

Online Appendix for

A Model of Secular Stagnation: Theory and Quantitative Evaluation

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A Derivation of Simple Model

A.1 Households' Problem

In this section, we specify and solve the household's problem in the general case of income received in all periods and taxes paid in all periods. For household i , the objective function and budget constraints are given below:

$$\max_{C_t(i), C_{t+1}(i), C_{t+2}(i)} \mathbb{E}_t \{ \log(C_t(i)) + \beta \log(C_{t+1}(i)) + \beta^2 \log(C_{t+2}(i)) \} \quad (\text{A.1})$$

$$\text{s.t.} \quad C_t(i) = w_t L_t(i) - T_t^y + B_t(i) \quad (\text{A.2})$$

$$C_{t+1}(i) = Z_t + w_{t+1} L_{t+1}(i) - T_{t+1}^m + B_{t+1}(i) - \frac{(1+i_t)}{\Pi_{t+1}} B_t(i) \quad (\text{A.3})$$

$$C_{t+2}(i) = w_{t+2} L_{t+2}(i) - T_{t+2}^o - \frac{(1+i_{t+1})}{\Pi_{t+2}} B_{t+1}(i) \quad (\text{A.4})$$

$$B_{t+j}(i) \leq \mathbb{E}_{t+j} (1 + r_{t+j+1}) D_{t+j} \quad \text{for } j = 0, 1, \quad (\text{A.5})$$

where the household i has exogenous labor supply endowments in each period of life, D_{t+j} is an exogenous collateral constraint, and T_{t+j} are lump-sum taxes imposed by the government. We allow taxes to differ across household types and taxes to change over time.

We restrict ourselves to cases in which the collateral constraint is binding in the first period of life and possibly binding in the second period of life. In particular, we will assume two types of households — a household that has sufficiently low labor endowment in its middle period of life and remains credit constrained, and a household that has sufficiently high labor endowment in its middle period of life and is unconstrained. For the former, borrowing in the young and middle-aged generations is determined by the binding collateral constraints. For the latter, borrowing is determined by the collateral constraint only while young; in the middle-aged generation, an Euler equation determines the optimal level of saving:

$$\frac{1}{C_t^{m,h}} = \beta \mathbb{E}_t \frac{1+i_t}{\Pi_{t+1} C_{t+1}^{o,h}}. \quad (\text{A.6})$$

Let L^y be the labor endowment for the young generation, $L^{m,l}$ be the labor endowment for the poor middle-generation household, $L^{m,h}$ the labor endowment for the wealthy middle-generation household, and L^o the labor endowment in the last period. We adopt the normalization that

$L^y + \eta_s L^{m,l} + (1 - \eta_s) L^{m,h} + L^o = 1$. The budget constraints for each type of household alive at any point in time are given below:

$$C_t^y = \alpha Y_t \frac{L^y}{L_t^{flex}} - T_t^y + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1 + i_t} \quad (\text{A.7})$$

$$C_t^{m,l} = \alpha Y_t \frac{L^{m,l}}{L_t^{flex}} + (1 - \alpha) Y_t - T_t^m - D_{t-1} + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1 + i_t} \quad (\text{A.8})$$

$$C_t^{m,h} = \alpha Y_t \frac{L^{m,h}}{L_t^{flex}} + (1 - \alpha) Y_t - T_t^m - D_{t-1} - B_t^{m,h} \quad (\text{A.9})$$

$$C_t^{o,l} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o - D_{t-1} \quad (\text{A.10})$$

$$C_t^{o,h} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o + B_{t-1}^{m,h} \frac{1 + i_{t-1}}{\Pi_t}, \quad (\text{A.11})$$

where T_t^i are lump-sum taxes per capita and Y_t is output per middle-generation household.⁶⁴ Aggregate consumption in this economy is given by the following expression:

$$C_t = N_t C_t^y + N_{t-1} \left(\eta_s C_t^{m,l} + (1 - \eta_s) C_t^{m,h} \right) + N_{t-2} \left(\eta_s C_t^{o,l} + (1 - \eta_s) C_t^{o,h} \right).$$

A.2 Firms' Problem, Labor Supply, and Wage Determination

In this section, we specify the firm's problem in the baseline case with no capital accumulation. Firms choose labor to maximize profits subject to a standard decreasing returns to scale production function, taking wages as given:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t^d \quad (\text{A.12})$$

$$\text{s.t. } Y_t = A_t \left(L_t^d \right)^\alpha, \quad (\text{A.13})$$

where L_t^d is the firm's labor demand. Firms' labor demand is determined by equating the real wage to the marginal product of labor:

$$\frac{W_t}{P_t} = \alpha A_t \left(L_t^d \right)^{\alpha-1}. \quad (\text{A.14})$$

Each middle-generation household operates a firm and collects profits from its operation. The total measure of firms in the economy is N_{t-1} and therefore grows with the total population. All firms are identical, sharing the same labor share parameter α .

Labor supply is exogenous and fixed over a household's lifetime. When population is constant ($g = 0$), then labor supply is constant and can be normalized to unity. In the absence of downward nominal wage rigidity, the real wage equalizes labor supply to labor demand:

$$\left(N_t L^y + N_{t-1} \left(\eta_s L^{m,l} + (1 - \eta_s) L^{m,h} \right) + N_{t-2} L^o \right) = N_{t-1} L_t^{flex}, \quad (\text{A.15})$$

⁶⁴Output is not expressed in per capita terms to avoid a proliferation of population growth rate terms. In this economy, aggregate output is $N_{t-1} Y_t$ while the total population is $N_t + N_{t-1} + N_{t-2}$.

where w_t^{flex} defines the market-clearing real wage.

In the presence of downward nominal wage rigidity, the real wage may exceed the market-clearing real wage. In this case, labor is rationed with a proportional reduction in labor employed across all households (i.e., if total labor demand is 10% below the full-employment level, then labor falls 10% for all cohorts).

We assume that nominal wages are downwardly rigid, implying that real wages exceed the market-clearing level in the presence of deflation. The process determining the real wage is given below:

$$W_t = \max \left\{ \tilde{W}_t, P_t w_t^{flex} \right\} \quad \text{where} \quad \tilde{W}_t = \gamma W_{t-1} + (1 - \gamma) P_t w_t^{flex}. \quad (\text{A.16})$$

A.3 Monetary and Fiscal Policy

Monetary and fiscal policies are straightforward. We assume a monetary policy rule of the following form:

$$1 + i_t = \max \left(1, (1 + i^*) \left(\frac{\Pi_t}{\Pi^*} \right)^{\phi_\pi} \right), \quad (\text{A.17})$$

where i^* is the targeted natural rate and Π^* is the central bank's gross inflation target. If the central bank has the correct natural rate target i^* , then inflation is stabilized at $\Pi = 1$ in steady state.

Taxation is determined by the government's budget constraint and exogenous processes for government spending, the public debt, and taxation of young households. We typically assume that the ratio of taxes between the old and middle-aged households satisfies the following rule:

$$T_t^o = \beta \frac{1 + i_{t-1}}{\Pi_t} T_t^m. \quad (\text{A.18})$$

In steady state, this fiscal rule ensures that changes in taxation have no effect on loan supply. We consider exceptions to this fiscal rule where taxes are levied only on old or middle-aged households, respectively. The government's budget constraint, together with the fiscal rule, determines T_t^m and T_t^o in response to the other exogenous fiscal processes:

$$B_t^g + T_t^y (1 + g_t) + T_t^m + \frac{1}{1 + g_{t-1}} T_t^o = G_t + \frac{1}{1 + g_{t-1}} \frac{1 + i_{t-1}}{\Pi_t} B_{t-1}^g, \quad (\text{A.19})$$

where all fiscal variables are all normalized in terms of middle-generation quantities.

A.4 Market Clearing and Equilibrium

Asset market clearing requires that total lending from savers equals total borrowing from credit-constrained young households and poor middle-generation households. This condition is given

below:

$$(1 - \eta_s) N_{t-1} B_t^{m,h} = N_t \frac{D_t}{1 + r_t} + \eta_s N_{t-1} \frac{D_t}{1 + r_t} \quad (\text{A.20})$$

$$(1 - \eta_s) B_t^{m,h} = (1 + g_t + \eta_s) \frac{D_t}{1 + r_t}. \quad (\text{A.21})$$

It can be verified that asset market clearing implies that aggregate consumption equals aggregate output less aggregate government purchases:

$$C_t = N_{t-1} Y_t - (N_t + N_{t-1} + N_{t-2}) G_t. \quad (\text{A.22})$$

A competitive equilibrium is a set of aggregate allocations $\{Y_t, C_t^{m,h}, C_t^{o,h}, B_t^{m,h}, L_t^{flex}, T_t^m, T_t^o\}_{t=0}^\infty$, price processes $\{i_t, \Pi_t, w_t, w_t^{flex}\}_{t=0}^\infty$, exogenous processes $\{G_t, g_t, D_t, T_t^y, B_t^g\}_{t=0}^\infty$ and initial values of household saving, nominal interest rate, real wage, and the public debt $\{B_{-1}^{m,h}, i_{-1}, w_{-1}, B_{-1}\}$ that jointly satisfy:

1. Household Euler equation (A.6)
2. Household budget constraints (A.9) and (A.11)
3. Asset market clearing (A.21)
4. Fiscal policy rule (A.18)
5. Government budget constraint (A.19)
6. Monetary policy rule (A.17)
7. Full-employment labor supply (A.15)
8. Full-employment wage rate: $w_t^{flex} = \alpha A_t (L_t^{flex})^{\alpha-1}$
9. Labor demand condition: $w_t = \alpha A_t \left(\frac{Y_t}{A_t}\right)^{\frac{\alpha-1}{\alpha}}$
10. Wage process: $w_t = \max\{\tilde{w}_t, w_t^{flex}\}$, where $\tilde{w}_t = \gamma \frac{w_{t-1}}{\Pi_t} + (1 - \gamma) w_t^{flex}$

B Inequality and the Natural Rate

There is no general result about how an increase in inequality affects the real rate of interest. This relationship will depend on how changes in income affect the relative supply and demand for loans. Here we show a simple example illustrating that there are relatively plausible conditions under which higher inequality will in fact reduce the natural rate of interest.

