

Online Appendix for Bubbly Recessions

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DERIVATIONS

A1. Bubbleless equilibrium

The credit market clearing condition is $\int_0^1 d_t^j dj = 0$. By substituting (III.4), we get

$$F(\bar{a}_t)\beta e_t = \frac{\theta\beta}{1-\theta} \int_0^1 1_{a^j > \bar{a}_t} e_t^j dj,$$

where 1 is the indicator function. Since net worth e_t^j is a function of past productivity a_{t-1}^j and the productivity shocks are i.i.d., the above equality can be rewritten as

$$F(\bar{a}_t)\beta e_t = \frac{\theta\beta}{1-\theta} \int_0^1 1_{a^j > \bar{a}_t} dj \times \underbrace{\int e_t^j dj}_{e_t},$$

which yields (III.8), as stated in the main text. Recall that the CDF for the Pareto distribution over $[1, \infty)$ with shape parameter σ is $F(a) = 1 - a^{-\sigma}$. Thus, the solution to equation (III.9) for the bubbleless cutoff threshold is given by (III.10), i.e.:

$$\bar{a}_n = \left(\frac{1}{1-\theta} \right)^{1/\sigma}.$$

A closed-form expression for \mathcal{A}_n is $\mathcal{A}_n = \frac{\beta}{1-\theta} \frac{\bar{a}_n^{1-\sigma} \sigma}{\sigma-1}$. Thus, a closed-form expression for the interest rate in the bubbleless steady state is

$$(A1) \quad R_n = \frac{\sigma-1}{\beta\sigma},$$

and for the capital stock is:

$$K_n = \left(\frac{\alpha\sigma\beta(1-\theta)^{-1/\sigma}}{\sigma-1} \right)^{\frac{1}{1-\alpha}}.$$

A2. Bubbly equilibrium

Suppose the bubble persists in period t . By substituting (IV.1) into credit market clearing condition $\int_0^1 d_t^j dj = 0$, we get:

$$\beta(F(\bar{a}_t) - \phi_t)e_t = \frac{\theta\beta}{1-\theta} \int_0^1 1_{a^j > \bar{a}_t} e_t^j dj.$$

As the productivity shocks are i.i.d., the right-hand side again reduces to $\frac{\theta\beta}{1-\theta} \int_0^1 1_{a^j > \bar{a}_t} dj \times \int e_t^j dj$. We thus arrive at (IV.6). The closed-form solution to this equation is $\bar{a}_t = \left(\frac{1}{(1-\theta)(1-\phi_t)}\right)^{1/\sigma}$, leading to the steady-state value of \bar{a}_b as in (IV.10). The capital accumulation equation is then given by (IV.7).

We now determine $R_{t,t+1}$ and R_{t+1}^k . From Proposition 3, we know that the slump only begins one period after the bubble collapses. That is, even if the bubble collapses in $t+1$, the slump only begins in $t+2$ and the labor market still clears in $t+1$, leading to $L_{t+1} = 1$. Hence, the rental rate of capital is given by $R_{t+1}^k = \alpha K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} = \alpha K_{t+1}^{\alpha-1}$ in period $t+1$, regardless of whether the bubble collapses or persists in $t+1$. Thus, from the perspective of the marginal investors, both the options of lending and investing in capital are safe. As a consequence, their indifference condition (IV.4) reduces to

$$(A2) \quad R_{t,t+1} = \bar{a}_t R_{t+1}^k = \bar{a}_t \alpha K_{t+1}^{\alpha-1}.$$

We now derive the bubble growth. Indifference condition (IV.3) gives:

$$\rho \frac{1}{c_{t+1}^{j,\rho}} \frac{p_{t+1}^b}{p_t^b} = \left(\rho \frac{1}{c_{t+1}^{j,\rho}} + (1-\rho) \frac{1}{c_{t+1}^{j,1-\rho}} \right) R_{t,t+1}$$

for all j such that $a^j < \bar{a}_t$, where $c_{t+1}^{j,\rho}$ and $c_{t+1}^{j,1-\rho}$ denote the consumption of entrepreneur j when the bubble persists and when the bubble collapses in $t+1$, respectively. By substituting out the consumption values, this equation can be algebraically simplified to

$$(A3) \quad \frac{\rho \frac{p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1-\rho} = \frac{p_{t+1}^b b_t^j}{\beta e_t^j - p_t^b b_t^j},$$

for all j such that $a^j < \bar{a}_t$. Furthermore, recall the following fact from algebra: if

