Online Appendix: A Simple Planning Problem for COVID-19 Lockdown, Testing and Tracing*

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Abstract

This appendix contains additional documentation to solve the planning problem using a finite differences. It also describes in detail the extension of the problem to implement the trace-test-quarantine protocol and the case where the period cost of lockdown is quadratic in the total number of hours in lockdown.

*First draft, March 23, 2020. We thank Tom Phelan who shared his code of our problem using a similar version of the finite difference method, substantially improving the speed on our earlier method of computation.

A Planning Problem

In this appendix we provide more details on how we use the finite difference method to approximate the solution of the continuous time Bellman equation. It begins describing the problem without the trace-test-quarantine (TTQ) protocol and providing a description of the finite difference method; it describes how it can be implemented using the same grid size for the two state variables of the problem and its application when the grid size of the two state variables differs, which is how is implemented in the code for computational reasons. It then describes the required assumptions to implement the TTQ protocol and an extension where we consider the case where the period cost of lockdown is quadratic in the total number of hours in lockdown.

The replication files include matlab codes that solve the planning problem for COVID-19 lockdown, testing and tracing. We include files that use the finite differences method. We also include matlab files that, after loading a pre-existing output file, run a time-path simulation and constructs the figures in the paper. The files are provided for academic use. Users of the files (or modified version of it) should cite Alvarez, Argente and Lippi (2020).

B Finite Difference Method without TTQ

The main idea of the finite difference method is to recast the problem of the planner as a stochastic control problem, where the domain is partitioned in space and in time, and whose solution is sure to converge to the continuous time version of the problem and the appropriate regularity conditions. We opted to numerically solve the value function of this problem for two reasons. First, the planning problem is not a convex problem. Second, we want to have a solution over the entire state space for counterfactuals.[1] We rely on discretization methods for stochastic control problems in continuous time described in Kushner and Dupuis (2013).

[1] Alternatively, we could have used the shooting algorithm for the system of ordinary differential equations given by the state and co-states defined by the Hamiltonian, but that will provide us with conditions that are necessary but, in general, not sufficient.

In this note, we first describe how the finite difference method can be applied when the same grid size is used for both state variables of the problem in a triangular domain. We also use that the value function is known when either of the two state variables is zero (i.e. we solve the problem in the remaining part of the state space). We then describe how it can be implemented when the grid size is different. For computational reasons described in this note, this is our preferred methodology and the one that is implemented in the replication code. Since the original problem is deterministic, we use the "upwind" approximation method both in the interior and in the boundary of the state space, where we do not know the value function. We also discuss how to easily carry the minimization even when the first order conditions may not apply. We provide the details of the implementation in the rest of the note. We first consider the problem without TTQ so that the state space is simply S and I, see Section C for details on the implementation of the TTQ protocol.

The planner solves the following Bellman-Hamilton-Jacobi equation:

$$\begin{aligned}(r+\nu)V(S,I) = \min_{L\in[0,\bar{L}]} & \; wL\Big[\tau(S+I)+1-\tau\Big] + I\phi(I)\,vsl\, + \\ & + \left[\beta\, S\, I\,(1-\theta L)^2\right]\left[\partial_I V(S,I) - \partial_S V(S,I)\right] \\ & - \gamma I \partial_I V(S,I)\end{aligned} \tag{B1}$$

The domain of V is $(S,I)\in\mathbb{R}^2$ such that $S+I\le 1$. Note that $V(S,I)$ can be interpreted as the minimum expected discounted cost of following the optimal policy in units of output loss. Recall that if $\phi(I)=0$, then the value of the objective of the planner at time $t=0$ will give $N(0)w$. Thus, the quantity $rV(S,I)/w$ converts the stock-value of the value function into a ratio of the flow cost relative to output at time $t=0$, when there is no virus. Finally, we notice that the value function has simple analytic expressions on the boundary of its domain, where the lockdown policy is not exercised. In particular, on the $I=0$ axis we have $V(S,0)=0$, for all $S\in(0,1)$, so that the cost is zero if nobody is infected. On the $S=0$ axis we have $V(0,I)=vsl\,\left(\frac{\varphi\gamma}{r+\nu+\gamma}+\frac{\kappa\gamma I}{r+\nu+2\gamma}\right) I$ for all $I\in(0,1)$.

Let S_i be the values on a grid of for S and I_j the values on a grid for I. Then we can try at each interior grid point:

$$(r+\nu)V(S_i, I_j) = \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] + I_j\phi(I_j)\,vsl \tag{B2}$$
$$+ \left[\beta\, S_i I_j(1-\theta L)^2\right] \left[V_I^+(i,j) - V_S^-(i,j)\right] - \gamma I_j V_I^-(i,j)$$

where for the first term we use

$$V_S^-(i,j) = \frac{V(S_i, I_j) - V(S_{i-1}, I_j)}{S_i - S_{i-1}} \text{ and } V_I^+(i,j) = \frac{V(S_i, I_{j+1}) - V(S_i, I_j)}{I_{j+1} - I_j} \tag{B3}$$

For instance, assuming that $S_i - S_{i-1} = I_{j+1} - I_j = \Delta$

$$V_I^+(i,j) - V_S^-(i,j) = \frac{1}{\Delta}\left[V(S_i, I_{j+1}) - V(S_i, I_j) - V(S_i, I_j) + V(S_{i-1}, I_j)\right] \tag{B4}$$
$$= \frac{1}{\Delta}\left[V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)\right] \tag{B5}$$

