Online Appendix: Long-Term Finance and Investment with Frictional Asset Markets

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Appendix

A Proofs

This section provides the proofs of the main results of the paper.

A.1 Lenders

A.1.1 Distribution of financiers

Proof of Lemma 1. Let $\mu(y) = \left[\mu^H(y), \mu^L(y)\right]$. Matching implies $\mu^B \beta \frac{\mu^L(y)}{\mu^S} = \lambda \mu^L(y)$. Then, (5)-(6) imply that $\dot{\mu}(y) = A\mu(y)$ with

$$A = \begin{bmatrix} \delta + \eta & -\lambda \\ -\eta & \delta + \lambda \end{bmatrix}.$$

The boundary condition is $\mu(\tau) = [\mu^0, 0]$. Note that A has two real and distinct eigenvalues. Let R be the vector with the eigenvalues and V be the matrix with eigenvectors of A. Define $B = (V)^{-1} \mu(\tau)$ so

$$V = \begin{bmatrix} -1 & \frac{\lambda}{\eta} \\ 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \eta + \lambda + \delta \\ \delta \end{bmatrix} \qquad B = \frac{\eta \mu^0}{\eta + \lambda} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

It is standard to show that

$$\mu^H(y) = \sum_{i=1}^{2} e^{R_i(y-\tau)} B_i V(1,i) \qquad \mu^L(y) = \sum_{i=1}^{2} e^{R_i(y-\tau)} B_i V(2,i).$$

Finally, a few lines of algebra deliver

$$\mu^H(y) = \frac{\mu^0 \eta}{\eta + \lambda} \left(e^{\delta(y-\tau)} \frac{\lambda}{\eta} + e^{(\eta+\lambda+\delta)(y-\tau)} \right)$$
$$\mu^L(y) = \frac{\mu^0 \eta}{\eta + \lambda} \left(e^{\delta(y-\tau)} - e^{(\eta+\lambda+\delta)(y-\tau)} \right).$$

□

A.1.2 Value functions

Proof of Proposition 1. Replace the price of the asset in the secondary market $P^S(y;\lambda)$ in (9)-(10) so

$$(\rho+\delta) D^H(y;\lambda) = -\frac{\partial D^H(y;\lambda)}{\partial y} + \eta\left(D^L(y;\lambda) - D^H(y;\lambda)\right)$$
$$(\rho+\delta) D^L(y;\lambda) = -h - \frac{\partial D^L(y;\lambda)}{\partial y} + \lambda\gamma\left(D^H(y;\lambda) - D^L(y;\lambda)\right).$$

Let $H(y;\lambda) = D^H(y;\lambda) - D^L(y;\lambda)$, then

$$(\rho+\delta+\eta+\lambda\gamma) H(y;\lambda) = h - \frac{\partial H(y;\lambda)}{\partial y},$$

with $H(0;\lambda) = 0$. It is straight forward to see that $H(y;\lambda) = h\frac{1-e^{-c_1 y}}{c_1}$ where $c_1 = \rho+\delta+\eta+\lambda\gamma$.

Next, solve for $D^H(y;\lambda)$ as

$$(\rho+\delta) D^H(y;\lambda) = -\frac{\partial D^H(y;\lambda)}{\partial y} - \eta h\frac{1-e^{-c_1 y}}{c_1},$$

with boundary $D^H(0;\lambda) = 1$. The solution is $D^H(y;\lambda) = A + Be^{-(\rho+\delta)y} + Ce^{-c_1 t}$ with constants

$$A = -\frac{1}{\rho+\delta}\frac{\eta h}{c_1} \qquad C = -\frac{1}{\eta+\lambda\gamma}\frac{\eta h}{c_1} \qquad B = 1 + \frac{\eta h}{(\eta+\lambda\gamma)(\rho+\delta)}.$$

Finally, a few lines of algebra deliver

$$D^H(y;\lambda) = e^{-(\rho+\delta)y} - \mathcal{L}(y,\lambda)$$
$$\mathcal{L}(y,\lambda) = \frac{\eta h}{\eta+\lambda\gamma}\left(\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta} - \frac{1-e^{-(\rho+\delta+\eta+\lambda\gamma)y}}{\rho+\delta+\eta+\lambda\gamma}\right)$$
$$\mathcal{L}(\tau,\lambda) = h \quad \frac{\eta}{\eta+\lambda\gamma} \quad \int_0^\tau e^{-(\rho+\delta)y}\left(1-e^{-(\eta+\lambda\gamma)y}\right)dy.$$

The value of a low-valuation agent is $D^L(y;\lambda) = D^H(y;\lambda) - H(y;\lambda)$, that is

$$D^L(y,\lambda) = e^{-(\rho+\delta)y} - h\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta} + \frac{\lambda\gamma}{\eta}\mathcal{L}(y,\lambda).$$

Properties of the illiquidity cost:

1. **Positive:** $\mathcal{L}(\tau,\lambda)$ is positive as $\rho+\delta+\eta+\lambda\gamma \geq \rho+\delta$.

2. **Sensitivity with respect to maturity τ:**

(a) $\mathcal{L}(\tau,\lambda)$ is increasing in τ:

$$\frac{\partial \mathcal{L}\left(\tau,\lambda\right)}{\partial \tau} = h \quad \frac{\eta}{\eta+\lambda\gamma} \quad e^{-(\rho+\delta)\tau}\left(1-e^{-(\eta+\lambda\gamma)\tau}\right) \geq 0.$$

(b) The limit of $\mathcal{L}(\tau,\lambda)$ when τ goes to infinity is

$$\begin{aligned}\lim_{\tau\to\infty} \mathcal{L}\left(\tau,\lambda\right) &= h\frac{\eta}{\eta+\lambda\gamma}\left(\frac{1}{\rho+\delta}-\frac{1}{\rho+\delta+\eta+\lambda\gamma}\right)\\ &= h\frac{\eta}{\left(\rho+\delta\right)\left(\rho+\delta+\eta+\lambda\gamma\right)}.\end{aligned}$$