$\frac{x}{y} = \frac{x'}{y'}$, then $\frac{x}{y} = \frac{x'}{y'} = \frac{x+x'}{y+y'}$. Thus,

$$\frac{\rho \frac{p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1 - \rho} = \frac{p_{t+1}^b b_t^j}{\beta e_t^j - p_t^b b_t^j} = \frac{p_{t+1}^b \int 1_{a^j < \bar{a}_t} b_t^j dj}{\beta \int 1_{a^j < \bar{a}_t} e_t^j dj - p_t^b \int 1_{a^j < \bar{a}_t} b_t^j dj}.$$

Because of the bubble market clearing conditions, we then get:

$$(A4) \quad \frac{\rho \frac{p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1 - \rho} = \frac{\phi_t}{F(\bar{a}_t) - \phi_t} \frac{p_{t+1}^b}{p_t^b}.$$

The growth of net worth is given by:

$$\frac{e_{t+1}}{e_t} = \frac{R_{t+1}^k K_{t+1} + p_{t+1}^b}{e_t} = \frac{R_{t+1}^k K_{t+1}}{e_t} + \phi_{t+1} \beta \frac{e_{t+1}}{e_t}.$$

Combined with (IV.7) and (A2), the equation above yields:

$$(A5) \quad \frac{e_{t+1}}{e_t} = \frac{R_{t+1}^k \frac{\beta}{1-\theta} \int_{\bar{a}_t} a dF(a)}{1 - \beta \phi_{t+1}}.$$

Combining (A4) and (A5) yields equation (IV.9).

Finally, from the analysis above of the equilibrium dynamics, the bubbly steady state can be straightforwardly characterized as in the main text.

A3. Welfare functions

The expected lifetime utility of a representative worker in the bubbly equilibrium in each period t is given by:

$$W_b(K_t, \phi_t) = \log(c_t^w) + \beta \rho W_b(K_{t+1}, \phi_{t+1}) + \beta(1 - \rho) W_{burst}(K_{t+1})$$

where the first term is the instantaneous utility, with

$$c_t^w = w_t = (1 - \alpha) K_t^\alpha,$$

and the second term is the continuation value conditional on the bubble persisting, and the last term is the continuation value conditional on the bubble collapsing in $t + 1$. From Proposition 3, we can calculate this last term to be:

$$W_{burst}(K_{t+1}) = \Gamma_0(s^*) + \Gamma_1(s^*) \log K_{t+1},$$

where

$$\begin{aligned}\Gamma_0(s^*) &= \frac{1}{1-\beta} \log(1-\alpha) \\ &\quad - \frac{1-\alpha}{\alpha} \left(\sum_{s=0}^{s^*-1} \frac{\beta^s s(s+1)}{2} + \frac{\alpha}{1-\beta} \frac{\beta^{s^*} s^* (s^*-1)}{2} \right) \log \gamma \\ &\quad + (1-\alpha) \left(\sum_{s=0}^{s^*} \beta^s s - \beta^{s^*} s^* \left(\frac{1-\alpha-\beta}{1-\beta} \right) \right) \log K_n \\ \Gamma_1(s^*) &= \sum_{s=0}^{s^*-1} \beta^s (\alpha - (1-\alpha)s) + \frac{\beta^{s^*} \alpha (1 - (1-\alpha)s^*)}{1-\beta}.\end{aligned}$$

Thus, the worker's welfare in the bubbly steady state is as given by (V.2).

