The Difficulty of Easy Projects
Online Appendix

Wioletta Dziuda[*] Arda Gitmez[†] Mehdi Shadmehr[‡]

August 2020

In this Appendix, we restate and proof Theorem 2. Theorem 2 tells us how the probability of success changes with $N-q$, and its proof follows the same logic as the proof of Theorem 1. Then we state and prove Proposition 2, which is a generalization of Proposition 1.

For general q and N with $q \leq N$, the indifference condition (which is a generalization of (1)) is:

$$\frac{c^*_{q,N}(u)}{u} = \binom{N-1}{q-1} \left(F\left(c^*_{q,N}(u)\right)\right)^{q-1} \left(1 - F(c^*_{q,N}(u))\right)^{N-q} \tag{1}$$

Similarly, the probability of success for general q and N (the generalization of (2) and (3)) is:

$$S_{q,N}(c^*_{q,N}(u)) = \sum_{k=q}^{N} \binom{N}{k} \left(F\left(c^*_{q,N}(u)\right)\right)^{k} \left(1 - F\left(c^*_{q,N}(u)\right)\right)^{N-k} \tag{2}$$

$$= 1 - \sum_{k=0}^{q-1} \binom{N}{k} \left(F\left(c^*_{q,N}(u)\right)\right)^{k} \left(1 - F\left(c^*_{q,N}(u)\right)\right)^{N-k} \tag{3}$$

We now restate Theorem 2 from the main text.

Theorem 2. *Take q, N, q' and N' such that $q \leq N$ and $q' \leq N'$.*

1. *If $q \leq q'$ and $N - q \geq N' - q'$, with at least one inequality strict, then $S_{q,N}(c^*_{q,N}(u)) > S_{q',N'}(c^*_{q',N'}(u))$ for sufficiently small u.*

[*]Harris School of Public Policy, University of Chicago. E-mail: wdziuda@uchicago.edu

[†]Harris School of Public Policy, University of Chicago, and Department of Economics, Bilkent University. E-mail: arda@uchicago.edu

[‡]Harris School of Public Policy, University of Chicago, and Department of Economics, University of Calgary. E-mail: mshadmeh@gmail.com

2. *Suppose the support of costs is bounded from above, or $1 - F(x)$ is log-concave for sufficiently large x. If $N - q < N' - q'$, then $S_{q,N}(c^*_{q,N}(u)) > S_{q',N'}(c^*_{q',N'}(u))$ for sufficiently large u.*

Before we provide the proof for Theorem 2, we provide a Lemma that will be crucial for the second part of the proof.

Lemma 2. *Take q, N, q' and N' such that $q < N$ and $q' < N'$. If the support of costs is bounded from above, or if $1 - F(x)$ is log-concave for sufficiently large x,*

1. *For all $\alpha > 0$, $\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{u^\alpha} = 0$.*

2. $\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)} < \infty$.

Proof of Lemma 2. Suppose first that the support of costs is bounded from above. Then, by (1), for any $q < N$ and $u > 0$, $c^*_{q,N}(u)$ is strictly smaller than the upper bound of the cost distribution, which proves the first part. Also by (1), $\lim_{u\to\infty} c^*_{q',N'}(u) > 0$, which proves the second part.

Next, suppose the support of costs is not bounded, but $1 - F(x)$ is log-concave for sufficiently high x.

To prove the first part, we first show that $\lim_{u\to\infty} c^*_{q,N}(u) = \infty$. Suppose towards a contradiction that $\lim_{u\to\infty} c^*_{q,N}(u) < \infty$. This implies that $\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{u} = 0$. But the right-hand side of (1) converges to a strictly positive number, a contradiction.

Then, using L'Hôpital's rule,

$$\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{u^\alpha} = \lim_{u\to\infty} \frac{\frac{dc^*_{q,N}(u)}{du}}{\alpha u^{\alpha-1}}. \tag{4}$$

Differentiating (1) with respect to u yields (we omit argument u here for brevity)

$$\frac{dc^*_{q,N}}{du} = \frac{\binom{N-1}{q-1}\left(F\left(c^*_{q,N}\right)\right)^{q-1}\left(1-F\left(c^*_{q,N}\right)\right)^{N-q}}{1-u\binom{N-1}{q-1}\left(F\left(c^*_{q,N}\right)\right)^{q-2}\left(1-F\left(c^*_{q,N}\right)\right)^{N-q-1}\left((q-1)\left(1-F\left(c^*_{q,N}\right)\right)-(N-q)F(c^*_{q,N})\right)f\left(c^*_{q,N}\right)}. \tag{5}$$