We can also use in the last term, since the drift is negative:

$$V_I^-(i,j) = \frac{1}{\Delta}\left[V(S_i, I_j) - V(S_i, I_{j-1})\right] \tag{B6}$$

The rule used for the derivatives, whether to use forward or backward difference, is whether the drift of the corresponding term is positive or negative. This is the rule used in Kushner and Dupuis (2013), "Numerical Methods for Stochastic Control Problems in Continuous Time" chapter 5 for the explanation in the case of drift (b is the drift of the state in the book notation) and no diffusion or degenerate case ($a = 0$ in the book notation). For instance, see Example 2, for "Uncontrolled Deterministic Case", or see section 5.3 "The general Finite Difference Method", The Diagonal Case, equations (3.4) and (3.5). The important point is that the direction of the change on the finite difference approximating the derivatives is given by the sign of the drift itself.

We will use the grids:

$$Grid_S = \{0, \Delta, 2\Delta, \ldots, (N-1)\Delta\} \tag{B7}$$

$$Grid_I = \{0, \Delta, 2\Delta, \ldots, (N-1)\Delta\} \tag{B8}$$

with $S_1 = 0, S_2 = \Delta, \ldots, S_i = (i-1)\Delta, \ldots, S_N = (N-1)\Delta = 1$, so $\Delta = \frac{1}{N-1}$. Likewise, $I_1 = 0, I_2 = \Delta, \ldots, I_j = (j-1)\Delta, \ldots, I_N = (N-1)\Delta = 1$. We require $S_i + I_j \leq 1$ so $(i+j-2)/(N-1) \leq 1$ or $(i+j-2) \leq N-1$ or $i+j \leq N+1$. We know the value function at $V(S_1, I_j)$ for all j given by the quadratic expression above $V(S_1, I_j) = vsl \left(\frac{\varphi\gamma}{r+\nu+\gamma} + \frac{\kappa\gamma I_j}{r+\nu+2\gamma} \right) I_j$. We also know that $V(S_i, I_1) = 0$ for all i. In both cases, i.e. when $I_1 = 0$ or when $S_1 = 0$, the optimal policy is no lockdown, i.e $L(1,j) = L(i,1) = 0$ for all i, j, where we use $L(i,j)$ to denote the optimal lockdown policy when $S = S_i$ and $I = I_j$. Thus the expression for $V_I^-(i,j)$, and for $V_I^+(i,j) - V_S^-(i,j)$ are only computed for pairs (i,j): $2 \leq i \leq N-1$ and $2 \leq j \leq N+1-i$.

Denote the right hand side of the HJB equation evaluated at (i,j) as $F(L; i,j)$ given by.

$$F(L; i,j) = Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] + I_j \phi(I_j)\, vsl \tag{B9}$$

$$+ \left[\beta\, S_i I_j (1-\theta L)^2\right] \left[V_I^+(i,j) - V_S^-(i,j)\right] - \gamma I_j V_I^-(i,j) \tag{B10}$$

The first derivative is:

$$\frac{dF(L;i,j)}{dL} = w\Big[\tau(S_i + I_j) + 1 - \tau\Big] - \theta(1-\theta L) 2 \left[\beta\, S_i I_j\right] \left[V_I^+(i,j) - V_S^-(i,j)\right] \tag{B11}$$

and the second derivative is

$$\frac{d^2 F(L;i,j)}{dL^2} = \theta^2 2 \left[\beta\, S_i I_j\right] \left[V_I^+(i,j) - V_S^-(i,j)\right] \tag{B12}$$

The following gives the optimal L:

1. If $V_I^+(i,j) - V_S^-(i,j) < 0$, then F is concave in L, and the minimum must be at either $L(i,j) = 0$ or $L(i,j) = \bar{L}$, and must be located by evaluating F at these two values.

2. If $V_I^+(i,j) - V_S^-(i,j) \geq 0$, then F is convex in L, and the minimum can be located using the first order condition $\frac{dF(L;i,j)}{dL} = 0$ and complementary slackness. Define

$$L^0(i,j) = \frac{-w\Big[\tau(S_i + I_j) + 1 - \tau\Big] + 2\theta\,[\beta\, S_i I_j]\,\big[V_I^+(i,j) - V_S^-(i,j)\big]}{2\theta^2\,[\beta\, S_i I_j]\,\big[V_I^+(i,j) - V_S^-(i,j)\big]} \tag{B13}$$

$$= -\frac{w\Big[\tau(S_i + I_j) + 1 - \tau\Big]}{2\theta^2\,[\beta\, S_i I_j]\,\big[V_I^+(i,j) - V_S^-(i,j)\big]} + \frac{1}{\theta} \tag{B14}$$

and

$$L(i,j) = \min\big\{\bar{L}\ ,\ \max\big\{0\ ,\ L^0(i,j)\big\}\big\} \tag{B15}$$

Note we can first multiply our discrete HJB equation by dt in both sides, and after add $V(S_i, I_j)$ to obtain:

$$\begin{aligned}[1 + (r+\nu)dt]\,V(S_i, I_j) = \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\,vsl\ dt \\ \big[\beta\, S_i I_j(1-\theta L)^2\big]\ dt\ \big[V_I^+(i,j) - V_S^-(i,j)\big] - \gamma I_j\,dt\,V_I^-(i,j) + V(S_i, I_j)\end{aligned}$$

