Online Appendix for: A Note on Temporary Supply Shocks with Aggregate Demand Inertia

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A. Omitted Proofs

This appendix contains the results and the derivations omitted from the main text.

A.1. Omitted proofs and extensions for Section 2

We first present the proofs omitted from Section 2. We then characterize the equilibrium for the remaining case in which the expansionary policy constraint might bind also in the low-supply state.

A.1.1. Omitted proofs

Proof of Lemma 1. Suppose the economy switched to the high-supply state $s = H$ with $y_{t-1} < y_H^*$. We verify that the conjectured allocation is an equilibrium.

We first show that the expansionary policy constraint in (2) binds along the conjectured equilibrium path. Suppose the constraint does not bind. Then, the central bank would target a zero gap, $y_t = y_H^*$, by setting the interest rate in (4),

$$i_t = \rho - \frac{\eta}{1-\eta}\left(y_H^* - y_{t-1}\right).$$

Along the conjectured path, we have $y_{t-1} < y_H^*$ and the required interest rate satisfies, $i_t < \rho$. However, since the policy targets a zero output gap, $y_t = y_H^*$, the policy constraint implies $i_t \geq \rho$. This provides a contradiction and implies that the policy constraint binds. In particular, the policy effectively follows the Taylor rule in (7).

We next characterize the evolution of output. Combining the IS curve in (1) and the Taylor rule in (7), output follows the difference equation,

$$y_t = \eta y_{t-1} + (1-\eta)\left(-\phi\left(y_t - y_H^*\right) + y_{t+1}\right).$$

We drop the expectations since there is no (residual) uncertainty. Let $\tilde{y}_t = y_t - y_H^*$ denote the output gap. Then, we can rewrite the difference equation as,

$$\tilde{y}_t = \eta \tilde{y}_{t-1} + (1-\eta)\left(-\phi \tilde{y}_t + \tilde{y}_{t+1}\right).$$

In matrix notation, we have the system,

$$\begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{y}_t \end{bmatrix} = M \begin{bmatrix} \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} \text{ where } M = \begin{bmatrix} \frac{1}{1-\eta} + \phi & -\frac{\eta}{1-\eta} \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix M is given by

$$P(x) = x^2 - x\left(\frac{1}{1-\eta} + \phi\right) + \frac{\eta}{1-\eta}.$$

This polynomial has two roots that satisfy

$$0 < \gamma_1 < 1 < \gamma_2.$$

Since $\tilde{y}_{t-1}$ is predetermined and $\tilde{y}_t$ is not, this condition ensures the system is saddle path stable. Moreover, letting $\gamma_H \equiv \gamma_1 \in (0,1)$ denote the stable eigenvalue, the solution converges to zero at a constant rate:

$$\tilde{y}_{t+h} = \gamma_H \tilde{y}_{t+h-1} = \gamma_H^{h+1} \tilde{y}_{t-1}.$$

This proves (8).

We can then solve for the value function over the region $y_{t-1} < y_H^*$ as

$$V_H = \sum_{h=0}^{\infty} -\beta^h \frac{\left(\tilde{y}_{t+h}\right)^2}{2} = \sum_{h=0}^{\infty} -\beta^h \frac{\left(\gamma_H^{h+1} \tilde{y}_{t-1}\right)^2}{2} = -\frac{\gamma_H^2}{1-\beta\gamma_H^2} \frac{\left(\tilde{y}_{t-1}\right)^2}{2}.$$

This establishes (9).

Note that $\frac{d\theta_H}{d\eta} > 0$ as long as $\frac{d\gamma_H}{d\eta} > 0$. To establish the latter inequality, let $\tilde{\eta} = \frac{\eta}{1-\eta}$ and note that γ_H is the solution to the following equation over the range $(0,1)$:

$$P(x, \tilde{\eta}, \phi) = x^2 - x(1 + \tilde{\eta} + \phi) + \tilde{\eta} = 0.$$

Implicitly differentiating with respect to $\tilde{\eta}$ and evaluating around $x = \gamma_H$, we obtain

$$\frac{dx}{d\tilde{\eta}} = -\left.\frac{\partial P/\partial\tilde{\eta}}{\partial P/\partial x}\right|_{x=\gamma_H} = \frac{1-\gamma_H}{1+\tilde{\eta}+\phi-2\gamma_H} > 0.$$

Here, the inequality follows since $\gamma_H < 1$ and $2\gamma_H < \gamma_1 + \gamma_2 = 1 + \tilde{\eta} + \phi$ (since γ_H is the smaller of the two roots γ_1, γ_2). Since $\tilde{\eta} = \frac{\eta}{1-\eta}$ is increasing in η, we also have $\frac{dx}{d\eta} > 0$. This completes the proof.

For completeness, consider also the case in which the initial output is above its potential $y_{t-1} \geq y_H^*$. In this case, the expansionary policy constraint does not bind and output converges to its potential immediately. The central bank sets the policy rate, $i_t = \rho - \frac{\eta}{1-\eta}\left(y_H^* - y_{t-1}\right)$, and implements $y_t = y_H^*$. The interest rate constraint does not bind because $i_t > \rho$ and $\underline{i}_t\left(y_t\right) = \rho + \phi\left(y_t - y_H^*\right) = \rho$. Over this range $\left(y_{t-1} \geq y_H^*\right)$, the value function satisfies $V_H\left(y_{t-1}\right) = 0$. $\square$

Proof of Proposition 1. Suppose y_{t-1} is sufficiently high that the expansionary constraint does not bind in state L. Then, we can write the central bank's problem as

$$\begin{aligned} V_L &= \max_{y_t} -\frac{\left(y_t - y_L^*\right)^2}{2} + \beta\left(\left(1-\lambda\right)V_L - \lambda\theta_H V_H\left(y_t\right)\right) \qquad \text{(A.1)} \\ \text{where } V_H\left(y_t\right) &= \begin{cases} -\frac{\left(y_t - y_H^*\right)^2}{2} & \text{if } y_t < y_H^* \\ 0 & \text{if } y_t \geq y_H^* \end{cases}. \end{aligned}$$

The second line combines the two cases analyzed in Lemma 1. This is a concave optimization problem. Any y_t that satisfies the first order optimality condition is an optimum. In the main text, we show that an interior solution (with $y_L < y_H^*$) satisfies the optimality condition in (11). Solving this condition, we obtain

$$y_L = \frac{y_L^* + \beta\lambda\theta_H y_H^*}{1+\beta\lambda\theta_H} \in \left(y_L^*, y_H^*\right).$$

It follows that the optimum output is interior and given by y_L. This also implies that solving problem $(A.1)$ is equivalent to solving problem (10) in the main text.

Next consider the interest rate that implements this output level. The IS curve (1) implies

$$i_{t,L} = \rho + \lambda\left(Y_H\left(y_t\right) - y_{t,L}\right) + \left(1-\lambda\right)\left(y_{t+1,L} - y_{t,L}\right) - \frac{\eta}{1-\eta}\left(y_{t,L} - y_{t-1}\right).$$

After substituting $y_{t+1,L} = y_{t,L} = y_L$, we obtain (13).

We also need to verify that this rate does not violate the expansionary policy constraint in (2). Using Lemma 1, we obtain $Y_H(y_L) = \gamma_H y_L + (1-\gamma_H) y_H^*$. Substituting this into (13), we have

$$i_t = \rho + \lambda(1-\gamma_H)(y_H^* - y_L) - \frac{\eta}{1-\eta}(y_L - y_{t-1}).$$

Since $y_t = y_L$, the policy constraint holds as long as:

$$i_t \geq \rho + \phi(y_L - y_H^*).$$

Combining these observations, we verify that the constraint holds as long as the past output gap satisfies the condition in (12),

$$y_{t-1} \geq \overline{y}_L = y_L - \frac{1-\eta}{\eta}(\lambda(1-\gamma_H)+\phi)(y_H^* - y_L).$$

This completes the proof of the proposition. □

A.1.2. Omitted extensions

Proposition 1 characterizes the equilibrium when the past output is not too low so that the expansionary constraint does not bind in the low-supply state. We next characterize the equilibrium in the other case in which the expansionary constraint binds for at least one period. In this case, the output gradually converges to the target level y_L after finitely many periods (absent transition to the high-supply state). Once the output reaches y_L, the equilibrium is the same as in Proposition 1.

