Online Appendix for: "Optimal Forward Guidance"

By Florin O. Bilbiie

Appendix A: Optimal FG in the Representative-Agent Model: Derivations

This Appendix contains the derivations referred to in text for the representative-agent model.

A1. Proofs for equal-weights case

The second-order condition is:

$$\left(\frac{dc_L}{dq}\right)^2 + c_L\frac{d^2c_L}{dq^2} + \left(\frac{dc_F}{dq}\right)^2 + c_F\frac{d^2c_F}{dq^2} > 0.$$

The second derivatives are:

$$\begin{aligned}
\frac{d^2c_F}{dq^2} &= \nu\left(\frac{\frac{dc_F}{dq}(1-q\nu)+\nu c_F}{(1-q\nu)^2}\right) = \frac{2\nu^2 c_F}{(1-q\nu)^2} = \frac{2\nu}{1-q\nu}\frac{dc_F}{dq} \\
\frac{d^2c_L}{dq^2} &= \frac{(1-p)}{1-p\nu}\frac{2\nu^2 c_F}{(1-q\nu)^2} = \frac{(1-p)}{1-p\nu}\frac{d^2c_F}{dq^2}
\end{aligned}$$

Replacing:

$$\begin{aligned}
&\left(\frac{(1-p)}{1-p\nu}\frac{dc_F}{dq}\right)^2 + c_L\frac{(1-p)}{1-p\nu}\frac{d^2c_F}{dq^2} + \left(\frac{dc_F}{dq}\right)^2 + c_F\frac{d^2c_F}{dq^2} \\
= &\left(\left(\frac{(1-p)}{1-p\nu}\right)^2+1\right)\left(\frac{dc_F}{dq}\right)^2 + \left(c_L\frac{(1-p)}{1-p\nu}+c_F\right)\frac{2\nu}{1-q\nu}\frac{dc_F}{dq} > 0
\end{aligned}$$

Since the derivative is positive:

$$\left(\left(\frac{(1-p)}{1-p\nu}\right)^2+3\right)c_F + 2\frac{(1-p)}{1-p\nu}c_L > 0,$$

for global convexity, we need:

$$\Delta_L < \frac{1}{2}\left(1-p+3\frac{(1-p\nu)^2}{(1-p)}+2q\nu(1-p)\right)\frac{1}{1-q\nu}.$$

For local convexity, we know that at q^*:

$$\left(\frac{(1-p)^2}{(1-p\nu)^2}q^*\nu+1\right)c_F\left(q^*\right)=\frac{(1-p)}{1-p\nu}\frac{\sigma}{1-p\nu}\left(-\rho_L\right).$$

Using in SOC:

$$\begin{aligned}\left(\left(\frac{(1-p)}{1-p\nu}\right)^2+1\right)c_F\left(q^*\right)+2\left(1+q^*\nu\left(\frac{(1-p)}{1-p\nu}\right)^2\right)c_F\left(q^*\right)+2\frac{(1-p)}{(1-p\nu)^2}\sigma\rho_L &> 0\\ \left(\left(\frac{(1-p)}{1-p\nu}\right)^2+1\right)c_F\left(q^*\right) &> 0\end{aligned}$$

which proves the result.

A2. Derivatives of q^*

$$\frac{dq^*}{dp}=\frac{1}{\nu}\frac{2\left(1-p\nu\right)\left(1-p\right)\left(\nu-1\right)+\left(1-p\right)^2\Delta_L+\left(2\nu\left(1-p\right)-1+p\nu\right)\left(1-p\nu\right)\Delta_L}{\left(\left(1-p\right)^2+\Delta_L\left(1-p\right)\right)^2}>0$$

$$\frac{dq^*}{d\Delta_L}=\frac{(1-p)}{\nu}\frac{\left(1-p\right)^2+\left(1-p\nu\right)^2}{\left(\left(1-p\right)^2+\Delta_L\left(1-p\right)\right)^2}>0$$

$$\frac{dq^*}{d\nu}=\frac{1}{\nu^2}\frac{1}{1-p+\Delta_L}\left(\frac{1-(p\nu)^2}{1-p}-\Delta_L\right)$$

A3. Optimal discounting: OFG versus Ramsey

Figure A1 Here.

A4. Optimal FG with Forward-Looking Pricing

When aggregate supply is given by the more general NKPC with discounting (equation 2 in text), the solution at the zero lower bound without FG (under the same IS equation as before) is standard:

$$c_L=\frac{\left(1-\beta_e p\right)\sigma}{\Gamma_p}\rho_L;\ \pi_L=\frac{\kappa}{1-\beta q}c_L$$

where $\Gamma_p=\left(1-\beta_e p\right)\left(1-p\right)-\sigma\kappa p>0$ by restriction (this is the equivalent of $p>1/\nu$ in text, see also footnote on sunspot equilibria).

