisfy (up to a zero-measure subset of times)*

$$\int_T^{T'} e^{-r(t-T)}\ y^*\left(\overline{X}, q^*\left(\overline{X}\right)\right) dt = \Delta/r \text{ and } (q_t)_{t=T}^{T'} = \left(q^*\left(\overline{X}\right)\right)_{t=T}^{T'}.$$

Therefore, the optimized phase 2 profits are $e^{-rT}\Delta/r$.

Proof. After manipulation, constraint (2) is

$$\int_T^{T'} e^{-r(t-T)}\ y^*\left(\overline{X}, q_t\right) dt = \Delta/r + \int_T^{T'} e^{-r(t-T)}\left[y^*\left(\overline{X}, q_t\right) - c\left(q_t\right) - v\left(\overline{X}\right)\right] dt.$$

Since the left-hand side is the principal's objective (measured in period T dollars) and $y^*\left(\overline{X}, q_t\right) - c\left(q_t\right)$ is uniquely maximized at $q_t = q^*\left(\overline{X}\right)$, the unique solution (up to a zero-measure subset of times) is to set $(q_t)_{t=T}^{T'} = \left(q^*\left(\overline{X}\right)\right)_{t=T}^{T'}$ and therefore $\int_T^{T'} e^{-r(t-T)}\ y^*\left(\overline{X}, q_t\right) dt = \Delta/r$. ■

Having solved for T' and $(q_t)_{t=T}^{T'}$, the principal's problem simplifies to

$$\max_{u_0, T, (q_t)_{t=0}^T} \underbrace{\int_0^T e^{-rt}\ y^*\left(\phi\left(u_t\right), q_t\right) dt}_{\text{phase 1 profits}} + \underbrace{e^{-rT}\Delta/r}_{\text{phase 2 profits}}$$

subject to (1).

Other that the second term in the objective and the modified constraints (1) on the state, this problem is identical to the original one. We now solve it by treating T as the terminal time.

Lemma S4 *In the model with a certificate, except for the new constraints (1) on the state variable, and the first-order condition for T, the conclusion in Lemma A3 remains valid. The first-order condition for T now takes the more general form*

$$\mathcal{H}_T = e^{-rT}\Delta.$$

Proof. Define $\phi(T) := e^{-rT}\Delta/r$, $\psi(u_0) := u_0 - \max\left\{v\left(\underline{X}\right), \underline{v} + \Delta\right\}$ and $\Phi(u_0, T) := \phi(T) + \lambda_0\psi(u_0)$. The only difference relative to the proof of Lemma A3 is that Chachuat (2007) Theorem 3.18 now requires that $\mathcal{H}_T = -\Phi_T(u_0, T) = e^{-rT}\Delta$. ■

Lemma S5 *In the model with a certificate, the conclusions in Lemmas A4 and A5 in the main appendix remain valid.*

Proof. From Lemma S4, the first-order condition for T is

$$e^{-rT}y^*\left(\overline{X},q_T\right)-\lambda_T r\left[v\left(\overline{X}\right)-\Delta+c\left(q_T\right)\right]=e^{-rT}\Delta,$$

and the first-order condition for q_T implies that $\lambda_T r=\left[e^{-rT}\frac{\partial y^*}{\partial q}\left(\overline{X},q_T\right)-\eta_T\right]/c'\left(q_T\right).$ By combining these two equalities we obtain

$$\begin{aligned}&c'\left(q_T\right)y^*\left(\overline{X},q_T\right)-\frac{\partial y^*}{\partial q}\left(\phi\left(u_T\right),q_T\right)\left[v\left(\overline{X}\right)+c\left(q_T\right)\right]=\\&\Delta\left[c'\left(q_T\right)-\frac{\partial y^*}{\partial q}\left(\phi\left(u_T\right),q_T\right)\right]-e^{rT}\eta_T\left[v\left(\overline{X}\right)+c\left(q_T\right)\right].\end{aligned}$$

Since the left-hand side is strictly increasing in q_T, and equal to zero when $q_T=q^*\left(\overline{X}\right)$, the unique solution is $\eta_T=0$, $q_T=q^*\left(\overline{X}\right)$, and $e^{rT}\lambda_T=\frac{1}{r}$. The proof is otherwise identical to that of Lemmas A4 and A5. ■

Proposition S1 *In the model with a certificate, for phase 1 of the contract the conclusions in Theorem 1 and 2 remain valid, but with state equation*

$$\frac{1}{r}\frac{d}{dt}v\left(X_t\right)=v\left(X_t\right)-\Delta+c\left(q_t\right),$$

and with the optimal initial knowledge level X_0 and contract length T now satisfying either:

$$X_0>\max\left\{\underline{X},v^{-1}\left(\underline{v}+\Delta\right)\right\}\quad \textit{and}\quad \int_0^T\rho_t dt=\frac{1}{r}\quad (\textbf{positive knowledge gift});$$

or

$$X_0=\max\left\{\underline{X},v^{-1}\left(\underline{v}+\Delta\right)\right\}\quad \textit{and}\quad \int_0^T\rho_t dt\leq\frac{1}{r}\quad (\textbf{zero knowledge gift}).$$

Proof. The proof is identical to the proofs of Theorems 1 and 2, but with state equation $\dot{u}_t=r\left[u_t-\Delta+c\left(q_t\right)\right]$ and with the ex-ante outside option $\max\left\{v\left(\underline{X}\right),\underline{v}+\Delta\right\}$ taking the place of $v\left(\underline{X}\right)$. ■

3 Training costs

Here we derive the optimal contract in the extended model with training costs.