Replacing the finite difference approximations:

$$\begin{aligned}[1 + (r+\nu)dt]\,V(S_i, I_j) = \min_{L\in[0,\bar{L}]} & Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\,vsl\ dt \\ & + \big[\beta\, S_i I_j(1-\theta L)^2\big]\ \frac{dt}{\Delta}\,[V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \\ & - \gamma I_j\,\frac{dt}{\Delta}\,[V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j)\end{aligned}$$

Moving some terms from the left to the right-and side:

$$
\begin{aligned}
V(S_i, I_j) = & \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\,vsl\ dt \\
& + \big[\beta\, S_i I_j (1-\theta L)^2\big]\ \frac{dt}{\Delta}\,[V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \\
& - \gamma I_j \,\frac{dt}{\Delta}\,[V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j)\,[1 - (r+\nu)dt]
\end{aligned}
$$

Collecting terms on the RHS

$$
\begin{aligned}
V(S_i, I_j) = & \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\,vsl\ dt \\
& + [1 - (r+\nu)dt]\Big\{\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt}\ \frac{dt}{\Delta}\,[V(S_i, I_{j+1}) - 2V(S_i, I_j) + V(S_{i-1}, I_j)] \\
& - \frac{\gamma I_j}{1-(r+\nu)dt}\,\frac{dt}{\Delta}\,[V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j)\Big\}
\end{aligned}
$$

Further rearranging the terms:

$$
\begin{aligned}
V(S_i, I_j) = & \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\,vsl\ dt \\
& + [1 - (r+\nu)dt]\left[1 - 2\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt}\ \frac{dt}{\Delta} - \frac{\gamma I_j}{1-(r+\nu)dt}\ \frac{dt}{\Delta}\right] V(S_i, I_i) \\
& + [1 - (r+\nu)dt]\left[\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt}\ \frac{dt}{\Delta}\right] [V(S_i, I_{j+1}) + V(S_{i-1}, I_j)] \\
& + [1 - (r+\nu)dt]\ \frac{\gamma I_j}{1-(r+\nu)dt}\ \frac{dt}{\Delta} V(S_i, I_{j-1})
\end{aligned}
$$

which can be rearrange to obtain the following expression of interest:

$$
\begin{aligned}
V(S_i, I_j) = & \min_{L \in [0, \bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] dt + I_j \phi(I_j)\, vsl\ dt \\
& + [1 - (r+\nu)dt] \left[1 - 2 \frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_i) \\
& + [1 - (r+\nu)dt] \left[\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j+1}) \\
& + [1 - (r+\nu)dt] \left[\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_{i-1}, I_j) \\
& + [1 - (r+\nu)dt] \left[\frac{\gamma I_j}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j-1})
\end{aligned}
$$

When i, j are such that $S_i + I_j = 1$, so that the state is in the boundary, or equivalently that $i + j = N + 1$, we can't evaluate $V(S_i, I_{j+1})$, so we use a different scheme for the difference. We approximate the difference in the derivatives as follows: $\partial_I V(S_i, I_j) - \partial_S V(S_i, I_j) = (V(S_{i-1}, I_{j+1}) - V(S_i, I_j))/\Delta$. The resulting operator is:

$$
\begin{aligned}
V(S_i, I_j) = & \min_{L \in [0, \bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] dt + I_j \phi(I_j)\, vsl\ dt \\
& + [1 - (r+\nu)dt] \left[1 - \frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt} \frac{dt}{\Delta} - \frac{\gamma I_j}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_i) \\
& + [1 - (r+\nu)dt] \left[\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_{i-1}, I_{j+1}) \\
& + [1 - (r+\nu)dt] \left[\frac{\gamma I_j}{1-(r+\nu)dt} \frac{dt}{\Delta} \right] V(S_i, I_{j-1})
\end{aligned}
$$

Note that the expressions for $V(S_i, I_j)$ can be considered as the Bellman equation corresponding to a finite state stochastic discrete time programming problem. Indeed, we can use it to define a corresponding operator and use the method of successive approximations to approximate its fixed point. The optimal value of $L(i, j)$ was obtained above, as function of the difference of the finite difference approximation on derivatives. On the one hand, note that for any fixed grid side $\Delta > 0$, the last terms of the left hand side can be considered a weighted average of the value function at different points. For dt small enough all the weights are non-negative. On the other hand, this operator contracts are rate $[1-(r+\nu)dt]$, which is closer to one as dt gets smaller.

To make sure that $\left[1 - 2\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt}\,\frac{dt}{\Delta} - \frac{\gamma I_j}{1-(r+\nu)dt}\,\frac{dt}{\Delta}\right] > 0$ note that

$$\begin{aligned} 2\frac{\beta\, S_i I_j (1-\theta L)^2}{1-(r+\nu)dt}\,\frac{dt}{\Delta} &\leq \frac{\beta\, dt\,(N-1)}{2[1-(r+\nu)dt]} \\ \frac{\gamma I_j}{1-(r+\nu)dt}\,\frac{dt}{\Delta} &\leq \frac{\gamma\, dt(N-1)}{1-(r+\nu)dt} \end{aligned}$$

Thus we require dt to satisfy:

$$\begin{aligned} 1 &\geq [r+\nu+(N-1)\,(\beta/2+\gamma)]\,dt \text{ or} \\ dt &\leq \frac{1}{r+\nu+(N-1)\,(\beta/2+\gamma)} \end{aligned}$$

Different S and I grid sizes. Since for the initial conditions of interest, the state spends a large time around low values of I. We consider a non-equal spaced grid for I, while keeping an equally spaced grid for S such that:

$$\begin{aligned} S_{i+1} &= S_i + \Delta_S = i\Delta_S \text{ for } i = 1, 2, ..., N_I \\ I_{j+1} &= I_j + \Delta_I = j\Delta_I \text{ for } j = 1, 2, ..., N_S \\ \Delta_S &= k\,\Delta_I \text{ where } k \text{ is a strictly positive integer} \end{aligned}$$

Thus, we have $\Delta_S = \frac{1}{N_S-1}$ and $\Delta_I = \frac{1}{N_I-1}$ and $(N_S-1)k = N_I-1$ or $N_S = (N_I-1)/k+1$.