Proposition 3. *Suppose the economy is in the temporary supply shock state, $s = L$, with past output y_{t-1} that violates (12), that is: $y_{t-1} < \overline{y}_L$.Then the expansionary policy constraint binds in $s = L$ for at least one period. The initial interest rate is constrained, $i_t = \rho + \phi(y_t - y_H^*)$, and the initial output is below its unconstrained level, $Y_L(y_{t-1}) < y_L$. The output function $Y_L(y_{-1})$ is continuous, piecewise linear, and strictly increasing. Absent a transition to the high-supply state, output converges to the target level y_L after finitely many periods.*

Proof of Proposition 3. Suppose $y_{t-1} < \overline{y}_L$. Then, the interest rate is given by $i_t = \rho + \phi(y_t - y_H^*)$. Using (1) and $Y_H(y_t) = y_H^* + \gamma_H(y_t - y_H^*)$ [see Lemma 1], output follows the recursive equation:

$$
\begin{aligned}
y_t &= \eta y_{t-1} + (1-\eta)\left(\phi\left(y_H^* - y_t\right) + \lambda Y_H\left(y_t\right) + (1-\lambda)\, y_{t+1}\right) \\
&= \eta y_{t-1} + (1-\eta)\left(\phi\left(y_H^* - y_t\right) + \lambda\left(y_H^* + \gamma_H\left(y_t - y_H^*\right)\right) + (1-\lambda)\, y_{t+1}\right)
\end{aligned}
\tag{A.2}
$$

After rearranging terms, this implies

$$
y_t = \frac{\eta y_{t-1} + (1-\eta)\left(\phi y_H^* + \lambda\left(1-\gamma_H\right) y_H^* + (1-\lambda)\, y_{t+1}\right)}{1 + (1-\eta)\left(\phi - \lambda\gamma_H\right)}. \tag{A.3}
$$

Let $\overline{y}_{L,-1} = y_L$ and $\overline{y}_{L,0} = \overline{y}_L < y_L$. We recursively define a sequence of cutoffs $\left\{\overline{y}_{L,k}\right\}$ as follows: given $\overline{y}_{L,k-1}$ and $\overline{y}_{L,k}$, let $\overline{y}_{L,k+1}$ denote the unique solution to:

$$
\overline{y}_{L,k} = \frac{\eta \overline{y}_{L,k+1} + (1-\eta)\left(\phi y_H^* + \lambda\left(1-\gamma_H\right) y_H^* + (1-\lambda)\, \overline{y}_{L,k-1}\right)}{1 + (1-\eta)\left(\phi - \lambda\gamma_H\right)}.
$$

Using $(A.3)$, the output function maps a lower cutoff into the higher cutoff:

$$
Y_L\left(\overline{y}_{L,k+1}\right) = \overline{y}_{L,k}. \tag{A.4}
$$

By induction, we can also show that the cutoffs satisfy $\overline{y}_{L,k+1} < \overline{y}_{L,k} - \frac{1-\eta}{\eta}\phi\left(y_H^* - y_L\right)$. Therefore, there exists K_L such that $\overline{y}_{L,K_L} < 0$. Then, the cutoffs $\left\{\overline{y}_{L,k}\right\}_{k=-1}^{K_L}$ cover the entire region $[0, y_L]$.

We can then define the output function recursively over the intervals $\left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right]$. Let $Y_{L,0}\left(y_{-1}\right) = y_L$ and define a sequence of functions with:

$$
Y_{L,k}\left(y_{-1}\right) = \frac{\eta y_{-1} + (1-\eta)\left(\phi y_H^* + \lambda\left(1-\gamma_H\right) y_H^* + (1-\lambda)\, Y_{L,k-1}\left(Y_{L,k}\left(y_{-1}\right)\right)\right)}{1 + (1-\eta)\left(\phi - \lambda\gamma_H\right)}. \tag{A.5}
$$

These functions are uniquely defined, linear, and strictly increasing over $[0, \overline{y}_L]$. Then, Eq. $(A.4)$ implies that for each interval the output function agrees with the corresponding function in the sequence

$$
Y_L\left(y_{-1}\right) = Y_{L,k}\left(y_{-1}\right) \text{ for } y_{-1} \in \left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right].
$$

In particular, the output function is the piecewise-linear function that maps each interval $\left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right]$ into the higher interval $\left[\overline{y}_{L,k-1}, \overline{y}_{L,k-2}\right]$. This implies that, absent transition to the high-supply state, output converges to the target level y_L after finitely many periods (at most $K_L + 1$ periods). This completes the proof of the proposition. $\square$

A.2. Omitted results and proofs for Section 3

We first consider the case with the NKPC and present the formal results omitted from Section 3.1 along with their proofs. We then consider the case with an inertial Phillips curve analyzed in Section 3.2 and present the omitted results and proofs.

A.2.1. Overheating with a New-Keynesian Phillips Curve

Suppose inflation is determined according to the NKPC (17)

$$\pi_t = \kappa \left(y_t - y^*_{s_t}\right) + \beta E_t \left[\pi_{t+1}\right].$$

Let $\Pi_s \left(y_{t-1}\right), Y_s \left(y_{t-1}\right), V_s \left(y_{t-1}\right)$ denote the inflation, the output, and the value function level when the current state is $s \in \{H, L\}$, and the most recent output is y_{t-1}.

We first characterize the equilibrium in the high supply state $s = H$. To state the result, we define the polynomial:

$$P\left(x\right) = x^3 - x^2 \left(\frac{1}{1-\eta} + \phi_y + \frac{1+\kappa}{\beta}\right) + x\left(\left(\frac{1}{1-\eta} + \phi_y\right)\frac{1}{\beta} + \phi_\pi \frac{\kappa}{\beta} + \frac{\eta}{1-\eta}\right) - \frac{1}{\beta}\frac{\eta}{1-\eta}. \tag{A.6}$$

Lemma 2. *Consider the setup with inflation determined by the NKPC* (17)*. Suppose the polynomial in* (*A*.6) *has exactly one stable root that satisfies* $\gamma_H \in (0,1)$ *(a sufficient condition is* $\phi_y (1-\beta) + (\phi_\pi - 1)\kappa > 0$ *and* $\beta\phi_\pi \leq 1$*). Suppose the economy has switched to the high-supply state,* $s = H$*, with past output* y_{t-1}*. Then, the output gap and the inflation functions are given by:*

$$Y_H \left(y_{t-1}\right) - y^*_H = \gamma_H \left(y_{t-1} - y^*_H\right) \tag{A.7}$$

$$\Pi_H \left(y_{t-1}\right) = \boldsymbol{\pi}_h \left(y_{t-1} - y^*_H\right) \text{ where } \boldsymbol{\pi}_h = \frac{\kappa\gamma_H}{1-\beta\gamma_H}. \tag{A.8}$$

The output gap and inflation both converge to zero at a constant rate γ_H*. The value function is given by*

$$V_H \left(y_{t-1}\right) = -\theta_H \frac{\left(y_{t-1} - y^*_H\right)^2}{2} \text{ where } \theta_H = \frac{\gamma_H^2}{1-\beta\gamma_H^2}\left(1 + \psi\left(\frac{\kappa}{1-\beta\gamma_H}\right)^2\right). \tag{A.9}$$

In the high-supply state, the equilibrium is determined by the IS curve, the NKPC, and the Taylor rule in (15). Under appropriate parametric conditions, the Taylor rule ensures that the output and inflation gaps converge to zero. As before, the convergence is not immediate. Due to inertial demand, *past* output, y_{t-1}, affects the output and inflation gaps in the high-supply state.

Next consider the equilibrium in the low supply state $s = L$. Using Lemma 2, the central bank solves the following version of problem (10):

$$\begin{aligned} V_L(y_{t-1}) &= \max_{y_t,\pi_t} -\frac{(y_t - y_L^*)^2}{2} - \psi\frac{\pi_t^2}{2} + \beta\left((1-\lambda)V_L(y_t) - \lambda\theta_H\frac{(y_t - y_H^*)^2}{2}\right) \quad \text{(A.10)} \\ \text{s.t. } \pi_t &= \kappa(y_t - y_L^*) + \beta((1-\lambda)\Pi_L(y_t) + \lambda\boldsymbol{\pi}_H(y_t - y_H^*)). \end{aligned}$$

Here, the functions, $V_L(y_{t-1})$ and $\Pi_L(y_{t-1}) \equiv \pi_L$, are also both independent of y_{t-1}. Using this observation, the optimality condition is given by

$$\begin{aligned} y_L - y_L^* + \psi\frac{d\pi_t}{dy_t}\pi_L &= \beta\lambda\theta_H(y_H^* - y_L) \\ \text{where } \frac{d\pi_t}{dy_t} &= \kappa + \beta\lambda\boldsymbol{\pi}_H \\ \text{and } \pi_L &= \frac{\kappa(y_L - y_L^*) + \beta\lambda\boldsymbol{\pi}_H(y_L - y_H^*)}{1-\beta(1-\lambda)}. \end{aligned}$$

Here, the last line uses the NKPC to solve for the inflation in the low-supply state. Combining these observations, the optimum is given by the unique solution to:

$$\left[1 + \frac{\psi(\kappa + \beta\lambda\boldsymbol{\pi}_H)\kappa}{1-\beta(1-\lambda)}\right](y_L - y_L^*) = \beta\lambda\left[\theta_H + \frac{\psi(\kappa+\beta\lambda\boldsymbol{\pi}_H)\boldsymbol{\pi}_H}{1-\beta(1-\lambda)}\right](y_H^* - y_L). \quad \text{(A.11)}$$

This leads to the following result, which generalizes Proposition 1 to this setting.

Proposition 4. *Consider the setup with inflation determined by the NKPC* (17) *and the parametric conditions described in Lemma 2. Suppose the economy is in the temporary supply shock state,* $s = L$, *with past output* y_{t-1}. *The central bank implements the constant output level* $y_L \in (y_L^*, y_H^*)$ *that solves* (A.11) *along with the constant inflation*

$$\pi_{t,L} = \pi_L \equiv \frac{\kappa(y_L - y_L^*) + \beta\lambda\boldsymbol{\pi}_H(y_L - y_H^*)}{1-\beta(1-\lambda)}. \quad \text{(A.12)}$$

The associated real and nominal interest rates are given by

$$\begin{aligned} r_{t,L} &= \rho + \lambda(Y_H(y_L) - y_L) - \frac{\eta}{1-\eta}(y_L - y_{t-1}) \quad \text{(A.13)} \\ i_{t,L} &= r_{t,L} + \lambda\Pi_H(y_L) + (1-\lambda)\pi_L. \quad \text{(A.14)} \end{aligned}$$

The central bank chooses a level of output that induces positive output gaps in the current low-supply state (current overheating), $y_L > y_L^*$, and negative output gaps and disinflation after transition to the high-supply state (future demand shortages), $Y_H(y_L) < y_H^*$ and $\Pi_H(y_L) < 0$.

