Online Appendix for

A Linear Panel Model with Heterogeneous Coefficients and Variation in Exposure

Liyang Sun, *UC Berkeley and CEMFI*
Jesse M. Shapiro, *Harvard University and NBER*[8]

April 2022

This appendix formalizes claims made in the paper.

Claim 1. In the setting of Section "The Possibility of Heterogeneous Coefficients," the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data $x = \{x_{10}, ..., x_{S0}\}$ for states $s \in \{1, ..., S\}$, is given by

$$\mathrm{E}\left(\hat{\beta}|x\right) = \frac{\mathrm{Cov}\left(\beta_s\left(1 - x_{s0}\right), \left(1 - x_{s0}\right)\right)}{\mathrm{Var}\left(1 - x_{s0}\right)}$$

where $\mathrm{Cov}\left(\cdot, \cdot\right)$ and $\mathrm{Var}\left(\cdot\right)$ denote the sample covariance and variance, respectively, and the expectation $\mathrm{E}\left(\hat{\beta}|x\right)$ is taken with respect to the distribution of the errors ε_{st} conditional on the data $x = \{x_{10}, ..., x_{S0}\}$.

Proof. With only two time periods the TWFE estimator of the exposure model is equivalent to an OLS estimator of the first-differenced model

$$y_{s1} - y_{s0} = \delta_1 - \delta_0 + \beta\left(1 - x_{s0}\right) + \varepsilon_{s1} - \varepsilon_{s0}.$$

Therefore the TWFE estimator based on the given sample is

$$\hat{\beta} = \frac{\mathrm{Cov}\left(y_{s1} - y_{s0}, 1 - x_{s0}\right)}{\mathrm{Var}\left(1 - x_{s0}\right)}.$$

[8] E-mail: lsun20@berkeley.edu, jesse_shapiro@fas.harvard.edu.

From the heterogeneous model we have that

$$y_{s1} - y_{s0} = \delta_1 - \delta_0 + \beta_s (1 - x_{s0}) + \varepsilon_{s1} - \varepsilon_{s0}$$

and therefore

$$\hat{\beta} = \frac{\mathrm{Cov}(\beta_s (1 - x_{s0}), 1 - x_{s0})}{\mathrm{Var}(1 - x_{s0})} + \frac{\mathrm{Cov}(\varepsilon_{s1} - \varepsilon_{s0}, 1 - x_{s0})}{\mathrm{Var}(1 - x_{s0})}.$$

If $(\varepsilon_{s1} - \varepsilon_{s0})$ is mean zero conditional on $(1 - x_{s0})$ then the expected value of $\hat{\beta}$ conditional on the data $x = \{x_{10}, ..., x_{S0}\}$ is

$$\mathrm{E}\left(\hat{\beta}|x\right) = \frac{\mathrm{Cov}(\beta_s (1 - x_{s0}), 1 - x_{s0})}{\mathrm{Var}(1 - x_{s0})}.$$

$\square$

Corollary 1. *In the setting of Section "The Possibility of Heterogeneous Coefficients," if β_s is independent of x_{s0} across states s, then the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data $x = \{x_{10}, ..., x_{S0}\}$ for states $s \in \{1, ..., S\}$, is given by*

$$\mathrm{E}\left(\hat{\beta}|x\right) = \mathrm{E}(\beta_s)$$

for $\mathrm{E}(\beta_s)$ the expected value of β_s. Here the expectation $\mathrm{E}\left(\hat{\beta}|x\right)$ is taken with respect to the distribution of the errors ε_{st} and coefficients β_s conditional on the data x.

