Robot Arithmetic: New Technology and Wages

ONLINE APPENDIX

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Proof of Results and Specific Examples

Proof of Result 1

Stack the prices of consumption and investment goods into a single vector ρ. Combine the cost functions into a single vector as well – continue to denote this by c. Write the stacked prices as:

$$\rho = c\left(w, \rho, \theta\right) \tag{1}$$

Taking logs and differentiating leads to:

$$\frac{\partial \log \rho}{\partial \theta} = \Lambda^{\rho} \frac{\partial \log \rho}{\partial \theta} + \Lambda^{w} \frac{\partial \log w}{\partial \theta} + \frac{\partial \log c}{\partial \theta} \tag{2}$$

Where Λ^{ρ} is a non-negative matrix whose ijth element, γ_{ij}^{ρ}, is given by:

$$\gamma_{ij}^{\rho} = \frac{\rho_j}{c_i} \frac{\partial c_i}{\partial \rho_j} \tag{3}$$

From Shephard's Lemma we know that the derivative of the cost function with respect to a price is the per output demand for that input. Hence γ_{ij}^{ρ} is the share of the cost of input j in the production of good i. Similarly, Λ^{w} is a non-negative matrix whose ijth element, γ_{ij}^{w}, is given by:

$$\gamma_{ij}^{w} = \frac{w_j}{c_i} \frac{\partial c_i}{\partial w_j} \tag{4}$$

γ_{ij}^{w} is the share of the cost of type of labor j in the production of good i. The ith row of Λ^{ρ} must sum to one minus the share of labor costs in the production of good i, and the ith row of Λ^{w} must sum to the share of labor in the production of good i. Denote the vector of shares of labor costs by s.

Denote the maximum goods price change as $\frac{\partial \log \rho^{\max}}{\partial \theta}$ and the maximum wage change as $\frac{\partial \log w^{\max}}{\partial \theta}$. Then (2) implies that, for all goods, we must have:

$$\begin{aligned} \frac{d \log \rho}{d\theta} &\leq \Lambda^{\rho} \frac{d \log \rho^{\max}}{d\theta} + \Lambda^{w} \frac{d \log w^{\max}}{d\theta} + \frac{\partial \log c}{\partial \theta} \\ &= \left(1-s\right) \frac{d \log \rho^{\max}}{d\theta} + s \frac{d \log w^{\max}}{d\theta} + \frac{\partial \log c}{\partial \theta} \end{aligned} \tag{5}$$

With equality only for goods which are produced only use goods and labor with the maximum price and wage changes. (5) applies for all goods, including goods with prices increasing at the fastest rate. For these goods we can re-arrange (5) to yield:

$$\frac{d \log \rho^{\max}}{d\theta} \leq \frac{d \log w^{\max}}{d\theta} + \frac{1}{s} \frac{\partial \log c}{\partial \theta} \leq \frac{d \log w^{\max}}{d\theta} \tag{6}$$

Where s is the labor share for that good. This proves the result but is only valid if $s > 0$. What happens if the good with the highest price increase is produced using no labor? If this good is produced using some goods with price increases below the maximum then it cannot be the good with the highest price index as (5) will be a strict inequality leading to a contradiction if $s = 0$. If it is only produced using goods with the highest price increase, this is a contradiction if there is any technical change in that sector. If there is not, there is a set of goods with no technical change produced with no labor and only each other as intermediate or capital goods. Because these goods are produced without fixed factors, there is no limit to the supply of them so the price of them will always be zero, contradicting the fact that they have the highest inflation rate.

Ultimately workers are only interested in the price of consumption goods and this result seems to leave open the possibility that prices only fall for investment goods. But if these investment goods are used, directly or indirectly (meaning it might only be used to produce investment goods but those investment goods are ultimately used in the production of consumption goods through some chain), this must be transmitted to the price of some consumption good. There will be no production, in equilibrium, of a set of investment goods only used to produce themselves in which case the result says that technological change in goods that are not produced will have no benefit.

Result 2 when the wage of some types of workers are zero

It is possible that the wages of some types of labor fall to zero and there is possibly some unemployment for those types. This case can be allowed for in the following way. Remove these types of workers from the cost functions as the zero wage allows us to do this. The result above then goes through for the set of workers with non-zero wages. But the average wage result applies to all workers if we include the unemployed as having zero earnings. The set of types of workers with zero wages may change with the technology but the formulae above remain valid even for this case.

If all types of workers have zero wages then we are in a situation where all prices (as well as wages) will be zero, i.e. this is a situation of total abundance. Our static analysis is not well-suited to this case – it is discussed at the end of the paper.

An example where the average wage of workers falls

The example outlined here shows how the real wage of workers can fall if new technology causes the price of investment goods to rise relative to consumption goods i.e. the condition of Result 2 is not satisfied. Such an example must have at least two types of goods (to allow relative prices to change) and two types of labor (otherwise Result 1 would imply that real wages would rise).

Assume that there are two sectors, a consumption good sector and an investment good sector. The consumption good is assumed to be produced by one type of labor – call it c-labor – and capital goods, according to the production function:

$$X = L_c f\left(\frac{K_c}{L_c},\theta\right) = L_c f\left(k_c,\theta\right) \tag{7}$$

The investment good is assumed to be produced by a different type of labor – call it i-labor – and capital goods according to the production function:

$$I = L_i g\left(\frac{K_i}{L_i},\theta\right) = L_i g\left(k_i,\theta\right) \tag{8}$$

Assume the price of consumption good is numeraire – set it equal to 1.