From (1), the numerator is equal to $\frac{c^*_{q,N}(u)}{u}$, and also

$$\binom{N-1}{q-1}\left(F\left(c^*_{q,N}(u)\right)\right)^{q-2}\left(1-F\left(c^*_{q,N}(u)\right)\right)^{N-q-1} = \frac{c^*_{q,N}(u)}{u}\frac{1}{F(c^*_{q,N}(u))\left(1-F(c^*_{q,N}(u))\right)}. \tag{6}$$

Substituting $\frac{c^*_{q,N}(u)}{u}$ and (6) into (5) yields:

$$\begin{aligned}\frac{dc^*_{q,N}(u)}{du} &= \frac{\frac{c^*_{q,N}(u)}{u}}{1-u\frac{c^*_{q,N}(u)}{u}\frac{1}{F(c^*_{q,N}(u))\left(1-F(c^*_{q,N}(u))\right)}\left((q-1)\left(1-F\left(c^*_{q,N}(u)\right)\right)-(N-q)F(c^*_{q,N}(u))\right)f\left(c^*_{q,N}(u)\right)} \\ &= \frac{1}{u}\frac{1}{\frac{1}{c^*_{q,N}(u)}-\left((q-1)\frac{1-F(c^*_{q,N}(u))}{F(c^*_{q,N}(u))}-(N-q)\right)\frac{f(c^*_{q,N}(u))}{1-F(c^*_{q,N}(u))}}. \qquad (7)\end{aligned}$$

Substituting (7) into (4),

$$\lim_{u\to\infty}\frac{c^*_{q,N}(u)}{u^\alpha} = \lim_{u\to\infty}\frac{1}{\alpha u^\alpha}\frac{1}{\frac{1}{c^*_{q,N}(u)}-\left((q-1)\frac{1-F(c^*_{q,N}(u))}{F(c^*_{q,N}(u))}-(N-q)\right)\frac{f(c^*_{q,N}(u))}{1-F(c^*_{q,N}(u))}}. \qquad (8)$$

Recall that $\lim_{u\to\infty}c^*_{q,N}(u)=\infty$, and thus $\lim_{u\to\infty}F(c^*_{q,N}(u))=1$. Since $1-F(x)$ is log-concave, $\lim_{u\to\infty}\frac{f(c^*_{q,N}(u))}{1-F(c^*_{q,N}(u))}>0$. Equation (8) then implies that $\lim_{u\to\infty}\frac{c^*_{q,N}(u)}{u^\alpha}=0$.

To prove the second part by contradiction, suppose that $\lim_{u\to\infty}\frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)}=\infty$. Then, $\lim_{u\to\infty}c^*_{q,N}(u)>\lim_{u\to\infty}c^*_{q',N'}(u)$. Because $1-F(x)$ is log-concave, $\frac{f(x)}{1-F(x)}$ is increasing, and hence

$$\lim_{u\to\infty}\frac{\frac{f\left(c^*_{q',N'}(u)\right)}{1-F\left(c^*_{q',N'}(u)\right)}}{\frac{f\left(c^*_{q,N}(u)\right)}{1-F\left(c^*_{q,N}(u)\right)}}<\infty. \qquad (9)$$

Using L'Hôpital's rule,

$$\begin{aligned}\lim_{u\to\infty}\frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)} &= \lim_{u\to\infty}\frac{\frac{dc^*_{q,N}(u)}{du}}{\frac{dc^*_{q',N'}(u)}{du}} \\ &= \lim_{u\to\infty}\frac{\frac{1}{c^*_{q',N'}(u)}-\left((q'-1)\frac{1-F(c^*_{q',N'}(u))}{F(c^*_{q',N'}(u))}-(N'-q')\right)\frac{f(c^*_{q',N'}(u))}{1-F(c^*_{q',N'}(u))}}{\frac{1}{c^*_{q,N}(u)}-\left((q-1)\frac{1-F(c^*_{q,N}(u))}{F(c^*_{q,N}(u))}-(N-q)\right)\frac{f(c^*_{q,N}(u))}{1-F(c^*_{q,N}(u))}} \quad \text{(from (7))} \\ &= \lim_{u\to\infty}\frac{N'-q'}{N-q}\frac{\frac{f\left(c^*_{q',N'}(u)\right)}{1-F\left(c^*_{q',N'}(u)\right)}}{\frac{f\left(c^*_{q,N}(u)\right)}{1-F\left(c^*_{q,N}(u)\right)}}<\infty \quad \text{(from (9))},\end{aligned}$$