Proof. Based on a similar proof for Claim 1, we have that

$$\mathrm{E}\left(\hat{\beta}|x\right) = \frac{\mathrm{E}(\mathrm{Cov}(\beta_s (1 - x_{s0}), 1 - x_{s0}))}{\mathrm{Var}(1 - x_{s0})}$$

where the expectation is now taken with respect to the distribution of the errors ε_{st} as well as β_s conditional on the data $x = \{x_{10}, ..., x_{S0}\}$. By the independence of β_s and x_{s0}, we have that

$$\mathrm{E}(\mathrm{Cov}(\beta_s (1 - x_{s0}), 1 - x_{s0})) = \mathrm{Cov}(\mathrm{E}(\beta_s)(1 - x_{s0}), 1 - x_{s0}) = \mathrm{E}(\beta_s)\,\mathrm{Var}(1 - x_{s0}),$$

and therefore that

$$\mathrm{E}\left(\hat{\beta}|x\right) = \mathrm{E}\left(\beta_s\right).$$

$\square$

Corollary 2. *In the numerical example of Section "The Possibility of Heterogeneous Coefficients," the expected value of the two-way fixed effects (TWFE) estimator of the exposure model, given the data* $x = \{x_{10}, ..., x_{S0}\}$ *for states* $s \in \{1, ..., S\}$*, lies outside the range of coefficients* $[\min_s \beta_s, \max_s \beta_s]$ *if and only if* $\lambda \neq 0$*. The same continues to hold when the sample is extended to include a totally unaffected state.*

Proof. From Claim 1 we have that

$$\mathrm{E}\left(\hat{\beta}|x\right) = \frac{\mathrm{Cov}\left(\beta_s\left(1 - x_{s0}\right), 1 - x_{s0}\right)}{\mathrm{Var}\left(1 - x_{s0}\right)}.$$

Because in the numerical example $\beta_s = 1 + 0.5\lambda - \lambda x_{s0}$, we have that

$$\mathrm{E}\left(\hat{\beta}|x\right) = 1 + 0.5\lambda - \lambda C$$

for

$$C = \frac{\mathrm{Cov}\left(x_{s0}\left(1 - x_{s0}\right), \left(1 - x_{s0}\right)\right)}{\mathrm{Var}\left(1 - x_{s0}\right)}.$$

In the setting of Section "The Possibility of Heterogeneous Coefficients," given the data $x = \{x_{10}, ..., x_{S0}\}$ where $x_{s0} = 0.245 + s/100$ for $s = 1, \ldots, 50$, by direct calculation we have that $C = 0$, which means that

$$\mathrm{E}\left(\hat{\beta}|x\right) = 1 + 0.5\lambda.$$

If we add to the sample a totally unaffected state $s = 0$ with $x_{00} = 1$, and the remaining states $s = 1, \ldots, 50$ continue to follow $x_{s0} = 0.245 + s/100$, by direct calculation we have that $C \approx 0.087$, which means that

$$\mathrm{E}\left(\hat{\beta}|x\right) \approx 1 + 0.413\lambda.$$

Therefore, with or without a totally unaffected state, when $\lambda > 0$ we have $\mathrm{E}\left(\hat{\beta}|x\right) > \beta_s$ for all s because $\max_s \beta_s = 1 + 0.245\lambda$. Similarly, with or without a totally unaffected state, when $\lambda < 0$ we have $\mathrm{E}\left(\hat{\beta}|x\right) < \beta_s$ for all s because

$\min_s \beta_s = 1 + 0.245\lambda$. Finally, with or without a totally unaffected state, when $\lambda = 0$ we have $\mathrm{E}\left(\hat{\beta}|x\right) = 1 = \mathrm{E}\left(\beta_s\right) = \max_s \beta_s = \min_s \beta_s$. □

Claim 2. In the setting of Section "The Possibility of Heterogeneous Coefficients," there exists no estimator $\hat{\beta}'$ that can be expressed as a function of the data $\{(x_{s0}, y_{s0}, y_{s1})\}_{s=1}^{S}$ and whose expected value is guaranteed to be contained in $[\min_s \beta_s, \max_s \beta_s]$ for any heterogeneous model and any $\{x_{s0}\}_{s=1}^{S}$.