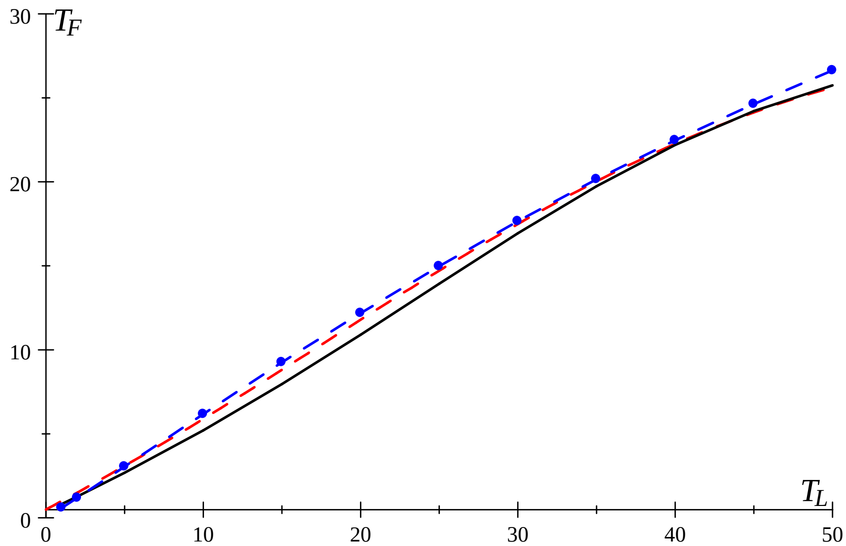

FIGURE A1. FG DURATION IMPLIED BY RAMSEY POLICY (BLUE DOT-DASH), ALONG WITH OPTIMAL FG DURATION UNDER EQUAL WEIGHTS (RED DASH) AND OPTIMAL DISCOUNTING (BLACK SOLID), AS A FUNCTION OF TRAP DURATION.

Modelling FG in exactly the same way, the equilibrium is state F is now given by:

$$c_F = \frac{(1-\beta_e q)\,\sigma}{\Gamma_q}\rho;\;\; \pi_F = \frac{\kappa}{1-\beta q}c_F,$$

where $\Gamma_q = (1-\beta_e q)(1-q) - \sigma\kappa q$.

During the LT state, taking into account the FG equilibrium solved for above, the closed-form solution is

$$c_L = \frac{(1-p)\,q}{\Gamma_p}\frac{(1-\beta_e p)(1-\beta_e q)+\sigma\kappa}{\Gamma_q}\sigma\rho + \frac{(1-\beta_e p)\,\sigma}{\Gamma_p}\rho_L$$

Notice that FG has very similar effects to the ones found in the simpler case covered in text, namely:

$$\frac{dc_F}{dq} = \frac{(1-\beta_e q)^2+\sigma\kappa}{\Gamma_q^2}\sigma\rho = \left(\frac{(1-\beta_e q)+\frac{\sigma\kappa}{1-\beta_e q}}{\Gamma_q}\right)c_F > 0$$

$$\frac{dc_L}{dq} = \frac{1-p}{\Gamma_p}\left(\frac{q\left(1-\beta_e p+\frac{\sigma\kappa}{1-\beta_e q}\right)\left(1-\beta_e q+\frac{\sigma\kappa}{1-\beta_e q}\right)}{\Gamma_q}+1-\beta_e p+\frac{\sigma\kappa}{1-\beta_e q}+\frac{q\beta_e\sigma\kappa}{2\left(1-\beta_e q\right)^2}\right)c_F > 0$$

Optimal FG consists of the persistence probability q that solves the first-order condition:

$$c_L\frac{dc_L}{dq}+\omega\left(q\right)c_F\frac{dc_F}{dq}+\frac{1}{2}\omega'\left(q\right)c_F^2=0,$$

given the equilibrium c_F and c_L solved above. Given those equilibrium values and the expression for $\omega\left(q\right)$ given in text, it can be easily seen that this is a sixth-order polynomial equation in q. We solve this numerically for the baseline calibration, under the restrictions $\Gamma_p, \Gamma_q > 0$ and plot the solution as a function of key parameters in the main text.

Appendix B: Ramsey-Optimal Policy and Forward Guidance: Markov Shocks

The solution to the Ramsey problem is described by the two conditions in equation 19 in text; combining them to eliminate consumption c_t we obtain:

$$\left[-\nu E_t\phi_{t+1}+\left(1+\beta^{-1}\nu^2\right)\phi_t-\beta^{-1}\nu\phi_{t-1}+\sigma\rho_t\right]\phi_t\geq 0$$

Call T the stopping time of the exogenous shock (under Markov shocks, this is a stochastic variable with expected value $(1-p)^{-1}$) and T_F^R the (unknown, to be determined) number of periods for which the ZLB binds, determined implicitly by the boundary condition $\phi_{T+T_F^R-1}=0$. First notice that once ZLB stopped binding (for any $t\geq T+T_F^R$) the economy is back at steady state, $i_t=\rho$, $c_t=0$, $\phi_t=0$. Therefore, we only need to solve for ϕ_t when ZLB binds, for $t\leq T+T_F^R-1$, case in which we have the second-order difference equation:

$$\nu E_t\phi_{t+1}-\left(1+\beta^{-1}\nu^2\right)\phi_t+\beta^{-1}\nu\phi_{t-1}=\sigma\rho_t,$$

or written with lag operators (recall $L^{-j}x_t=E_tx_{t+j}$):

$$\left[L^{-2}-\left(\nu^{-1}+\beta^{-1}\nu\right)L+\beta^{-1}\right]\phi_{t-1}=\sigma\nu^{-1}\rho_t$$