which is a contradiction. □

Proof of Theorem 2. To prove the first part, note that from (1), $c^*_{q,N}(0)=0$. Therefore, when $u = 0$, the number of citizens contributing is:

$$X \sim Binomial(N, F(0))$$

and the probability of success is:

$$\begin{aligned} S_{q,N}(c^*_{q,N}(0)) = S_{q,N}(0) &= \Pr\{X \geq q\} \\ &= 1 - \Pr\{X \leq q-1\} \\ &= 1 - I_{1-F(0)}\left(N-q+1, q\right), \end{aligned}$$

where $I_p\left(a,b\right)$ is the regularized beta function, defined as:

$$I_p\left(a,b\right) = \frac{\int_0^p t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt} \qquad p \in [0,1], a,b > 0.$$

Since $I_p\left(.,.\right)$ is strictly decreasing in its first argument and strictly increasing in its second argument, $S_{q,N}(0)$ is strictly increasing in $N-q+1$ and strictly decreasing in q. As a result, $S_{q,N}(c^*_{q,N}(0)) > S_{q',N'}(c^*_{q',N'}(0))$ if $N-q \geq N'-q'$ and $q \leq q'$, with at least one inequality strict. The first part of the theorem follows from the continuity of $c^*_{q,N}(u)$ and $S^*_{q,N}(c^*_{q,N}(u))$ in u.

Now, we prove the second part of the theorem. Take q, N, q' and N' such that $N-q < N'-q'$. We will consider three separate cases.

Case 1: $N-q > 0$. Write the difference between success probabilities as:

$$S_{q,N}(c^*_{q,N}(u)) - S_{q',N'}(c^*_{q',N'}(u)) = \left(1 - S_{q',N'}(c^*_{q',N'}(u))\right)\left(1 - \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))}.\right)$$

Thus, it suffices to show

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} < 1.$$

From (3),

$$\begin{aligned} \frac{1 - S_{q,N}(u)}{1 - S_{q',N'}(u)} &= \frac{\sum_{k=0}^{q-1}\binom{N}{k}\left(F\left(c^*_{q,N}(u)\right)\right)^k\left(1 - F\left(c^*_{q,N}(u)\right)\right)^{N-k}}{\sum_{k=0}^{q'-1}\binom{N'}{k}\left(F\left(c^*_{q',N'}(u)\right)\right)^k\left(1 - F\left(c^*_{q',N'}(u)\right)\right)^{N'+1-k}} \\ &= \frac{\sum_{k=0}^{q-1}\binom{N}{k}\left(F\left(c^*_{q,N}(u)\right)\right)^k\left(1 - F\left(c^*_{q,N}(u)\right)\right)^{q-1-k}}{\sum_{k=0}^{q'-1}\binom{N+1}{k}\left(F\left(c^*_{q',N'}(u)\right)\right)^k\left(1 - F\left(c^*_{q',N'}(u)\right)\right)^{q'-1-k}} \frac{\left(1 - F\left(c^*_{q,N}(u)\right)\right)^{N+1-q}}{\left(1 - F\left(c^*_{q',N'}(u)\right)\right)^{N'+1-q'}} \end{aligned} \tag{10}$$

Moreover, since $\lim_{u\to\infty} F(c^*_{q,N}(u)) = 1$,

$$\lim_{u\to\infty} \sum_{k=0}^{q-1} \binom{N}{k} \left(F\left(c^*_{q,N}(u)\right)\right)^k \left(1 - F\left(c^*_{q,N}(u)\right)\right)^{q-1-k} = \binom{N}{q-1}. \tag{11}$$

Substituting (11) into (10) yields

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} = \lim_{u\to\infty} \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}} \frac{\left(1 - F\left(c^*_{q,N}(u)\right)\right)^{N+1-q}}{\left(1 - F\left(c^*_{q',N'}(u)\right)\right)^{N'+1-q'}}. \tag{12}$$