Consider first one form of income inequality, that of inequality across generations. As we have seen in (11), the relative endowment of the old versus the middle-aged generation affects the real interest rate; moving resources from the old to the middle-aged will increase savings and thereby put downward pressure on the real interest rate. The conclusion here is that redistribution that raises savings increases downward pressure on the real rate. Consider now the alternative: If all generations receive the same endowment $Y_t^y = Y_t^m = Y_t^o$, then it is easy to see that there is no incentive to borrow or lend, and, accordingly, the real interest rate is equal to the inverse of the discount factor $1 + r_t = \beta^{-1}$. It is thus inequality of income across generations that is responsible for our results and triggers possibly negative real interest rates.

Generational inequality, however, is typically not what people have in mind when considering inequality; instead, commentators often focus on unequal income of individuals within the working-age population. This type of inequality can also have a negative effect on the real interest rate. Before getting there, however, let us point out that this need not be true in all cases. Consider, for example, an endowment distribution $Y_t(z)$, where z denotes the type of an individual who is known once he is born. Suppose further that once again $Y_t^y(z) = Y_t^m(z) = Y_t^o(z)$ for all z . Once again, income is perfectly smoothed across ages and the real interest rate is given by β^{-1} . The point is that we can choose any distribution of income $Y(z)$ to support that equilibrium so, in this case, income distribution is irrelevant for the type of preferences we have assumed.⁶⁵

Let us now consider the case when inequality in a given cohort can in fact generate negative pressure on real interest rates. When authors attribute demand slumps to a rise in inequality, they typically have in mind — in the language of old Keynesian models — that income gets redistributed from those with a high propensity for consumption to those who instead wish to save their income. We have already seen how this mechanism works in the case of inequality across generations. But we can also imagine that a similar mechanism applies if income gets redistributed within a cohort as long as some of the agents in that cohort are credit constrained.

Again, let us assume that only the middle-aged and old generations receive an income endowment. Now, however, suppose that some fraction of households receives a larger endowment in their middle years (i.e., high-income households) while the remaining households (i.e., low-income households) receive a very small endowment in the middle period of life. For simplicity, all households receive the same income endowment in old age (this could be thought of as some sort of state-provided pension like Social Security). For sufficiently low levels of the middle-period endowment and a sufficiently tight credit constraint, low-income households will remain credit constrained in the middle period of life. These households will roll over their debt in the middle years and repay their debts only in old age, consuming any remaining endowment. In this situation, only the high-income households will save in the middle period and will, therefore, supply savings to both credit-constrained middle-aged households and the youngest generation.

⁶⁵More general preference specifications, however, can easily make a difference.

As before, we can derive an explicit expression for the real interest rate in this richer setting with multiple types of households. Under the conditions described above, the only operative Euler equation is for the high-income households who supply loans in equilibrium. The demand for loans is obtained by adding together the demand from young households and from the credit constrained low-income households. The expression we obtain is a generalization of the case obtained in equation (11):

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t + \eta_s) D_t}{(1 - \eta_s) (Y_t^{m,h} - D_{t-1})} + \frac{1}{\beta} \frac{Y_{t+1}^o}{(Y_t^{m,h} - D_{t-1})}, \quad (\text{A.23})$$

where η_s is the fraction of low-income households, $Y_t^{m,h}$ is the income of the high-income middle generation, and Y_{t+1}^o is the income of these households in the next period (i.e., the pension income received by all households). If $\eta_s = 0$, we recover the expression for the real interest rate derived in (11).

Total income for the middle-aged generation is a weighted-average of high-and low-income workers: $Y_t^m = \eta_s Y_t^{m,l} + (1 - \eta_s) Y_t^{m,h}$. Let us then define an increase in inequality as a redistribution of middle-generation income from low-to high-income workers, without any change in Y_t^m . While this redistribution keeps total income for the middle generation constant by definition, it must necessarily lower the real interest rate by increasing the supply of savings, which is only determined by the income of the wealthy. This can be seen in equation (A.23), where the real interest rate is decreasing in $Y_t^{m,h}$ without any offsetting effect via $Y_t^{m,l}$.⁶⁶

As this extension of our model suggests, the secular rise in wage inequality in recent decades in the US and other developed nations may have been one factor in exerting downward pressure on the real interest rate. Labor market polarization — the steady elimination of blue-collar occupations and the consequent downward pressure on wages for a large segment of the labor force — could show up as increase in income inequality among the working-age population, lowering the real interest rate in the manner described here (for evidence on labor market polarization, see, e.g., [Autor and Dorn \(2013\)](#) and [Goos, Manning and Salomons \(2009\)](#)). Several other theories have been suggested for the rise in inequality. To the extent that they imply an increase in savings, they could fit into our story as well.

⁶⁶As emphasized earlier, not all forms of income inequality should be expected to have a negative effect on the real interest rate. If middle-generation income is drawn from a continuous distribution, a mean-preserving spread that raises the standard deviation of income could be expected to have effects on both the intensive and extensive margin. That is, the average income among savers would rise, but this effect would be somewhat offset by an increase in the fraction of credit-constrained households (i.e, an increase in η_s). The extensive margin boosts the demand for loans and would tend to increase the real interest rate. Whether inequality raises or lowers rates would depend on the relative strength of these effects.

C Linearization and Solution

In this section, we detail the linearization and general solution to the model without capital but with income received in all periods. For simplicity, we do not consider the effect of population growth shocks, which greatly complicate the linearization and the computation of analytical solutions.

The generalized model with income received in all three periods and credit-constrained middle-aged households can be summarized by the following linearized AD curve and linearized AS curve:

$$i_t = E_t \pi_{t+1} - s_y (y_t - g_t) + (1 - s_w) E_t (y_{t+1} - g_{t+1}) + s_w d_t + s_d d_{t-1} \quad (\text{A.24})$$

$$y_t = \gamma_w y_{t-1} + \gamma_w \frac{\alpha}{1 - \alpha} \pi_t, \quad (\text{A.25})$$

where various coefficients are given in terms of their steady-state values:

$$\begin{aligned} \gamma_w &= \frac{\gamma}{\bar{\pi}} \\ s_y &= \frac{\bar{Y}_{m,h}}{\bar{Y}_{m,h} - \bar{D}} \\ s_d &= \frac{\bar{D}}{\bar{Y}_{m,h} - \bar{D}} \\ s_w &= \frac{1 + \beta}{\beta} \frac{(1 + \bar{g} + \eta_s) \bar{D}}{\bar{i}/\bar{\pi} (\bar{Y}_{m,h} - \bar{D})}. \end{aligned}$$

The exogenous shocks are the collateral shock d_t and the government spending shock g_t , which means that a solution to this linear system takes the form:

$$y_t = \beta_y y_{t-1} + \beta_g g_t + \beta_d d_t + \beta_{d,l} d_{t-1} \quad (\text{A.26})$$

$$\pi_t = \alpha_y y_{t-1} + \alpha_g g_t + \alpha_d d_t + \alpha_{d,l} d_{t-1}. \quad (\text{A.27})$$

Solving by the method of undetermined coefficients, we obtain the following expressions for the coefficients that determine equilibrium output and inflation in response to collateral and gov-

ernment spending shocks:

$$\beta_y = 0 \tag{A.28}$$

$$\alpha_y = -\frac{1-\alpha}{\alpha} \tag{A.29}$$

$$\beta_{d,l} = \frac{s_d}{s_y + \frac{1-\alpha}{\alpha}} \tag{A.30}$$

$$\alpha_{d,l} = \frac{1-\alpha}{\gamma_w \alpha} \beta_{d,l} \tag{A.31}$$

$$\beta_d = \frac{s_w + \beta_{d,l} \left(\frac{1-\alpha}{\gamma_w \alpha} + (1-s_w) \right)}{s_y + (1-s_w) \rho_d + \frac{1-\alpha}{\alpha} (1-1/\gamma_w \rho_d)} \tag{A.32}$$

$$\alpha_d = \frac{1-\alpha}{\gamma_w \alpha} \beta_d \tag{A.33}$$

$$\beta_g = \frac{s_y + (1-s_w) \rho_g}{s_y + (1-s_w) \rho_g + \frac{1-\alpha}{\alpha} (1-1/\gamma_w \rho_g)} \tag{A.34}$$

$$\alpha_g = \frac{1-\alpha}{\gamma_w \alpha} \beta_g. \tag{A.35}$$

By substituting (A.25) into (A.24), we can obtain a first-order difference equation in output. This forward-looking difference equation implies that inflation and output will be determinate if and only if the following condition obtains:

$$s_y - (1-s_w) > \frac{1-\alpha}{\alpha} \frac{1-\gamma_w}{\gamma_w}.$$

When $s_w = 1$, this condition is the same determinacy condition as discussed in the main text. When the above condition holds, there is a unique rational expectations equilibrium in the deflation steady state. The left-hand side is always positive, so in the case of perfect price rigidity (i.e., $\gamma_w = 1$), this condition is satisfied and the deflation steady state is locally unique.

D Properties of Secular Stagnation Equilibrium

Here we provide a formal proof for various properties of the secular stagnation equilibrium described in the body of the text.