The eigenvalues being obvious $\beta^{-1}\nu>1$ and $\nu^{-1}<1$we can solve by (e.g.) factorizing:

$$\left(L^{-1}-\beta^{-1}\nu\right)\left(L^{-1}-\nu^{-1}\right)\phi_{t-1}=\sigma\nu^{-1}\rho_t,$$

which delivers the backward-forward solution

$$\phi_t=\nu^{-1}\phi_{t-1}-\sigma\beta\nu^{-2}E_t\sum_{j=0}^{T+T_F^R-1}\left(\beta\nu^{-1}\right)^j\rho_{t+j}, \tag{B1}$$

where the forward summation goes on only as long as the solution applies, i.e. up to $T+T_F^R-1$, where T_F^R is unknown.

Consider first what happens *after the shock has been absorbed*, i.e. between the (stochastic) T and $T+T_F^R-1$, whereby $E_t\rho_{t+j}=\rho$. Solving for $\phi_{T+T_F^R-1}$ we obtain

$$\phi_{T+T_F^R-1}=\nu^{-T_F^R}\phi_{T-1}-\frac{\sigma\beta\nu^{-2}\rho}{1-\beta\nu^{-1}}\left(\frac{1-\nu^{-T_F^R}}{1-\nu^{-1}}-\beta\nu^{-1}\frac{1-\left(\beta\nu^{-2}\right)^{T_F^R}}{1-\beta\nu^{-2}}\right),$$

which combined with the boundary condition (equation 20 in text) $\phi_{T+T_F^R-1}=0$ delivers the Lagrange multiplier in the moment of absorption

$$\text{(B2)}\qquad \begin{aligned}\frac{1-\beta\nu^{-1}}{\sigma\beta\nu^{-2}\rho}\phi_{T-1} &= \tilde{\phi}\left(T_F^R\right)\\ \text{where } \tilde{\phi}\left(T_F^R\right) &\equiv \frac{\nu^{T_F^R}-1}{1-\nu^{-1}}-\beta\nu^{-1}\frac{\nu^{T_F^R}-\left(\beta\nu^{-1}\right)^{T_F^R}}{1-\beta\nu^{-2}}\end{aligned}$$

This defines an increasing function $\tilde{\phi}\left(T_F^R\right)$ that is independent of the exogenous random stopping time T.

The print Appendix covers the case of perfect foresight, here we focus on Markov shocks. Consider what happenswhen uncertainty prevails, i.e. *before the shock has been absorbed*: between 0 and $T-1$. We now solve for ϕ_{T-1} "backward", recalling that the starting state between 0 and T is ρ_L and hence $E_t\rho_{t+j}=p^j\rho_L+\left(1-p^j\right)\rho$. For any t between 0 and T-1 we have thus, replacing the expectation

$$\phi_t=\nu^{-1}\phi_{t-1}+\sigma\beta\nu^{-2}\rho\left(\left(\Delta_L+1\right)\frac{1-\left(\beta\nu^{-1}p\right)^{T+T_F^R-t}}{1-\beta\nu^{-1}p}-\frac{1-\left(\beta\nu^{-1}\right)^{T+T_F^R-t}}{1-\beta\nu^{-1}}\right),$$

recalling that we sum up to the (endogenous, to be solved for) stopping time. Iterate backwards, denoting $X_t=\sigma\beta\nu^{-2}\rho\left(\left(\Delta_L+1\right)\frac{1-\left(\beta\nu^{-1}p\right)^{T+T_F^R-t}}{1-\beta\nu^{-1}p}-\frac{1-\left(\beta\nu^{-1}\right)^{T+T_F^R-t}}{1-\beta\nu^{-1}}\right)$

$$\begin{aligned}\phi_t &= \nu^{-(t+1)}\phi_{-1}+\sum_{i=0}^{t}\nu^{-i}X_{t-i}\\ &= \sigma\beta\nu^{-2}\rho\left(\frac{\Delta_L+1}{1-\beta\nu^{-1}p}\sum_{i=0}^{t}\nu^{-i}\left[1-\left(\beta\nu^{-1}p\right)^{T+T_F^R-t+i}\right]-\frac{1}{1-\beta\nu^{-1}}\sum_{i=0}^{t}\nu^{-i}\left[1-\left(\beta\nu^{-1}\right)^{T+T_F^R-t+i}\right]\right)\end{aligned}$$