The wage of c-labour will be given by:

$$w_c = f - k_c f_k \tag{9}$$

And the demand for capital in consumption good sector will be given by:

$$\left(r+\delta\right) p_i = f_k \tag{10}$$

The wage of i-labour will be given by:

$$w_i = p_i \left[g - k_i g_k \right] \tag{11}$$

And the demand for capital in the i-sector will be given by:

$$\left(r+\delta\right) p_i = p_i g_k \tag{12}$$

Note that (12) implies that the capital-labour ratio in the i-sector solves the equation:

$$g_k\left(k_i, \theta\right) = \left(r+\delta\right) \tag{13}$$

Which, conveniently, is independent of prices. Given the inelastic supply of i-labour this also fixes the amount of i-capital. Now the total supply of c-capital must satisfy the equation

$$\delta\left(K_c + L_i k_i\right) = L_i g \tag{14}$$

Which can be re-arranged to give:

$$K_c = L_i \frac{g - \delta k_i}{\delta} \tag{15}$$

Which implies that the amount of c-capital can also be solved for independent of prices. This then implies that the total capital stock is given by:

$$K = L_i \frac{g}{\delta} \tag{16}$$

Note that the production function for the consumer good plays no role in determining the total level of capital or its allocation across sectors.

Total income to workers must be the difference between the production of consumption goods and the consumption of capitalists which is the part of their income not used to cover depreciation i.e. $rp^i K$ of capitalists. This implies:

$$\begin{aligned} Lw &= L_c w_c + L_i w_i = L_c f - rp^i K \\ &= L_c f - \frac{r}{r+\delta} f_k K = L_c f - \frac{r}{r+\delta} \frac{g}{\delta} f_k L_i \end{aligned} \tag{17}$$

Now suppose that the nature of the new technology is that it does not affect production of the investment good (this is an example so this is not meant to be plausible). In this case g is fixed and the capital-labor ratios in the two sectors are unaffected by the new technology. Differentiating (17) we have that:

$$\frac{d\left(Lw\right)}{d\theta}=L_c f_\theta-\frac{r}{r+\delta}\frac{g}{\delta}f_{k\theta}L_i \tag{18}$$

The first term is positive but the second term can outweigh it if new technology heavily raises the marginal product of capital in the consumption goods sector.

Note the link to the condition in Result 1. (10) implies that the relative price of investment goods rises (resp. falls) if $f_{k\theta}>(<)0$. If $f_{k\theta}<0$ the relative price of investment goods falls and (18) says that average wages must rise, consistent with Result 1. But if $f_{k\theta}>0$ then the relative price of investment goods rises and and (18) says that average wages can fall.

Proof of Result 3

If there are many types of labor but they are in perfectly elastic supply then relative wages are constant. The cost functions can be written as a function of the wage of one type of labor chosen as numeraire and the relative wages which are exogenous. The model is then reduced to one in which there is only one wage and as result 1 implies the real wage must rise for one type of worker, they must rise for all types of labor.

Increasing Returns and Imperfect Competition

Many current models of the economy assume that individual firms have increasing returns to scale. This section considers what happens if that is the case. Continue to use c to denote marginal costs but now assume that firms have to pay a fixed cost $c^f\left(w,\rho,\theta\right)$ to enter an industry – for simplicity here we use the stacked price approach of Result 1 rather than distinguish between consumption and investment goods.

Increasing returns at the individual firm level is not compatible with perfect competition so we assume that price is a mark-up, μ, possibly varying across sectors, on marginal costs i.e. we have:

$$\rho=\left(1+\mu\right)c \tag{19}$$

We treat μ as exogenously given though it is usually derived from other parameters in the model – for our purpose this is not important.

Free entry into an industry implies that total revenue of the industry must equal total costs which can be written as:

$$\left(\rho - c\right) X = N c^{f} \tag{20}$$

Where X is gross output and N is the number of firms. Using (19) this can be written as:

$$N = X \frac{\mu c}{c^{f}} \tag{21}$$

Now consider input demands. Total demand from this sector for input j can be written as:

$$X \frac{\partial c}{\partial p_j} + N \frac{\partial c^{f}}{\partial p_j} \tag{22}$$

Using (21), (22) can be written as:

$$X \frac{\partial c}{\partial p_j} + X \mu c \frac{\partial \log c^{f}}{\partial p_j} = X \frac{\partial c}{\partial p_j} \left[1 + \mu \frac{\dfrac{\partial \log c^{f}}{\partial p_j}}{\dfrac{\partial \log c}{\partial p_j}} \right] \tag{23}$$

If marginal costs and fixed costs using inputs in the same proportions this implies that $\frac{\partial \log c^{f}}{\partial p_j} = \frac{\partial \log c}{\partial p_j}$ the total factor demands can be written as:

$$X \frac{\partial c}{\partial p_j} \left[1 + \mu \right] \tag{24}$$

Which is completely isomorphic to our standard model using $\left(1 + \mu\right) c$ as the cost function. One can derive a similar expression for the demand for a type of labor replacing the price with the wage.

This leaves open the possibility that technology might harm workers if it affects fixed costs in a different way to marginal costs. And it does assume that all firms within an industry are identical – many models assume heterogeneity which gives rents to the more productive firms. It is possible that new technology might disproportionately advantage these firms. The analysis of these models is left for later research.