We will assume that N_S, γ/β satisfies:

$$\Delta_S = \frac{1}{N_S-1} \leq \frac{\gamma}{\beta}$$

We will consider two cases:

1. In the "interior" of the state space we will estimate the derivatives by:

$$V_S^-(i,j) = \frac{1}{\Delta_S}\left[V(S_i, I_j) - V(S_{i-1}, I_j)\right] \tag{B16}$$

$$V_I^+(i,j) = \frac{1}{\Delta_I}\left[V(S_i, I_{j+1}) - V(S_i, I_j)\right] \tag{B17}$$

$$V_I^-(i,j) = \frac{1}{\Delta_I}\left[V(S_i, I_j) - V(S_i, I_{j-1})\right] \tag{B18}$$

Replacing them in the value function, for values $S_i > 0$ and $I_{j-1} > 0$ are in the state space, but also so that $S_i + I_{j+1} \leq 1$. This is corresponds to:

$$2 \leq i \leq N_S - 1$$
$$2 \leq j \leq N_I - 1 - (i-1)k$$

since $S_i + I_{j+1} \leq 1$ is equivalent to $j\Delta_I + (i-1)\Delta_S \leq 1$ or $j \leq \frac{1}{\Delta_I} - (i-1)\frac{\Delta_S}{\Delta_I}$ or $j \leq N_I - 1 - (i-1)k$.

Replacing the finite difference approximation into the value function:

$$\begin{aligned}(r+\nu)V(S_i, I_j) = \min_{L\in[0,\bar{L}]} & Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] + I_j\phi(I_j)\, vsl \\ & + \left[\beta\, S_i I_j (1-\theta L)^2\right] \frac{1}{\Delta_I}\left[V(S_i, I_{j+1}) - V(S_i, I_j)\right] \\ & - \left[\beta\, S_i I_j (1-\theta L)^2\right] \frac{1}{\Delta_S}\left[V(S_i, I_j) - V(S_{i-1}, I_j)\right] \\ & - \gamma I_j \frac{1}{\Delta_I}\left[V(S_i, I_j) - V(S_i, I_{j-1})\right]\end{aligned}$$

Multiplying by dt and adding $V(S_i, I_j)$ on each side:

$$\begin{aligned}
V(S_i, I_j) = \min_{L\in[0,\bar{L}]} & Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
& + \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_I}\,[V(S_i, I_{j+1}) - V(S_i, I_j)] \\
& - \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_S}\,[V(S_i, I_j) - V(S_{i-1}, I_j)] \\
& - \gamma I_j\, \frac{dt}{\Delta_I}\,[V(S_i, I_j) - V(S_i, I_{j-1})] + V(S_i, I_j)\,[1 - (r+\nu)dt]
\end{aligned}$$

Grouping common terms for the value functions evaluated at the same points:

$$\begin{aligned}
& V(S_i, I_j) = \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
& + \left\{[1-(r+\nu)dt] - \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_I} - \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_S} - \gamma I_j\, \frac{dt}{\Delta_I}\right\} V(S_i, I_j) \\
& + \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_I} V(S_i, I_{j+1}) \\
& + \left[\beta\, S_i I_j (1-\theta L)^2\right]\ \frac{dt}{\Delta_S} V(S_{i-1}, I_j) \\
& + \gamma I_j\, \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

taking $[1-(r+\nu)dt]$ as a common factor:

$$\begin{aligned}
& V(S_i, I_j) = \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
& + [1-(r+\nu)dt]\left\{1 - \frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_I} - \frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_S} - \frac{\gamma I_j}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_I}\right\} V(S_i, I_j) \\
& + [1-(r+\nu)dt]\, \frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_I} V(S_i, I_{j+1}) \\
& + [1-(r+\nu)dt]\, \frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_S} V(S_{i-1}, I_j) \\
& + [1-(r+\nu)dt]\, \frac{\gamma I_j}{[1-(r+\nu)dt]}\ \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

2. Now we write the approximation for those (i, j) for which $S_i + I_j = 1$ so that $S_i + I_{j+1} >$

1. We will first consider the case where

$$2 \leq i \leq N_S - 1 \text{ and } j = N_I - (i-1)k$$

For this case we use

$$V(S - \Delta_S, I + k\Delta_I) - V(S, I) = -\partial_S V(S,I)\Delta_S + \partial_I V(S,I)\Delta_I$$
$$= \Delta_S \left[\partial_I V(S,I) - \partial_S V(S,I)\right] + o(||(\Delta, \Delta_I)||)$$