Proposition 1. *If $\gamma > 0$, $\Pi^* = 1$, and $i^* = r^f < 0$, then there exists a unique determinate secular stagnation equilibrium.*

Proof. Under the assumptions of the proposition, the inflation rate at which the zero lower bound binds, given in equation (29), is strictly greater than unity. Let Y_{AD} denote the level of output implied by the aggregate demand relation and Y_{AS} denote the level of output implied by the

aggregate supply relation. For gross inflation rates less than unity, Y_{AD} and Y_{AS} are given by

$$Y_{AD} = D + \psi\Pi \quad (\text{A.36})$$

$$Y_{AS} = \left(\frac{1 - \frac{\gamma}{\Pi}}{1 - \gamma} \right)^{\frac{\alpha}{1-\alpha}} Y^f, \quad (\text{A.37})$$

where $\psi = \frac{1+\beta}{\beta} (1+g) D > 0$. The AD curve is upward sloping because $\Pi < 1 < \Pi_{kink}$ under our assumptions and, therefore, the zero lower bound binds.

When $\Pi = \gamma$, $Y_{AD} > Y_{AS} = 0$. When $\Pi = 1$, the real interest rate equals $\Pi^{-1} = 1 > r^f$. Thus, when $\Pi = 1$, $Y_{AD} < Y^f$. Furthermore, from the equations above, when $\Pi = 1$, $Y_{AS} = Y^f$. Therefore, it must be the case that $Y_{AD} < Y_{AS}$ when $\Pi = 1$. Since the AS and AD curves are both continuous functions of inflation, it must be the case that there exists a Π_{ss} at which $Y_{AD} = Y_{AS}$.

To establish uniqueness, we first assume that there exist multiple distinct values of Π_{ss} at which $Y_{AD} = Y_{AS}$. In inflation-output space (output on the x-axis), the AS curve lies above the AD curve when inflation equals γ and the AS curve lies below the AD curve for inflation at unity — see equation (26). Thus, if multiple steady states exist, given that AS is a continuous function, there must exist at least three distinct points at which the AS and AD curves intersect.

At the first intersection point, the slope of the AS curve crosses the AD line from above and, therefore, at the second intersection the AS curve crosses the AD curve from below. Since the AD curve is a line, the AS curve as a function of output is locally convex in this region. Similarly, between the second and third intersections, the AS curve is locally concave. Thus, given an increase in Y , the AS curve must first have a positive second derivative followed by a negative second derivative.

We compute the second derivative of inflation with respect to output of the AS curve and derive the following expression:

$$\frac{d^2\Pi}{dY^2} = G(Y) \left((1 + \phi)(1 - \gamma) \left(\frac{Y}{Y^f} \right)^\phi + (\phi - 1) \right) \quad (\text{A.38})$$

$$G(Y) = \frac{\phi\gamma(1 - \gamma) \left(\frac{Y}{Y^f} \right)^\phi}{Y^2 \left(1 - (1 - \gamma) \left(\frac{Y}{Y^f} \right)^\phi \right)} \quad (\text{A.39})$$

$$\phi = \frac{1 - \alpha}{\alpha}. \quad (\text{A.40})$$

As can be seen, over the region considered, the function $G(Y)$ is positive and, therefore, the convexity of the AS curve is determined by the second term. This term may be negative if $\phi < 1$, but this expression is increasing in Y between 0 and Y^f . Therefore, the second derivative cannot switch signs from positive to negative. Thus, we have derived a contradiction by assuming multiple steady states. Therefore, there must exist a unique intersection point.

As established before, it must be the case that the AS curve has a lower slope than the AD

curve at the point of intersection. The slope of the AS curve is

$$\frac{d\Pi}{dY} = \frac{1 - \alpha}{\alpha} \frac{1}{\gamma} \frac{\Pi}{Y} (\Pi - \gamma). \quad (\text{A.41})$$

If the slope of the AS curve is less than the slope of the AD curve at the intersection point, then it must be the case that

$$\begin{aligned} \frac{1 - \alpha}{\alpha} \frac{\Pi}{Y} \left(\frac{\Pi}{\gamma} - 1 \right) &< \psi^{-1} \\ \frac{1 - \alpha}{\alpha} \frac{\psi \Pi}{Y} \left(\frac{\Pi}{\gamma} - 1 \right) &< 1 \\ \frac{1 - \alpha}{\alpha} \frac{Y - D}{Y} \left(\frac{\Pi}{\gamma} - 1 \right) &< 1 \\ s_y \frac{\alpha}{1 - \alpha} + 1 &> \frac{\Pi}{\gamma} \\ \frac{\gamma}{\Pi} \left(s_y \frac{\alpha}{1 - \alpha} + 1 \right) &> 1. \end{aligned}$$

The last inequality here is precisely the condition for determinacy discussed in Section 6. Thus, the unique secular stagnation steady state is always determinate as required. \square

E Calvo Pricing

In this section, we modify the aggregate supply block of our model to consider product market frictions instead of downward nominal wage rigidity. As in our baseline model, we assume that middle-aged households supply a constant level of labor \bar{L} . However, wages adjust frictionlessly to ensure that labor is fully employed in all periods.

Monopolistically competitive firms produce a differentiated good l and set nominal prices periodically. Households consume a Dixit-Stiglitz aggregate of these differentiated goods, implying that each firm faces the following demand schedule:

$$y_t(l) = Y_t \left(\frac{p_t(l)}{P_t} \right)^{-\theta} \quad (\text{A.42})$$

$$P_t = \left(\int p_t^{1-\theta} dl \right)^{\frac{1}{1-\theta}}, \quad (\text{A.43})$$

where θ is the elasticity of substitution in the Dixit-Stiglitz aggregator and P_t is the price level of the consumption bundle consumed by households. Production depends only on labor, and labor market clearing requires total labor demand to equal labor supply:

$$y_t(l) = L_t(l) \quad (\text{A.44})$$

$$\bar{L} = \int L_t(l) dl. \quad (\text{A.45})$$

Combining labor market clearing with the demand for each product (A.44), we can derive an expression for output in terms of exogenous labor supply and a term that reflects losses due to misallocation from pricing frictions:

$$Y_t = \frac{\bar{L}}{\Delta_t} \quad (\text{A.46})$$

$$\Delta_t = \int \left(\frac{p_t(l)}{P_t} \right)^{-\theta} dl, \quad (\text{A.47})$$

Under Calvo pricing, firms are periodically able to reset their prices and will choose a single optimal reset price irrespective of the time since their last price change. Under the Calvo assumption, we can derive dynamic expressions for inflation and the misallocation term Δ_t in terms of the reset price p_t^* :

$$1 = \chi \Pi_t^{\theta-1} + (1 - \chi) \left(\frac{p_t^*}{P_t} \right)^{1-\theta} \quad (\text{A.48})$$

$$\Delta_t = \chi \Pi_t^\theta \Delta_{t-1} + (1 - \chi) \left(\frac{p_t^*}{P_t} \right)^{-\theta}, \quad (\text{A.49})$$

where χ is the Calvo parameter — the fraction of firms that do not adjust prices in the current period. Equations (A.46), (A.48), and (A.49) collectively define the aggregate supply block of the model with monopolistic competition and price friction. The IS curve and monetary policy rule close the model.

We can derive the long-run Phillips curve by combining the steady-state cases of equations (A.46), (A.48), and (A.49). The steady-state AS curve is given below:

$$Y = \bar{L} \frac{1 - \chi \Pi^\theta}{1 - \chi} \left(\frac{1 - \chi}{1 - \chi \Pi^{\theta-1}} \right)^{\frac{\theta}{\theta-1}}$$

F Incorporating Money

In this section, we extend our baseline model to explicitly introduce a role for money and a money demand function. Households now have preferences for real money balances to capture the value of money in easing transactions frictions. For simplicity, we assume that households only hold money in the middle period of life and utility over real money balances is separable. We also assume that there exists a level of real money balances \bar{m} at which households are satiated: that is, $v'(\bar{m}) = 0$.

We specify and characterize the household's problem in the case of income received in the middle period only and taxes paid in all periods. For household i , the objective function and

budget constraints are given below:

$$\max_{C_t(i), C_{t+1}(i), M_{t+1}(i), C_{t+2}(i)} \mathbb{E}_t \left\{ \log(C_t(i)) + \beta \log(C_{t+1}(i)) + \beta v(M_{t+1}(i)) + \beta^2 \log(C_{t+2}(i)) \right\} \quad (\text{A.50})$$

$$\text{s.t.} \quad C_t(i) = B_t(i) - T_t^y \quad (\text{A.51})$$

$$C_{t+1}(i) = Y_{t+1} - T_{t+1}^m + B_{t+1}(i) - M_{t+1}(i) - \frac{(1+i_t)}{\Pi_{t+1}} B_t(i) \quad (\text{A.52})$$

$$C_{t+2}(i) = \frac{1}{\Pi_{t+1}} M_{t+1}(i) - T_{t+2}^o - \frac{(1+i_{t+1})}{\Pi_{t+2}} B_{t+1}(i) \quad (\text{A.53})$$

$$B_{t+j}(i) \leq \mathbb{E}_{t+j}(1+r_{t+j+1}) D_{t+j} \quad \text{for } j = 0, 1, \quad (\text{A.54})$$

where $M_{t+1}(i)$ are real money balances demanded by household i . Money earns zero interest and carries a liquidity premium on bonds away from the zero lower bound. The household's money demand condition is given below:

$$C_{t+1}(i) v'(M_{t+1}(i)) = \frac{i_{t+1}}{1+i_{t+1}}. \quad (\text{A.55})$$

The above expression implicitly defines a money-demand equation. The given monetary policy rule determines real money balances via (A.55). Given a representative middle-aged cohort and given that only the middle aged demand money, we can drop the i , and money demand per middle-generation household is

$$M_t = v'^{-1} \left(\frac{i_t}{1+i_t} \frac{1}{C_t^m} \right). \quad (\text{A.56})$$

The issuance of money by the central bank modifies the government's budget constraint in (A.19). The government's consolidated budget constraint expressed in real terms is given below:

$$B_t^g + M_t + T_t^y(1+g_t) + T_t^m + \frac{1}{1+g_{t-1}} T_t^o = G_t + \frac{1}{1+g_{t-1}} \left(\frac{1+i_{t-1}}{\Pi_t} B_{t-1}^g + \frac{1}{\Pi_t} M_{t-1} \right). \quad (\text{A.57})$$

We assume a fiscal policy that adjust taxes T_t^y , T_t^m , and T_t^o to keep the government's consolidated liabilities, $M_t + B_t^g$, at some constant target level. In particular, this means that, in periods of deflation, the nominal stock of government liabilities is being reduced in proportion to the fall in the price level. In the steady state of a stagnation equilibrium featuring a constant rate of deflation, nominal government liabilities are contracting at the rate of deflation. Under a fiscal policy that keeps real government liabilities constant, the presence of money does not materially alter our conclusions.