We now calculate the welfare of entrepreneurs in the bubbleless steady state. Recall from the main text that in the bubbleless steady state, the lifetime expected utility of an entrepreneur j who starts the period with a net worth e^j , denoted by $V_n(e^j)$, satisfies equation (V.3). We solve $V_n(\cdot)$ by the guess and verify method. We conjecture that $V_n(e^j) = g_0 + g_1 \log K_n + g_2 \log e^j$. By plugging into the equation above and solving for g_0, g_1, g_2 , we get:

$$\begin{aligned}g_0 &= \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2} \left(\log(\bar{a}_n \beta \alpha) + \int_{\bar{a}_n} \log \left(\frac{a - \bar{a}_n \theta}{\bar{a}_n (1-\theta)} \right) dF(a) \right) \\ &\quad - \left(\frac{1-\alpha}{1-\beta} \right)^2 \frac{\beta \log K_n}{1-\beta \alpha} \\ g_1 &= - \frac{1-\alpha}{1-\beta} \frac{\beta \alpha}{1-\beta \alpha} \\ g_2 &= \frac{1}{1-\beta}.\end{aligned}$$

Thus

$$\begin{aligned}V_n(e^j) &= \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2} \left[\log(\alpha \bar{a}_n \beta) + \int_{\bar{a}_n} \log \left(\frac{a - \bar{a}_n \theta}{\bar{a}_n (1-\theta)} \right) dF(a) \right] \\ &\quad - \frac{\beta(1-\alpha)}{(1-\beta)^2} \log K_n + \frac{1}{1-\beta} \log(e^j).\end{aligned}$$

Under the definition $V_n \equiv V_n(\alpha K_n^\alpha)$, we then get:

(A6)

$$V_n = \frac{\log \alpha (1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2} \left[\log(\alpha \bar{a}_n \beta) + \int_{\bar{a}_n} \log \left(\frac{a - \bar{a}_n \theta}{\bar{a}_n (1-\theta)} \right) dF(a) \right] - \frac{\beta-\alpha}{(1-\beta)^2} \log K_n.$$

Finally, we compute the welfare of entrepreneurs in the bubbly steady state. Recall that the lifetime expected utility of an entrepreneur j who starts the period with a net worth e^j , denoted by $V_b(e^j)$, satisfies equation (V.4). From Proposition 3, we can calculate the post-bubble continuation value as given by:

$$(A7) \quad V_{burst}(e^j) = \kappa_0(s^*, \bar{a}_n) - \kappa_1(s^*) \log K_b + \frac{\log((1-\beta)e^j)}{1-\beta},$$

where:

$$\begin{aligned} \kappa_0(s^*, \bar{a}_n) &= \frac{1-\beta^{s^*}}{1-\beta} \log(1-\beta) + \beta^{s^*} \left[\Gamma_n(\bar{a}_n) - \left(\frac{1-\alpha}{1-\beta} \right)^2 \frac{\beta}{1-\beta\alpha} \log K_n \right] \\ &+ \frac{\beta - \beta^{s^*+1}}{(1-\beta)^2} \left[\log(\alpha \bar{a}_n \beta) + \int_{\bar{a}_n} \log \left(\frac{a - \bar{a}_n \theta}{\bar{a}_n (1-\theta)} \right) dF(a) \right] \\ &- \frac{\beta^{s^*+1} s^* \alpha (1-\alpha)}{1-\beta} \log \left[\frac{\alpha \beta}{1-\theta} \int_{\bar{a}_n} a dF(a) \right] \\ &- \frac{(1-\alpha)}{2\alpha} \left[\left(\sum_{s=0}^{s^*-1} \beta^s s (s+1) \right) + \frac{\beta^{s^*} s^*}{(1-\beta)} \frac{1-\beta\alpha^2 + s^* (1-2\beta\alpha + \beta\alpha^2)}{1-\beta\alpha} \right] \log \gamma \\ \kappa_1(s^*) &= (1-\alpha) \left[\sum_{s=0}^{s^*-1} \beta^s s + \frac{\beta^{s^*}}{1-\beta} \left(\frac{\beta\alpha + s^* (1-2\beta\alpha + \beta\alpha^2)}{1-\beta\alpha} \right) \right]. \end{aligned}$$