Lemma S6 *In the model with training costs, the conclusions in Lemmas A1 and A2 remain valid.*

Proof. With the exception of Claims 1 and 4, it is easy to see that the proofs of Lemmas A1 and A2 extend to this case. Claim 1 states that the principal obtains a strictly positive profit by contracting with the agent. To see why this is still true, consider a contract in which $X_0 = \underline{X}$ and $X_T = \overline{X}$, and at any time $0 \leq t \leq T$ effort is $q_t = q^* (X_t)$, wages are zero, and the agent receives training dX_t/dt such that

$$v' (X_t) \frac{dX_t}{dt} = r [v (X_t) + c (q^* (X_t))] = ry^* (X_t, q^* (X_t)) .$$

This contract satisfies the agent's participation constraints with equality at all times and delivers profits

$$\int_0^T e^{-rt} \left[y^* (X_t, q^* (X_t)) - k\frac{1}{r}v' (X_t) \frac{dX_t}{dt} \right] dt = \int_0^T e^{-rt} (1-k) \, y^* (X_t, q^* (X_t)) \, dt > 0.$$

Claim 4 states that any finite-duration contract with $W_\infty = 0$ and $X_T < \overline{X}$ is strictly dominated by some finite-duration contract with $W_\infty = 0$ and $X_T = \overline{X}$. To see why this is still true, notice that if a finite-duration contract with $W_\infty = 0$ had $X_T < \overline{X}$, then the principal could obtain strictly higher profits by extending the contract to date $T' > T$, setting $X_{T'} = \overline{X}$, and for all $T < t \leq T'$ offering the same arrangement as above. ■

It follows from this lemma that with the exception of the principal's objective, the optimal control problem is the same as in the original model. The principal's objective is now

$$\int_0^T e^{-rt} \left[y^* (\phi (u_t), q_t) - k\frac{1}{r}\dot{u}_t \right] dt - k\frac{1}{r} [u_0 - v (\underline{X})] ,$$

where the second term in the objective is the cost of the initial gift. The Hamiltonian is now $\mathcal{H} = e^{-rt} \left[y^* (\phi (u_t), q_t) - k\frac{1}{r}\dot{u}_t \right] - \lambda_t \dot{u}_t$, with $\dot{u}_t =$

$r\left[u_t + c\left(q_t\right)\right]$. Assign to the ex-ante participation constraint $u_0 \geq v\left(\underline{X}\right)$ multiplier ζ.

Lemma S7 *In the model with training costs, except for the transversal condition, the conclusion in Lemma A3 remains valid. The transversal condition is now* $\mathcal{H}_T = 0$, $\lambda_0 = -\frac{1}{r}k + \zeta$, $\zeta \geq 0$, and $\zeta\left[u_0 - v\left(\underline{X}\right)\right] = 0$.

Proof. Define $\varphi\left(u_0\right) := -k\frac{1}{r}\left[u_0 - v\left(\underline{X}\right)\right]$, $\psi\left(u_0\right) := u_0 - v\left(\underline{X}\right)$ and $\Phi\left(u_0\right) := \varphi\left(u_0\right) + \zeta\psi\left(u_0\right)$. The only difference relative to the proof of Lemma A3 is that Chachuat (2007) Theorem 3.18 now requires that $\lambda_0 = \Phi'\left(u_0\right) = -\frac{1}{r}k + \zeta$. ■

Lemma S8 *In the model with training costs, the conclusions in Lemmas A4 and A5 remain valid, but with co-state equation*

$$\lambda_t = e^{-rt}\left[e^{rT}\lambda_T - \int_t^T \left[\rho_\tau - k\right] d\tau\right],$$

and with $\lambda_T = e^{-rT}\left[1-k\right]/r$.