G Productivity Growth and Hysteresis

In this section, we extend the baseline model to include trend productivity growth and offer a simple extension to model hysteresis — where output gaps feed back onto the productivity growth

process. The extension of the model to include productivity growth does not greatly alter the basic features of the model, but will allow the model to better match the dynamics of real GDP per capita. The aggregate demand block is still summarized by an asset market clearing that relates middle-generation income and the real interest rate:

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} \quad (\text{A.58})$$

$$= \frac{1 + \beta}{\beta} \frac{(1 + g_t) \tilde{D}_t}{\tilde{Y}_t^m - \tilde{D}_{t-1} \frac{A_{t-1}}{A_t}}, \quad (\text{A.59})$$

where $\tilde{X}_t = X_t/A_t$ are detrended variables. So long as the collateral constraint grows at the same rate as productivity growth, there exists a balanced growth path with a constant real interest rate in steady state and quantities growing at the rate of productivity growth. Relative to the AD curve in the baseline model, the only difference is that higher productivity growth increases saving by lowering the value of debt incurred when young.

Trend productivity growth also impacts the wage norm as the flexible-price real wage rises over time. Now, deflation must exceed the rate of productivity growth for the wage norm to bind. More generally, if nominal wages are indexed to the inflation target, the shortfall of inflation below target must exceed the growth rate of productivity. The wage norm indexed to inflation is given below along with the flexible-price real wage:

$$W_t = \max \left\{ \gamma \Pi^* W_{t-1} + (1 - \gamma) P_t w_t^{flex}, P_t w_t^{flex} \right\} \quad (\text{A.60})$$

$$w_t^{flex} = \alpha A_t \bar{L}^{\alpha-1}. \quad (\text{A.61})$$

Real wages and output can be detrended by productivity growth to obtain stationary variables. Trend stationary real wages are given by the following expression:

$$\tilde{w}_t = \max \left\{ \gamma \Pi^* \frac{\tilde{w}_{t-1} A_{t-1}}{\Pi_t A_t} + (1 - \gamma) \tilde{w}^{flex}, \tilde{w}^{flex} \right\}, \quad (\text{A.62})$$

where \tilde{w}_t are detrended real wages and $\tilde{w}^{flex} = \alpha \bar{L}^{\alpha-1}$. In steady state, the wage norm binds when $\Pi^* > \bar{\Pi} \mu$, where μ is the steady-state growth rate of productivity. The AS curve can be derived by substituting the following expressions for output and the full-employment level of output:

$$\tilde{w}_t = \alpha \tilde{Y}_t^{\frac{\alpha-1}{\alpha}} \quad (\text{A.63})$$

$$\tilde{w}^{flex} = \alpha \tilde{Y}_e^{\frac{\alpha-1}{\alpha}}. \quad (\text{A.64})$$

Equations (A.59), (A.62), and (A.63) along with the Fisher relation and the monetary policy rule jointly determine $\left\{ \tilde{Y}_t, \tilde{w}_t, r_t, \Pi_t, i_t \right\}$.⁶⁷

⁶⁷Income for the middle-aged household \tilde{Y}_t^m is assumed to be a constant fraction of total income.

The modified equilibrium conditions presented in this section have simply taken productivity growth as an exogenous process. One possibility is that prolonged output gaps feed back into slower productivity growth. Productivity growth could be related to the output gap simply by positing a simple feedback process. This feedback process represents a reduced-form mechanism whereby prolonged output gaps reduced productivity growth by, for example, reducing investment, technology adoption, and expenditures in public or private research and development, or by limiting the degree of firm entry. We posit the following feedback rule:

$$\frac{A_t}{A_{t-1}} = \mu_0 \left(\frac{\tilde{Y}_t}{\tilde{Y}_{fe}} \right)^\kappa,$$

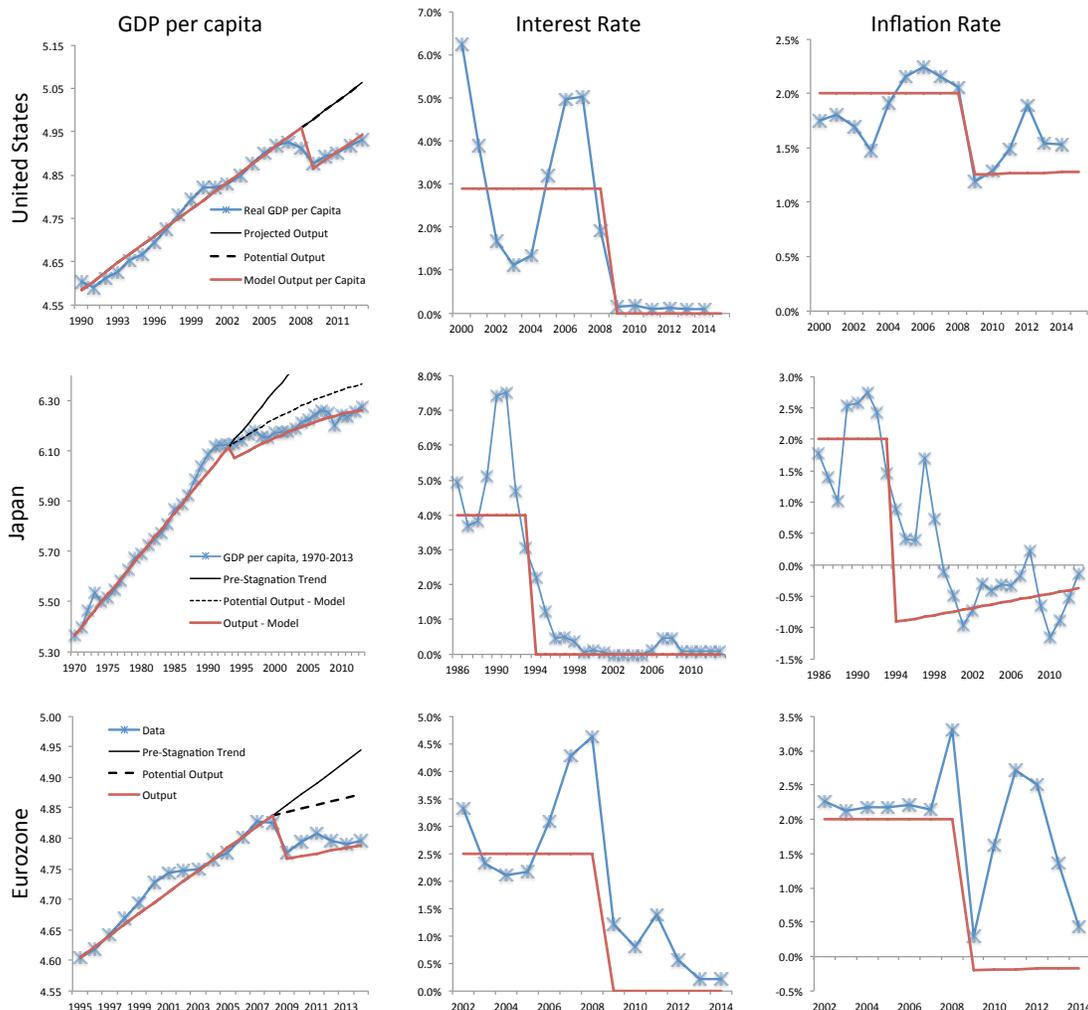
where κ determines the strength of the hysteresis effect.

H Quantitative Calibration: US, Europe, and Japan

The simple three-period OLG model captures the salient features of secular stagnation: persistently low levels of inflation and interest rates and below-trend output. Though our model is obviously highly stylized, we still think it is of value to explicitly parameterize it and examine its capacity to explain recent stagnation episodes. Figure A.1 displays the key series whose behavior our theory is trying to explain. In all these episodes, we have witnessed a drop in the short-term nominal interest rate to close to zero and a decline in inflation below the implicit inflation target of the central bank. The size of the fall in inflation has varied. Japan has experienced outright deflation, but in the US and Europe, deflation was very short-lived while inflation has remained persistently below the target of the central banks. The size of the output gap is more controversial. Here, we calibrate the model to recent estimates, but, conceptually, even without any remaining output gap, low natural rates of interest pose an ongoing challenge for monetary policy.

To evaluate the capacity of our model to match the behavior of GDP per capita, interest rates, and inflation, we choose parameters to target steady-state output gaps and deviations of inflation from target for each region. As discussed in the previous section, we modify the equilibrium conditions of our model to allow for trend productivity growth and wage indexation to the inflation target. In the case of Japan and the Eurozone, we assume that trend productivity growth fell at the onset of secular stagnation to fit the slower trend rate of GDP growth in each region. We log-linearize the model around the secular stagnation steady state, and plot transition paths for output per capita, nominal interest rates, and inflation in response to a shock to the collateral constraint in each region. Given the three-period OLG structure, each period is taken to be 20 years.