Substituting the indifference conditions (1) into (12) yields

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} = \lim_{u\to\infty} \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}} \frac{\left(\frac{c^*_{q,N}(u)}{u} \frac{1}{\binom{N-1}{q-1}\left(F\left(c^*_{q,N}(u)\right)\right)^{q-1}}\right)^{\frac{N+1-q}{N-q}}}{\left(\frac{c^*_{q',N'}(u)}{u} \frac{1}{\binom{N'-1}{q'-1}\left(F\left(c^*_{q',N'}(u)\right)\right)^{q'-1}}\right)^{\frac{N'+1-q'}{N'-q'}}}$$

$$= \lim_{u\to\infty} \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}} \frac{\left(\frac{1}{\binom{N-1}{q-1}\left(F\left(c^*_{q,N}(u)\right)\right)^{q-1}}\right)^{\frac{N+1-q}{N-q}}}{\left(\frac{1}{\binom{N'-1}{q'-1}\left(F\left(c^*_{q',N'}(u)\right)\right)^{q'-1}}\right)^{\frac{N'+1-q'}{N'-q'}}} \left(\frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)}\right)^{\frac{N'+1-q'}{N'-q'}} \left(\frac{c^*_{q,N}(u)}{u}\right)^{\frac{N+1-q}{N-q} - \frac{N'+1-q'}{N'-q'}}$$

$$= \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}} \frac{\left(\frac{1}{\binom{N-1}{q-1}}\right)^{\frac{N+1-q}{N-q}}}{\left(\frac{1}{\binom{N'-1}{q'-1}}\right)^{\frac{N'+1-q}{N'-q'}}} \left(\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)}\right)^{\frac{N'+1-q'}{N'-q'}} \left(\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{u}\right)^{\frac{N+1-q}{N-q} - \frac{N'+1-q'}{N'-q'}}.$$

By Lemma 2, $\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{c^*_{q',N'}(u)} < \infty$ and $\lim_{u\to\infty} \frac{c^*_{q,N}(u)}{u} = 0$. Thus,

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} = 0$$

if

$$\frac{N+1-q}{N-q} - \frac{N'+1-q'}{N'-q'} > 0 \iff N - q < N' - q'.$$

Case 2: $N - q = 0$ **and** $N' - q' = 1$**.** If the support of costs is bounded from above, then

$1 - F(c^*_{q,N}(u)) = 0$ for sufficiently large u, while $1 - F(c^*_{q',N'}(u)) > 0$ for all u. As a result, $S_{q,N}(c^*_{q,N}(u)) = 1 > S_{q',N'}(c^*_{q',N'}(u))$. If the support of costs has no upper bound, (12) applies and

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} = \lim_{u\to\infty} \frac{N}{\binom{N'}{N'-2}} \frac{\left(1 - F\left(c^*_{q,N}(u)\right)\right)}{\left(1 - F\left(c^*_{q',N'}(u)\right)\right)^2}.$$

By (1), $\lim_{u\to\infty} \frac{c^*_{N,N}(u)}{u} = 1$. By Lemma 2, $\lim_{u\to\infty} \frac{c^*_{q',N'}(u)}{u^\alpha} = 0$ for any $\alpha > 0$. Therefore, it suffices to show:

$$\lim_{u\to\infty} \frac{1 - F(u)}{(1 - F(u^\alpha))^2} = 0, \text{ for some } \alpha \in (0,1). \tag{13}$$

This is precisely condition (13), demonstrated in the proof of Theorem 1.

Case 3: $N - q = 0$ **and** $N' - q' > 1$**.** For sufficiently large u,

$$S_{N,N}(c^*_{N,N}(u)) > S_{N-1,N}(c^*_{N-1,N}(u)) > S_{q',N'}(c^*_{q',N'}(u)).$$

where the first inequality follows from Case 2 and the second inequality follows from Case 1.

□

We state and prove now the equivalent of Proposition 1.

Proposition 2. *Suppose* $1 - F(x) = \beta/x^\alpha$*,* $\alpha, \beta > 0$*, for sufficiently large* x*.*

- *If* $\alpha > 1$*, the likelihood of success is decreasing in* $N - q$ *for sufficiently large* u*:* $S_{q,N}(c^*_{q,N}(u)) > S_{q',N'}(c^*_{q',N'}(u))$ *when* $N - q < N' - q'$ *for sufficiently large* u*,*
- *if* $\alpha < 1$*, the likelihood of success is increasing in* $N - q$ *for sufficiently large* u*:* $S_{q,N}(c^*_{q,N}(u)) > S_{q',N'}(c^*_{q',N'}(u))$ *when* $N - q > N' - q'$ *for sufficiently large* u*.*

Proof of Proposition 2. Substituting $1 - F(x) = \frac{\beta}{x^\alpha}$ in (12),

$$\lim_{u\to\infty} \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))} = \lim_{u\to\infty} \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}} \frac{\beta^{N-q+1}}{\beta^{N'-q'+1}} \left(\frac{\left(c^*_{q',N'}(u)\right)^{N'-q'+1}}{\left(c^*_{q,N}(u)\right)^{N-q+1}} \right)^\alpha. \tag{14}$$