Again, by applying the guess and verify method to equation (V.4), we get the following solution for V_b :

$$(A8) \quad V_b(e^j) = \Gamma_b(s^*) - \frac{\beta}{1-\beta\rho} \left(\frac{1-\alpha}{1-\beta} + (1-\rho) \kappa_1(s^*) \right) \log K_b + \frac{1}{1-\beta} \log(e^j),$$

where

$$\begin{aligned} \Gamma_b(s^*) &= \frac{1}{1-\beta\rho} \log(1-\beta) + \frac{\beta(1-\rho)}{(1-\beta\rho)} \kappa_0(s^*, \bar{a}_n) \\ &+ \frac{\beta}{(1-\beta\rho)(1-\beta)} \left(\log(\alpha \bar{a}_b \beta) + \int_{\bar{a}_b} \log \left(\frac{a - \bar{a}_b \theta}{\bar{a}_b (1-\theta)} \right) dF(a) \right) \\ &+ \frac{\beta F(\bar{a}_b)}{(1-\beta\rho)(1-\beta)} \left(\rho \log \left(\frac{\theta(1-F(\bar{a}_b))}{\theta - (\theta + \beta(1-\rho)) F(\bar{a}_b)} \rho \right) + (1-\rho) \log \left(\frac{\theta(1-F(\bar{a}_b))}{\beta F(\bar{a}_b)} \right) \right). \end{aligned}$$

Under the definition $V_b \equiv V_b\left(\frac{\alpha K_b^\alpha}{1-\beta\phi}\right)$, we then get:

$$(A9) \quad V_b = \Gamma_b(s^*) - \frac{\beta}{1-\beta\rho} \left(\frac{1-\alpha}{1-\beta} + (1-\rho) \kappa_1(s^*) \right) \log K_b + \frac{1}{1-\beta} \log \left(\frac{\alpha K_b^\alpha}{1-\beta\phi} \right).$$

A4. *Bubble dynamics and steady state with macroprudential tax*

In the presence of a macroprudential tax, the equilibrium dynamics are similar to that of the bubble equilibrium, except that τ will affect the first-order condition with respect to bubbly investment of entrepreneurs. The indifference condition between investing in the bubbly asset and lending for entrepreneurs with productivity shock below \bar{a}_t is now given by:

$$E_t \left[u'(\bar{c}_{t+1}) \frac{(1-\tau)p_{t+1}^b}{p_t^b} \right] = E_t [u'(\bar{c}_{t+1})R_{t,t+1}], \quad \text{if } a_t^j < \bar{a}_t,$$

which can be reduced to:

$$\frac{\rho \frac{(1-\tau)p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1-\rho} = \frac{p_{t+1}^b b_t^j}{\beta e_t^j - p_t^b b_t^j}.$$

By integrating across $a^j < \bar{a}_t$, and with more algebraic manipulations, we then get an aggregate expression:

$$(1-\tau) \frac{\phi_{t+1}}{\phi_t} = \frac{(1-\beta\phi_{t+1})\bar{a}_t}{\frac{\beta}{1-\theta} \int_{\bar{a}_t} adF(a)} \frac{F(\bar{a}_t) - \phi_t}{\rho F(\bar{a}_t) - \phi_t}.$$

This equation gives a new expression that determines the bubble size in the bubbly steady state:

$$1-\tau = \frac{(1-\beta\phi)\bar{a}_b}{\underbrace{\frac{\beta}{1-\theta} \int_{\bar{a}_b} adF(a)}_{R_b}} \frac{F(\bar{a}_b) - \phi}{\rho F(\bar{a}_b) - \phi}.$$

The closed-form expression for the bubble ratio is:

$$\phi(\tau) = \frac{\theta}{\beta} \cdot \frac{1 - \sigma(1 - \beta\rho(1 - \tau))}{\theta - \sigma\rho(1 - \theta)(1 - \tau) + \sigma(1 - \tau - \theta)}$$

Note how ϕ is a decreasing function of τ . The bubble exists ($\phi > 0$) if and only if $\tau < \bar{\tau} \equiv 1 - \frac{\sigma-1}{\beta\sigma\rho}$. The capital stock in the bubbly steady state is then given by:

$$K_b(\tau) = \left(\frac{\frac{\beta}{1-\theta}}{1 - \beta\phi(\tau)} \alpha \int_{\bar{a}_b(\tau)} adF(a) \right)^{\frac{1}{1-\alpha}},$$

where the cutoff threshold is $\bar{a}_b(\tau) = \left(\frac{1}{(1-\theta)(1-\phi(\tau))} \right)^{1/\sigma}$.