Comparing $(A.11)$ and (11) shows that inflation affects the policy trade-off in two ways. On the one hand, positive output gaps in the low-supply phase increase current inflation. This raises the cost of overheating, captured by the second term inside the brackets on the left side of $(A.11)$. On the other hand, negative output gaps expected in the *future* high-supply phase reduce *current* inflation. Since overheating helps shrink future gaps, this effect raises the benefit of overheating, captured by the second term inside the brackets on the right side of $(A.11)$. It follows that inflation affects the cost as well as the benefit of overheating, but it does not change the *qualitative* aspects of optimal policy.

The equilibrium with the NKPC has one subtlety: The central bank does not *necessarily* induce positive inflation in the low-supply state: that is, π_L is not necessarily positive (even though $y_L > y_L^*$). This effect is driven by the forward-looking term in the NKPC, together with the fact that the economy experiences disinflation after transition to the high-supply state, $\boldsymbol{\pi}_H(y_L - y_H^*) < 0$ (see $(A.12)$). Nonetheless, in our simulations this effects is typically weak and the central bank implements $\pi_L > 0$ along with $y_L > y_L^*$.

Proof of Lemma 2. Combining the NKPC, the IS curve, and the Taylor policy rule, the dynamic system that characterizes the equilibrium is given by

$$\begin{aligned} y_t &= \eta y_{t-1} + (1-\eta)\left(-\phi_y(y_t - y_H^*) - \phi_\pi \pi_t + E_t[\pi_{t+1}] + E_t[y_{t+1}]\right) \\ \pi_t &= \kappa(y_t - y_H^*) + \beta E_t[\pi_{t+1}]. \end{aligned}$$

We drop the expectations since there is no (residual) uncertainty. Let $\tilde{y}_t = y_t - y_H^*$ denote the output gap. Then, we can rewrite the system as

$$\begin{aligned} \tilde{y}_t &= \eta \tilde{y}_{t-1} + (1-\eta)\left(-\phi_y \tilde{y}_t - \phi_\pi \pi_t + \pi_{t+1} + \tilde{y}_{t+1}\right) \\ \pi_t &= \kappa \tilde{y}_t + \beta \pi_{t+1}. \end{aligned}$$

In matrix notation, we have

$$\begin{bmatrix} \tilde{y}_{t+1} \\ \pi_{t+1} \\ \tilde{y}_t \end{bmatrix} = M \begin{bmatrix} \tilde{y}_t \\ \pi_t \\ \tilde{y}_{t-1} \end{bmatrix} \text{ where } M = \begin{bmatrix} \frac{1}{1-\eta} + \phi_y + \frac{\kappa}{\beta} & \phi_\pi - \frac{1}{\beta} & -\frac{\eta}{1-\eta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of the matrix M is

$$\begin{aligned} P(x) &= -\det\left(\begin{bmatrix} \frac{1}{1-\eta}+\phi_y+\frac{\kappa}{\beta}-x & \phi_\pi-\frac{1}{\beta} & -\frac{\eta}{1-\eta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta}-x & 0 \\ 1 & 0 & -x \end{bmatrix}\right) \\ &= x^3-x^2\left(\frac{1}{1-\eta}+\phi_y+\frac{1+\kappa}{\beta}\right)+x\left(\left(\frac{1}{1-\eta}+\phi_y\right)\frac{1}{\beta}+\phi_\pi\frac{\kappa}{\beta}+\frac{\eta}{1-\eta}\right)-\frac{1}{\beta}\frac{\eta}{1-\eta}. \end{aligned}$$

This is the polynomial we define in $(A.6)$. We assume the parameters are such that this polynomial has a single stable root that satisfies $\gamma_H \in (0,1)$. The conditions in the propositions are sufficient (but not necessary). To check sufficiency, note that we have $P(0) < 0$. We also have

$$P(1) = \frac{\phi_y(1-\beta)+(\phi_\pi-1)\kappa}{\beta} > 0$$

in view of the first part of the sufficient condition, $\phi_y(1-\beta)+(\phi_\pi-1)\kappa$. We also have

$$P\left(\frac{1}{\beta}\right) = -\frac{\kappa}{\beta^3}+\phi_\pi\frac{\kappa}{\beta^2} \leq 0$$

in view of the second part of the sufficient condition, $\beta\phi_\pi \leq 1$. Thus, with these conditions the roots of the polynomial satisfy

$$0 < \gamma_1 < 1 < \gamma_2 \leq \frac{1}{\beta} \leq \gamma_3.$$

In particular, the polynomial has exactly one stable root that satisfies $\gamma_H \equiv \gamma_1 \in (0,1)$.

Since $\tilde{y}_{t-1}$ is predetermined but $\tilde{y}_t, \pi_t$ are not, the system is saddle path stable. Moreover, the solution converges to zero at the constant rate $\gamma_H \in (0,1)$, that is:

$$\begin{aligned} \tilde{y}_{t+h} &= \gamma_H \tilde{y}_{t+h-1} = \gamma_H^{h+1}\tilde{y}_{t-1} \\ \tilde{\pi}_{t+h} &= \gamma_H \tilde{\pi}_{t+h-1}. \end{aligned}$$

This establishes $(A.7)$. To solve for the initial inflation, we use the NKPC to obtain

$$\pi_t = \sum_{h=0}^{\infty}\beta^h\kappa\tilde{y}_{t+h} = \sum_{h=0}^{\infty}\beta^h\gamma_H^h\kappa\gamma_H\tilde{y}_{t-1} = \frac{\kappa\gamma_H\tilde{y}_{t-1}}{1-\beta\gamma_H}.$$

This establishes $(A.8)$.

Finally, we calculate the value function as:

$$
\begin{aligned}
V_H &= -\sum_{h=0}^{\infty} \beta^h \left(\frac{\tilde{y}_{t+h}^2}{2} + \psi \frac{{\pi_{t+h}}^2}{2} \right) \\
&= -\sum_{h=0}^{\infty} \left(\beta \gamma_H^2\right)^h \left(\frac{{\tilde{y}_t}^2}{2} + \psi \frac{{\pi_t}^2}{2} \right) \\
&= -\frac{1}{1-\beta\gamma_H^2} \left(\gamma_H^2 + \psi \left(\frac{\kappa \gamma_H}{1-\beta\gamma_H} \right)^2 \right) \frac{{\tilde{y}_{t-1}}^2}{2}.
\end{aligned}
$$

Here, the second line uses the fact that inflation and the output gap converge to zero at rate $\gamma_H \in (0,1)$ and the last line substitutes $\tilde{y}_t$ and π_t in terms of the past output gap $\tilde{y}_{t-1}$. This establishes $(A.9)$ and completes the proof of the lemma. □

Proof of Proposition 4. The proof is mostly presented earlier in the section. To solve for the real interest rate, note that the IS curve (14) implies

$$
r_{t,L} = \rho + \lambda \left(Y_H \left(y_t \right) - y_{t,L} \right) + (1-\lambda) \left(y_{t+1,L} - y_{t,L} \right) - \frac{\eta}{1-\eta} \left(y_{t,L} - y_{t-1} \right).
$$

After substituting $y_{t,L} = y_{t+1,L} = y_L$, this implies $(A.13)$. The nominal interest rate is then

$$
i_{t,L} = r_{t,L} + E_t \left[\pi_{t+1} \right] = r_{t,L} + \lambda \Pi_H \left(y_L \right) + (1-\lambda) \pi_L.
$$

This establishes $(A.14)$ and completes the proof. □

A.2.2. Overheating with an inertial Phillips Curve

Suppose inflation is determined according to the inertial Phillips curve (18)

$$
\pi_t = \kappa \left(y_t - y_{s_t}^* \right) + b \pi_{t-1}.
$$

Suppose also that the parameters satisfy the simplifying assumptions described in the main text. We first state the lemma that characterizes the equilibrium in the high supply-state $s = H$. We then present the proof of Proposition 2, which characterizes the optimal policy in the low-supply state $s = L$.

Lemma 3. *Consider the setup with an inertial Phillips curve. Suppose the parameters satisfy $\phi_\pi = b$ and $\phi_y > \kappa$.*

Suppose the economy has switched to the high-supply state, $s = H$, with past output y_{t-1}. Let $\gamma_H \in (0,1)$ denote the smaller root of the polynomial $P(x) = (1+\kappa) x^2 -$

$\left(\frac{1}{1-\eta}+\phi_y\right)x+\frac{\eta}{1-\eta}$. *Then the output gap and the inflation functions are given by:*

$$Y_H\left(y_{t-1},\pi_{t-1}\right)-y_H^* = \gamma_H\left(y_{t-1}-y_H^*\right) \tag{A.15}$$
$$\Pi_H\left(y_{t-1},\pi_{t-1}\right) = \kappa\gamma_H\left(y_{t-1}-y_H^*\right)+b\pi_{t-1}. \tag{A.16}$$

The value function in the first period after transition (with $s_{t-1}=L$) is given by:

$$V_H\left(y_{t-1,}\pi_{t-1}\right)=-\frac{\theta_H}{2}\left(y_{t-1}-y_H^*\right)^2-\frac{\Psi_H}{2}\pi_{t-1}^2-\mathcal{I}_H\left(y_{t-1}-y_H^*\right)\pi_{t-1}, \tag{A.17}$$

where the coefficients $\Psi_H,\mathcal{I}_H,\theta_H$ are given by

$$\begin{aligned}\Psi_H &= \frac{b^2}{1-\beta b^2}\psi \\ \mathcal{I}_H &= \frac{\gamma_H b}{1-\beta\gamma_H b}\left(\psi+\beta\Psi_H\right)\kappa \\ \theta_H &= \frac{\gamma_H^2}{1-\beta\gamma_H^2}\left(1+\left(\psi+\beta\Psi_H\right)\kappa^2+2\beta\mathcal{I}_H\kappa\right).\end{aligned} \tag{A.18}$$