Applying this to the last period $t = T - 1$

$$\text{(B3)} \qquad \frac{1-\beta\nu^{-1}}{\sigma\beta\nu^{-2}\rho}\phi_{T-1} = \hat{\phi}\left(T_F^R, T\right)$$

with

$$\begin{aligned}\hat{\phi}\left(T_F^R, T\right) \quad \equiv \quad & \frac{\left(\Delta_L+1\right)\left(1-\beta\nu^{-1}\right)}{1-\beta\nu^{-1}p}\left(\frac{1-\nu^{-T}}{1-\nu^{-1}} - \left(\beta\nu^{-1}p\right)^{1+T_F^R}\frac{1-\left(\beta\nu^{-2}p\right)^T}{1-\beta\nu^{-2}p}\right) \\ & -\frac{1-\nu^{-T}}{1-\nu^{-1}} + \left(\beta\nu^{-1}\right)^{1+T_F^R}\frac{1-\left(\beta\nu^{-2}\right)^T}{1-\beta\nu^{-2}}.\end{aligned}$$

This defines another schedule for ϕ_{T-1} as a function of T_F^R and T, call it $\hat{\phi}\left(T_F^R, T\right)$.

The optimal stopping time, aka Ramsey duration of forward guidance, is found by requiring the two solutions for ϕ_{T-1} (the value of the constraint when the exogenous shock converges) coincide, i.e. the intersection of the two schedules $\phi_{T-1}\left(T_F^R\right)$ defined by (B3) and (B2) namely:

$$\begin{aligned}&\frac{\nu^{T_F^R}-1}{1-\nu^{-1}} - \beta\nu^{-1}\frac{\nu^{T_F^R}-\left(\beta\nu^{-1}\right)^{T_F^R}}{1-\beta\nu^{-2}} = \frac{\left(\Delta_L+1\right)\left(1-\beta\nu^{-1}\right)}{1-\beta\nu^{-1}p}\left(\frac{1-\nu^{-T}}{1-\nu^{-1}} - \left(\beta\nu^{-1}p\right)^{1+T_F^R}\frac{1-\left(\beta\nu^{-2}p\right)^T}{1-\beta\nu^{-2}p}\right) \\ &-\frac{1-\nu^{-T}}{1-\nu^{-1}} + \left(\beta\nu^{-1}\right)^{1+T_F^R}\frac{1-\left(\beta\nu^{-2}\right)^T}{1-\beta\nu^{-2}}\end{aligned}$$

This defines a nonlinear equation for the stopping time T_F^R as a function of the model parameters. In Figure B1, I plot the solution (solved for numerically) as a function of the expected duration $T_L = E(T) = (1-p)^{-1}$ by solving this equation under rational expectations, $T = T_L$.[1]

Note: Recall that durations are stochastic with probability distribution $p^T(1-p)$.

Appendix C: Simple FG Rule under Perfect Foresight

A simple rule for FG when the shock duration is known can be derived as follows. Let us start from Ramsey policy: the optimality conditions in the limit

[1] The domain is restricted by the requirements for equilibrium uniqueness and no-starvation, translated in durations T<50. This threshold can be relaxed (and the feasile duration increased) by considering even smaller values of ν, closer to 1 (so more sticky prices and/or less intertemporal substitution). Notice that by definition in that case the approximation implied by the simple rule becomes even better.

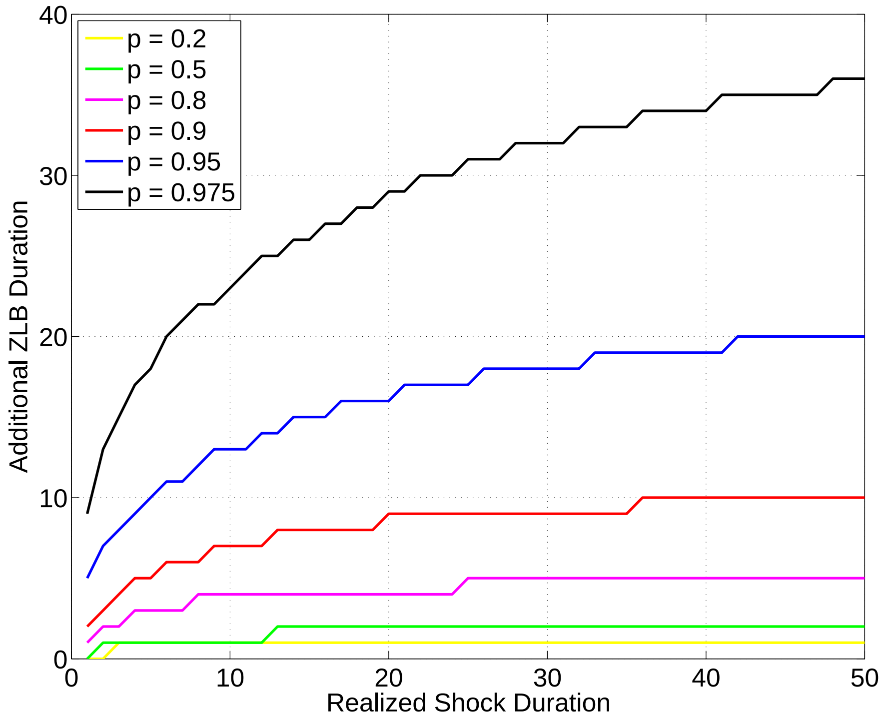

Figure B1. Extra ZLB duration as a function of actual shock duration, for different values of LT persistence from 0.2 to 0.975.