Similar to the analysis in the main text and Section A.A3, the duration of the post-bubble slump is given by (IV.20) and the bubbly steady-state welfare expression for workers is given by (V.2), where we note that $K_b = K_b(\tau)$ is now a function of τ . The bubbly steady-state welfare expression for entrepreneurs is given by $V_b \equiv V_b\left(\frac{\alpha K_b^\alpha}{1-\beta\phi(\tau)}\right)$, where $V_b(e^j)$ is as defined in (A8), except that $\Gamma_b(s^*)$ is now defined as:

$$\begin{aligned} \Gamma_b(s^*) &= \frac{1}{1-\beta\rho} \log(1-\beta) + \frac{\beta(1-\rho)}{(1-\beta\rho)} \kappa_0(s^*, \bar{a}_n) \\ &+ \frac{\beta}{(1-\beta\rho)(1-\beta)} \left(\log(\alpha \bar{a}_b \beta) + \int_{\bar{a}_b} \log\left(\frac{a - \bar{a}_b \theta}{\bar{a}_b(1-\theta)}\right) dF(a) \right) \\ &+ \frac{\beta F(\bar{a}_b)}{(1-\beta\rho)(1-\beta)} \left(\rho \log\left(\frac{(1-\tau)(F(\bar{a}_b) - \phi) + \tau\phi\left(\frac{F(\bar{a}_b) - \phi}{F(\bar{a}_b)}\right)}{(1-\tau)\rho F(\bar{a}_b) - (1-\tau\rho)\phi}\right) \rho \right) + (1-\rho) \log\left(\frac{F(\bar{a}_b) - \phi}{F(\bar{a}_b)}\right). \end{aligned}$$

A5. Constrained efficiency

Following the macroprudential literature (e.g., Bianchi 2011; Bianchi and Mendoza 2018), we define constrained efficiency as follows. Consider a benevolent social planner who cares about both entrepreneurs and workers. The planner has restricted planning abilities: it chooses allocations subject to the resource, implementability, and credit constraints but allows the markets to clear competitively. Its policy instrument is limited to a constant tax on bubbly speculation. Since prices remain market-determined, the first-order conditions for private agents enter the planner's problem as implementability constraints. The key difference between the planner's problem and private agents' problems is that the *planner internalizes how its decisions affect prices*.

The implementability constraints for the planner will come from the first-order conditions of individual entrepreneurs. Recall that with the constant tax τ , the optimization problem of each entrepreneur j is:

$$\max_{\{c_t^j, d_t^j, b_t^j, l_t^j\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

subject to

$$\begin{aligned} c_t^j + I_t^j + p_t^b b_t^j &= R_t^k a_{t-1}^j I_{t-1}^j + (1 - \tau) p_t^b b_{t-1}^j - R_{t-1,t} d_t^j + d_t^j + T_t^j \\ I_t^j, b_t^j &\geq 0 \\ d_t^j &\leq \theta \cdot I_t^j. \end{aligned}$$

It is more convenient to rewrite this problem in the following equivalent form: instead of choosing the amount of the bubbly asset b_t^j , each entrepreneur chooses the amount of (before-tax) bubbly investment $B_t^j \equiv p_t^b b_t^j$. Then the problem becomes:

$$\max_{\{c_t^j, d_t^j, B_t^j, I_t^j\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

subject to

$$\begin{aligned} c_t^j + I_t^j + B_t^j &= F_K(K_t, L_t) a_{t-1}^j I_{t-1}^j + (1 - \tau) R_t^b B_{t-1}^j - R_{t-1,t} d_t^j + d_t^j + T_t^j \\ I_t^j, b_t^j &\geq 0 \\ d_t^j &\leq \theta \cdot I_t^j, \end{aligned}$$

where $R_t^b \equiv \frac{p_t^b}{p_{t-1}^b}$ denotes the return on the bubbly asset. In the budget constraint above, we have also replaced R_t^k with the marginal product of capital.