Proof of Lemma 3. Combining the inertial Phillips curve, the IS curve, and the Taylor policy rule, the dynamic system that characterizes the equilibrium is given by

$$\begin{aligned}y_t &= \eta y_{t-1}+(1-\eta)\left(-\phi_y\left(y_t-y_H^*\right)-\phi_\pi\pi_t+E_t\left[\pi_{t+1}\right]+E_t\left[y_{t+1}\right]\right) \\ \pi_t &= \kappa\left(y_t-y_H^*\right)+b\pi_{t-1}.\end{aligned}$$

We drop the expectations since there is no (residual) uncertainty. Let $\tilde{y}_t=y_t-y_H^*$ denote the output gap. Then, we can rewrite the system as

$$\begin{aligned}\tilde{y}_t &= \eta\tilde{y}_{t-1}+(1-\eta)\left(-\phi_y\tilde{y}_t-\phi_\pi\pi_t+\pi_{t+1}+\tilde{y}_{t+1}\right) \\ \pi_t &= \kappa\tilde{y}_t+b\pi_{t-1}.\end{aligned}$$

After rewriting the second equation and substituting the first equation, we obtain

$$\begin{aligned}\tilde{y}_{t+1} &= \frac{1}{1+\kappa}\left(\frac{\tilde{y}_t-\eta\tilde{y}_{t-1}}{1-\eta}+\phi_y\tilde{y}_t+\left(\phi_\pi-b\right)\pi_t\right) \\ \pi_t &= \kappa\tilde{y}_t+b\pi_{t-1}.\end{aligned}$$

This system is in general complicated, because there are two state variables $\tilde{y}_{t-1},\pi_{t-1}$. However, in the special case $\phi_\pi=b$, inflation drops out of the first equation and the system becomes block-recursive. In particular, the output gap satisfies the difference equation:

$$\tilde{y}_{t+1} = \frac{1}{1+\kappa}\left(\left(\frac{1}{1-\eta}+\phi_y\right)\tilde{y}_t - \frac{\eta}{1-\eta}\tilde{y}_{t-1}\right).$$

This is a standard difference equation with the characteristic polynomial given by

$$P(x) = (1+\kappa)x^2 - \left(\frac{1}{1-\eta}+\phi_y\right)x + \frac{\eta}{1-\eta} = 0.$$

Note that $P(0) > 0$ and $P(1) < 0$ in view of the parametric condition $\phi_y > \kappa$. Thus, the polynomial has a single stable root that satisfies $\gamma_H \in (0,1)$. It follows that the output gap converges to zero at a constant rate

$$\tilde{y}_{t+h} = \gamma_H \tilde{y}_{t+h-1} = \gamma_H^{h+1}\tilde{y}_{t-1}.$$

This establishes $(A.15)$. Substituting $\tilde{y}_t$ into the inertial Phillips curve, we solve for inflation as:

$$\pi_t = \kappa\tilde{y}_t + b\pi_{t-1} = \kappa\gamma_H\tilde{y}_{t-1} + b\pi_{t-1}.$$

This establishes $(A.16)$.

Finally, consider the value function. The value function satisfies the recursive relation

$$\begin{aligned} V_H(y_{t-1,}\pi_{t-1}) &= -\frac{1}{2}\tilde{y}_t^2 - \frac{\psi}{2}\pi_t^2 + \beta V_H(y_t,\pi_t) \\ \text{where } \tilde{y}_t &= \gamma_H\tilde{y}_{t-1} \\ \text{and } \pi_t &= \kappa\gamma_H\tilde{y}_{t-1} + b\pi_{t-1}. \end{aligned}$$

We conjecture that the value function has the quadratic functional form in $(A.17)$. After substituting the functional form, and dropping the H subscripts, we obtain:

$$\begin{aligned} -\theta\tilde{y}_{t-1}^2 - \Psi\pi_{t-1}^2 - 2\mathcal{I}\tilde{y}_{t-1}\pi_{t-1} &= -\tilde{y}_t^2 - \psi\pi_t^2 + \beta\left(-\theta\tilde{y}_t^2 - \Psi\pi_t^2 - 2\mathcal{I}\tilde{y}_t\pi_t\right) \\ &= -(1+\beta\theta)\tilde{y}_t^2 - (\psi+\beta\Psi)\pi_t^2 - 2\beta\mathcal{I}\tilde{y}_t\pi_t \\ &= \begin{bmatrix} -(1+\beta\theta)(\gamma\tilde{y}_{t-1})^2 \\ -(\psi+\beta\Psi)(\kappa\gamma\tilde{y}_{t-1}+b\pi_{t-1})^2 \\ -2\beta\mathcal{I}(\gamma\tilde{y}_{t-1})(\kappa\gamma\tilde{y}_{t-1}+b\pi_{t-1}) \end{bmatrix} \\ &= \begin{bmatrix} -(1+\beta\theta+(\psi+\beta\Psi)\kappa^2+2\beta\mathcal{I}\kappa)\gamma^2\tilde{y}_{t-1}^2 \\ -(\psi+\beta\Psi)b^2\pi_{t-1}^2 \\ -(2\beta\mathcal{I}+2(\psi+\beta\Psi)\kappa)\gamma b\tilde{y}_{t-1}\pi_{t-1} \end{bmatrix}. \end{aligned}$$

Here, the third line substitutes $\tilde{y}_t, \pi_t$ in terms of $\tilde{y}_{t-1}, \pi_{t-1}$ and the last line collects terms. After matching the coefficients for the terms $\tilde{y}_{t-1}^2, \pi_{t-1}^2, \tilde{y}_{t-1}\pi_{t-1}$, we obtain

$$\begin{aligned}
\theta &= \left(1 + \beta\theta + (\psi + \beta\Psi)\kappa^2 + 2\beta\mathcal{I}\kappa\right)\gamma^2 \\
\Psi &= (\psi + \beta\Psi) b^2 \\
\mathcal{I} &= (\beta\mathcal{I} + (\psi + \beta\Psi)\kappa)\gamma b.
\end{aligned}$$

Solving these equations and substituting back the H subscripts, we establish $(A.18)$, completing the proof. □

Proof of Proposition 2. Consider problem (19), which we replicate here

$$\begin{aligned}
V_L(y_{t-1}, \pi_{t-1}) &= \max_{y_t, \pi_t} -\frac{(y_t - y_L^*)^2}{2} - \psi\frac{\pi_t^2}{2} + \beta\left((1-\lambda) V_L(y_t, \pi_t) + \lambda V_H(y_t, \pi_t)\right) \\
\pi_t &= \kappa(y_t - y_L^*) + b\pi_{t-1}
\end{aligned}$$

In this case, the value function $V_L(y_{t-1}, \pi_{t-1})$ depends on past inflation, π_{t-1}, but it is still independent of past output, y_{t-1}. Using this observation, we can write the problem as

$$\begin{aligned}
V_L(\pi_{t-1}) &= \max_{\pi_t} F(\pi_{t-1}, \pi_t) + \beta(1-\lambda) V_L(\pi_t) \\
\text{where } F(\pi_{t-1}, \pi_t) &= -\frac{(\pi_t - b\pi_{t-1})^2}{2\kappa^2} - \psi\frac{\pi_t^2}{2} + \beta\lambda V_H\left(y_L^* + \frac{\pi_t - b\pi_{t-1}}{\kappa}, \pi_t\right).
\end{aligned}$$

This is a standard dynamic optimization problem. The first order condition is given by the Euler equation:

$$\frac{\partial F(\pi_{t-1}, \pi_t)}{\partial \pi_t} + \beta(1-\lambda)\frac{\partial F(\pi_t, \pi_{t+1})}{\partial \pi_t} = 0. \tag{A.19}$$

We calculate the derivatives as:

$$\begin{aligned}
\frac{\partial F(\pi_{t-1}, \pi_t)}{\partial \pi_t} &= -\frac{(\pi_t - b\pi_{t-1})}{\kappa^2} - \psi\pi_t + \beta\lambda\left(\frac{\partial V_H(y_t, \pi_t)}{\partial y_t}\frac{1}{\kappa} + \frac{\partial V_H(y_t, \pi_t)}{\partial \pi_t}\right), \\
\frac{\partial F(\pi_t, \pi_{t+1})}{\partial \pi_t} &= \frac{b}{\kappa}\left(\frac{\pi_{t+1} - b\pi_t}{\kappa} - \beta\lambda\frac{\partial V_H(y_{t+1}, \pi_{t+1})}{\partial y_{t+1}}\right).
\end{aligned}$$

Combining these observations, and using $y_t - y_L^* = \frac{\pi_t - b\pi_{t-1}}{\kappa}$, the Euler equation $(A.19)$

implies

$$\begin{aligned} & y_t - y_L^* + \kappa\psi\pi_t - \beta\lambda\left(\frac{\partial V_H(y_t,\pi_t)}{\partial y_t} + \kappa\frac{\partial V_H(y_t,\pi_t)}{\partial \pi_t}\right) \\ = \; & \beta(1-\lambda)\,b\left(y_{t+1} - y_L^* - \beta\lambda\frac{\partial V_H(y_{t+1},\pi_{t+1})}{\partial y_{t+1}}\right). \end{aligned}$$

We next use Eq. $(A.17)$ to calculate the partial derivatives of $V_H(y_t,\pi_t)$ as follows:

$$\begin{aligned} \frac{\partial V_H(y_t,\pi_t)}{\partial y_t} &= -\theta_H(y_t - y_H^*) - \mathcal{I}_H\pi_t \\ &= -\theta_H(y_t - y_L^*) - \mathcal{I}_H\pi_t + \theta_H(y_H^* - y_L^*) \\ &\quad \text{and} \\ \frac{\partial V_H(y_t,\pi_t)}{\partial \pi_t} &= -\Psi_H\pi_t - \mathcal{I}_H(y_t - y_H^*) \\ &= -\Psi_H\pi_t - \mathcal{I}_H(y_t - y_L^*) + \mathcal{I}_H(y_H^* - y_L^*). \end{aligned}$$