Proof. The co-state evolution equation is $\dot{\lambda}_t = -r\lambda_t + e^{-rt}\left[\rho_t - k\right]$, the first-order condition for T is $e^{-rT}\, y^*\left(\phi\left(u_T\right), q_T\right) - \left(\lambda_T r + e^{-rT}k\right)\left[u_T + c\left(q_T\right)\right] = 0$ and the first-order condition for q_T implies that $\lambda_T r + e^{-rT}k = \left[e^{-rT}\frac{\partial y^*}{\partial q}\left(\overline{X}, q_T\right) - \eta_T\right]/c'\left(q_T\right)$. Therefore, after replacing $\lambda_T r$ with $\lambda_T r + e^{-rT}k$, the proof of this lemma is identical to the proofs of Lemmas A4 and A5. ■

Proposition S2 *In the model with training costs, the conclusions in Theorems 1 and 2 remain valid, but with*

$$\frac{\partial y^*}{\partial q}\left(X_t, q_t\right)/c'\left(q_t\right) = \max\left\{1 - r\int_t^T \left[\rho_\tau - k\right] d\tau, \frac{\partial y^*}{\partial q}\left(X_t, 1\right)/c'\left(1\right)\right\}$$

and with the optimal initial knowledge level X_0 *and contract length* T *now satisfying either*

$$X_0 > \underline{X} \;\; \textit{and} \;\; \int_0^T \left[\rho_t - k\right] dt = \frac{1}{r} \;\; (\textbf{positive knowledge gift})$$

or

$$X_0 = \underline{X} \;\; \textit{and} \;\; \int_0^T \left[\rho_t - k\right] dt \leq \frac{1}{r} \;\; (\textbf{zero knowledge gift}).$$

Proof. We begin with two observations. First, Lemma S8 implies that the co-state evolution equation is $\lambda_t = e^{-rt}\left[\frac{1}{r}[1-k] - \int_t^T [\rho_\tau - k]\, d\tau\right]$, and therefore the effort path satisfies, for all $s \geq 0$,

$$c'(q_{T-s}) = \begin{cases} \min\left\{\frac{\frac{\partial y^*}{\partial q}(X_{T-s}, q_{T-s})}{1 - r\int_0^s [\rho_{T-\tau} - k]\, d\tau},\ c'(1)\right\} & \text{when } r\int_0^s [\rho_{T-\tau} - k]\, d\tau \leq 1, \\ c'(1) & \text{otherwise.} \end{cases}$$

Second, whenever the ex-ante participation constraint is slack ($\zeta = 0$), Lemma S7 implies that $\lambda_0 = -\frac{1}{r}k$, and so the co-state evolution equation implies that $\lambda_0 = \frac{1}{r}[1-k] - \int_0^T [\rho_t - k]\, dt = -\frac{1}{r}k$. Consequently, the optimal unconstrained terminal date T satisfies $\int_0^T [\rho_t - k]\, dt = \frac{1}{r}$.

It follows from these two observations that after replacing ρ_t with $[\rho_t - k]$ for all t, the proof of the present proposition is identical to the proofs of Theorems 1 and 2. ■

4 Multiplicative output

Here we show that the optimal contract prescribes $q_t \geq q^*\left(\overline{X}\right)$ for all t when $y^*(X, q) = Xq$, $c(q) = q^2/2$, and the effort upper bound is sufficiently large.

Notice that $q^*(X) = X$, $v(X) = X^2/2$, and $\rho(X, q) = q/X$. Moreover from Theorem 2, when the effort constraint is slack,

$$\frac{\frac{dy^*}{dq}(X_t, q_t)}{c'(q_t)} = \frac{X_t}{q_t} = 1 - r\int_t^T \rho_\tau d\tau.$$

Because $\rho_t = \rho(X_t, q_t) = q_t/X_t$, this equality implies $1/\rho_t = 1 - r\int_t^T \rho_\tau d\tau$, and so $\rho_T = 1$ and $\dot{\rho}_t = -r\rho_t^3$. Thus $\rho_t = [1 - 2r(T-t)]^{-\frac{1}{2}}$.

Next, the ongoing participation constraint implies $\dot{u}_t = r[u_t + c(q_t)]$, which specializes to $\dot{X}_t = \frac{r}{2}[1 + \rho_t^2] X_t$. As a result

$$X_t = \overline{X} e^{-r\int_t^T \left[\frac{1 - r(T-\tau)}{1 - 2r(T-\tau)}\right] d\tau},$$

and since $q_t = \rho_t X_t$,

$$q_t = \frac{\overline{X}}{[1 - 2r(T-t)]^{\frac{1}{2}}} e^{-r\int_t^T \left[\frac{1 - r(T-\tau)}{1 - 2r(T-\tau)}\right] d\tau}.$$

Note that dq_t/dt is

$$-r\left(\overline{X}e^{-r\int_t^T\left[\frac{1-r(T-\tau)}{1-2r(T-\tau)}\right]d\tau}\right)\left[1-2r\left(T-t\right)\right]^{-\frac{1}{2}}\left[\left[1-2r\left(T-t\right)\right]^{-1}-\frac{1-r\left(T-t\right)}{1-2r\left(T-t\right)}\right],$$

which is strictly negative, and so $q_t \geq q_T = q^*\left(\overline{X}\right)$. Notice moreover that moving backward in time, X_t goes to 0 as t goes to $T - 1/\left(2r\right)$. Because the participation constraint requires that $X_t \geq \underline{X} > 0$ at all times, it follows that the effort upper bound does not bind whenever this bound is sufficiently large. Thus the solution to the relaxed program is optimal.