The entrepreneurs' first-order conditions are:

$$(A10) \quad u'(c_t^j) = \beta a_t^j F_K(K_{t+1}, L_{t+1}) E_t u'(c_{t+1}^j) + \lambda_{I,t}^j + \lambda_{d,t}^j \theta$$

$$(A11) \quad u'(c_t^j) = \beta \rho (1 - \tau) R_{t+1}^{b,\rho} u'(c_{t+1}^{j,\rho}) + \lambda_{b,t}^j / p_t^b$$

$$(A12) \quad u'(c_t^j) = \beta R_{t,t+1} \cdot E_t u'(c_{t+1}^j) + \lambda_{d,t}^j,$$

$$(A13) \quad \lambda_{I,t}^j I_t^j = \lambda_{b,t}^j b_t^j = \lambda_{d,t}^j (\theta \cdot I_t^j - d_t^j) = 0,$$

where $R_t^{b,\rho}$ and $c_t^{j,\rho}$ are the return on the bubbly asset and consumption conditional on the state that the bubble persisting in t , $\lambda_{I,t}^j$, $\lambda_{b,t}^j$, $\lambda_{d,t}^j$ are the Lagrange multipliers associated with the nonnegativity constraints on I_t^j and b_t^j and the credit constraint, respectively.

We can now define the planner's problem:

Definition 1. The constrained central planner's problem is as follows:

$$\max_{\{c_t^j, B_t^j \geq 0, I_t^j \geq 0, d_t^j, R_{t-1,t}, R_t^{b+}, \theta_{I,t}^j, \theta_{b,t}^j, \theta_{d,t}^j, \tau\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda \cdot u(c_t^w) + (1 - \lambda) \cdot \int_{j \in J} u(c_t^j) \right),$$

where $\lambda \in [0, 1]$ is the Pareto weight the planner assigns to the representative worker and $1 - \lambda$ is that for entrepreneurs, subject to the following individual budget constraints:

$$\begin{aligned} c_t^w &= F_L(K_t, L_t)L_t \\ c_t^j &= F_K(K_t, L_t)a_{t-1}^j I_{t-1}^j + ((1 - \tau)R_t^b B_{t-1}^j - B_t^j) - R_{t-1,t}d_{t-1}^j + d_t^j - I_t^j + T_t^j, \end{aligned}$$

the planner's transfer being given by:

$$T_t^j = \tau B_t^j, \forall j \in J,$$

credit constraint:

$$d_t^j \leq \theta \cdot I_t^j,$$

implementability constraints (A10-A13), and the labor market conditions:

$$\begin{aligned} \text{(A14)} \quad & F_L(K_t, L_t) \geq \gamma F_L(K_{t-1}, L_{t-1}) \\ & L_t \leq 1 \\ & (1 - L_t)(F_L(K_t, L_t) - \gamma F_L(K_{t-1}, L_{t-1})) = 0. \end{aligned}$$

As in the main text, we focus on the planner's problem in the bubbly steady state and assume that each entrepreneur begins the steady state with the same net worth.

Note that the planner takes as given the exogenous sunspot process of bubble bursting. The key thing to notice is that unlike individual agents, *the planner internalizes the downward wage rigidity condition (A14) in its optimization problem.* As a consequence, the first-order conditions of the planner will contain a Lagrange multiplier associated with this constraint, which would be otherwise absent in the laissez-faire first-order conditions of individual entrepreneurs. This formalizes the notion that the competitive equilibrium allocations are constrained inefficient.