Figure A.1: Data versus model transition paths: US, Japan, and Eurozone



H.1 US Calibration

Table 1 shows parameters annualized for the US (hence in computing steady state, these values need to be converted to 20 years). We pick an inflation target of 2% per year, consistent with the Federal Reserve’s implicit inflation target. Moreover, we assume that the wage norm is indexed at the inflation rate times the growth rate of productivity. In this case, the wage norm binds whenever inflation falls below target. We assume trend productivity growth of 2.1% per year to match average GDP per capita growth from 1990-2007. The rate of time preference β is set at a conventional value of 0.96. We set US population growth of 0.7% per year based upon UN population projections. We set α at 0.7 to match the labor share.

The two novel parameters to choose are the degree of wage rigidity determined by γ and the collateral constraint on the young D . We must also set an initial value for the collateral constraint

Table A.1: Parameter values for the US

Description	Parameter	Value
Population growth	g	0.7%
Discount rate	β	0.96
Labor share	α	0.7
Inflation target	Π^*	2.0%
Pre-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ini}} \frac{D_{ini}}{Y}$	126%
Post-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ss}} \frac{D_{ss}}{Y}$	100%
Wage adjustment	γ	0.94
Youth income share	Y_y/Y	0.4

as the initial point for the determining the transition path of the economy. To find realistic values of the collateral constraint, it is necessary to include other sources of demand. We do this by assuming that some income is received by the young, the government issues public debt, and the government absorbs some output through purchases that are financed by taxes levied on the middle aged. Government spending as a percentage of GDP is set at 20%, and public debt as a percentage of annual GDP is set at 100%. The government budget constraint determines the level of taxes.

Given government spending, taxes, and public debt, the collateral constraint D , the income distribution between young and old, and the wage rigidity parameter are set to match the following targets: an output gap of 13%, an inflation rate of 1.4%, and household debt of 100% of GDP in 2014. The output gap represents the deviation of output per capita from its 1990-2007 trendline. The inflation rate is based on the growth rate of core PCE in 2014. The household debt target is taken from the Federal Reserve Flow of Funds and is set at 100% of GDP in order to match the sum of loans to households and nonprofits (13.7 trillion in 2014) and loans to nonfinancial non-corporate businesses (\$4.4 trillion in 2014). Finally, the pre-shock level of the collateral constraint is set to match the average nominal interest of 2.9% between 2001 and 2007.

How reasonable are the values we set for these parameters? The implied weight on last-period wages (γ multiplied by the inflation target and productivity growth) is 0.98 per year. [Schmitt-Grohé and Uribe \(2016\)](#) provide an authoritative overview of the evidence on downward nominal wage rigidity and estimate a very similar wage adjustment curve to ours. The implied income distribution implies that 60% of national income is received by the middle aged while 40% of national income is received by the young. Our calibration implies an initial household debt level of 126% of GDP, which is quite close to the pre-2008 level of loans to households and small businesses of 121% of GDP. Therefore, the collateral shock we choose is quite close in magnitude to the contraction in lending experienced during the Great Recession.

Table A.2: Parameter values for Japan and Eurozone

Description	Parameter	Japan	Eurozone
Population growth	g	0%	0%
Discount rate	β	0.96	0.96
Labor share	α	0.7	0.7
Inflation target	Π^*	2.0%	2.0%
Pre-shock collateral	D_0	0.37	0.27
Post-shock collateral	D_1	0.23	0.21
Wage adjustment	γ	0.90	0.89
Hysteresis elasticity	κ	4.8	3.0

The top row of Figure A.1 plots the transition paths for GDP per capita, inflation, and the interest rate over the relevant data series. A collateral shock can indeed explain the drop in inflation and the interest rate in our model. The nominal interest rate drops to the zero lower bound on impact while the inflation rate slightly overshoots, gradually rising towards its steady-state value of 1.4%. Output per capita falls by less on impact than in the long-run. Therefore, the output gap is widening slightly along the transition path. Indeed, this behavior is consistent with that of US GDP per capita, which is drifting slightly away from the pre-recession trendline. Broadly speaking, our fairly simple model with a shock that matches the magnitude of the contraction in lending can capture the behavior of GDP per capita, inflation, and interest rates in the US during the Great Recession.

H.2 Eurozone and Japan Calibration

With hysteresis effects, we can calibrate the model to analyze stagnation episodes in the Eurozone and Japan. As with the US calibration, we set the rate of time preference β and the labor share α to standard values. Population growth in both regions is set to zero to reflect recent population trends. The inflation target is set at 2% in both regions. The key remaining parameters are the collateral constraint D , the degree of wage rigidity γ , and the hysteresis parameter κ . For both regions, we set these parameters to match the output gap, inflation rate, and change in trend output growth. The pre-shock level of the collateral constraint is set to match the nominal interest rate prior to the stagnation episode.⁶⁸ In each case, output is normalized to unity.

In the case of Japan, we target an output gap of 10%, a rate of deflation of -0.25%, and a reduction in trend productivity growth from 3.3% to 0.7%. The last value is determined by trend GDP per capita growth from 1970 to 1994 and from 1994 to 2008, respectively. Data on real GDP

⁶⁸For this calibration, we set government debt/spending to zero, assume income is received only in the middle years, and do not seek to match the collateral constraint to measures of household debt in the Eurozone or Japan.

per capita come from the World Economic Indicators. The output gap is based on estimates from [Hausman and Wieland \(2014\)](#), and the inflation rate is the growth rate of Japan’s CPI as reported in the OECD main economic indicators. As Table A.2 shows, the implied degree of wage rigidity is low (relevant parameters are annualized). Because of hysteresis effects in our model, real wages are relatively flexible in comparison to the values found in [Schmitt-Grohé and Uribe \(2011\)](#). Hysteresis effects limit downward pressure on prices due to the paradox of toil, and therefore limit the degree of deflation despite a persistent output gap.

In the case of the Eurozone, we target an output gap of 10%, an inflation rate of 0%, and a slowdown in productivity growth from 1.8% to 0.2%. The output gap is chosen to best fit the observed decline in output per capita during the Great Recession. The inflation rate reflects the growth rate of the harmonized CPI in the Eurozone during 2014. The slowdown in productivity growth is estimated by simple trendlines for GDP per capita growth from 1995 to 2008 and 2009 to 2014, respectively. The degree of wage rigidity in the Eurozone is comparable to values for Japan, and the hysteresis effects are about half as strong, reflecting the more modest slowdown in trend GDP growth before and after the stagnation episode. As in the US, the output gap in the Eurozone is gradually increasing over time along the transition path, consistent with the behavior of EU GDP per capita post Great Recession.

I Derivation of the Quantitative Life Cycle Model

I.1 Demographics and Labor Supply

The economy consists of a large number of households with identical utility parameters. Households enter economic maturity at age 26, after which they work, consume, have children, and participate in markets. Finally they die at age J , which we take to be 81 years. Households have children at age 26, and the population growth rate is determined by the total fertility rate (Γ) of every family. Individuals have a probability of dying stochastically before reaching maximum age J . The probability of surviving between age j and $j + 1$ is denoted by s_j .⁶⁹ The unconditional probability of reaching age j is denoted with a superscript s^j .⁷⁰

The total population alive at any given time, N_t , is the sum of the population of the individual ages, $N_{j,t}$. The population size of a given generation $N_{j,t}$ is the population of the generation the previous year that has survived. But what about at age 26 (the first year of economic maturity in our model)? The total population of a generation entering economic majority at time t is equal to the total population of their parents when they entered economic maturity at time $t - 25$, times the total fertility rate of their parents’ generation. Thus population evolves in the model according to

⁶⁹Age-specific survival rates may also vary over time t ; however, for notational simplicity we omit these additional subscripts.

⁷⁰This can be calculated as the production on one-period survival probabilities: $s^j = \prod_{m=26}^{j-1} s_m$.

the law of motion given below:

$$N_t = \sum_{j=26}^J N_{j,t}$$

$$N_{j+1,t+1} = s_{j,t} N_{j,t} \text{ for } j \in \{27, J\}$$

$$N_{26,t} = N_{26,t-25} * \Gamma_{26,t-25}.$$

The total fertility rate along with the age at which households have children determines the rate of population growth. In a steady state, for a given total fertility rate Γ , the rate of population growth n is equal to

$$n = \Gamma^{\frac{1}{25}} - 1. \quad (\text{A.65})$$

In a steady state with a rate of population growth n , each generation is $(1+n)$ times larger than the previous. Thus the total population size can be calculated as (normalizing the total population to 1)

$$N = \sum_{j=26}^J N_j \quad (\text{A.66})$$

$$N_{j+1} = s_j \frac{N_j}{(1+n)} \text{ for } j \in \{27, J\} \quad (\text{A.67})$$

$$N_{26} = \bar{N}_{26}, \quad (\text{A.68})$$

where \bar{N}_{26} is the normalized population in period 1, equal to

$$\bar{N}_1 = \frac{1}{\sum_{j=26}^J \frac{s^j}{(1+n)^{j-1}}}.$$

Each household has an identical schedule of lifetime exogenous labor productivity, or human capital, denoted by hc_j , which varies by age. Households receive no wage income after retirement, which in our model occurs after age 65 (model age 40). We assume labor is supplied inelastically. Therefore, wage income at full employment is equal to the wage multiplied by the individual age-specific labor productivity hc_j net of labor taxes $(1 - \tau^w)$. In a secular stagnation, labor demand falls below labor supply, and labor is rationed proportionally for each cohort:

$$L_t^s = \sum_{j=26}^J N_{j,t} hc_j. \quad (\text{A.69})$$

I.2 Household's Problem

Households receive utility from two sources: (i) consumption, which is given by a time-separable constant elasticity of substitution (CES) utility function $u(\cdot)$ with an elasticity of intertemporal substitution parameter ρ , and (ii) bequests, which are divided equally among all descendants.