Thus we approximate $\partial_I V(S,I) - \partial_S V(S,I)$ as

$$V_I^+(i,j) - V_S^-(i,j) \equiv \frac{1}{\Delta_S}\left[V(S_{i-1}, I_{j+k}) - V(S_i, I_j)\right]$$

where k is the ratio of Δ_S/Δ_I so $S_{j-1} + I_{j+k} \leq 1$ whenever $S_i + I_j = 1$. For $\partial_I V(S,I)$ we still use:

$$V_I^-(i,j) = \frac{1}{\Delta_I}\left[V(S_i, I_{j-1}) - V(S_i, I_j)\right]$$

Replacing the finite difference into the value function:

$$(r+\nu)V(S_i, I_j) = \min_{L \in [0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big] + I_j \phi(I_j)\, vsl$$
$$+ \left[\beta\, S_i I_j (1 - \theta L)^2\right] \left[V_I^+(i,j) - V_S^-(i,j)\right] - \gamma I_j\, V_I^-(i,j)$$

Using the expressions for the finite differences, multiplying by dt and adding $V(S_i, I_j)$

on each side:

$$\begin{aligned}
V(S_i, I_j) = \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
+ \left[\beta\, S_i I_j (1-\theta L)^2\right] \frac{dt}{\Delta_S}\left[V(S_{i-1}, I_{j+k}) - V(S_i, I_j)\right] \\
- \gamma I_j \frac{dt}{\Delta_I}\left[V(S_i, I_j) - V(S_i, I_{j-1})\right] + V(S_i, I_j)\left[1 - (r+\nu)dt\right]
\end{aligned}$$

Grouping common terms for the value functions evaluated at the same points:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
&+ \left\{[1-(r+\nu)dt] - \left[\beta\, S_i I_j (1-\theta L)^2\right] \frac{dt}{\Delta_S} - \gamma I_j \frac{dt}{\Delta_I}\right\} V(S_i, I_j) \\
&+ \left[\beta\, S_i I_j (1-\theta L)^2\right] \frac{dt}{\Delta_S} V(S_{i-1}, I_{j+k}) \\
&+ \gamma I_j \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

taking $[1-(r+\nu)dt]$ as a common factor:

$$\begin{aligned}
V(S_i, I_j) &= \min_{L\in[0,\bar{L}]} Lw\Big[\tau(S_i + I_j) + 1 - \tau\Big]dt + I_j\phi(I_j)\, vsl\ dt \\
&+ [1-(r+\nu)dt]\left\{1 - \frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]} \frac{dt}{\Delta_S} - \frac{\gamma I_j}{[1-(r+\nu)dt]} \frac{dt}{\Delta_I}\right\} V(S_i, I_j) \\
&+ [1-(r+\nu)dt]\frac{[\beta\, S_i I_j (1-\theta L)^2]}{[1-(r+\nu)dt]} \frac{dt}{\Delta_S} V(S_{i-1}, I_{j+k}) \\
&+ [1-(r+\nu)dt]\frac{\gamma I_j}{[1-(r+\nu)dt]} \frac{dt}{\Delta_I} V(S_i, I_{j-1})
\end{aligned}$$

See Section D for instructions on how to implement this version in the code.

C Allowing for the Trace-Test-Quarantine policy

This section explains how the the basic model can be extended by endowing the planner with an additional policy instrument: a trace-test-quarantine (TTQ) protocol. In this case, the planner will choose two controls as a function of the state: the rate of tracing-testing-quarantining as well as the lockdown rate. The codes provided already allow for this extension.

State Space Reduction. To reduce the state space define X as the stock of those infected, not in quarantine:

$$X = I - Q \tag{C19}$$

so that $\dot{X}_t = \dot{I}_t - \dot{Q}_t$, thus we can write:

$$\dot{S}_t = -\beta S_t X_t (1 - \theta L)^2 \tag{C20}$$

$$\dot{X}_t = \beta S_t X_t (1 - \theta L)^2 - T_t - \gamma X_t \tag{C21}$$

The initial conditions of interest are $X_0 = I_0$ and $S_0 = 1 - X_0$, since there is quarantine, and note that $S + X \leq 1$.

To eliminate $\{Q_t\}$ from the state, we rewrite the objective function. The expected discounted cost of output forgone for those in quarantine, denoted by $\mathbb{C}(\{Q\})$, can be written, using integration by parts and the law of motion of Q, as follows:

$$\mathbb{C}(\{Q\}) \equiv \int_0^\infty e^{-(r+\nu)t} Q_t dt = \frac{Q_0}{r + \nu + \gamma} + \int_0^\infty \frac{e^{-(r+\nu)t}}{r + \nu + \gamma} T_t dt$$

which is equivalent to "booking" the expected discounted cost every time someone is traced and put in quarantine. The advantage of this formulation is that we can keep track of the forgone cost of output due to the quarantine using the contemporaneous control T_t.

Two comments about the boundaries of the state space. First, if $Q_t = I_t$, i.e $X_0 = 0$, then $\dot{S}_t = 0$ and $\dot{I}_t - \dot{Q}_t = -T_t$. In this case, it will be optimal to set $L_t = 0$ and $T_t = 0$, and hence $I_r - Q_r = I_t - Q_t$, and $S_r = S_t$ for all the future $r \geq t$. Second, note that with a positive but finite value of $T_r = \hat{T} > 0$ applied for a long enough time, then $X_t = 0$ in finite time $t < \infty$.