(c) $\mathcal{L}$ is concave in τ if $\lambda \geq \eta$: The second derivative of liquidity with respect to maturity is

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial \tau^2} &= -\left(\rho+\lambda\right)h\frac{\eta}{\eta+\lambda\gamma}e^{-(\rho+\lambda)\tau}\left(1-e^{-(\eta+\lambda\gamma)\tau}\right)\\ &+\left(\eta+\lambda\gamma\right)h\frac{\eta}{\eta+\lambda\gamma}e^{-(\rho+\lambda)\tau}\left(e^{-(\eta+\lambda\gamma)\tau}\right)\end{aligned}$$

that is

$$\frac{\partial^2 \mathcal{L}}{\partial \tau^2} = h\frac{\eta}{\eta+\lambda\gamma}e^{-(\rho+\lambda)\tau}\left(e^{-(\eta+\lambda\gamma)\tau}\left(\eta+\lambda\gamma-\rho-\lambda\right)-\rho-\lambda\right)$$

so, the curvature depends on the sign of

$$F\left(x,y,\tau\right) = e^{-x\tau}\left(x-y\right)-y$$

where $x=\eta+\lambda\gamma$ and $y=\rho+\lambda$. We want to find a sufficient condition for $\mathcal{L}$ to be concave, i.e. $F \leq 0$. First, note that F is decreasing in τ. So, the maximum value occurs be at $\tau=0$. Then $F\left(x,y,0\right)=x-2y$. That is

$$\begin{aligned}F\left(x,y,0\right) &= \eta+\lambda\gamma-2\left(\rho+\lambda\right)\\ &= -2\rho-\lambda\left(2-\gamma\right)+\eta\end{aligned}$$

Recall $\gamma\in[0,1]$ so, for a sufficient condition we can consider the case of $\gamma=1$ and

$$F\left(x,y,0\right) = -2\rho-\lambda+\eta$$

Finally, a sufficient condition is that $\lambda \geq \eta$. This condition means that it is faster to find a trading counterpart than receiving a preference shock, which is strongly supported by the data. Recall that this is a sufficient condition, but depending on other parameter values, such as γ, it is not a necessary condition. Moreover, note that as τ increases, the first term of $F(x,y,\tau)$ goes to zero and F becomes negative, independently of the parameter values.

3. **Sensitivity with respect to liquidity shocks η:**

   (a) If there are no liquidity shocks ($\eta = 0$), then $\mathcal{L}(\tau,\lambda) = 0$.

   (b) If $\eta \to \infty$ (i.e., always has to pay the cost h), then

$$\lim_{\eta\to\infty} \mathcal{L}(\tau,\lambda) = h\frac{1-e^{-(\rho+\delta)\tau}}{\rho+\delta}.$$

4. **Sensitivity with respect to liquidity of the secondary market λ:**

   (a) $\mathcal{L}(\tau,\lambda)$ is decreasing in λ. Note that the illiquidity cost is

$$\begin{aligned}\mathcal{L}(\tau,\lambda) = \eta h &\left(\frac{1}{(\rho+\delta)(\rho+\delta+\eta+\lambda\gamma)} - \frac{e^{-(\rho+\delta)\tau}}{(\eta+\lambda\gamma)(\rho+\delta)}\right)\\ &+ \eta h\frac{e^{-(\rho+\delta+\eta+\lambda\gamma)\tau}}{(\eta+\lambda\gamma)(\rho+\delta+\eta+\lambda\gamma)},\end{aligned}$$

so

$$\begin{aligned}\frac{\partial\mathcal{L}(\tau,\lambda)}{\partial\lambda} = \eta h &\left(-\frac{1}{(\rho+\delta)(\rho+\delta+\eta+\lambda\gamma)^2} + \frac{e^{-(\rho+\delta)\tau}}{(\rho+\delta)(\eta+\lambda\gamma)^2}\right)\\ &- \eta h\left(\frac{\tau e^{-(\rho+\delta+\eta+\lambda\gamma)\tau}}{(\eta+\lambda\gamma)(\rho+\delta+\eta+\lambda\gamma)} + \frac{e^{-(\rho+\delta+\eta+\lambda\gamma)\tau}}{(\eta+\lambda\gamma)^2(\rho+\delta+\eta+\lambda\gamma)}\right)\\ &- \eta h\frac{e^{-(\rho+\delta+\eta+\lambda\gamma)\tau}}{(\eta+\lambda\gamma)(\rho+\lambda^D+\eta+\lambda\gamma)^2}.\end{aligned}$$

Let $a = \eta+\lambda\gamma$ and $b = \rho+\delta$ so

$$\frac{\partial\mathcal{L}(\tau,\lambda)}{\partial\lambda} = \eta h\left(-\frac{1}{b(a+b)^2} + \frac{e^{-b\tau}}{ba^2}\right) - \eta h\frac{e^{-(a+b)\tau}}{a(a+b)}\left(\tau + \frac{2a+b}{a(a+b)}\right).$$

We want to show that

$$\frac{e^{-b\tau}}{ba^2} \le \frac{1}{b(a+b)^2} + \frac{e^{-(a+b)\tau}}{a(a+b)}\left(\tau + \frac{2a+b}{a(a+b)}\right). \tag{23}$$

Define $L(\tau)$ and $R(\tau)$ as the left- and right-hand-sides of (23), respectively. Note that $R(0) = L(0) = \frac{1}{ba^2}$. Hence, it is sufficient to show that the slope of $L(\tau)$ is lower than the slope of $R(\tau)$ for all τ. Note that

$$\frac{\partial L(\tau)}{\partial \tau} = -\frac{e^{-b\tau}}{a^2} \qquad \frac{\partial R(\tau)}{\partial \tau} = -\frac{e^{-(a+b)\tau}}{a}\left(\tau + \frac{1}{a}\right).$$

Hence, the slope of L is lower than the slope of R because $a\tau \geq \log(a\tau + 1)$.