as $\nu \to 1$ and in addition $\beta \to 1$ boil down to, between T and $T + T^* - 1$:

$$\begin{aligned} 0 &= \phi_{T+T^*-1} = \phi_{T-1} - \sigma\rho \sum_{j=0}^{T^*-1} (T^* - j) \\ &= \phi_{T-1} - \sigma\rho \frac{T^*(T^*+1)}{2} \end{aligned}$$

while before, between 0 and $T - 1$:

$$\begin{aligned} \phi_t &= \phi_{t-1} - \sigma\beta \sum_{j=t}^{T-1} \beta^{j-t}\rho_L - \sigma\beta \sum_{j=T}^{T+T^*-1} \beta^{j-t}\rho \\ &= \phi_{t-1} - \sigma(T-t)\rho_L - \sigma T^*\rho \end{aligned}$$

Now solve backwards, letting $X_t = \sigma\rho\left(\left(T-t\right)\Delta_L - T^*\right)$

$$\phi_t = \sum_{i=0}^{t} X_{t-i} = \sigma\rho\left(\Delta_L \sum_{i=0}^{t}\left(T-t+i\right) - \sum_{i=0}^{t} T^*\right)$$

Evaluating at the last period of negative shock $t = T-1$

$$\begin{aligned}\phi_{T-1} &= \sigma\rho\left(\Delta_L \sum_{i=0}^{T-1}\left(1+i\right) - \sum_{i=0}^{T-1} T^*\right)\\ &= \sigma\rho T\left(\Delta_L \frac{\left(T+1\right)}{2} - T^*\right)\end{aligned}$$

The stopping time solves:[2]

$$T_S^* = \sqrt{\left(\frac{1}{2}+T\right)^2 + \Delta_L T\left(1+T\right)} - \left(\frac{1}{2}+T\right)$$

Appendix D: Heterogeneous Beliefs about Duration

Let F be FG time. Realized equilibrium is from T to T+F-1

$$c_{T+j} = \left(\nu^{T^*-j} - 1\right)\kappa^{-1}\rho.$$

and, as before $\frac{dc_{T+j}}{dT^*} = \nu^{T^*-j}\kappa^{-1}\rho\ln\nu$. But for 0 to T-1 things are different now ex ante for any t<T: while for optimists the expectation is correct $E_t c^o_{T+j} = c_{T+j} = \left(\nu^{T^*-j}-1\right)\kappa^{-1}\rho$, for pessimists we have instead:

$$\begin{aligned}E_t c^m_{T+j} &= \left(\nu^{T^*-j} - 1\right)\kappa^{-1}\rho_L.\\ E_t c^m_T &= \left(\nu^{T^*} - 1\right)\kappa^{-1}\rho_L.\end{aligned}$$

Thus

$$E_t c_T = \alpha c^m_T + \left(1-\alpha\right)c^o_T = \left(\nu^{T^*} - 1\right)\kappa^{-1}\left(\alpha\rho_L + \left(1-\alpha\right)\rho\right)$$

[2] Evidently, by virtue of our equivalence proposition the same solution is obtained by solving for OFG; namely from T to T+T*-1 we have $c_{T+j} = \left(T^* - j\right)\sigma\rho$ while for 0 to T-1 $c_j = \left(T^* - \left(T-j\right)\Delta_L\right)\sigma\rho$. Maximizing welfare with $\beta = 1$ delivers the same solution.

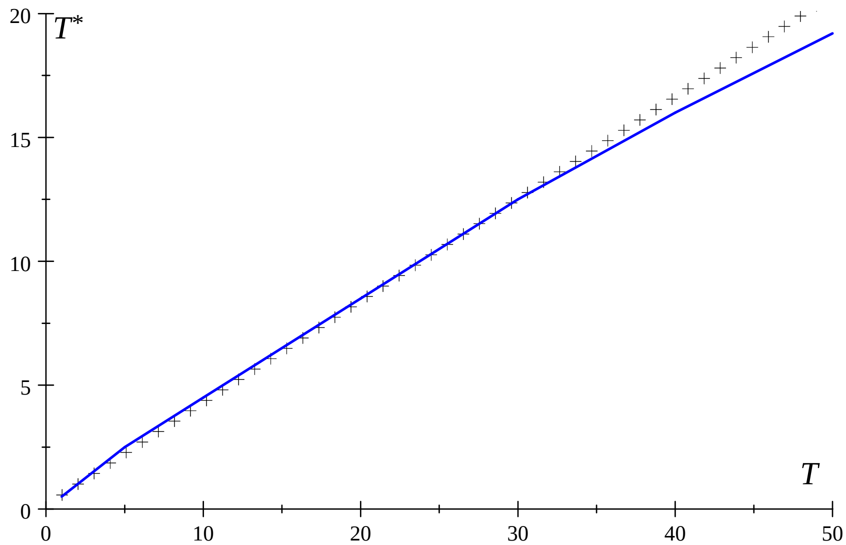

Figure C1. FG durations under perfect foresight implied by Ramsey and OFG (blue solid), along with simple rule (crosses).