Substituting these expressions into the Euler equation, we obtain

$$\begin{aligned} & y_t - y_L^* + \kappa\psi\pi_t + \beta\lambda\begin{pmatrix} (\theta_H + \kappa\mathcal{I}_H)(y_t - y_L^*) \\ +(\mathcal{I}_H + \kappa\Psi_H)\pi_t \\ -(\theta_H + \kappa\mathcal{I}_H)(y_H^* - y_L^*) \end{pmatrix} \\ = \; & \beta(1-\lambda)\,b\begin{pmatrix} (1+\beta\lambda\theta_H)(y_{t+1} - y_L^*) \\ +\beta\lambda\mathcal{I}_H\pi_{t+1} \\ -\beta\lambda\theta_H(y_H^* - y_L^*) \end{pmatrix}. \end{aligned}$$

Rearranging terms, we have

$$\begin{aligned} A(y_t - y_L^*) + B\pi_t &= C(y_{t+1} - y_L^*) + D\pi_{t+1} + E(y_H^* - y_L^*) \text{ where} \\ A &= 1 + \beta\lambda(\theta_H + \kappa\mathcal{I}_H) \\ B &= \kappa\psi + \beta\lambda(\mathcal{I}_H + \kappa\Psi_H) \\ C &= \beta(1-\lambda)\,b(1+\beta\lambda\theta_H) \\ D &= \beta(1-\lambda)\,b\beta\lambda\mathcal{I}_H \\ E &= \beta\lambda[\theta_H + \kappa\mathcal{I}_H - \beta(1-\lambda)\,b\theta_H]. \end{aligned}$$

Here $A, B, C, D, E > 0$ are the derived parameters in (20). Note also that $A > C$ and $B > D$.

Combining the equation for y_t with the NKPC, we obtain the system:

$$\begin{aligned} A\left(y_t - y_L^*\right) + B\pi_t &= C\left(y_{t+1} - y_L^*\right) + D\pi_{t+1} + E\left(y_H^* - y_L^*\right) \\ \pi_t &= \kappa\left(y_t - y_L^*\right) + b\pi_{t-1}. \end{aligned} \tag{A.20}$$

We next calculate the steady-state, denoted by $(\overline{y}_L, \overline{\pi}_L)$. From the second equation, the steady-state inflation satisfies $\overline{\pi}_L = \frac{\kappa\left(\overline{y}_L - y_L^*\right)}{1-b}$. Substituting this into the first equation, we solve for the steady-state output as:

$$\overline{y}_L - y_L^* = \frac{E\left(y_H^* - y_L^*\right)}{A - C + (B - D)\frac{\kappa}{1-b}}.$$

Note that $\overline{y}_L > y_L^*$ (and thus $\overline{\pi}_L > 0$) since $E > 0, A > C$, and $B > D$. This establishes $(21 - 22)$.

We next characterize the transition dynamics away from the steady-state. Let $\tilde{y}_t = y_t - \overline{y}_L$ and $\tilde{\pi}_t = \pi_t - \overline{\pi}_t$ denote the deviations from the steady state (these variables are different than the output and inflation gaps). With this notation, we write $(A.20)$ as

$$\begin{aligned} A\tilde{y}_t + B\tilde{\pi}_t &= C\tilde{y}_{t+1} + D\tilde{\pi}_{t+1} \\ \tilde{\pi}_t &= \kappa\tilde{y}_t + b\tilde{\pi}_{t-1}. \end{aligned}$$

After substituting $\tilde{\pi}_{t+1} = \kappa\tilde{y}_{t+1} + b\tilde{\pi}_t$ and $\tilde{\pi}_t = \kappa\tilde{y}_t + b\tilde{\pi}_{t-1}$ in the first equation, we can write this system as

$$\begin{aligned} (C + D\kappa)\,\tilde{y}_{t+1} &= (A + (B - Db)\,\kappa)\,\tilde{y}_t + (B - Db)\,b\tilde{\pi}_{t-1} \\ \tilde{\pi}_t &= \kappa\tilde{y}_t + b\tilde{\pi}_{t-1}. \end{aligned}$$

In matrix notation, we have

$$\begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{\pi}_t \end{bmatrix} = \begin{bmatrix} \frac{A+(B-Db)\kappa}{C+D\kappa} & \frac{(B-Db)b}{C+D\kappa} \\ \kappa & b \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{\pi}_{t-1} \end{bmatrix}.$$

The characteristic polynomial is given by

$$\begin{aligned}
P(x) &= \det\left(\begin{bmatrix} \frac{A+(B-Db)\kappa}{C+D\kappa} - x & \frac{(B-Db)b}{C+D\kappa} \\ \kappa & b-x \end{bmatrix}\right) \\
&= x^2 - \left(\frac{A+(B-Db)\kappa}{C+D\kappa} + b\right)x + \frac{Ab}{C+D\kappa} \\
&= x^2 - \frac{A+B\kappa+bC}{C+D\kappa}x + \frac{Ab}{C+D\kappa}.
\end{aligned}$$

Note that $P(0) > 0$ and

$$\begin{aligned}
P(b) &= b^2 - \frac{A+B\kappa+bC}{C+D\kappa}b + \frac{Ab}{C+D\kappa} \\
&= -\frac{(B-bD)}{C+D\kappa}\kappa b < 0.
\end{aligned}$$

This implies there is a stable root that satisfies $\gamma_L \equiv \gamma_1 \in (0,b)$. We also claim that $P(1) < 0$, which holds iff

$$P(1) = \frac{(C-A)(1-b) + (D-B)\kappa}{C+D\kappa} < 0.$$

The inequality holds because $A > C$ and $B > D$. This inequality implies that there is also an unstable root that satisfies $\gamma_2 > 1$.

These observations prove that the system is saddle path stable. Starting with the inflation deviation $\tilde{\pi}_{t-1}$, both the output deviation and inflation deviation converge to zero at a constant rate γ_L

$$\tilde{y}_{t+1} = \gamma_L \tilde{y}_t \text{ and } \tilde{\pi}_t = \gamma_L \tilde{\pi}_{t-1} \text{ for each } t.$$

To characterize the output in terms of past inflation, note the Phillips curve implies

$$\tilde{\pi}_t = \gamma_L \tilde{\pi}_{t-1} = \kappa \tilde{y}_t + b\tilde{\pi}_{t-1} \Longrightarrow \tilde{y}_t = -\left(\frac{b-\gamma_L}{\kappa}\right)\tilde{\pi}_{t-1}.$$

This establishes $(23-24)$.

Finally, we calculate the interest rate the central bank needs to set to implement the optimal output and inflation path. First consider the real interest rate. Using the IS curve (14),

$$
\begin{aligned}
r_t &= \rho + E_t\left[y_{t+1}\right] - \frac{y_t}{1-\eta} + \frac{\eta}{1-\eta} y_{t-1} \\
&= \rho + \lambda Y_H\left(y_t\right) + (1-\lambda) y_{t+1} - \frac{y_t}{1-\eta} + \frac{\eta}{1-\eta} y_{t-1} \\
&= \rho + \lambda\left(Y_H\left(y_t\right) - y_t\right) + (1-\lambda)\left(y_{t+1} - y_t\right) - \frac{\eta}{1-\eta}\left(y_t - y_{t-1}\right). \qquad \text{(A.21)}
\end{aligned}
$$

Here, y_{t+1} denote the future output if the economy stays in the low-supply state (characterized earlier). Likewise, the nominal interest rate is given by

$$
\begin{aligned}
i_t &= E_t\left[\pi_{t+1}\right] + r_t \\
&= \begin{array}{c} \lambda \Pi_H\left(y_t\right) + (1-\lambda)\pi_{t+1} + \\ \rho + \lambda\left(Y_H\left(y_t\right) - y_t\right) + (1-\lambda)\left(y_{t+1} - y_t\right) - \frac{\eta}{1-\eta}\left(y_t - y_{t-1}\right). \end{array} \qquad \text{(A.22)}
\end{aligned}
$$

Here, π_{t+1} is the inflation if the economy stays in state L. This completes the proof. $\square$

B. Alternative model with a ZLB constraint

In the main text, we formalize the expansionary policy constraints by assuming that the central bank is subject to a Taylor-rule type lower bound on the nominal interest rate (see (2)). In this appendix, we analyze an alternative model in which the central bank is subject to a zero lower bound (ZLB) constraint. We show that our main result holds also in this more realistic scenario. We relegate the proofs to the end of the appendix.

Environment with a ZLB constraint. Consider the setup in Section 1 with the difference that the lower bound on the interest rate is zero [cf. (2)]

$$
i_t \geq \underline{i}_t\left(y_t\right) = 0. \qquad \text{(B.1)}
$$

As before, the central bank sets policy without commitment, and it minimizes the present discounted value of quadratic output gaps. We can then formulate the policy problem recursively as

$$
\begin{aligned}
V_{s_t}\left(y_{t-1}\right) &= \max_{i_t, y_t} -\frac{\left(y_t - y_{s_t}^*\right)^2}{2} + \beta E_t\left[V_{s_{t+1}}\left(y_t\right)\right] \qquad \text{(B.2)} \\
\text{s.t. } y_t &= \eta y_{t-1} + (1-\eta)\left(-\left(i_t - \rho\right) + E_t\left[Y_{s_{t+1}}\left(y_t\right)\right]\right) \\
i_t &\geq 0.
\end{aligned}
$$

As in our main setup, $Y_s(y_{-1})$ and $V_s(y_{-1})$ denote the central bank's optimal output choice and optimal value, respectively, when the current state is $s \in \{H, L\}$ and the most recent output is y_{-1}. The central bank takes its future interest rate decisions and output choices as given and sets the current interest rate and output to minimize quadratic gaps, subject to the inertial IS curve and the ZLB constraint.