Substituting $1 - F(x) = \frac{\beta}{x^\alpha}$ in (1),

$$c^*_{q,N}(u) = \binom{N-1}{q-1} \left(1 - \frac{\beta}{(c^*_{q,N}(u))^\alpha}\right)^{q-1} \left(\frac{\beta}{(c^*_{q,N}(u))^\alpha}\right)^{N-q} u.$$

Thus,

$$(c^*_{q,N}(u))^{N-q+1} = \left(\binom{N-1}{q-1}\beta^{N-q}\left(1-\frac{\beta}{(c^*_{q,N}(u))^\alpha}\right)^{q-1}u\right)^{\frac{N-q+1}{\alpha(N-q)+1}}. \tag{15}$$

Substituting this in (14),

$$\lim_{u\to\infty}\frac{1-S_{q,N}(c^*_{q,N}(u))}{1-S_{q',N'}(c^*_{q',N'}(u))} = \lim_{u\to\infty}\frac{\binom{N}{q-1}}{\binom{N'}{q'-1}}\beta^{N-q-(N'-q')}\left(\frac{\left(\binom{N'-1}{q'-1}\beta^{N'-q'}\left(1-\frac{\beta}{(c^*_{q',N'}(u))^\alpha}\right)^{q'-1}u\right)^{\frac{N'-q'+1}{\alpha(N'-q')+1}}}{\left(\binom{N-1}{q-1}\beta^{N-q}\left(1-\frac{\beta}{(c^*_{q,N}(u))^\alpha}\right)^{q-1}u\right)^{\frac{N-q+1}{\alpha(N-q)+1}}}\right)^\alpha.$$

Since $\lim_{u\to\infty} c^*_{q,N}(u)=\infty$,

$$\begin{aligned}\lim_{u\to\infty}\frac{1-S_{q,N}(c^*_{q,N}(u))}{1-S_{q',N'}(c^*_{q',N'}(u))} &= \lim_{u\to\infty}\frac{\binom{N}{q-1}}{\binom{N'}{q'-1}}\beta^{N-q-(N'-q')}\left(\frac{\left(\binom{N'-1}{q'-1}\beta^{N'-q'}u\right)^{\frac{N'-q'+1}{\alpha(N'-q')+1}}}{\left(\binom{N-1}{q-1}\beta^{N-q}u\right)^{\frac{N-q+1}{\alpha(N-q)+1}}}\right)^\alpha \\ &= \frac{\binom{N}{q-1}}{\binom{N'}{q'-1}}\beta^{N-q-(N'-q')}\left(\frac{\left(\binom{N'-1}{q'-1}\beta^{N'-q'}\right)^{\frac{N'-q'+1}{\alpha(N'-q')+1}}}{\left(\binom{N-1}{q-1}\beta^{N-q}\right)^{\frac{N-q+1}{\alpha(N-q)+1}}}\right)^\alpha \lim_{u\to\infty} u^{\left(\frac{\alpha(N'-q')+\alpha}{\alpha(N'-q')+1}-\frac{\alpha(N-q)+\alpha}{\alpha(N-q)+1}\right)}.\end{aligned}$$

Now, suppose:

- $\alpha>1$ and $N-q<N'-q'$, or,
- $\alpha<1$ and $N-q>N'-q'$.

In both cases,

$$\frac{\alpha(N'-q')+\alpha}{\alpha(N'-q')+1}-\frac{\alpha(N-q)+\alpha}{\alpha(N-q)+1}<0.$$

Therefore,

$$\lim_{u\to\infty}\frac{1-S_{q,N}(c^*_{q,N}(u))}{1-S_{q',N'}(c^*_{q',N'}(u))}=0.$$

Thus, for sufficiently large u, $\frac{1-S_{q,N}(c^*_{q,N}(u))}{1-S_{q',N'}(c^*_{q',N'}(u))} < 1$ and

$$S_{q,N}(c^*_{q,N}(u)) - S_{q',N'}(c^*_{q',N'}(u)) = \big(1 - S_{q',N'}(c^*_{q',N'}(u))\big)\left(1 - \frac{1 - S_{q,N}(c^*_{q,N}(u))}{1 - S_{q',N'}(c^*_{q',N'}(u))}\right) > 0$$

$\square$