The bequest motive is also characterized by a CES function, $v(\cdot)$, whose argument is the amount of bequests left *per descendent*⁷¹, denoted by x . The utility of bequests is multiplied by a parameter $\mu \geq 0$ that determines the strength of the bequest motive. Denoting consumption of a household of age j at time t by $C_{j,t}$ and the discount rate by β , a household born at time t then maximizes its lifetime expected utility:

$$U_t = \sum_{j=26}^J s^j \beta^{j-1} u(C_{j,t+j-1}) + s^J \beta^{J-1} \mu v(x_{J,t+J-1}). \quad (\text{A.70})$$

The household of age j can purchase or borrow a real asset $a_{j,t}$ at price ξ_t at time t , which is used as productive capital. At time $t + 1$, it will pay the return r_{t+1}^k , which is the rental rate of capital, and has a resell value (net of depreciation) $(1 - \delta)\xi_{t+1}$, where ξ_{t+1} is the relative price of capital in terms of the consumption good. All households, prior to the terminal period, participate in a perfectly competitive one-period annuity market as in [Ríos-Rull \(1996\)](#). For a given cohort, the assets of households that die are distributed evenly to the surviving members of the cohort. Households also receive income from the pure profits from firms, denoted by $\Pi_{j,t}$.⁷² Finally, the household may receive a bequest $q_{j,t}$. Bequest received are assumed to be zero at all times except at age $J - 24$. Bequests given are zero at all times, except at age J .

The flow budget constraint of a household of age j at time t can then be written as

$$c_{j,t} + \xi_t a_{j+1,t+1} + \Gamma_{26,t-j+26} \cdot x_{j,t} = (1 - \tau^w) w_t h c_j + \Pi_{j,t} + [r_t^k + \xi_t (1 - \delta)] \left(a_{j,t} + q_{j,t} + \frac{1 - s_j}{s_j} a_{j,t} \right). \quad (\text{A.71})$$

Households may wish to borrow against future income and face a borrowing constraint of the same form as in the three-period model⁷³:

$$a_{j,t} \geq \frac{D_t}{1 + r_t}. \quad (\text{A.72})$$

There is one further complication with bequests. Since mortality is stochastic, not all parents will survive to their maximum age and be able to give bequests to their children. Thus, absent any insurance markets, there would be stochastic within-generation inequality, as some households would receive bequests and others would not. In order to remove this channel, we assume all generations participate in a form of bequest insurance markets. At the maximum age, all surviving members of a generation pool their optimal bequests and divide them equally among their surviving children. Thus the relationship between bequests given (at age J) and those received by children (at model age k) at time t is given by

$$q_{k,t} = \frac{N_{J,t-1} x_{J,t-1} \cdot \Gamma_{26,t-J+26}}{N_{k,t}}. \quad (\text{A.73})$$

⁷¹Thus the total size of the bequest left by households is the bequest x multiplied by the fertility of the household.

⁷²We assume that these are distributed proportional to labor income.

⁷³We assume that D_t grows at the rate of productivity growth and is expressed in terms of consumption goods to ensure balanced growth.

I.3 Final Goods Firms

There exists a continuum of final goods firms of type i of measure one that costlessly differentiate an intermediate good and resell to the representative household. The final good composite is the CES aggregate of these differentiated final goods:

$$Y_t = \left[\int_0^1 y_t^f(i)^{\frac{\theta_t-1}{\theta_t}} di \right]^{\frac{\theta_t}{\theta_t-1}}.$$

These firms are monopolistically competitive, set prices in each period, and face a demand curve that takes the following form:

$$y_t^f(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\theta_t},$$

where θ_t is a time-varying shock to the firm's market power. An increase in θ_t decreases a firm's market power and lowers equilibrium markups. Each final goods producer uses y_t^m of intermediate goods to produce output, according to a linear technology function $y_t^f = y_t^m$. A final goods firm chooses real prices $\frac{p_t(i)}{P_t}$ and $y_t^f(i)$ to maximize real profits, subject to the production constraint:

$$\begin{aligned} \max \quad & \frac{p_t(i)}{P_t} y_t^f(i) - \frac{p_t^{int}}{P_t} y_t^f(i) \\ \text{subject to} \quad & y_t^f(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\theta_t}, \end{aligned}$$

where $\frac{p_t^{int}}{P_t}$ is the price of the intermediate good taken as given by the firm.

The optimality condition for the real price of the firm's good is a time-varying markup over the price of the intermediate good:

$$\frac{p_t(i)}{P_t} = \frac{\theta_t}{\theta_t - 1} \frac{p_t^{int}}{P_t} \tag{A.74}$$

The nominal price index implies the following expression for the price of intermediate goods:

$$P_t = \left(\int p_t(i)^{1-\theta_t} di \right)^{\frac{1}{1-\theta_t}}.$$

Since the price of the intermediate good is the same, all final goods firms make the same pricing decisions (no pricing frictions), and thus $p_t(i) = P_t$, yielding

$$\frac{p_t^{int}}{P_t} = \frac{\theta_t - 1}{\theta_t}. \tag{A.75}$$

With a retail elasticity of substitution of θ_t , aggregate profits in equilibrium will be given by

$$\Pi_t = \frac{Y}{\theta}. \tag{A.76}$$

Profits from monopolistically competitive firms are distributed according to wage income. In equilibrium, the total distributed profit must equal total profits:

$$\frac{Y_t}{\theta_t} = \sum_{j=26}^J N_{j,t} \Pi_{j,t} \quad (\text{A.77})$$

I.4 Intermediate Goods Firms

There exists a perfectly competitive intermediate goods sector that sells its production to the final goods sector at real price $\frac{p_t^{int}}{P_t}$. These firms operate a CES production function with an elasticity of substitution σ , hire labor, and rent capital. The representative intermediate goods firms maximize static real profits given the following production function:

$$\Pi_t^{int} = \max \frac{p_t^{int}}{P_t} Y_t - w_t L_t - r_t^k K_t \quad (\text{A.78})$$

$$Y_t = \left(\alpha (A_{k,t} K_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) (A_{l,t} L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \quad (\text{A.79})$$

Labor productivity $A_{L,t}$ grows at each period at the rate of g_t .

The first-order conditions that determine labor and capital demand are given below:

$$w_t = \frac{p_t^{int}}{P_t} (1-\alpha) A_{l,t} \left(\frac{Y_t}{L_t} \right)^{\frac{1}{\sigma}} \quad (\text{A.80})$$

$$r_t^k = \frac{p_t^{int}}{P_t} \alpha A_{k,t} \left(\frac{Y_t}{K_t} \right)^{\frac{1}{\sigma}}. \quad (\text{A.81})$$

The risk-free real rate is related to the return on capital by a standard no-arbitrage condition:

$$1 + r_t = \frac{r_t^k + (1-\delta) \xi_t}{\xi_{t-1}}. \quad (\text{A.82})$$

I.5 Relative Price of Capital Goods

Investment-specific productivity is defined as the amount of capital goods that can be produced with one unit of investment. In particular, we will assume that capital goods are produced by perfectly competitive firms in an investment-specific production sector, which converts the final composite goods into capital goods. These firms maximize the following profit function subject to a linear production function:

$$\Pi^K = \xi_t * K_t - Y_t^K \quad (\text{A.83})$$

$$K_t = z_t Y_t^K, \quad (\text{A.84})$$

where ξ_t is the relative price of capital goods. Here, z_t is the productivity of the capital-producing sector. The zero profit conditions mean that, in equilibrium, $\xi_t = \frac{1}{z_t}$. Thus, the more productive the investment goods sector is, the lower the relative price of capital goods.

The aggregate capital stock evolves according to the standard law of motion:

$$K_{t+1} = (1 - \delta)K_t + \frac{I_t}{\xi_t}, \quad (\text{A.85})$$

where δ is the rate of depreciation, I_t is investment, and ξ_t is the relative price of capital goods.

The government spends an exogenous G_t and may accumulate debt. The budget constraint is given by

$$b_{g,t} = G_t + (1 + r_t)b_{g,t-1} - T_t, \quad (\text{A.86})$$

where the total tax bill is collected with labor income tax. For the purpose of our simulations, fiscal policy will be specified as an exogenous sequence of two variables: government debt to GDP and government spending to GDP. The wage income tax will then be endogenously determined by the model in order for the budget constraint to hold.

We have economized on notation by omitting real and nominal bonds as assets; they enter in the same way as in the simpler model, so there is both a well-defined real interest rate r_t on a risk-free one-period bond and a nominal interest rate i_t .

I.6 Equilibrium

A competitive equilibrium is a set of household allocations: $\{c_{j,t}, a_{j,t}, \Pi_{j,t}, q_{j,t}, x_{j,t}\}_{j=26}^J\}_{t=0}^\infty$, a set of aggregate quantities $\{Y_t, K_t, L_t, \Pi_t\}_{t=0}^\infty$, a set of prices $\{w_t, r_t^k, r_t, \frac{p_t^{int}}{P_t}, \xi_t\}_{t=0}^\infty$, a set of government variables $\{b_t, \tau_t^w, \tau_t^r\}_{t=0}^\infty$, and a set of exogenous processes $\{N_{j,t}, p_{j,t+1}\}_{j=0}^J\}_{t=0}^\infty$ that jointly satisfy:

1. Consumption and bequests maximizes (A.70) subject to (A.71) and (A.72).
2. Asset holdings satisfy (A.71), with $a_1 = 0$ given.
3. Profits are distributed proportionally to labor income.
4. Bequests received equal bequests given by the surviving parents according to (A.73).
5. Output is given by aggregate production function (A.78).
6. Population by age group is given by (A.66).
7. Aggregate labor supply is given by (A.69).
8. Aggregate profits are given by (A.76).
9. There are perfect factor markets, and thus (A.81), (A.80), and (A.82) hold.
10. Markup condition: (A.74).
11. The government satisfies budget equation (A.86).