Overheating with a ZLB constraint. Recall that, in the first-best benchmark without expansionary policy constraints, the central bank sets a relatively low interest rate in the first period after transition to the high-supply state [see (5)]. We assume the parameters are such that this interest rate is negative: In the first-best benchmark, the ZLB constraint is violated in the first period after transition. Thus, a central bank that is subject to a ZLB constraint cannot achieve zero gaps in all periods and states.

Assumption 1. $\rho - \frac{\eta}{1-\eta}(y_H^* - y_L^*) < 0$.

Our first result characterizes the equilibrium after the economy transitions to the absorbing state $s = H$.

Lemma 4. *Suppose Assumption 1 holds and the economy has switched to the high-supply state, $s = H$, with past output $y_{-1} \equiv y_{t-1}$. Let $\overline{y}_H = y_H^* - \frac{1-\eta}{\eta}\rho \in (y_L^*, y_H^*)$.*

- *If $y_{-1} \geq \overline{y}_H$, then the ZLB constraint does not bind and the central bank can achieve zero gaps, $Y_H(y_{-1}) = y_H^*$ and $V_H(y_{-1}) = 0$. The interest rate is given by*

$$i_{t,H} = \rho - \frac{\eta}{1-\eta}(y_H^* - y_{t-1}). \tag{B.3}$$

- *If $y_{-1} < \overline{y}_H$, then the ZLB constraint binds and the output gap is negative for at least one period, $Y_H(y_{-1}) < y_H^*$ and $V_H(y_{-1}) < 0$. The output and the value functions are characterized in the proof and satisfy the following:*
  - *$Y_H(y_{-1}) \geq y_{-1}$ is continuous, strictly increasing, and piecewise linear (it is linear except for a finite number of kink points). Output converges to the efficient level y_H^* after finitely many periods.*
  - *$V_H(y_{-1})$ is continuous, strictly concave and increasing, and piecewise differentiable. At the ZLB cutoff, $y_{-1} = \overline{y}_H$, the value function is differentiable with a zero derivative, $\frac{dV_H(\overline{y}_H)}{dy_{-1}} = 0$.*

Lemma 4 says that, after the supply recovers, the ZLB constraint binds when output is sufficiently low relative to potential. Technically, the ZLB constraint introduces a finite number of kink points into the solution, but the optimal output and the value function satisfy intuitive properties. Starting with a sufficiently low output level, the output gradually recovers and eventually reaches its potential level, y_H^*. Similar to our baseline analysis in Lemma 1, a greater past output increases the current output as well as the value function (over the relevant range $y_{-1} < \overline{y}_H$).

We next establish the analogue of our main result (Proposition 1) in this alternative setup with a ZLB constraint. Consider the optimal policy in the temporary low-supply state, $s = L$. For now, suppose past output y_{-1} is high enough so that the ZLB constraint does not bind in the low-supply state (we consider the case with a binding ZLB in this state subsequently). Then, we can rewrite problem (3) as

$$V_L(y_{-1}) = \max_y -\frac{(y - y_L^*)^2}{2} + \beta((1-\lambda)V_L(y) + \lambda V_H(y)). \tag{B.4}$$

The value function in the future low-supply state does not depend on past output, $V_L(y) \equiv V_L$ (as long as the ZLB does not bind, which we will verify). The value function in the future high-supply state $V_H(y)$ is concave. Therefore, the optimality condition is

$$y - y_L^* = \beta\lambda\delta; \quad \text{where } \delta \in \nabla V_H(y). \tag{B.5}$$

Here, δ is a subgradient of the value function. It is equal to the derivative, except possibly at kink points, where it lies in an interval between the left and the right derivatives. Let y_L denote the optimum that solves $(B.5)$.

Eq. $(B.5)$ establishes our main result with the ZLB constraint: the (unique) optimum satisfies $y_L \in (y_L^*, \overline{y}_H)$ and thus $y_L > y_L^*$ and $Y_H(y_L) < Y_H(\overline{y}_H) = y_H^*$. *In the temporary low-supply state, the central bank chooses a level of output that induces positive output gaps in the current low-supply state (current overheating), and negative output gaps after transition to the high-supply state (future demand shortages).* The intuition is the same as in Section 2. As before, the central bank overheats the current output to accelerate the recovery in future periods after transition to high supply.

We can now solve for the associated interest rate:

$$i_t = \rho + \lambda(Y_H(y_L) - y_L) - \frac{\eta}{1-\eta}(y_L - y_{t-1}). \tag{B.6}$$

Recall that $Y_H(y_L) > y_L$. This shows that the ZLB constraint does not bind in the low-supply state ($i_t > \rho > 0$) when past output is already equal to the target level, $y_{t-1} = y_L$. However, there is a sufficiently low level of past output (y_{t-1}) below which the ZLB constraint binds in the low-supply state for at least one period:

$$\overline{y}_L = y_L - \frac{1-\eta}{\eta}\left(\rho + \lambda\left(Y_H\left(y_L\right) - y_L\right)\right). \tag{B.7}$$

The following proposition summarizes the discussion in this appendix and completes the characterization of equilibrium in $s = L$.

Proposition 5. *Suppose Assumption 1 holds and the economy is in the temporary supply shock state, $s = L$, with past output $y_{-1} \equiv y_{t-1}$. Let $\overline{y}_L$ be given by $(B.7)$.*

- *If $y_{-1} \geq \overline{y}_L$, then the ZLB constraint does not bind in $s = L$ and the central bank chooses the output level y_L that is the unique solution to $(B.5)$. The output choice satisfies $y_L \in (y_L^*, \overline{y}_H)$. In the temporary supply shock state, the economy experiences overheating, $y_L > y_L^*$. At the transition to the high-supply state, the economy experiences demand shortages, $Y_H\left(y_L\right) < Y_H\left(\overline{y}_H\right) = y_H^*$. The interest rate in $s = L$ is given by $(B.6)$.*

- *If $y_{-1} < \overline{y}_L$, then the ZLB constraint binds in $s = L$ for at least one period. The initial interest rate is zero, $i_t = 0$, and the initial output is below its unconstrained level, $Y_L\left(y_{-1}\right) < y_L$. The output function $Y_L\left(y_{-1}\right)$ (characterized in the proof) is continuous and strictly increasing. Absent a transition to the high-supply state, output converges to the target level y_L after finitely many periods.*

Numerical illustration. Figure B.1 simulates the equilibrium for a numerical example. The figure resembles Figure 1 in the main text. The solid lines plot the equilibrium with the ZLB constraint and illustrate the main result. As before, the optimal policy induces *overheating* in the low-supply state. The policy achieves this by *cutting* the rate aggressively in the earlier periods while the economy is in the low-supply state. In fact, in this simulation the policy runs into the ZLB constraint in the first period. Once the policy brings the output in the low-supply state to a target level above the potential (denoted by $y_L > y_L^*$ in the figure), it raises the interest rate to keep the output constant until the economy transitions to the high-supply state. After the transition, the policy cuts the interest rate once again to raise aggregate demand toward the higher aggregate supply level. However, the policy runs into the ZLB constraint. Due to the binding ZLB, the recovery in the high-supply state takes several periods to complete.

The figure illustrates several other cases to illustrate different properties of the equilibrium with the optimal policy. Compared to the first-best benchmark without the ZLB constraint (dotted lines), the optimal policy frontloads the interest rate cuts. Compared to the myopic benchmark where the central bank closes the current output gaps (the dashed line), the optimal policy generates some overheating in the low-supply phase but accelerates the recovery once the economy transitions to the high-supply phase. Finally, compared to a case with less inertia (dash-dotted line), the baseline case with higher inertia results in higher gaps both before and after transition to the supply recovery. These comparisons highlight that our results in this section (as in the main text) are driven by the *interaction* of the aggregate demand inertia and expansionary policy constraints.

Proof of Lemma 4. If $y_{-1} \geq \overline{y}_H$, then the central bank can achieve a zero gap, $Y_H(y_{-1}) = y_H^*$ and $V_H(y_{-1}) = 0$. Using the IS curve (1) with $y_t = y_{t+1} = y_H^*$, the interest rate is given by $(B.3)$. The interest rate is nonnegative, $i_{t,H} \geq 0$. In this case, the ZLB constraint does not bind.

In contrast, if $y_{-1} < \overline{y}_H$, then the ZLB constraint binds and the output gap is negative for at least one period, $Y_H(y_{-1}) < y_H^*$ and $V_H(y_{-1}) < 0$.

Consider the constrained range, $y_{-1} \leq \overline{y}_H$. In this range, the IS curve with $i_{t,H} = 0$ implies that output satisfies the recursive relation

$$Y_H(y_{-1}) = \eta y_{-1} + (1-\eta)\left(\rho + Y_H\left(Y_H(y_{-1})\right)\right). \tag{B.8}$$

We first solve this relation over a sequence of cutoff points for past output. Given $\overline{y}_{H,-1} \equiv y_H^*$ and $\overline{y}_{H,0} = \overline{y}_H$, we recursively define a sequence of cutoffs with:

$$\overline{y}_{H,k+1} = \overline{y}_{H,k} - \frac{1-\eta}{\eta}\left(\rho + \overline{y}_{H,k-1} - \overline{y}_{H,k}\right). \tag{B.9}$$

Using $(B.8)$, it is easy to check that the output function maps a lower cutoff into the higher cutoff:

$$Y_H\left(\overline{y}_{H,k+1}\right) = \overline{y}_{H,k}. \tag{B.10}$$

Note also that the cutoffs satisfy $\overline{y}_{H,k+1} \leq \overline{y}_{H,k} - \frac{(1-\eta)\rho}{\eta}$. Therefore, there exists K_H such that $\overline{y}_{H,K_H} < 0$. Then, the cutoffs $\left\{\overline{y}_{H,k}\right\}_{k=-1}^{K_H}$ cover the entire region $[0, y_H^*]$.