(b) If there are no secondary markets; i.e., $\lambda = 0$, then the illiquidity cost represents the expected holding costs; i.e.,

$$\mathcal{L}(\tau, 0) = h\int_0^\tau e^{-\left(\rho+\lambda^D\right)y}\left(1 - e^{-\eta y}\right)dy.$$

(c) If secondary markets are totally liquid (i.e., $\lambda \to \infty$), then $\mathcal{L}(\tau, \lambda) = 0$.

5. **Liquidity is more important for long-term assets:** Recall that

$$\frac{\partial \mathcal{L}(\tau, \lambda)}{\partial \tau} = h\frac{\eta}{\eta + \lambda\gamma}e^{-(\rho+\delta)\tau}\left(1 - e^{-(\eta+\lambda\gamma)\tau}\right)$$
$$\frac{\partial \mathcal{L}(\tau, \lambda)}{\partial \tau} = \eta h e^{-(\rho+\delta)\tau}\int_0^\tau e^{-(\eta+\lambda\gamma)y}dy,$$

therefore,

$$\frac{\partial \mathcal{L}(\tau, \lambda)}{\partial \tau\lambda} = -\eta h e^{-(\rho+\delta)\tau}\int_0^\tau y e^{-(\eta+\lambda\gamma)y}dy \leq 0.$$

□

Finally, note that the price is positive if $e^{-(\rho+\delta)\tau} \geq \mathcal{L}(\tau)$. Recall that $\mathcal{L}(\tau) \leq h\frac{\eta}{(\rho+\delta)(\rho+\delta+\eta+\gamma\lambda)}$. Hence, a sufficient condition for having a positive price is

$$e^{-(\rho+\delta)\tau} \geq h\frac{\eta}{(\rho+\delta)(\rho+\delta+\eta+\gamma\lambda)},$$
$$\tau \leq \frac{1}{\rho+\delta}\log\left(\frac{(\rho+\delta)(\rho+\delta+\eta+\gamma\lambda)}{h\eta}\right).$$

A.1.3 Liquidity spread

Proof of Lemma 2. We show the following:

1. **The liquidity spread $cs^{liq}(\tau,\lambda)$ is increasing in maturity τ:**

$$\frac{\partial cs^{liq}(t,\lambda)}{\partial t}=\frac{1}{t^2}\log\left(1-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)\right)+\frac{e^{(\rho+\delta)t}}{t}\frac{(\rho+\delta)\mathcal{L}(t,\lambda)+\frac{\partial\mathcal{L}(t,\lambda)}{\partial t}}{1-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)}.$$

Recall that $\log(x)\geq\frac{x-1}{x}$. Hence

$$\log\left(1-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)\right)\geq\frac{-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)}{1-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)},$$

which implies that

$$\frac{\partial cs^{liq}(t,\lambda)}{\partial t}\geq\frac{1}{t^2}\frac{e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)}{1-e^{(\rho+\delta)t}\mathcal{L}(t,\lambda)}\left(t(\rho+\delta)+\frac{\partial\mathcal{L}(t,\lambda)}{\partial t}\frac{t}{\mathcal{L}(t,\lambda)}-1\right).$$

Let $\varepsilon_{\mathcal{L},t}=\frac{\partial\mathcal{L}(t,\lambda)}{\partial t}\frac{t}{\mathcal{L}(t,\lambda)}$, and note that

$$\varepsilon_{\mathcal{L},t}=t\left[e^{-(\rho+\delta)\tau}-e^{-(\rho+\delta+\eta+\lambda\gamma)t}\right]\left[\frac{1-e^{-(\rho+\delta)t}}{\rho+\delta}-\frac{1-e^{-(\rho+\delta+\eta+\lambda\gamma)t}}{\rho+\delta+\eta+\lambda\gamma}\right]^{-1}.$$

A sufficient condition is $t(\rho+\delta)+\varepsilon_{\mathcal{L},t}-1\geq 0$. Let $a=\rho+\delta$ and $b=\eta+\lambda\gamma$, and define

$$E(t,a,b)=t\left(a+\left[e^{-at}-e^{-(a+b)t}\right]\left[\frac{1-e^{-at}}{a}-\frac{1-e^{-(a+b)t}}{a+b}\right]^{-1}\right)-1.$$

It is easy to show numerically that $E(t,a,b)\geq 0$ for all $t,a,b\geq 0$. Hence, the liquidity spread is increasing in maturity. Finally, it is straightforward to see that the liquidity spread is decreasing in liquidity λ.

2. **The liquidity spread is increasing in the default intensity δ:** Note that

$$e^{(\rho+\delta)\tau}\mathcal{L}(\tau,\lambda)=\frac{\eta}{\eta+\lambda\gamma}\int_0^\tau e^{(\rho+\delta)(\tau-t)}\left(1-e^{-(\eta+\lambda\gamma)t}\right)dt$$

$$\frac{\partial\left(e^{(\rho+\delta)\tau}\mathcal{L}(\tau,\lambda)\right)}{\partial\delta}=\frac{\eta}{\eta+\lambda\gamma}\int_0^\tau(\tau-t)\,e^{(\rho+\delta)(\tau-t)}\left(1-e^{-(\eta+\lambda\gamma)t}\right)dt>0.$$

□

Proof of Lemma 3. The mid-price is

$$\frac{1}{2}\left(D^H(y;\lambda)+D^L(y;\lambda)\right)=e^{-(\rho+\delta)y}-\frac{1}{2}\left(h\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}+\left(\frac{\eta-\lambda\gamma}{\eta}\right)\mathcal{L}(y,\lambda)\right),$$

where

$$\left(\frac{\eta-\lambda\gamma}{\eta}\right)\mathcal{L}\left(y,\lambda\right)=h\frac{\eta-\lambda\gamma}{\lambda^{H}+\lambda\gamma}\left(\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}-\frac{1-e^{-\left(\rho+\delta+\lambda^{H}+\lambda\gamma\right)y}}{\rho+\delta+\eta+\lambda\gamma}\right).$$