The introduction of X as a state variable, and the use of the current control T to represent $\mathbb{C}(\{Q\})$ allow us to eliminate one state variable in the law of motion of the state. However, this is not yet enough to write the problem as a two-state variable problem, since the return function still requires to have (S, X, Q) or alternatively the original (S, I, Q). To see this note that the period cost is given by:

$$wL\Big[\tau(S+X) + (1-\tau)(1-Q)\Big] + \frac{wT}{r+\nu+\gamma} + c(T; S, I, Q) + vsl\, \phi(X+Q)\,(X+Q)$$

Finally note that the function $c(T; S, I, Q)$ gives the cost of tracing-testing T agents, which are put in quarantine when the state is (S, I, Q). This cost does not include the forgone output of the quarantine. We allow the cost function c to depend the flow of those traced-tested-quarantined, T, as well as on the composition of the state (S, I, Q). We assume that the function $c(T; S, I, Q)$ is increasing and convex in T, for fixed (S, I, Q), and that $c(0; S, I, Q) = 0$. Below, we elaborate more on the parameterization and interpretation of the cost function c and the presence of the state (S, I, Q) in it.

We will add two assumptions, which will allow to eliminate Q as part of the state, which we discuss in three steps:

(a) The first term in the flow cost contains the forgone output cost of lockdown if there is no test, i.e. the term $wL(1-Q)$ if $\tau = 0$. This term can be dispensed of by focusing on the case with an antibody test, i.e. the case with $\tau = 1$. We will assume this from now on.

(b) The second term where we have a Q is the specification of the tracing-testing cost

$c(\cdot)$. We discuss the proposed formulation, and how it dispensed from the use of Q. The cost of finding a number of people T that are infected and aren't currently in quarantine, i.e. a member of the population X, should depend on the size of X in the population that is being trace-tested. If we assume that $\tau = 1$, that population is of size $S + X$. In one extreme, if testing is random, the number of people that have to be tested to identify T is $T(S + X)/X$. Simply put, it is harder to find someone infected if there are very few infected in the population and we search at random. If, instead, there is a smart tracing technology, the cost can scale at a lower rate relative to the composition of the pool. This motivates the following functional form:

$$c(T, S, X) = \eta \left(T \left(\frac{S + X}{X} \right)^{1-\zeta} \right) \tag{C22}$$

where η is a weakly increasing, positive, and convex function, and where $\zeta \in [0, 1]$ indexes how smart the tracing is. If $\zeta = 0$ then there is no tracing, and it is just random sampling. If $\zeta = 1$ then tracing is very powerful, the fraction in the population is immaterial, and the cost depends only on the number to be traced. Summarizing, the cost function $\eta(z)$ depends on the number of "tasks" that have to be carried out to identify T infected, and we parameterize the number of tasks as $z = T\left((S + X)/X\right)^{1-\zeta}$, where each "task" is a combination of tracing and testing.

(c) The third term where Q shows up is the number of deaths per unit of time $\phi(X + Q)\,(X + Q) = \phi(I)\,I$, which depends on the total number of infected $I = Q + X$, regardless of whether they are in quarantine or not. This can be dispensed with if we consider the case in which the fatality rate function $\phi(\cdot)$ is constant, i.e. $\kappa = 0$, so that $\phi(X + Q) = \varphi\gamma(X + Q)$, where φ and γ are constant parameters.

Combining these assumptions, i.e. $\tau = 1$ and $\kappa = 0$, and again using integration by parts and the law of motion of Q to rewrite $vsl\,\varphi\gamma \int_0^\infty e^{-(r+\nu)t} Q_t dt$, we obtain the following per period flow cost:

$$wL\,(S+X) + T\frac{w + vsl\,\varphi\gamma}{r+\nu+\gamma} + \eta\left(T\left(\frac{S+X}{X}\right)^{1-\zeta}\right) + vsl\,\varphi\gamma X$$

Two-state-problem. Thus, if $\tau = 1$ and $\kappa = 0$ we can formulate the problem with only two state variables, and with two controls: $L \leq \bar{L}$ and $T \leq \bar{T}$.

$$\begin{aligned}(r+\nu)v(S,X) = &\min_{L\in[0,\bar{L}],T\in[0,\bar{T}]} wL\Big[S+X\Big] + T\frac{w + vsl\,\varphi\gamma}{r+\nu+\gamma} + \eta\left(T\left(\frac{S+X}{X}\right)^{1-\zeta}\right) + vsl\,\varphi\gamma X \\ &+ \left[\beta\, S\, X\,(1-\theta L)^2\right]\left[\partial_X v(S,X) - \partial_S v(S,X)\right] \\ &- \left[\gamma X + T\right]\partial_X v(S,X) \qquad \text{(C23)}\end{aligned}$$

We summarize the previous argument in the next proposition:

Proposition 1. Assume that there is an antibody test, so $\tau = 1$, and that the fatality rate function $\phi(I) = \varphi\,\gamma$ is constant, i.e. $\kappa = 0$. Let (S, I, Q) be the initial conditions for the original three state variable problem, with a minimized value $\mathcal{V}(S, I, Q)$ and associated optimal policies $\mathcal{L}(S, I, Q), \mathcal{T}(S, I, Q)$. Let (S, X) be the initial conditions for the modified two state variable defined in equation (C23), with a minimized value $v(S, X)$ and associated optimal policy $L(S, X), T(S, X)$. Then, for all $(S, I, Q) \in \mathbb{R}^3_+$ with $S + I + Q \leq 1$, and $I \geq Q$, we have:

$$\mathcal{V}(S,I,Q) = v(S, I-Q) + Q\frac{w + vsl\,\varphi\gamma}{r+\gamma+\nu} \qquad \text{(C24)}$$

$$\mathcal{L}(S,I,Q) = L(S, I-Q) \text{ and } \mathcal{T}(S,I,Q) = T(S, I-Q) \qquad \text{(C25)}$$

We note that the minimization problem in the right hand side of equation (C23) is a convex problem in T. Instead, the minimization problem with respect to L, as in the problem without TTQ, is convex at the point where $\partial_X v(S, X) \geq \partial_S v(S, X)$.