We next extend the solution to the intervals, $\left[\overline{y}_{H,k}, \overline{y}_{H,k-1}\right]$. Specifically, we claim that the output function is piecewise linear and strictly increasing. That is, there exist $\{a_k, b_k\}_{k=0}^{K_H}$ such that

$$Y_H(y_{-1}) = a_k y_{-1} + b_k \text{ for } y_{-1} \in \left[\overline{y}_{H,k}, \overline{y}_{H,k-1}\right]. \tag{B.11}$$

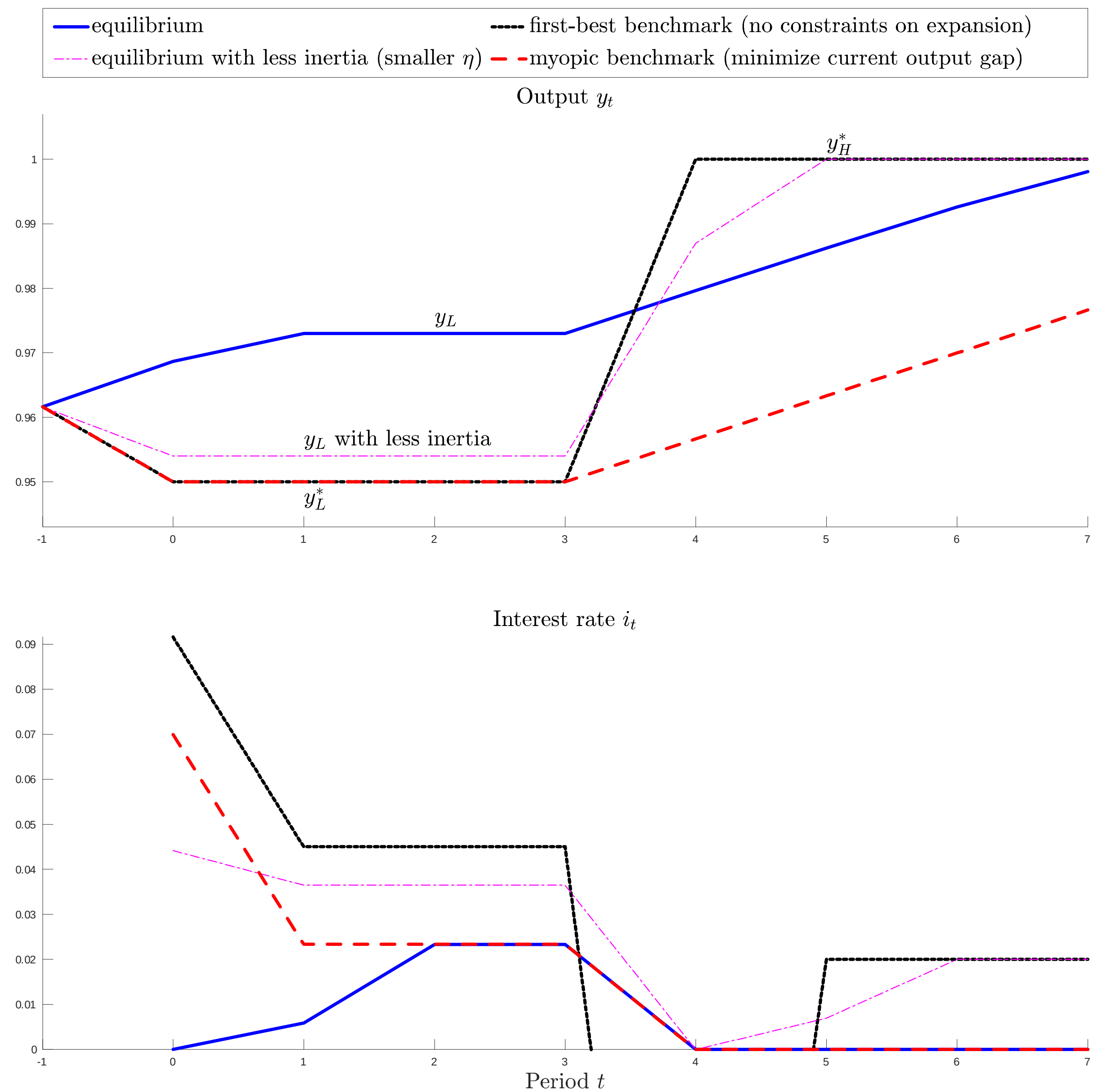

Figure B.1: A simulation of the equilibrium with a ZLB constraint. The economy starts in the low-supply state, $s_0 = L$, with the most recent output that satisfies $y_{-1} < y_L$. Solid lines: Equilibrium with the optimal policy. Dotted lines: First-best benchmark case without the ZLB constraint. Dashed lines: Myopic benchmark case in which the policy minimizes the current output gap. Dash-dotted lines: Equilibrium with a smaller inertia parameter (η). See Online Appendix C for the parameters used.

We also claim that the slope coefficients satisfy $a_k > a_{k-1} \geq 0$ and $a_k < \min\left(1, \frac{\eta}{1-\eta}\right)$.

Using the characterization for the unconstrained region, the claim holds for $k = 0$ with the coefficients

$$a_0 = 0 \text{ and } b_0 = y_H^*. \tag{B.12}$$

Suppose the claim holds for $k-1$ and consider it for k. Using Eq. $(B.8)$, we have

$$a_k y_{-1} + b_k = \eta y_{-1} + (1-\eta)\left(\rho + a_{k-1}\left(a_k y_{-1} + b_k\right) + b_{k-1}\right).$$

After rearranging terms, we obtain a recursive characterization for the coefficients

$$\begin{aligned} a_k &= \eta + (1-\eta)\, a_{k-1} a_k \qquad \text{(B.13)} \\ &\implies a_k = \frac{\eta}{1-(1-\eta)\, a_{k-1}} \\ b_k &= (1-\eta)\left(\rho + a_{k-1} b_k + b_{k-1}\right) \\ &\implies b_k = \frac{(1-\eta)\left(\rho + b_{k-1}\right)}{1-(1-\eta)\, a_{k-1}} = a_k \frac{1-\eta}{\eta}\left(\rho + b_{k-1}\right). \end{aligned}$$

Note that $a_{k-1} < 1$ implies $a_k = \frac{\eta}{1-(1-\eta)a_{k-1}} \in (0,1)$. Likewise, $a_{k-1} < \frac{\eta}{1-\eta}$ implies $a_k = \frac{\eta}{1-(1-\eta)a_{k-1}} < \frac{\eta}{1-\eta}$. We also need to check $a_k = \frac{\eta}{1-(1-\eta)a_{k-1}} > a_{k-1}$. Note that this is equivalent to $P\left(a_{k-1}\right) > 0$ where $P(x) = x^2 - \frac{1}{1-\eta}x + \frac{\eta}{1-\eta}$. This polynomial has roots $\frac{\eta}{1-\eta}$ and 1. Since $a_{k-1} < \min\left(1, \frac{\eta}{1-\eta}\right)$, we have $P\left(a_{k-1}\right) > 0$ and thus $a_k > a_{k-1}$. This proves the claim in $(B.11)$ by induction.

Eqs. $(B.10)$ and $(B.11)$ imply that the output function maps each interval $\left[\overline{y}_{H,k}, \overline{y}_{H,k-1}\right]$ into the higher interval $\left[\overline{y}_{H,k-1}, \overline{y}_{H,k-2}\right]$. This establishes the claim in the proposition that output converges to y_H^* after finitely many periods (at most $K_H + 1$ periods).

We next consider the value function $V_H\left(y_{-1}\right)$. Following similar steps, we can define the value function recursively over the intervals $\left[\overline{y}_{H,k}, \overline{y}_{H,k-1}\right]$. Let $V_{H,0}\left(y_{-1}\right) = 0$ and define a sequence of functions with:

$$V_{H,k}\left(y_{-1}\right) = -\frac{1}{2}\left(a_k y_{-1} + b_k - y_H^*\right)^2 + \beta V_{H,k-1}\left(a_k y_{-1} + b_k\right). \tag{B.14}$$

For each interval, the value function agrees with the corresponding function in the sequence:

$$V_H\left(y_{-1}\right) = V_{H,k}\left(y_{-1}\right) \text{ for } y_{-1} \in \left[\overline{y}_{H,k}, \overline{y}_{H,k-1}\right].$$

Note also that the functions in the sequence are differentiable with derivatives that satisfy:

$$\frac{dV_{H,k}\left(y_{-1}\right)}{dy_{-1}} = -\left(a_k y_{-1} + b_k - y_H^*\right) a_k + \beta \frac{dV_{H,k-1}\left(a_k y_{-1} + b_k\right)}{dy_{-1}} a_k. \tag{B.15}$$

Therefore, *inside* each interval, the value function is differentiable and its derivative agrees with the derivative of the corresponding function in the sequence:

$$\frac{dV_H\left(y_{-1}\right)}{dy_{-1}} = \frac{dV_{H,k}\left(y_{-1}\right)}{dy_{-1}} \text{ for } y_{-1} \in \left(\overline{y}_{H,k}, \overline{y}_{H,k-1}\right).$$

At each cutoff $\overline{y}_{H,k}$, the value function is left and right-differentiable with derivatives respectively given by $\frac{dV_{H,k+1}\left(\overline{y}_{H,k}\right)}{dy_{-1}}$ and $\frac{dV_{H,k}\left(\overline{y}_{H,k}\right)}{dy_{-1}}$.