The mid-price is

$$e^{-(\rho+\delta)y}-\frac{h}{\eta+\lambda\gamma}\left(\eta\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}-\frac{(\eta-\lambda\gamma)}{2}\frac{1-e^{-(\rho+\delta+\eta+\lambda\gamma)y}}{\rho+\delta+\eta+\lambda\gamma}\right).$$

Define the gains from trade as

$$GT\left(y\right)=h\frac{1-e^{-\left(\rho+\delta+\lambda^{H}+\lambda\gamma\right)y}}{\rho+\delta+\eta+\lambda\gamma},$$

so

$$BA\left(y\right)=GT\left(y\right)\left[e^{-\left(\rho+\lambda^{D}\right)y}-\frac{1}{\eta+\lambda\gamma}\left(h\eta\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}-\frac{(\eta-\lambda\gamma)}{2}GT\left(y\right)\right)\right]^{-1}$$

$$BA\left(y\right)=\left[\frac{e^{-(\rho+\delta)y}}{GT\left(y\right)}-h\frac{\eta}{\eta+\lambda\gamma}\frac{\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}}{GT\left(y\right)}+\frac{1}{2}\frac{\eta-\lambda\gamma}{\eta+\lambda\gamma}\right]^{-1}.$$

Note that $\frac{e^{-(\rho+\delta)y}}{GT(y)}$ is decreasing in y because of $e^{-(\rho+\delta)y}$ is decreasing and $GT\left(y\right)$ is increasing in y. Note that $\frac{\frac{1-e^{-(\rho+\delta)y}}{\rho+\delta}}{GT(y)}$ is increasing in y because the discount in GT is larger than in the numerator. Hence, with the negative sign it is decreasing. Therefore, everything in the square bracket is decreasing in y, and as it is to the power of -1, the $BA\left(y\right)$ is increasing in y. □

A.1.4 Free entry

Proof of Proposition 2. Gains from trade are

$$D^{H}\left(y;\lambda\right)-D^{L}\left(y;\lambda\right)=h\frac{1-e^{-c_{1}y}}{c_{1}}\qquad c_{1}=\rho+\delta+\eta+\lambda\gamma.$$

The buyer gets $(1-\gamma)$ of the gains from trade. Hence, the free entry condition reads

$$c=(1-\gamma)\int_{0}^{\tau}\beta\frac{\mu^{L}(y)}{\mu^{S}}h\frac{1-e^{-c_{1}y}}{c_{1}}dy,$$

and $\theta = \frac{\mu^S}{\mu^B}$. Also, recall that $\mu^S = \int_0^\tau \mu^L(y)\, dy$ Hence, the free entry condition is

$$c = \frac{(1-\gamma)\, h}{c_1} A\theta^\alpha \int_0^\tau \frac{\mu^L(y)}{\mu^S} \left(1 - e^{-c_1 y}\right) dy$$
$$c = \frac{(1-\gamma)\, h}{c_1} A\theta^\alpha \left(1 - \frac{\int_0^\tau e^{-c_1 y}\mu^L(y)\, dy}{\int_0^\tau \mu^L(y)\, dy}\right).$$

Define $c_2 = \eta + \delta + \lambda$ and note that

$$\int_0^\tau e^{-c_1 y}\mu^L(y)\, dy = \mu^0 \frac{\eta}{\eta+\lambda}\left(\frac{e^{-c_1\tau} - e^{-\delta\tau}}{\delta - c_1} - \frac{e^{-c_1\tau} - e^{-c_2\tau}}{c_2 - c_1}\right).$$

As a result, the ratio of integrals in the free-entry condition reads

$$\left(\frac{e^{-c_1\tau} - e^{-\lambda^D\tau}}{\delta - c_1} - \frac{e^{-c_1\tau} - e^{-c_2\tau}}{c_2 - c_1}\right)\left(\frac{1 - e^{-\tau\delta}}{\delta} - \frac{1 - e^{-(\eta+\delta+\lambda)\tau}}{\eta+\delta+\lambda}\right)^{-1}, \tag{24}$$

and the free-entry condition boils down to

$$c = \frac{(1-\gamma)\, h}{c_1} A\theta^\alpha \left(1 - \left(\frac{e^{-c_1\tau} - e^{-\delta\tau}}{\delta - c_1} - \frac{e^{-c_1\tau} - e^{-c_2\tau}}{c_2 - c_1}\right)\left(\frac{1 - e^{-\tau\delta}}{\delta} - \frac{1 - e^{-c_2\tau}}{c_2}\right)^{-1}\right).$$

First, note that it is easy to show that Equation (24) is increasing in τ. Next, consider $\tau = 0$. Note that the ratio of integrals in the free-entry condition is equal to 1 as

$$\begin{aligned}
&\lim_{\tau\to 0}\left(\frac{e^{-c_1\tau} - e^{-\delta\tau}}{\lambda^D - c_1} - \frac{e^{-c_1\tau} - e^{-c_2\tau}}{c_2 - c_1}\right)\left(\frac{1 - e^{-\delta\tau}}{\delta} - \frac{1 - e^{-c_2\tau}}{c_2}\right)^{-1} \\
&= \lim_{\tau\to 0}\left(\frac{-c_1 e^{-c_1\tau} + \delta e^{-\delta\tau}}{\delta - c_1} - \frac{-c_1 e^{-c_1\tau} + c_2 e^{-c_2\tau}}{c_2 - c_1}\right)\left(\frac{\delta e^{-\tau\delta}}{\delta} - \frac{c_2 e^{-c_2\tau}}{c_2}\right)^{-1} \\
&= \lim_{\tau\to 0}\left(\frac{(c_1)^2 e^{-c_1\tau} - (\delta)^2 e^{-\delta\tau}}{\delta - c_1} - \frac{(c_1)^2 e^{-c_1\tau} - (c_2)^2 e^{-c_2\tau}}{c_2 - c_1}\right)\left(\frac{-(\delta)^2 e^{-\tau\delta}}{\delta} - \frac{-(c_2)^2 e^{-c_2\tau}}{c_2}\right)^{-1} \\
&= \left(\frac{(c_1)^2 - (\delta)^2}{\delta - c_1} - \frac{(c_1)^2 - (c_2)^2}{c_2 - c_1}\right)\left(\frac{-(\delta)^2}{\lambda^D} - \frac{-(c_2)^2}{c_2}\right)^{-1} \\
&= \left(\frac{(c_1+\delta)\left(c_1 - \lambda^D\right)}{\delta - c_1} - \frac{(c_1+c_2)(c_1-c_2)}{c_2 - c_1}\right)(c_2 - \delta)^{-1} \\
&= \left(-(c_1+\delta) + (c_1+c_2)\right)(c_2-\delta)^{-1} = \left(c_2 - \lambda^D\right)(c_2-\delta)^{-1} = 1,
\end{aligned}$$