Proof. It is sufficient to establish this claim for a special case with $S = 2$, some x_{s0}'s with $0 < x_{20} \leq x_{10} < 1$, $\beta_1 < \beta_2$, and δ_0 known to be zero. The model for the data is then

$$\begin{aligned} y_{s0} &= \alpha_s + \beta_s \cdot x_{s0} + \varepsilon_{s0} \\ y_{s1} &= \alpha_s + \delta_1 + \beta_s + \varepsilon_{s1} \end{aligned}$$

with parameters $\theta = \left(\{(\alpha_s, \beta_s)\}_{s=1}^{2}, \delta_1, F_{\varepsilon|X}\right)$, for $F_{\varepsilon|X}$ the distribution of $(\varepsilon_{s0}, \varepsilon_{s1})$ conditional on x_{s0}. Pick some estimator $\hat{\beta}'$. Given any parameter θ, define the distinct parameter $\theta' = \left(\left\{\left(\alpha_s', \beta_s'\right)\right\}_{s=1}^{2}, \delta_1', F_{\varepsilon|X}\right)$ given by

$$\theta' = \left(\left\{\left(\alpha_s + \frac{\Delta \cdot x_{s0}}{1 - x_{s0}}, \beta_s - \frac{\Delta}{1 - x_{s0}}\right)\right\}_{s=1}^{2}, \delta_1 + \Delta, F_{\varepsilon|X}\right)$$

for some $\Delta > (\beta_2 - \beta_1) \cdot (1 - x_{20}) > 0$.

We show that the two parameter values θ and θ' are observationally equivalent, which means the expected value of $\hat{\beta}'$ must be the same under θ and θ'. To see this, note that the distribution of (y_{s0}, y_{s1}) conditional on x_{s0} is the same under θ and θ':

$$
\begin{aligned}
&F_{Y_0,Y_1|X}\left(y_0, y_1 \mid x_{s0}=x;\theta\right)\\
=&\Pr\left\{\varepsilon_{s0} \leq y_0-\alpha_s-\beta_s\cdot x,\ \varepsilon_{s1}\leq y_1-\alpha_s-\delta_1-\beta_s \mid x_{s0}=x;\theta\right\}\\
=&\Pr\left\{\varepsilon_{s0} \leq y_0-\alpha_s-\beta_s\cdot x,\ \varepsilon_{s1}-\varepsilon_{s0}\leq y_1-y_0-\delta_1-\beta_s\left(1-x\right)|x_{s0}=x;\theta\right\}\\
=&\Pr\left\{\begin{array}{c}\varepsilon_{s0}\leq y_0-\left(\alpha_s+\frac{\Delta\cdot x}{1-x}\right)-\left(\beta_s-\frac{\Delta}{1-x}\right)\cdot x,\\ \varepsilon_{s1}-\varepsilon_{s0}\leq y_1-y_0-\left(\delta_1+\Delta\right)-\left(\beta_s-\frac{\Delta}{1-x}\right)\left(1-x\right)\end{array}\middle| x_{s0}=x;\theta\right\}\\
=&\Pr\left\{\begin{array}{c}\varepsilon_{s0}\leq y_0-\alpha'_s-\beta'_s\cdot x,\\ \varepsilon_{s1}-\varepsilon_{s0}\leq y_1-y_0-\delta'_1-\beta'_s\left(1-x\right)\end{array}\middle| x_{s0}=x;\theta'\right\}\\
=&F_{Y_0,Y_1|X}\left(y_0,y_1\mid x_{s0}=x;\theta'\right).
\end{aligned}
$$

However, the Δ is chosen such that $\beta'_1=\beta_1-\frac{\Delta}{1-x_{10}}<\beta_2-\frac{\Delta}{1-x_{20}}=\beta'_2<\beta_1<\beta_2$. Therefore the expected value of $\hat{\beta}'$ cannot be contained in both $[\beta_1,\beta_2]$ and $[\beta'_1,\beta'_2]$, because these intervals do not intersect. □