We next prove that the value function, $V_H\left(y_{-1}\right)$, is strictly concave over the constrained range, $y_{-1} \leq \overline{y}_{H,0}$. For the interior points, $\left(\overline{y}_{H,k}, \overline{y}_{H,k-1}\right)$, it is easy to check that the derivative, $\frac{dV_H\left(y_{-1}\right)}{dy_{-1}}$, is strictly decreasing. Consider the cutoff points, $\overline{y}_{H,k}$. It suffices to check that the left derivative is greater than the right derivative:

$$\frac{dV_{H,k+1}\left(\overline{y}_{H,k}\right)}{dy_{-1}} > \frac{dV_{H,k}\left(\overline{y}_{H,k}\right)}{dy_{-1}}.$$

This claim is true for $k = 0$. Suppose it is true for $k-1$. Using Eq. $(B.15)$, we have

$$\begin{aligned}
\frac{dV_{H,k+1}\left(\overline{y}_{H,k}\right)}{dy_{-1}} &= -\left(\overline{y}_{H,k-1} - y_H^*\right) a_{k+1} + \beta \frac{dV_{H,k}\left(\overline{y}_{H,k-1}\right)}{dy_{-1}} a_{k+1} \\
\frac{dV_{H,k}\left(\overline{y}_{H,k}\right)}{dy_{-1}} &= -\left(\overline{y}_{H,k-1} - y_H^*\right) a_k + \beta \frac{dV_{H,k-1}\left(\overline{y}_{H,k-1}\right)}{dy_{-1}} a_k.
\end{aligned}$$

Since $\frac{dV_{H,k}\left(\overline{y}_{H,k-1}\right)}{dy_{-1}} > \frac{dV_{H,k-1}\left(\overline{y}_{H,k-1}\right)}{dy_{-1}}$ and $a_{k+1} > a_k$, we also have $\frac{dV_{H,k+1}\left(\overline{y}_{H,k}\right)}{dy_{-1}} > \frac{dV_{H,k}\left(\overline{y}_{H,k}\right)}{dy_{-1}}$. This proves the claim and shows that $V_H\left(y_{-1}\right)$ is strictly concave over the constrained range.

Finally, we prove that the value function is differentiable at the cutoff point at which starts to bind, $y_{-1} = \overline{y}_H = \overline{y}_{H,0}$, with derivative equal to zero, $\frac{dV_H\left(\overline{y}_{H,0}\right)}{dy_{-1}} = 0$. The right derivative is zero since $V_{H,0}\left(y_{-1}\right) = 0$. Recall that $Y_H\left(\overline{y}_{H,0}\right) = y_H^*$. Therefore, using Eq. $(B.15)$ for $k = 1$, we have

$$\frac{dV_{H,1}\left(\overline{y}_{H,0}\right)}{dy_{-1}} = -\left(Y_H\left(\overline{y}_{H,0}\right) - y_H^*\right) a_1 = 0.$$

This completes the proof of the proposition. Note also that Eqs. $(B.9 - B.15)$ enable a numerical characterization of equilibrium in the high-supply state. $\square$

Proof of Proposition 5. The case $y_{-1} > \overline{y}_L$ is analyzed before the proposition. Suppose $y_{-1} < \overline{y}_L$ so that the ZLB constraint binds. In this case, the IS curve with $i_{t,L} = 0$ implies the output function satisfies the recursive relation

$$Y_L(y_{-1}) = \eta y_{-1} + (1-\eta)\left(\rho + \lambda Y_H(Y_L(y_{-1})) + (1-\lambda) Y_L(Y_L(y_{-1}))\right). \tag{B.16}$$

The analysis follows similar steps as in the proof of Lemma 4. Given $\overline{y}_{L,0} = \overline{y}_L$ and $\overline{y}_{L,-1} \equiv y_L$, we recursively define a sequence of cutoffs with:

$$\overline{y}_{L,k+1} = \overline{y}_{L,k} - \frac{1-\eta}{\eta}\left(\rho + \lambda Y_H\left(\overline{y}_{L,k}\right) + (1-\lambda)\overline{y}_{L,k-1} - \overline{y}_{L,k}\right). \tag{B.17}$$

Using $(B.16)$, it is easy to check that the output function maps a lower cutoff into the higher cutoff:

$$Y_L\left(\overline{y}_{L,k+1}\right) = \overline{y}_{L,k}. \tag{B.18}$$

Using $Y_H(y_L) > y_L$, we also obtain $\overline{y}_{L,k+1} < \overline{y}_{L,k} - \frac{(1-\eta)\rho}{\eta}$. Therefore, there exists K_L such that $\overline{y}_{L,K_L} < 0$. Then, the cutoffs $\left\{\overline{y}_{L,k}\right\}_{k=-1}^{K_L}$ cover the entire region $[0, y_L]$.

We can then define the output function recursively over the intervals $\left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right]$. Let $Y_{L,0}(y_{-1}) = y_L$ and define a sequence of functions with:

$$Y_{L,k}(y_{-1}) = \eta y_{-1} + (1-\eta)\begin{pmatrix} \rho + \lambda Y_H\left(Y_{L,k}(y_{-1})\right) \\ +(1-\lambda) Y_{L,k-1}\left(Y_{L,k}(y_{-1})\right) \end{pmatrix} \text{ for } y_{-1} \in \left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right]. \tag{B.19}$$

These functions are uniquely defined and increasing over $[0, \overline{y}_L]$ (since the output function in the high-supply state, $Y_H(\cdot)$, is piecewise linear with slopes strictly less than one, as we characterized earlier). Then, Eq. $(B.18)$ implies that for each interval the output function agrees with the corresponding function in the sequence

$$Y_L(y_{-1}) = Y_{L,k}(y_{-1}) \text{ for } y_{-1} \in \left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right].$$

In particular, the output function maps each interval $\left[\overline{y}_{L,k}, \overline{y}_{L,k-1}\right]$ into the higher interval $\left[\overline{y}_{L,k-1}, \overline{y}_{L,k-2}\right]$. This establishes the claim in the proposition that, absent transition to the high-supply state, output converges to the target level y_L after finitely many periods (at most $K_L + 1$ periods). This completes the proof of the proposition. Note also that Eqs. $(B.17 - B.19)$ enable a numerical characterization of equilibrium in the low-supply state. $\square$

C. Parameters for the numerical examples

This appendix describes the parameters used for the numerical examples plotted in Figures 1-3 and B.1.

C.1. Parameters for Figure 1

We think of each period as a year. For the baseline model (analyzed in Section 2 and illustrated in Figure 1), we set the following parameters:

$$
\begin{array}{cc}
\text{Discount rate:} & \beta = \exp(-0.02) \\
\text{Inertia:} & \eta = 0.8 \\
\text{Potential output in states } H, L\text{:} & y_H^* = 1, y_L^* = 0.95 \\
\text{Probability of transition to } H\text{:} & \lambda = 0.5 \\
\text{Taylor rule coefficient:} & \lambda = 0.5 \\
\text{Initial past output:} & y_{-1} = \overline{y}_L = 0.96.
\end{array}
$$

These parameters are relatively standard. We set the discount rate so that the long-run *real* interest rate ("rstar") is about 2%. To make our results stark, we set the inertia parameter to a relatively high level, $\eta = 0.8$. The (magenta) dash-dotted lines in Figure 1 plot the equilibrium for an alternative case with lower inertia where we set, $\tilde{\eta} = 0.5$. We set $\lambda = 0.5$, which corresponds to expected supply recovery in about two years. In Figure 1 (as well as in other figures), the actual recovery is delayed relative to expectations and takes place in year four. We set the output gap coefficient in the Taylor rule to a relatively high level, $\phi = 1$ (see (2)). Finally, we start the economy with past output equal to the threshold level below which the lower bound constraint binds, $y_{-1} = \overline{y}_L < y_L$ (see (12)).

C.2. Parameters for Figure 2

For the model with inflation determined by the NKPC (analyzed in Section 3.1 and Online Appendix A.2.1 and illustrated in Figure 2), we adopt the same parameters in the previous Section 1 (except for ϕ). For the parameters specific to this model, we set:

$$
\begin{array}{cc}
\text{Inflation sensitivity to output gap:} & \kappa = 0.5 \\
\text{Generalized Taylor rule coefficients:} & \phi_y = 1, \phi_\pi = 1 \\
\text{Relative welfare weight on inflation gaps:} & \psi = 1.
\end{array}
$$

The inflation sensitivity to output gap is in line with the standard calibrations of the Phillips curve. For the Taylor rule, the coefficient on the output gap is the same as before, $\phi_y = \phi = 1$. The coefficient on inflation, $\phi_\pi = 1$, ensures the Taylor condition (marginally) holds. Finally, we assume the central bank puts the same welfare weight on inflation and output gaps, $\psi = 1$ (see (16)).

C.3. Parameters for Figure 3

For the model with inertial inflation (analyzed in Section 3.2 and Online Appendix A.2.2 and illustrated in Figure 3), we adopt the parameters in the previous Section 3, except for ϕ_π. We reset this parameter to satisfy the simplifying assumption in Lemma 3, $\phi_\pi = b$. For the parameters specific to this model, we set:

$$\begin{aligned} \text{Inflation inertia:} \quad & b = 0.9 \\ \text{Initial past inflation:} \quad & \pi_{-1} = 0. \end{aligned}$$

We set the inertia in the Phillips curve to a relatively high level, $b = 0.9$, to make our results stark (see (18)). We start the economy with past inflation equal to zero.

C.4. Parameters for Figure B.1

For the model with the zero lower bound constraint (analyzed in Section B and illustrated in Figure B.1), we adopt the same parameters in Section 1 for the baseline model with a Taylor rule constraint.