where we applied L'Hopital's rule in the second and third line. As a result, the free-entry condition is satisfied if and only if $\lim_{\tau\to 0}\theta = \infty$. Hence, $\lim_{\tau\to 0}\lambda = 0$. That is, $\lambda(0) = 0$.

Next, consider the case of $\tau \to \infty$. The ratio of integrals in the free-entry condition is equal to zero. Hence $c = \frac{h}{c_1}(1-\gamma) A\theta^{\alpha}$. Recall that $c_1 = \rho + \delta + \eta + \lambda\gamma$ and $\lambda = A\theta^{\alpha-1}$. Hence,

$$\rho + \delta + \eta + \gamma A\theta^{\alpha-1} = \frac{h(1-\gamma)}{c} A\theta^{\alpha}.$$

As $\alpha \in (0,1)$ the left-hand side is decreasing in θ and the right-hand side is increasing in θ. As a result, there exists a unique $\theta \in \mathbb{R}_+$. That is, $\lim_{\tau\to\infty} \lambda(\tau) = \overline{\lambda} \in \mathbb{R}_+$. □

A.2 Borrowers

Proof of Proposition 3. Let $J(\tau, \lambda)$ be the value of the firm with maturity τ and liquidity λ, and let $Z = \frac{\zeta}{\rho+\delta}$. The first-order condition is

$$\begin{aligned} J_{\tau}(\tau,\lambda) =& e^{-(\rho+\delta)\tau} Z \left(1 - (\rho+\delta)\tau\right) \\ &- e^{cs^{liq}(\lambda,\tau)\tau} \left[\frac{\partial I(\tau)}{\partial \tau} + (\Phi + I(\tau))\, cs^{liq}(\lambda,\tau) \left(1 + \varepsilon_{cs^{liq},\tau}(\lambda,\tau)\right)\right]. \end{aligned}$$

Note that τ is increasing in λ if the derivative of the first-order condition with respect to λ is positive

$$\begin{aligned} J_{\tau\lambda}(\tau,\lambda) =& -e^{cs^{liq}(\lambda,\tau)\tau} \frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} \tau \left[\frac{\partial I(\tau)}{\partial \tau} + (\Phi + I(\tau))\, cs^{liq}(\lambda,\tau) \left(1 + \varepsilon_{cs^{liq},\tau}(\lambda,\tau)\right)\right] \\ &- e^{cs^{liq}(\lambda,\tau)\tau} \frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} (\Phi + I(\tau)) \left(1 + \varepsilon_{cs^{liq},\tau}(\lambda,\tau)\right) \\ &- e^{cs^{liq}(\lambda,\tau)\tau} (\Phi + I(\tau))\, cs^{liq}(\lambda,\tau) \frac{\partial \varepsilon_{cs^{liq},\tau}(\lambda,\tau)}{\partial \lambda}. \end{aligned}$$

Recall that $\frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} \leq 0$, so the first and second terms are positive. However, the last term involves $\frac{\partial \varepsilon_{cs^{liq},\tau}(\lambda,\tau)}{\partial \lambda}$ for which we do not know the sign. We can write $J_{\tau\lambda}(\tau,\lambda)$ as

$$\begin{aligned} J_{\tau\lambda}(\tau,\lambda) =& -e^{cs^{liq}(\lambda,\tau)\tau} \frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} \frac{\partial I(\tau)}{\partial \tau} \tau - e^{cs^{liq}(\lambda,\tau)\tau} (\Phi + I(\tau)) \\ & \left[\frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} \left(1 + \varepsilon_{cs^{liq},\tau}(\lambda,\tau)\right) \left(\tau cs^{liq}(\lambda,\tau) + 1\right) + cs^{liq}(\lambda,\tau) \frac{\partial \varepsilon_{cs^{liq},\tau}(\lambda,\tau)}{\partial \lambda}\right]. \end{aligned}$$

The first term is positive. A sufficient condition for $J_{\tau\lambda}(\tau,\lambda) \geq 0$ is that the second term is also positive. This implies

$$\frac{\partial cs^{liq}(\lambda,\tau)}{\partial \lambda} \left(1 + \varepsilon_{cs^{liq},\tau}(\lambda,\tau)\right) \left(\tau cs^{liq}(\lambda,\tau) + 1\right) \leq -cs^{liq}(\lambda,\tau) \frac{\partial \varepsilon_{cs^{liq},\tau}(\lambda,\tau)}{\partial \lambda}.$$

This expression depends only on $cs^{liq}(\lambda,\tau)$. By Lemma 2 we can approximate the liquidity spread as a linear function increasing in τ and decreasing in λ. Let $cs^{liq}(\lambda,\tau) = c_\tau \tau + c_\lambda \lambda$ with $c_\tau \geq 0$ and $c_\lambda \leq 0$. Then $\varepsilon_{cs^{liq},\tau}(\lambda,\tau) = \frac{c_\tau \tau}{c_\tau \tau + c_\lambda \lambda}$ and $\frac{\partial \varepsilon_{cs^{liq},\tau}(\lambda,\tau)}{\partial \lambda} = -\frac{c_\tau \tau c_\lambda}{(c_\tau \tau + c_\lambda \lambda)^2}$. The sufficient condition reads

$$\left(c_\tau \tau + cs^{liq}(\lambda,\tau)\right) cs^{liq}(\lambda,\tau)\,\tau + cs^{liq}(\lambda,\tau) \geq 0,$$

which is satisfied. Therefore, $J_{\tau\lambda}(\tau,\lambda) \geq 0$ and $\frac{\partial \tau(\lambda)}{\partial \lambda} \geq 0$. Finally it is straightforward to see that $\underline{\tau} = \tau(0) \leq \lim_{\lambda\to\infty} \tau(\lambda) = \tau^* < \infty$. □

A.3 Existence of equilibrium

Proof of Proposition 4. First, Proposition 2 defines a schedule for the lenders $\tau^L(\lambda)$. Note that $\tau^L(0) = 0$, and there exists $\overline{\lambda}$ such that $\tau^L(\overline{\lambda}) = \infty$.