Using this expectation heterogeneity, the solution for j from 0 to T-1, which depends on $E_t c_T$ is

$$c_j = \kappa^{-1}\rho\left(\nu^{T-j}\left(\nu^{T^*}-1\right)(1-\alpha(1+\Delta_L)) - \left(\nu^{T-j}-1\right)\Delta_L\right).$$

with $\frac{dc_j}{dT^*} = \nu^{T+T^*-j}\kappa^{-1}\rho\,(1-\alpha(1+\Delta_L))\ln\nu$. As in the stochastic setup, FG power is diminished by the presence of pessimistic agents; furthermore, beyond the same threshold $\alpha > (1+\Delta_L)^{-1}$ FG becomes in fact self-defeating and has perverse effects.

We can find optimal FG using the same technique as in the representative-agent case, replacing the equilibria just found in the loss function and differentiating with respect to T^* to obtain the first-order condition:

$$\sum_{j=0}^{T-1}\beta^j c_j \frac{dc_j}{dT^*} + \sum_{j=0}^{T^*-1}\beta^{T+j} c_{T+j}\frac{dc_{T+j}}{dT^*} = 0.$$

Replacing and calculating the sums we obtain:

$$\begin{aligned}
& (1-\alpha(1+\Delta_L))\left(\left(\nu^{T^*}-1\right)(1-\alpha(1+\Delta_L))-\Delta_L\right)\nu^{2T}\frac{1-\left(\beta\nu^{-2}\right)^T}{1-\beta\nu^{-2}} \\
& +(1-\alpha(1+\Delta_L))\Delta_L\nu^T\frac{1-\left(\beta\nu^{-1}\right)^T}{1-\beta\nu^{-1}}+\nu^{T^*}\beta^T\frac{1-\left(\beta\nu^{-2}\right)^{T^*}}{1-\beta\nu^{-2}}-\beta^T\frac{1-\left(\beta\nu^{-1}\right)^{T^*}}{1-\beta\nu^{-1}} \\
= \; & 0
\end{aligned}$$

The following Table presents the numerical solution for different values of α

Table D1—Model outcomes, heterogeneous beliefs about duration

α	0	0.03	0.1	0.2	0.3	0.4	0.45	0.49	0.498
T^*	16.15	16.89	18.88	22.32	25.96	26.19	21.14	9.73	4.08
$[T^*]$	16	17	19	22	26	26	21	10	4
Loss	2.10	2.19	2.49	3.33	5.31	9.88	13.1	15.09	15.26

In the last row, I evaluate the lifetime loss under optimal odyssean policy:

$$\sum_{j=0}^{T-1}\left(\beta^j\left(\nu^{T-j}\left(\nu^{T^*}-1\right)(1-\alpha(1+\Delta_L))-\left(\nu^{T-j}-1\right)\Delta_L\right)^2\right)+\sum_{j=0}^{[T^*]-1}\left(\beta^{T+j}\left(\nu^{T^*-j}-1\right)^2\right)$$

and compare with "simple rule" which for deterministic case with $T=40$ is $T_S^*=17$

$$\sum_{j=0}^{T-1}\left(\beta^j\left(\nu^{T-j}\left(\nu^{T_S^*}-1\right)-\left(\nu^{T-j}-1\right)\Delta_L\right)^2\right)+\sum_{j=0}^{T_S^*-1}\left(\beta^{T+j}\left(\nu^{T_S^*-j}-1\right)^2\right)=2.1595$$

It can be easily checked that the value of α at which the lifetime loss under optimal policy becomes larger than this is a mere $\alpha=0.015$ even though the trap duration is large.

Appendix E: OFG in A Medium-Scale DSGE Model

This Appendix outlines for completion the medium-scale DSGE model used in Section V; the presentation is compact and consists of outlining the complete set of equilibrium conditions that are by now standard. Their detailed derivation can be found in Justiniano, Primiceri, and Tambalotti (2013, JPT); Bilbiie, Monacelli, and Perotti (2018) use the JPT model to compute the welfare effects of government spending at the ZLB and show that it replicates remarkably well the Great Recession episode.

I modify this model in two substantial respects. First, instead of Calvo stickiness of prices and wages, I use Rotemberg adjustment costs; this is because I solve the model nonlinearly and Rotemberg pricing gives very tractable nonlinear Phillips curves without introducing extra endogenous states that keep track of the cross-sectional dispersion of prices and wages (which instead appear with the Calvo formulation, augmenting an already large state space). Second, I replace the monetary-policy Taylor rule that they use by the different rule that suits our purposes: in particular, I assume like before that the nominal rate either equals zero (when the ZLB is binding and/or when the CB decides to do FG) or else it equals the discount rate.