D Implementation with and without TTQ of the code

The user studying the planning problem without the TTQ protocol must run the files provided choosing the maximum of the testing-tracing rate $T_{max} = 0$. In order to implement the code with the TTQ protocol, the user must choose $T_{max} = 1$ as well as $\tau = 1$ and $\kappa = 0$ in accordance to the assumptions required for the implementability of the problem described above. Furthermore, the accuracy of the solution increases as the number of grid points increase and as the number of iteration increase (or equivalently the tolerance to stop the iterations). To obtain accurate solutions even when the initial conditions are close to the boundary (i.e. X_0 =0.01 and S_0=0.97), for the benchmark case we ran the code with 300 grid points for S (i.e. N_S in the code) and $k \times 300$ grid points for X where $k = 5$ (i.e. N_X in the code). For higher values of vsl (i.e. vsl=70), we use N_S=650 grid points for S and $N_X = 5 \times 650 = 3250$ grid points for X . The file "SimplePlanningProblem.m" solves the planning problem with and without TTQ, this file should be run first. The file "SimplePlanningProblem_Figures_NoTTQ.m" plots the main figures in Alvarez, Argente and Lippi (2020) without TTQ and takes as an input the .mat file created in "SimplePlanningProblem.m." The file "SimplePlanning-Problem_Figures_TTQ.m" plots the figures with the TTQ protocol.

E Quadratic Lockdown Costs

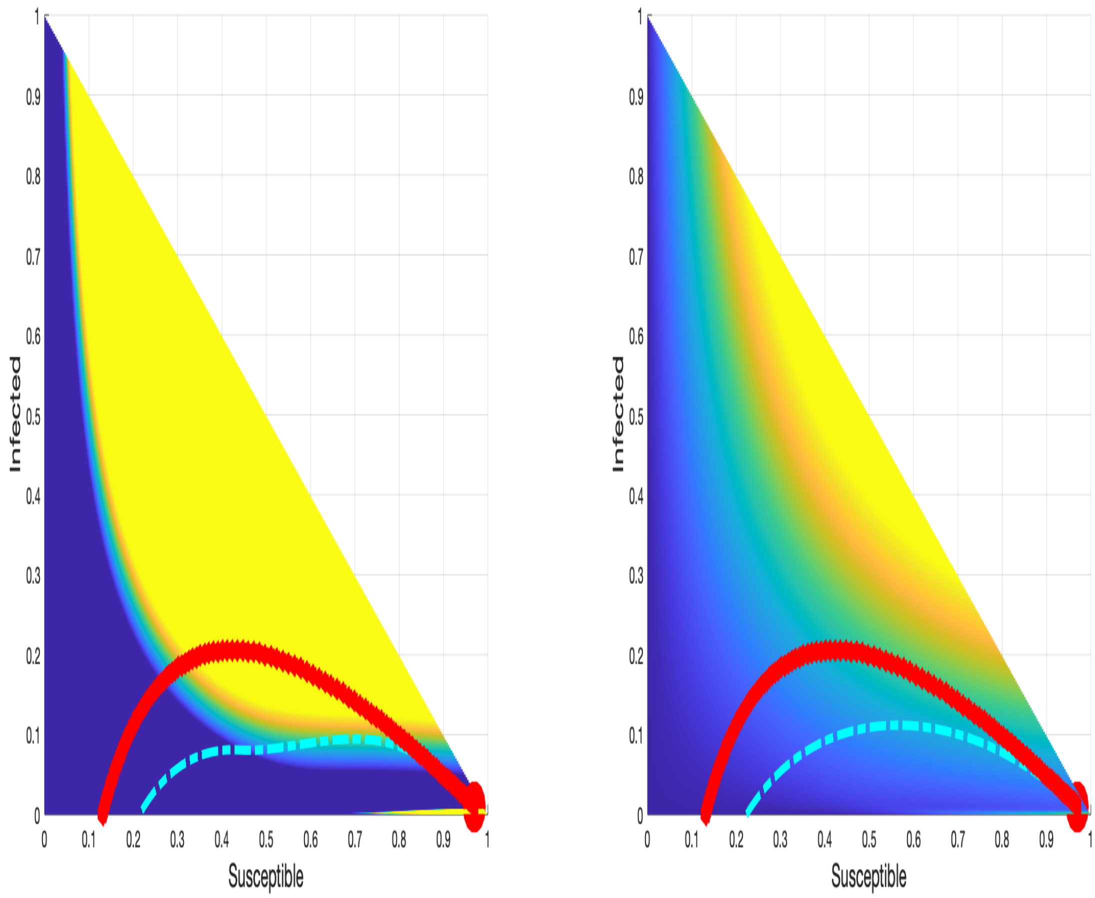
In this section, we consider the case briefly mentioned in the paper where the period cost of lockdown is quadratic in the total number of hours in lockdown. We consider only the case without TTQ. In particular, instead of having $w[L(1-\tau+\tau(I+S))]$ we use $w\frac{c}{2}[L(1-\tau+\tau(I+S))]^2$. The constant c is chosen as follows. Let L_t^b be the path that solves the benchmark case with either τ=0 or τ=1. Then c is chosen so that, if the same path for lockdown is followed in the benchmark and in the quadratic case, the expected discounted cost are equal i.e.

$$\int_0^\infty e^{-(r+\nu)t} w L_t^b \Big[\tau(S_t+I_t)+(1-\tau)\Big] dt = \int_0^\infty e^{-(r+\nu)t} w \frac{c}{2} \Big[L_t^b \tau(S_t+I_t)+(1-\tau)\Big]^2 dt$$

We compare the optimal policy of the benchmark case under τ=0 with the case where the output losses from a lockdown are quadratic. The panel on the right in Figure E1 shows that, in the quadratic cost case, the path of the lockdown policy is smoother through time. Figure E2 shows the time paths for the both the cases with and without anti-body test. In both instances, the lockdown policy starts immediately at positive values, but the share of the population under lockdown is smaller than in our benchmark cases with linear costs.