Second, Proposition 3 define $\tau^B(\lambda)$, and notice that $\tau^B(0) = \underline{\tau} > 0$ and $\tau^B(\lambda) \geq 0$ for all λ.

Finally, define $F(\lambda) = \tau^L(\lambda) - \tau^B(\lambda)$, and note that $F(0) = -\underline{\tau} < 0$ and $F(\overline{\lambda}) = \infty$. Hence, as F is continuous, Bolzano's theorem implies that there exits $\lambda^\star$ such that $F(\lambda^\star) = 0$, which defines the equilibrium. □

B Extensions

This Appendix extends the model in several dimension and complements the main analysis.

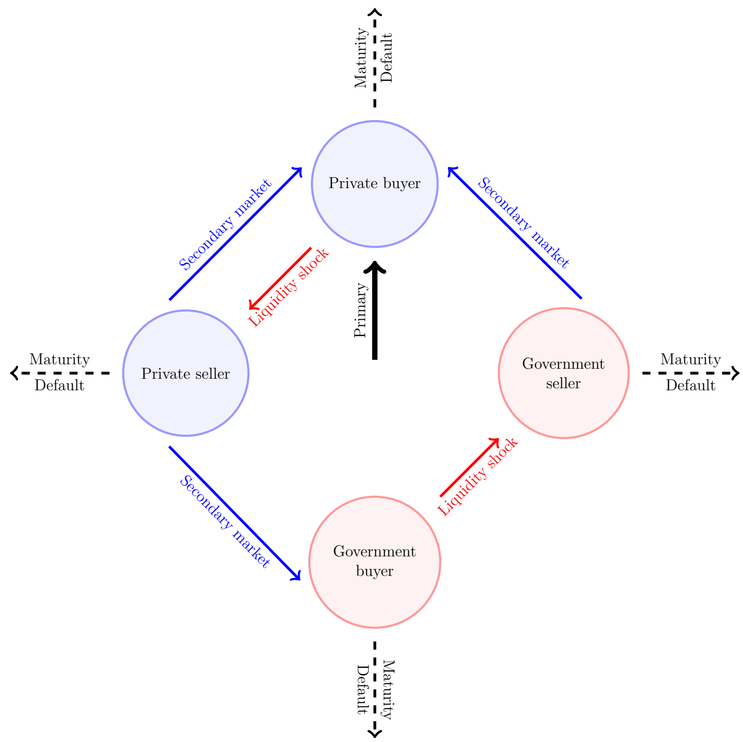
B.1 Government-Sponsored Intermediaries

The government agency intermediates assets in the secondary market to improve the liquidity of financial markets. Government agents are subject to the same idiosyncratic risk of holding costs as private agents, so they act as both buyers and sellers in the secondary market.[31] One possible interpretation for these idiosyncratic shocks for public agents can be balance sheet requirements, such as the priority sector lending in India discussed before. However, the government can choose different prices than those charged in private meetings. If they buy (sell) at a high (low) price they will run a deficit, which is financed by distortionary taxes on the corporate sector.

Figure 8 shows a schematic representation of the model with GSIs. Private sellers can sell both to private and government agents and private buyers can buy in the secondary market

[31] The formulation of government agents is similar to Aiyagari et al. (1996).

Figure 8: **Schematic representation of GSIs**



from either private or government sellers. In this section, we describe the key features of the model with GSIs while Appendix B.1.3 contains additional details.

The government has four instruments: the size of GSIs, prices for buying and selling for their trading agents, and the corporate tax rate. The objective is to maximize aggregate steady-state welfare subject to running a balanced budget and equilibrium conditions. Recall that both primary and secondary financial markets are competitive—i.e., participants make zero profits in expectation. However, the production sector—i.e., the borrowers—have positive profits in equilibrium. Hence, we define the objective of the government as maximizing the value of the corporate sector.

Matching There is random matching between sellers and buyers. In the benchmark policy we assume that both government and private agents have the same efficiency to find a counterpart. For robustness exercises in Section B.1.2, we consider a general formulation in which government and private agents may differ in their efficiency to find a counterpart. Let $e^{i,j}$ be the efficiency for $i = p, g$ (private and government agents, respectively) of $j = b, s$ (buy and sell, respectively). In the benchmark model we assume that $e^{i,j} = 1$ for all i, j.

The total mass of sellers, μ^s, is composed of private and government agents. Private sellers are $\mu^{p,s} = \int_0^\tau e^{p,s} \mu^{p,l}(y)dy$, where $\mu^{p,l}(y)$ is the measure of low-valuation private agents holding an asset and willing to sell. Similarly, government sellers are $\mu^{g,s} = \int_0^\tau e^{g,s}\mu^{l,g}(y)dy$, where $\mu^{l,g}(y)$ is the measure of low-valuation government agents holding an asset and willing to sell. The total measure of buyers includes private buyers $\mu^{p,b}$, determined by a free-entry condition, and government buyers $\mu^{g,b}$, which is a policy instrument chosen by the government. Hence, $\mu^b = e^{p,b}\mu^{p,b} + e^{g,b}\mu^{g,b}$.