The representative household's lifetime utility , which I use to compute welfare, is:
(E1)

$$\mathcal{V} = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t} (1+z_j) \left\{ \frac{(C_t - hC_{t-1})^{1-\gamma} - 1}{1-\gamma} - \frac{1}{1+\varphi} \left(\frac{N_t}{1 - \frac{\psi_w}{2} \tilde{\pi}_{w,t}^2} \right)^{1+\varphi} \right\},$$

where the modified labor disutility term comes about because of sticky wages: I already replaced labor supply using labor market clearing $N_t / \left(1 - \frac{\psi_w}{2} \tilde{\pi}_{w,t}^2\right)$ where N_t is total labor demand and the denominator is related to the labor cost of adjusting nominal wages—see below for details. The parameter h captures consumption habits and the exogenous z_t is a discount factor "shock", following a deterministic path: as in our previous deterministic setup, we assume it increases for T periods, making the ZLB bind.

The equilibrium conditions concern the set of endogenous variables: $\{\lambda_t, C_t, Y_t, N_t, mc_t, i_t, \pi_t, \tilde{\pi}_t, I_t, K_{t+1}, q_t, r_t^K, w_t, \pi_{w,t}, \tilde{\pi}_{w,t}\}$ and are as follows.

The marginal utility of consumption is defined as:

$$\lambda_t = (C_t - hC_{t-1})^{-\gamma}$$

The consumption Euler equation for holding nominal bonds is:

$$\lambda_t = \beta (1+z_t) \mathrm{E}_t \left(\frac{1+i_t}{1+\pi_{t+1}} \lambda_{t+1} \right)$$

Output is produced using both labor and capital; and aggregate "Production function" holds with an appropriately defined notion of GDP Y (that includes price adjustment costs) and input N (that includes wage adjustment costs):

$$Y_t = N_t^{1-\theta} K_t^{\theta}$$

Firms' cost minimization delivers (mc is real marginal cost) optimal demand for labor and capital respectively (with w and r^K the respective factor prices):

$$\begin{aligned} w_t &= (1-\theta)\, mc_t \frac{Y_t}{N_t} \\ r_t^K &= \theta mc_t \frac{Y_t}{K_t} \end{aligned}$$

Capital accumulation is subject to investment adjustment costs:

$$K_{t+1} = (1-\delta)K_t + I_t \left[1 - \frac{\psi_k}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right]$$

The price of capital (Tobin's q) is standard given the adjustment costs:

$$q_t = \beta\,(1+z_t)\,\mathrm{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t}\left[r_{t+1}^K + (1-\delta)q_{t+1}\right]\right\}.$$

Optimal investment is given by:

$$q_t \left[1 - \frac{\psi_k}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \psi_k \left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right] = 1 - \beta\,(1+z_t)\,\mathrm{E}_t \left[q_{t+1}\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2 \psi_k \left(\frac{I_{t+1}}{I_t} - 1\right)\right]$$

Equilibrium in the final good market requires that consumption and investment sump up to the economy's GDP, i.e. output net of the price adjustment cost:

$$C_t + I_t = \left(1 - \frac{\psi_p}{2}\tilde{\pi}_t^2\right) Y_t$$

where the variable $\tilde{\pi}$ is defined below.

The optimality condition for each monopolistic producer choosing the price of their individual variety (substitutable to others according to a Dixit-Stiglitz aggregator with elasticity ε) subject to a standard Rotemberg quadratic price adjustment cost with parameter ψ_p and to a potential sales subsidy s delivers a New Keynesian Phillips curve for the "net-of-indexation" (defined below) inflation rate:

$$(1+\tilde{\pi}_t)\,\tilde{\pi}_t = \beta\,(1+z_t)\,\mathrm{E}_t \left\{\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}\tilde{\pi}_{t+1}\,(1+\tilde{\pi}_{t+1})\right\} + \frac{\varepsilon - 1}{\psi_p}\left[\frac{\varepsilon}{\varepsilon - 1} mc_t - (1+s)\right]$$

Similarly, the optimality condition for each union setting wages for a differentiated labor type subject to a downward sloping labor demand with elasticity ε_w, Rotemberg adjustment costs paid in labor units ψ_w and a labor subsidy s_w delivers a NK wage Phillips curve for the "net-of-indexation" (defined below) wage inflation rate:

$$\begin{aligned}\frac{\tilde{\pi}_{w,t}\left(\tilde{\pi}_{w,t}+1\right)}{1-\frac{\psi_w}{2}\tilde{\pi}_{w,t}^2} &= \beta\left(1+z_t\right)\mathrm{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{N_{t+1}}{N_t}\frac{\tilde{\pi}_{w,t+1}\left(\tilde{\pi}_{w,t+1}+1\right)^2}{1-\frac{\psi_w}{2}\tilde{\pi}_{w,t+1}^2}\right] \\ &\quad +\frac{\varepsilon_w-1}{\psi_w}\left[\frac{\varepsilon_w}{\varepsilon_w-1}\frac{1}{\lambda_t w_t}\left(\frac{N_t}{1-\frac{\psi_w}{2}\tilde{\pi}_{w,t}^2}\right)^{\varphi}-\left(1+s_w\right)\right]\end{aligned}$$