EXTENSION: A GENERALIZED MODEL

We now extend the model in several directions. First, we allow for exogenous growth. Specifically, assume that the production technology is given by:

$$Y_t = K_t^\alpha \cdot (A_t L_t)^{1-\alpha},$$

where the technology term A_t grows at an exogenous growth rate $g \geq 0$:

$$A_t \equiv (1 + g)^t.$$

Throughout, we will use a lowercase letter to denote a detrended variable, for example:

$$y_t \equiv \frac{Y_t}{(1+g)^t}, \quad k_t \equiv \frac{K_t}{(1+g)^t}, \quad w_t \equiv \frac{W_t}{(1+g)^t} \dots$$

The downward wage rigidity condition is growth-adjusted, meaning:

$$w_t \geq (1+g)\gamma w_{t-1}.$$

Second, we allow for partial capital depreciation. Specifically, we assume that the capital stock depreciates at a rate $\delta \in [0, 1]$. As in Kocherlakota (2009), we assume that the capital good can be converted back one-to-one to the consumption good.

Remark 1. The main model in Section II is a special case of this extended model (where $g = 0$, $\delta = 1$, and F is Pareto over $[1, \infty)$). Note further that the calibration exercise in Section IV.B uses only the first two extended assumptions (i.e., $g > 0$ and $\delta < 1$).

The changes compared to the main model are as follows. We first consider the bubbleless equilibrium. The time-invariant cutoff threshold is implicitly determined by the credit-market clearing condition:

$$F(\bar{a}_n) = \frac{\theta}{1-\theta} (1 - F(\bar{a}_n)).$$

The detrended capital stock evolves according to the following new law of motion:

$$(1+g)k_{t+1} = \frac{\beta}{1-\theta} \int_{\bar{a}_n} a dF(a) (\alpha k_t^\alpha L_t^{1-\alpha} + (1-\delta)k_t).$$

The bubbleless steady-state capital stock is thus:

$$k_n = \left(\frac{\beta \alpha \int_{\bar{a}_n} a dF(a)}{(1-\theta)(1+g) - (1-\delta)\beta \int_{\bar{a}_n} a dF(a)} \right)^{\frac{1}{1-\alpha}}.$$

The bubbleless steady-state interest rate is:

$$\begin{aligned} R_n &= \bar{a}_n (\alpha k_n^{\alpha-1} + 1 - \delta) \\ &= \frac{(1-\theta)(1+g)\bar{a}_n}{\beta \int_{\bar{a}_n} a dF(a)}. \end{aligned}$$

We now consider the bubbly equilibrium. The cutoff threshold is implicitly

determined by the credit-market clearing condition:

$$F(\bar{a}_t) - \phi_t = \frac{\theta}{1 - \theta} \cdot (1 - F(\bar{a}_t)).$$

The detrended capital stock evolves according to the following new law of motion:

$$(1 + g) k_{t+1} = \frac{\beta}{1 - \theta} \int_{\bar{a}_t} a dF(a) \cdot \left(\frac{\alpha k_t^\alpha L_t^{1-\alpha} + (1 - \delta) k_t}{1 - \beta\phi} \right).$$

The new law of motion of the bubble price is given by:

$$(1 + g) \frac{p_{t+1}^b}{p_t^b} = \frac{F(\bar{a}_t) - \phi_t}{\rho F(\bar{a}_t) - \phi_t} R_{t,t+1}.$$

The bubbly steady state is characterized by the following new expression for the (detrended) capital stock:

$$k_b = \left(\frac{\beta\alpha \int_{\bar{a}_b} a dF(a)}{(1 - \theta)(1 + g)(1 - \beta\phi) - (1 - \delta)\beta \int_{\bar{a}_b} a dF(a)} \right)^{\frac{1}{1-\alpha}},$$

and the following new expression for the interest rate:

$$R_b = \frac{(1 - \theta)(1 + g)(1 - \beta\phi)\bar{a}_b}{\beta \int_{\bar{a}_b} a dF(a)},$$

where the cutoff threshold \bar{a}_b as given in the main model, and the following new equation that determines the bubble ratio:

$$\phi = \frac{(1 + g)\rho - R_b}{(1 + g) - R_b} F(\bar{a}_b).$$

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