The market tightness is $\theta = \frac{\mu^s}{\mu^b}$, which affects the buying and selling intensities $\beta = A\theta^\alpha$ and $\lambda = A\theta^{\alpha-1}$, respectively. Let λ^{s-b} be the intensity at which a seller of type $s = p, g$ meets a buyer of type $b = p, g$. Similarly, let $\beta^{s-b}(y)$ be the intensity at which a buyer of type $b = p, g$ meets a seller of type $s = p, g$ with an asset of time-to-maturity y. The matching technology implies that

$$\lambda^{p-p} = \lambda e^{p,s} e^{p,b} \frac{\mu^{p,b}}{\mu^b} \qquad \lambda^{p-g} = \lambda e^{p,s} e^{g,b} \frac{\mu^{g,b}}{\mu^b} \qquad \lambda^{g-p} = \lambda e^{g,s} e^{p,b} \frac{\mu^{p,b}}{\mu^b}$$

$$\beta^{p-p}(y) = \beta e^{p,s} e^{p,b} \frac{\mu^{l,p}(y)}{\mu^s} \quad \beta^{g-p}(y) = \beta e^{g,s} e^{p,b} \frac{\mu^{l,g}(y)}{\mu^s} \quad \beta^{p-g}(y) = \beta e^{p,s} e^{g,b} \frac{\mu^{l,p}(y)}{\mu^s}.$$

Finally, we have to specify what happens after a meeting between a government buyer and a government seller. The idea is to interpret the government as a large player and private agents as atomistic. However, for tractability, we assume that all investors can hold either zero or one asset. To bypass this restriction, we assume that a government seller cannot trade with a government buyer; i.e., $\lambda^{g-g} = 0$. Note that this is a conservative assumption as the cost of the policy is smaller if intra-government trades can occur. In fact, Section B.1.2 solves the model with this type of trade and shows that there are larger effects.

Prices in secondary markets There are three types of meetings in secondary markets. Let $P^{S,s-b}(y)$ be the price when a seller of an asset with time-to-maturity y of type $s = p, g$ meets a buyer of type $b = p, g$. In a meeting between private agents, the price is determined by Nash Bargaining in which the seller has bargaining power γ so

$$P^{S,p-p}(y) = D^L(y) + \gamma(D^H(y) - D^L(y)).$$

The prices that involve either a government buyer or seller are determined by the government. In the quantitative solution we restrict prices to be in the following parametric family:

$$P^{S,g-p}(y) = D^L(y) + \gamma^{g,s}(D^H(y) - D^L(y)), \tag{25}$$
$$P^{S,p-g}(y) = D^L(y) + \gamma^{g,b}\left(D^H(y) - D^L(y)\right), \tag{26}$$

and let the government choose $\gamma^{g,s}$ and $\gamma^{g,b}$ in $[0, 1]$. Note that prices are similar to those in private meetings but that the government can choose a different bargaining power. As we will show latter, it is optimal to set $\gamma^{g,s} = 0$ and $\gamma^{g,b} = 1$. This implies that the government gives all the bargaining power to the private sector—i.e., the government buys at a *high* price and sells at a *low* price.

Of course, this is an important restriction on government prices, but it follows from the objective of finding a lower bound on the effects of the policy. For example, one can use the model with segmented markets presented in Appendix B.3 and allow the government to set different prices according to maturity. However, we will show that even without this flexibility, the effects of GSIs are quantitatively significant and the extension of targeting prices according to maturity is likely to improve the results from the lower bound identified in this exercise. Importantly, note that this alternative specification would work through the same channel as the mechanism described in the benchmark policy.

Private valuations The value of holding an asset for a high-valuation private agent is equivalent to the benchmark model, Equation (9). However, the value of a low-valuation private agent is different as now the agent can sell the asset to both private and government buyers. Under the government prices specified in (25), the price the government offers is equivalent to that offered in private meetings but in which the seller has a different bargaining power. Hence, the value for a low-valuation agent is equal to the benchmark model, Equation (10), with an augmented selling intensity: $\lambda = \lambda^{p-p}\gamma + \lambda^{p-g}\gamma^{g,b}$.

Let λ^{GSI} and λ^{EQ} be the equilibrium liquidity in the economy with and without GSIs, respectively. If $\lambda^{GSI} > \lambda^{EQ}$, Lemma 2 implies that the liquidity spread will be lower in an economy with GSIs. However, borrowers have to pay a distortionary tax to finance the intervention. Hence, ex-ante, we don't know which policies will increase welfare for borrowers.

Private buyers can meet with private and government sellers. The free entry condition is

$$c = \int_0^\tau \beta^{p-p}(y) \left(D^H(y) - P^{S,p-p}(y)\right) dy + \int_0^\tau \beta^{g-p}(y) \left(D^H(y) - P^{S,g-p}(y)\right) dy.$$

Cost of GSIs The government runs a balanced budget. The constraint is

$$\begin{aligned} &\mu^\pi(\tau) x^c f(\tau) + \left[\mu^{g,h}(0) + \mu^{g,l}(0)\right] + \lambda^{g-p} \int_0^\tau \mu^{g,l}(y) P^{S,g-p}(y) y \\ =&\mu^{g,b} c + \mu^{g,b} \int_0^\tau \beta^{p-g}(y) P^{S,p-g}(y) dy + h \int_0^\tau \mu^{g,l}(y) dy. \end{aligned} \tag{27}$$

The left-hand side of Equation (27) represents the government's income. First, the government charges a proportional corporate tax x^c to producing firms μ^f, where flow profits are $\pi(\tau)$.

Second, some of the securities held by government agents mature. Third, some low-valuation government agents sell the securities to the private sector.

The right-hand side of Equation (27) captures the expenditures. A measure $\mu^{g,b}$ of agents are searching in secondary markets, and some of them buy a bond. Moreover, some government agents are low-valuation and have to pay the holding cost h.