Wage inflation is defined as:

$$1+\pi_{w,t}=\frac{w_t}{w_{t-1}}\left(1+\pi_t\right)$$

The "net-of-indexation" inflation rates for prices and wages respectively are, with ι the respective indexation parameters:

$$\begin{aligned}\tilde{\pi}_t &= \frac{1+\pi_t}{\left(1+\pi_{t-1}\right)^{\iota_p}}-1 \\ \tilde{\pi}_{w,t} &= \frac{1+\pi_{w,t}}{\left(1+\pi_{w,t-1}\right)^{\iota_w}}-1\end{aligned}$$

Finally, the model is closed by a specification of monetary policy. Because I want to isolate the role of FG and its effect on welfare, I choose a simple monetary policy specification that parallels the one used in the simple 3-equation model (but the results apply qualitatively to considering an empirically-relevant Taylor rule as in JPT):

$$1+i_t=\max\left(1,\beta^{-1}\left(1+z_t\right)^{-1}\right),$$

where z_t is a discount factor shock that, when large enough, makes the ZLB bind. As in the simple model, I assume that this shock has duration T. I model FG as a decision by the central bank to keep nominal rates at zero beyond the duration of the z shock and for T_F periods, i.e. between T and $T+T_F-1$ and calculate OFG duration T_F^* as described in text. The simple-rule duration T_F^s makes use of the disruption measure Δ_L now defined as

$$\Delta_L \equiv \frac{-\left(\ln\beta^{-1}-\ln\left(1+z\right)\right)}{\ln\beta^{-1}} \approx \frac{z}{\rho}-1.$$

Therefore under the baseline we have $\Delta_L=1$, while for the robustness checks we have "large shock" $\Delta_L=2$ and "small shock" $\Delta_L=0.5$.

The robustness exercises in Figure E1 redo the exercises reported in Figure 5 for two alternative calibrations, comparing them to the baseline case (reported

Table E1—Calibration of DSGE model

Parameter	*Description*	*Value*
γ	Relative risk aversion	1
φ	Curvature of labor disutility	2.36
θ	Capital share in production	0.22
ε	Elasticity of substitution (goods)	5
ε_w	Elasticity of substitution (labor)	5
ψ_p	Rotemberg parameter p	1000
ψ_w	Rotemberg parameter w	100
β	Discount factor at steady state	0.995
h	Consumption habits	0.8
δ	Capital Depreciation Rate	0.025
ψ_k	Investment adjustment cost	3.64
s	Sales subsidy	$1/(\varepsilon-1)$
s_w	Labor subsidy	$1/(\varepsilon_w-1)$
ι_p	Price indexation	0.04
ι_w	Wage indexation	0.15

for comparison in the middle column); all simulations are for the baseline shock value $z = 0.01$.

Note: OFG vs simple-rule durations (top row); welfare costs of simple rule and SIT relative to OFG

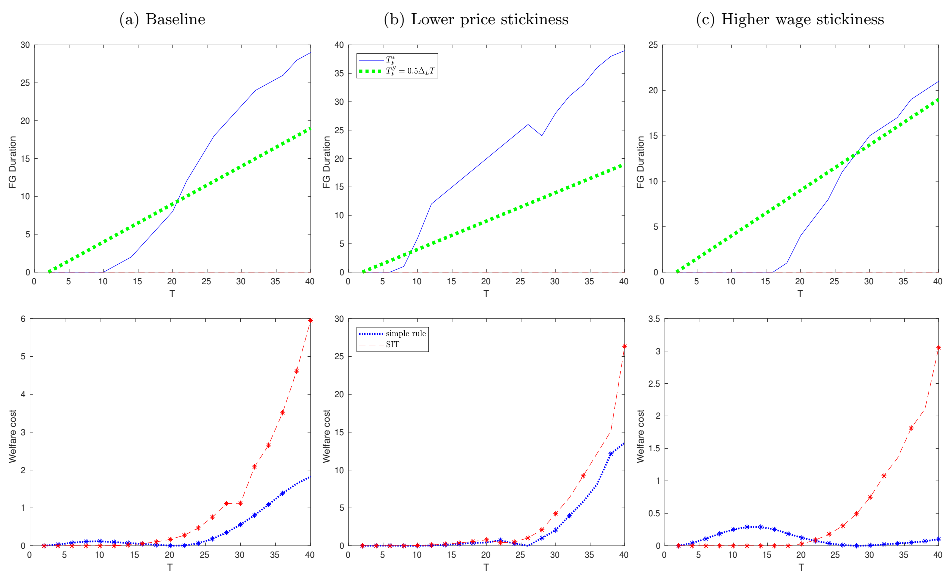

Figure E1. OFG in DSGE: lower price stickiness and higher wage stickiness.