12. Asset markets clear, and thus

$$\sum_{j=1}^J N_{j,t} \xi_t a_{j,t} = \xi_t K_t + b_t \quad \forall t.$$

I.7 Stationary Equilibrium

A steady-state equilibrium is the same as the above definition, except that all variables are constant rather than subscripted by time. With productivity and population growth greater than zero, however, the economy is not stationary. However, since preferences are of the CRRA variety, the economy can be rewritten as stationary by applying a transformation. All cohort level variables are divided by $(1 + g)^t$, aggregate variables are divided by $(1 + g)^t(1 + n)^t$, and wages are divided by $(1 + g)^t$.

In a steady-state equilibrium, there are $4 * J + 12$ unknowns: J consumption demands c_j , J asset demands a_j , J shares of aggregate profits Π_j , J age-specific population sizes N_j , 1 bequest given x , 1 bequest received q , aggregate output Y , aggregate capital K , aggregate labor L , aggregate profits Π , wage rate w , return on capital r^k , interest rate r , price of intermediate goods $\frac{p^{int}}{P}$, relative price of capital ξ , and tax rate τ^w .

There are also $4 * J + 12$ equations:

1. (J-1) Euler equations
2. (1) Optimal bequest given equation
3. (1) Equation determining size of bequest received
4. (J) Flow budget equations
5. (J) Profit share equations
6. (1) Initial assets are zero
7. (J) Population equations
8. (1) Aggregate production function
9. (1) Asset market clearing condition
10. (1) Aggregate labor supply equation
11. (1) Aggregate profit equation
12. (1) Wage equation
13. (1) Price of intermediate goods equation

14. (1) Rental rate of capital equation
15. (1) Relative price of capital is exogenously given
16. (1) Interest rate equation
17. (1) Government budget equation

J Computational Method

Solving the quantitative model is implemented much along the lines as described in chapter 4 of [Auerbach and Kotlikoff \(1987\)](#). We begin by describing how to solve for variables in the stationary equilibrium. In solving for the stationary equilibrium, the algorithm begins with a guess for a subset of endogenous variables. For the purposes of the rest of the iteration, this subset of variables is treated as exogenous. This simplification makes the resulting system easier to solve for the endogenous variables, including the variables for which guesses were made.

We begin with a guess for the aggregate capital stock, the wage tax, and bequests received. Given the aggregate capital and labor stock, we can calculate interest rates and wages, as well as aggregate output and profits. This allows us to calculate optimal household behavior for consumption, asset holdings, and bequests given. Since the optimization problem of the household includes a debt limit, we cannot solve the household's optimal decisions analytically, as in [Auerbach and Kotlikoff \(1987\)](#). Instead, we use Matlab's convex optimization solver "fmincon."

In the steady state, the population of each age group is given by equation ([A.66](#)). Using the population sizes of each age group, we aggregate the optimal asset supply to obtain a new value of the aggregate capital stock. Using aggregate capital and labor, we calculate aggregate output and profits in the economy. We also update the tax rate to satisfy the government's budget constraint. Finally, we set bequests received to be equal to total bequests given by the individuals that survive to 81 divided by the number of children who survived to the bequest age of 57. Typically, 20 to 30 iterations are necessary to converge to the stationary equilibrium.

The approach used to solve for the transition path is similar to solving for the initial and final steady states of the model. We assume that the economy will be in a steady state after 150 years in the transition. In the years between the initial and final steady state, we must solve for all endogenous variables, including the optimal consumption and bequests choices for individuals of all generations. We assume the year 1 is in steady state and that, in the year 2, agents are surprised by an unexpected change in the *path* of key economic variables, such as productivity growth, total fertility rate, mortality profiles, etc. Their asset choices were made previously in period 1, and they must now adjust their consumption and saving choices to the path of these variables. At the time of the shock, households have perfect foresight.

Our algorithm now begins with a guess for the same subset of endogenous variables, but now the guess for each variable is a 150×1 vector for each year of the transition path. Given the vector of guesses for the capital stock, we can calculate a vector of prices and wages and thus optimally solve for consumption and bequests of each generation in the transition period. After the year 151, agents assume the economy will be in the final steady state. We then aggregate the individual asset supply decisions to get a new value for capital supply and repeat the algorithm until convergence.

K Extension with Financial Frictions

All borrowing and lending is done through a perfectly competitive banking system. The banking system takes lending from individuals and matches them with borrowers, who can be either firms or other individuals. For every 1 unit of either investment or final good that is lent, it costs the bank ϕ . Because there is perfect competition in the banking industry, the rate the bank charges to borrowers, r_b , and the rate the bank gives to lenders, r_l are related by the following arbitrage equation:

$$r_b = r_l + \phi. \quad (\text{A.87})$$

K.1 Individuals

Individuals have the same utility function, budget constraint, and borrowing limit as they previously did. However, now the interest rate they face depends on whether they are a borrower or a lender: If $a_t < 0$, they face the borrowing interest rate r_b . If $a_t > 0$, they face the lending interest rate a_l .

K.2 Firms

Intermediate goods firms face the same profit maximization and production function as before. However, now all borrowing of capital (paying the rental rate r^k) must be intermediated through a bank as well. Firms rent capital services at rate ρ_b , and individuals can lend capital at rate ρ_l . The arbitrage equation for the real interest rate is now given by

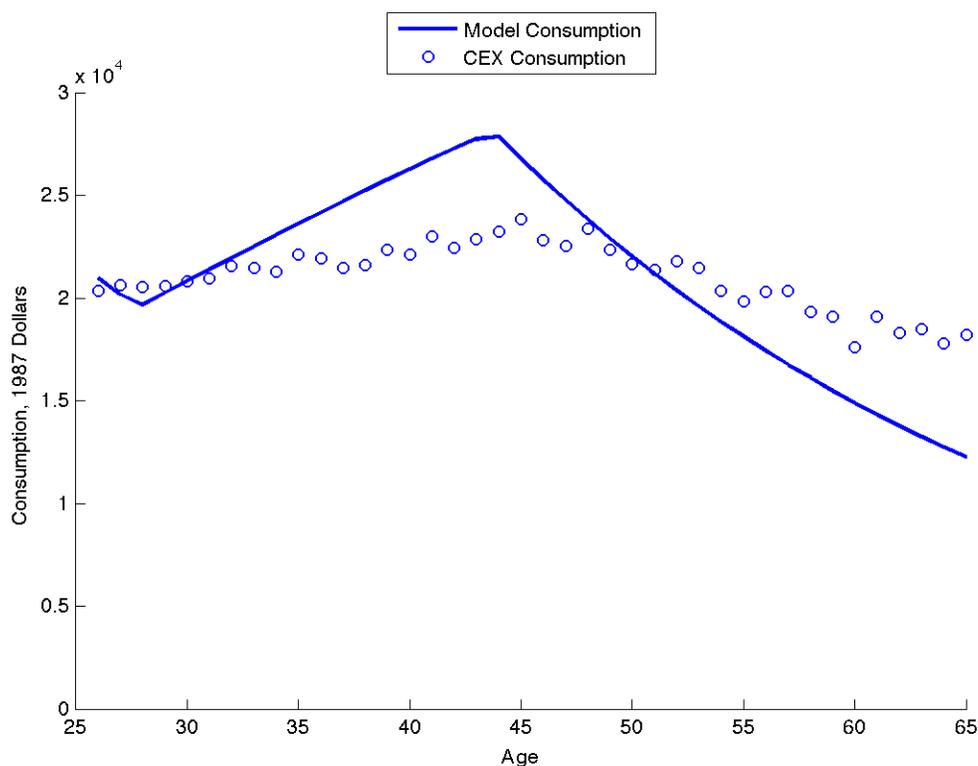
$$1 + r_{t,b} = \frac{r_{t,b}^k + (1 - \delta) \xi_t}{\xi_{t-1}} \quad (\text{A.88})$$

with a similar arbitrage equation for the interest rate for lending.

K.3 General Equilibrium

In general equilibrium, the supply for loanable funds must equal the demand:

Figure A.2: Consumption profile in stationary equilibrium



$$S(r_l) = D(r_b) \tag{A.89}$$

L Supplemental Figures and Tables

We perform several robustness exercises. The first alternate specification holds all variables at the levels in the main specification, but sets depreciation at 8%. The second alternate specification holds all variables at the level in the main specification, but sets the utility elasticity of substitution equal to 1. The third alternate specification holds all parameters at the level in the main specification, but sets the production elasticity equal to 1 (Cobb-Douglas). The fourth alternate specification sets the intertemporal elasticity of substitution to 0.5.

M Changes in the US Economy Since 1970

Our choice of 1970 as a starting point for our analysis is motivated primarily by the fact that there have been substantial changes in the fundamentals of the US economy since the 1970s. The baby