Claim 3. In the setting of Section "A Difference-in-Differences Perspective," the exposure-adjusted difference-in-differences estimator $\hat{\beta}^{DID}_{s,s'}$ is equivalent to the TWFE estimator $\hat{\beta}$ based on the two states s and s'. Moreover, the expected value of $\hat{\beta}^{DID}_{s,s'}$, given the data $x=\{x_{s0},x_{s'0}\}$ for states s and s', is given by

$$
\mathrm{E}\left(\hat{\beta}^{DID}_{s,s'}|x\right)=\frac{\left(1-x_{s0}\right)\beta_s-\left(1-x_{s'0}\right)\beta_{s'}}{x_{s'0}-x_{s0}}
$$

where the expectation $\mathrm{E}\left(\hat{\beta}^{DID}_{s,s'}|x\right)$ is taken with respect to the distribution of the errors ε_{st} conditional on the data $x=\{x_{s0},x_{s'0}\}$.

Proof. For the first part of the claim, note that from the proof of Claim 1 we have

$$
\hat{\beta}=\frac{\mathrm{Cov}\left(y_{s1}-y_{s0},1-x_{s0}\right)}{\mathrm{Var}\left(1-x_{s0}\right)}
$$

where $\mathrm{Cov}\left(\cdot,\cdot\right)$ and $\mathrm{Var}\left(\cdot\right)$ denote the sample covariance and variance, respectively.

Since the sample includes only two states s and s', for the numerator we have

$$
\begin{aligned}
&\mathrm{Cov}\left(y_{s1}-y_{s0},1-x_{s0}\right)\\
&=\frac{1}{4}\left(\left(y_{s1}-y_{s0}\right)-\left(y_{s'1}-y_{s'0}\right)\right)\left(1-x_{s0}\right)+\frac{1}{4}\left(\left(y_{s'1}-y_{s'0}\right)-\left(y_{s1}-y_{s0}\right)\right)\left(1-x_{s'0}\right)\\
&=\frac{1}{4}\left(\left(1-x_{s0}\right)-\left(1-x_{s'0}\right)\right)\left(\left(y_{s1}-y_{s0}\right)-\left(y_{s'1}-y_{s'0}\right)\right)
\end{aligned}
$$

where the first equality applies the definition of sample covariance and $a-\frac{a+b}{2}=\frac{a-b}{2}$. Similarly, for the denominator we have

$$
\mathrm{Var}\left(1-x_{s0}\right)=\frac{1}{4}\left(\left(1-x_{s0}\right)-\left(1-x_{s'0}\right)\right)^{2}.
$$

Plugging the above expressions into $\hat{\beta}$ gives the equivalence to $\hat{\beta}^{DID}_{s,s'}$.

Given the equivalence between $\hat{\beta}$ and $\hat{\beta}^{DID}_{s,s'}$ when the sample includes only two states s and s', we apply Claim 1 to derive the expected value of $\hat{\beta}^{DID}_{s,s'}$. Specifically, Claim 1 implies that given the data $x=\{x_{s0},x_{s'0}\}$ for states s and s', we have

$$
\mathrm{E}\left(\hat{\beta}^{DID}_{s,s'}|x\right)=\frac{\mathrm{Cov}\left(\beta_s\left(1-x_{s0}\right),1-x_{s0}\right)}{\mathrm{Var}\left(1-x_{s0}\right)}.
$$

Based on a similar simplification to the expression of $\hat{\beta}^{DID}_{s,s'}$, we have

$$
\mathrm{Cov}\left(\beta_s\left(1-x_{s0}\right),1-x_{s0}\right)=\frac{1}{4}\left(\left(1-x_{s0}\right)-\left(1-x_{s'0}\right)\right)\left(\left(1-x_{s0}\right)\beta_s-\left(1-x_{s'0}\right)\beta_{s'}\right)
$$

and therefore

$$
\frac{\mathrm{Cov}\left(\beta_s\left(1-x_{s0}\right),1-x_{s0}\right)}{\mathrm{Var}\left(1-x_{s0}\right)}=\frac{\left(1-x_{s0}\right)\beta_s-\left(1-x_{s'0}\right)\beta_{s'}}{x_{s'0}-x_{s0}}.
$$

□