Relative to the benchmark case, the code is very similar. The only changes in the code are the first order conditions for L in the right-hand side of the HBJ equation and the period return of the value function. The file "SimplePlanningProblem_quadratic.m" solves the planning problem without TTQ, this file should be run first and must be implemented choosing the maximum of the testing-tracing rate $T_{max}=0$. The file "SimplePlanningProblem_Figures_NoTTQ.m" plots the main figures and takes as an input the .mat file created in "SimplePlanningProblem_quadratic.m."

Figure E1: Optimal policy, benchmark case - linear and quadratic output costs



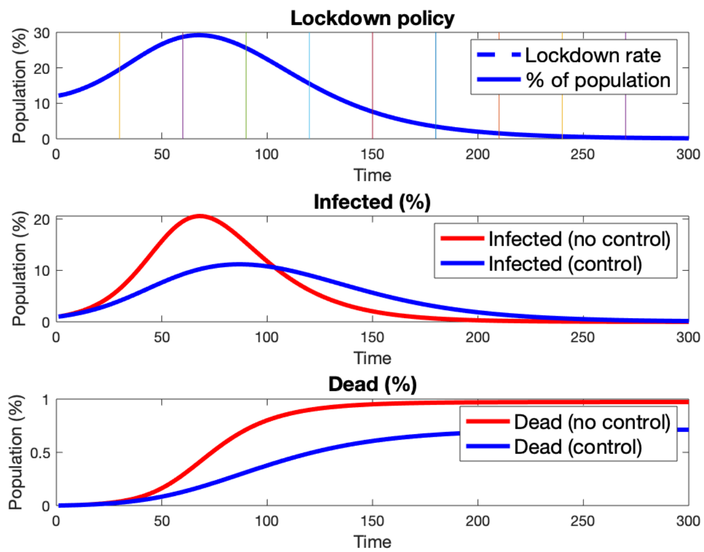
(a) Linear Cost

(b) Quadratic Cost

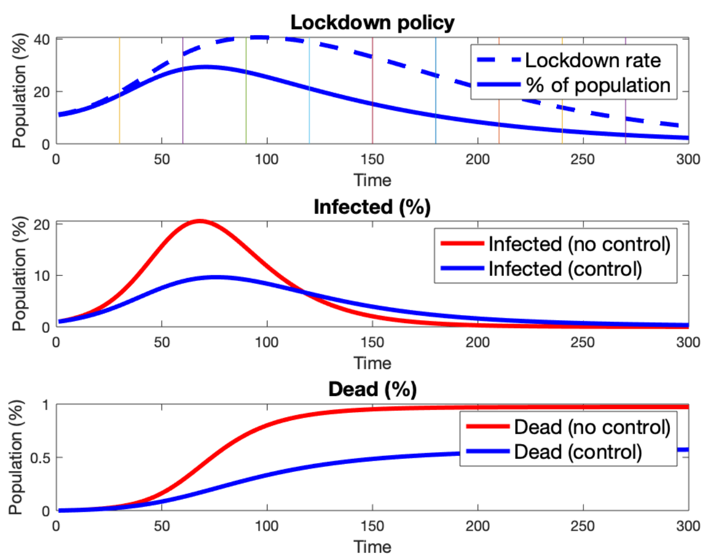
Note: The figure on the left shows the optimal policy for the benchmark parameter values. The blue area indicates lower values of lockdown (L=0) and the yellow color higher values (L=0.7). The figure on the right shows the optimal policy for the benchmark parameter values when the output costs of the lockdown are quadratic.

Figure E2: **Time paths under baseline parameters (quadratic output losses)**

Panel A – Case w/o testing ($\tau = 0$)



Panel B – Case w / testing ($\tau = 1$)



Note: Panel A considers the case where the test is not available ($\tau = 0$), Panel B the case where a test is available ($\tau = 1$). Both cases consider quadratic lockdown losses. The red lines correspond to the scenario where no Lockdown is exercised, the blue lines to the optimal control case. The parameters are: $r = 0.05$, $\nu = 1/1.5$, $w = 1$, $vsl = 40$, $\gamma = 1/18 \cdot 365$, $\beta = 0.13 \cdot 365$, $\varphi = 0.0068$, $\kappa = 0.034$, $\theta = 0.5$, $\bar{L} = 0.70$, $\bar{T} = 0$. The initial condition is $I_0 = 0.01$ and $S_0 = 0.97$.

References

Alvarez, Fernando E, David Argente, and Francesco Lippi. 2020. "A simple planning problem for covid-19 lockdown, testing and tracing."

Kushner, Harold, and Paul G Dupuis. 2013. *Numerical methods for stochastic control problems in continuous time.* Vol. 24, Springer Science & Business Media.