Figure A.3: Population pyramid in stationary equilibrium

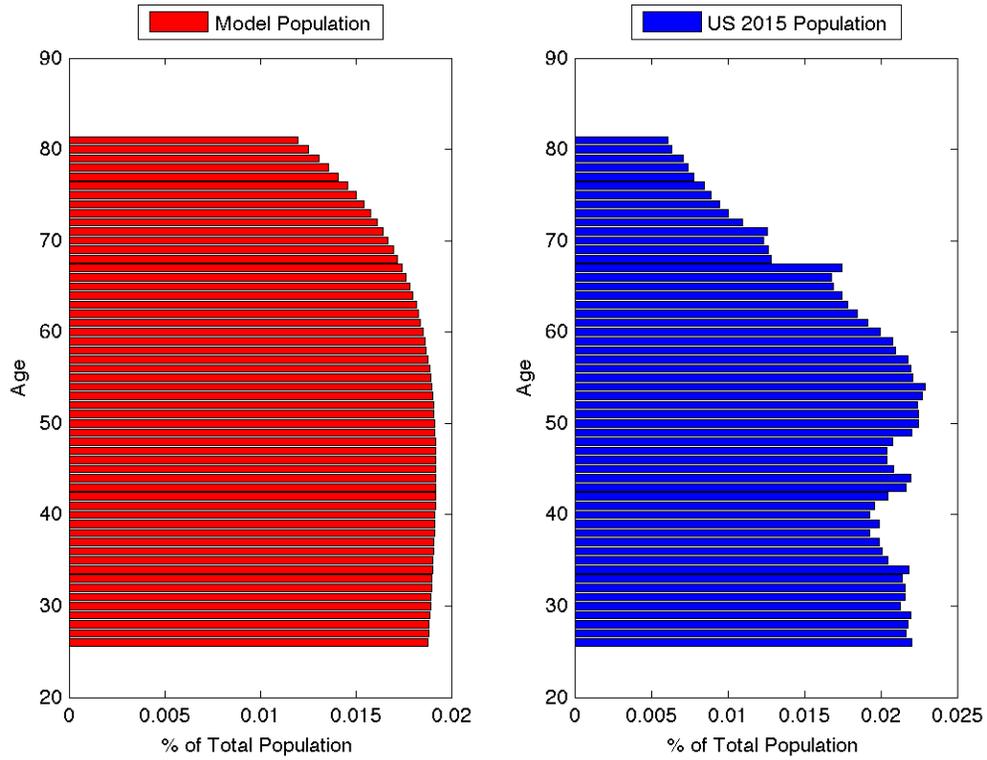


Table A.3: 2015 steady-state results, alternate specification 1

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.47%	-1.47%
Investment-to-output ratio	15.9%	15.9%
Consumer-debt-to-output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests-to-output	3%	3%

Table A.4: Decomposition, alternate specification 1

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-3.60%	100%
Mortality rate	-1.28	34%
Total fertility rate	-1.19	32%
Productivity growth	-1.67	45%
Government debt (% of GDP)	+1.13	-30%
Labor share	-.45	12%
Relative price of investment goods	-0.36	10%
Change in debt limit	+.08	-2%

Table A.5: Raising the rate of interest to 1%, alternate specification 1

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	5.40
Government debt (% of GDP)	118%	285%
Productivity growth	0.65%	2.65%
Relative price of investment goods	1.00	2.22

Table A.6: 2015 steady-state results, alternate specification 2

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.47%	-1.47%
Investment-to-output ratio	15.9%	15.9%
Consumer-debt-to output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests-to-output	3.01%	3%

Table A.7: Decomposition, alternate specification 2

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-2.86%	100%
Mortality rate	-1.22	43%
Total fertility rate	-1.36	47%
Productivity growth	-1.28	45%
Government debt (% of GDP)	+1.54	-54%
Labor share	-.33	12%
Relative price of investment goods	-0.30	10%
Change in debt limit	+.08	-3%

Table A.8: Raising the rate of interest to 1%, alternate specification 2

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	4.35
Government debt (% of GDP)	118%	279%
Productivity growth	0.65%	3.29%
Relative price of investment goods	1.00	1.78

Table A.9: 2015 steady-state results, alternate specification 3

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-.81%	-1.47%
Investment-to-output ratio	18.03%	15.9%
Consumer-debt-to-output ratio	6.3%	6.3%
Labor share	65.40%	65.99%
Bequests-to-output	3%	3%

Table A.10: Decomposition, alternate specification 3

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-2.85%	100%
Mortality rate	-1.45	46%
Total fertility rate	-1.41	45%
Productivity growth	-1.70	54%
Government debt (% of GDP)	+1.58	-50%
Labor share	-.3	10%
Relative price of investment goods	0	0%
Change in debt limit	+.13	-4%

Table A.11: Raising the rate of interest to 1%, alternate specification 3

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	3.18
Government debt (% of GDP)	118%	208%
Productivity growth	0.65%	2.11%
Relative price of investment goods	1.00	≠

Table A.12: 2015 steady-state results, alternate specification 4

<i>Moment</i>	<i>Model</i>	<i>US data</i>
Natural rate of interest	-1.43%	-1.47%
Investment-to-output ratio	15.9%	15.9%
Consumer-debt-to-output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests-to-output	2.34%	3%

Table A.13: Decomposition, alternate specification 4

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-7.07%	100%
Mortality rate	-3.36	38%
Total fertility rate	-3.10	35%
Productivity growth	-4.0	45%
Government debt (% of GDP)	+3.27	-37%
Labor share	-1.11	13%
Relative price of investment goods	-0.83	9%
Change in debt limit	+.29	-3%

Table A.14: Raising the rate of interest to 1%, alternate specification 4

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	2.56
Government debt (% of GDP)	118%	163%
Productivity growth	0.65%	1.51%
Relative price of investment goods	1.00	1.95

boom began soon after World War II, increasing fertility from 2.5 children per woman to a peak of 3.62 children per woman in 1960.⁷⁴ Although the baby boom started in 1945, the children of the baby boom did not enter economic maturity (age 26) until 1970; thus we are able to look at the impact of the baby boom on the evolution of interest rates in our model starting in 1970. Mortality has also changed substantially. In 1970 life expectancy at birth was 70.8 years, growing to 78.7 in 2010. The combination of a slower fertility rate and a decrease in mortality leads to the aging population we see today. This data on fertility and mortality are the raw data we put into the model. For every year from 1970 to 2015, we input the exact total fertility rate from UN fertility data, along with the exact age-specific mortality rate from CDC life tables. After 2015, we assume a steady state with respect to birth rates and mortality.

Gordon (2016) argues that productivity growth in the future will likely stagnate and that long-run growth rates might be half that of the 1.9% annual rate from 1960 to 2007. This prediction does not rely on a forecast of a future growth slowdown, for over the past four decades productivity growth has already slowed; since the 1970s, worker productivity has been slower by 0.8 percentage point per year than during the eight decades before. We incorporate this by using estimated

⁷⁴ After the well known “baby boom” came the less well-known “baby bust,” in which total fertility in the US gradually fell to 1.88 children per woman, below the replacement rate, before recovering roughly to the replacement rate seen today.

series for productivity growth rates from [Fernald \(2012\)](#). The exact productivity data are used as input in the model from the 1970s to present day, while from 2015 onward we hold it constant at 0.65.

Other important factors driving the real interest rate are the relative price of investment goods, the labor share of output, and government debt. [Greenwood, Hercowitz and Krusell \(1997\)](#) and [Fernald \(2012\)](#) have documented that the relative price of investment goods has fallen by 30% since 1970. Since the relative price of investment goods directly affects capital accumulations, this could potentially have a substantial effect on real interest rates. We set the relative price of capital goods in our model to match the decline since the 1970s. [Karabarbounis and Neiman \(2014\)](#) and [Elsby, Hobijn and Şahin \(2013\)](#) provide data indicating that the labor share has declined substantially since 1970. We model the declining labor share in our model very simply, through an increase in the profit share $\frac{1}{\theta_t}$. We choose the path for θ_t to match the decline in the labor share from [Elsby, Hobijn and Şahin \(2013\)](#). Government debt is also an important factor that has changed over the period. Since 1970, total government debt (including state and local debt) has increased to 118% of GDP. We set government debt to exactly match the data over the period.

N Literature Estimates

Table A.15: Literature estimates of parameters, Part A

<i>Parameter</i>	<i>Source</i>
Beta	
0.988-0.990	Kaplan (2012)
0.956-0.960	Gourinchas and Parker (2002)
1.011	Ríos-Rull (1996)
0.92	Krueger and Kubler (2006)
0.96-1.04	Constantinides, Donaldson and Mehra (2002)
Intertemporal Elasticity of Substitution	
0.54-0.62	Kaplan (2012)
0.71-2.00	Gourinchas and Parker (2002)
0.40	Glover et al. (2011)
0.25-1.00	Ríos-Rull (1996)
0.50	Krueger and Kubler (2006)
0.167-0.250	Constantinides, Donaldson and Mehra (2002)
Borrowing Limit (Various)	
0.18 (debt to GDP)	Guerrieri and Lorenzoni (2011)
30% average wage income	Kaplan (2012)
0	Gourinchas and Parker (2002)
80% of the value of risky assets they own	Hur (2016)
No borrowing limit	Ríos-Rull (1996)
No borrowing limit	Krueger and Kubler (2006)
0	Constantinides, Donaldson and Mehra (2002)
20% of value of home	Ríos-Rull and Sánchez-Marcos (2008)
0	Aiyagari (1994)
0	Krusell and Smith (1998)
25% of lifetime wage	Iacoviello and Pavan (2013)
Capital Share	
0.36	Ríos-Rull (1996)
0.25	Auerbach and Kotlikoff (1987)
0.33	Mankiw, Romer and Weil (1992)
0.30	Krueger and Kubler (2006)
0.30-0.34	Constantinides, Donaldson and Mehra (2002)

Table A.16: Literature estimates of parameters, Part B

<i>Parameter</i>	<i>Source</i>
Depreciation	
0.10-0.15	Jorgenson (1996)
0.05	Ríos-Rull (1996)
0.06-0.12	Nadiri and Prucha (1996)
Production Elasticity of Substitution	
0.67-0.70	Oberfield and Raval (2014)
1.00	Berndt (1976)
0.40-0.90	Antras (2004)
0.50-0.60	Klump, McAdam and Willman (2007)
1.25	Karabarbounis and Neiman (2014)
1.30-1.60	Piketty (2014)
0.84	Herrendorf, Herrington and Valentinyi (2015)
0.78	Alvarez-Cuadrado, Van Long and Poschke (2014)
Retailer Elasticity of Substitution	
6.00	Christiano, Eichenbaum and Evans (2005)
6.00	Galí (2015)
7.87	Woodford (2003)
4.00	Nakamura and Steinsson (2009)
4.00	Berry, Levinsohn and Pakes (1995)
3.00	Midrigan (2011)
7.00	Golosov and Lucas (2007)
Bequest Parameter	
Based on bequests as a percent of permanent income	Altig et al. (2001)
Based on a wealth transfer share of 60%	De Nardi (2004)