Optimal policy The objective of the government is to maximize steady-state profits of the production sector subject to the equilibrium conditions and the budget constraint (27)[32]

$$\max_{x^c,\mu^{g,b},\gamma^{g-b},\gamma^{g-s}} \mu^f(\tau)e^{-(\rho+\delta)\tau}\left((1-x^c)F(\tau)-I(\tau)e^{r(\tau)\tau}\right) \quad \text{s.t. (27) and equilibrium conditions.}$$

GSIs cause both a direct and an equilibrium effect. On the one hand, a larger intervention needs higher taxes, which lower welfare. On the other hand, if the policy increases the equilibrium liquidity, credit spreads for long-term borrowing decline, which benefits borrowers. Therefore, the optimal policy solves the trade-off between these two effects.

B.1.1 Optimal GSIs in the US

First, consider the optimal policy under the calibration for the US. The bargaining power when the government acts as a buyer, $\gamma^{g,b}$, directly affects the value of low- and high-valuation private agents. The optimal policy sets $\gamma^{g,b}=1$ so private sellers get more gains from trade when trading with the government. This generates a direct effect on increasing the value of private agents in the financial sector and reduces financial costs for the production sector.

The bargaining power when the government acts as a seller, $\gamma^{g,s}$, has a direct effect on the incentives of private agents to search in the secondary market. The optimal value is $\gamma^{g,s}=0$, i.e., the private buyer gets larger gains. Given the results in this section, the exercises set $\gamma^{g,b}=1$ and $\gamma^{g,s}=0$ and let the government choose $\mu^{g,b}$ and the tax rate.[33]

Finally, the measure of government agents searching in the secondary markets is optimally chosen to maximize the welfare gains. If $\mu^{g,b}=0$, the economy is equivalent to no intervention, while as $\mu^{g,b}$ increases, the tax rate also increases to balance the budget. Under the optimal policy, there is an increase in liquidity, which generates a drop on the five-year spread from 139 to 96 basis points. As a result, the optimal maturity increases from 5.00 to 5.07, and the welfare gains are about 0.77% (first and second row of Table 6).

[32]We consider steady-state welfare because the transitions involve manipulating the boundary conditions of the distributions. However, note that this is a conservative assumption because during a transition, old generations holding a security issued before the intervention are better off because asset prices increase.

[33]In fact, in all exercises we verified that if the government can choose bargaining power then it chooses these values. However, this restriction simplifies the description of the results without adding additional intuition.

B.1.2 Robustness: Alternative Policies

The intervention considered so far should be thought of as a lower bound on the effects of GSIs. Table 6 explores alternative assumptions that can improve the effects of government interventions for two levels of financial frictions. The first panel considers a matching efficiency at the level calibrated for the US, while the second panel considers an economy with trading frictions similar to Argentina.

First, consider government agents that are more efficient at searching for counterparts. The third and fourth rows of each panel of Table 6 show the result of increasing the search efficiency of government agents by 10% and 50%, respectively (i.e., $e^{g,b} = e^{g,s} = 1.1$ and $e^{g,b} = e^{g,s} = 1.5$, respectively). Overall, the results show that as the efficiency of the government increases, the intervention becomes more effective in increasing the liquidity of the economy; the yield curve flattens even more; and firms issue at longer maturities.

Finally, recall that the benchmark policy assumes that $\lambda^{g-g} = 0$; i.e., a government seller cannot trade with a government buyer. For a given size $\mu^{g,b}$, the cost of GSIs decreases if the government can reallocate securities among its trading agents. The last row of Table 6 shows that if government agents can trade among themselves, GSIs are more efficient and the effects on credit spreads, maturity, and welfare improve.

There are legitimate reasons to imagine that government agents might have more flexibility than private agents. Hence, the results of the benchmark policy should be considered as a lower bound on the implications for GSIs. For example, Table 6 shows that the gains from government intervention can be larger if government agents are more efficient at finding counterparts or can trade among themselves.

B.1.3 Proofs

This appendix describes how to solve the distribution of financiers with GSIs. The total assets with time-to-maturity t are $\mu(t) = \mu^0 e^{-\delta t}$. These assets are held by four types of agents: $\mu(t) = \mu^{p,h}(t) + \mu^{p,l}(t) + \mu^{g,h}(t) + \mu^{g,l}(t)$. The laws of motions for the private sector are

$$
\begin{aligned}
-\dot{\mu}^{p,h}(t) &= -(\eta+\delta)\,\mu^{p,h}(t) + \left(\beta^{p-p}(t) + \beta^{g-p}(t)\right)\mu^{p,b} \\
-\dot{\mu}^{p,l}(t) &= \eta\mu^{p,h}(t) - \left(\delta + \lambda^{p-p} + \lambda^{p-g}\right)\mu^{p,l}(t),
\end{aligned}
$$

Table 6: **GSI: Alternative policies**

	Liquidity	**5-year spread**	**Maturity**	**Welfare gains**	**Profitability**
Low trading frictions (US)					
No GSIs	13.00	139	5.00		
Benchmark policy	18.89	96	5.07	0.77	0.55
Gov. 10% more efficient	19.18	95	5.07	0.83	0.57
Gov. 50% more efficient	20.04	91	5.08	1.03	0.63
Gov. transactions	21.26	86	5.08	1.21	0.70
High trading frictions (Argentina)					
No GSIs	3.63	468	4.49		
Benchmark policy	5.25	332	4.68	1.61	2.04
Gov. 10% more efficient	5.34	326	4.69	1.78	2.13
Gov. 50% more efficient	5.62	311	4.71	2.29	2.39
Gov. transactions	6.03	291	4.75	3.37	2.77

with boundary conditions $\mu^{p,h}(\tau) = \mu^0$ and $\mu^{p,l}(\tau) = 0$. The law of motions for government agents are

$$
\begin{aligned}
-\dot{\mu}^{g,h}(t) &= -(\eta+\delta)\,\mu^{g,h}(t) + \left(\beta^{p-g}(t) + \beta^{g-g}(t)\right)\mu^{g,b} \\
-\dot{\mu}^{g,l}(t) &= \eta\mu^{g,h}(t) - \left(\delta + \lambda^{g-p} + \lambda^{g-g}\right)\mu^{g,l}(t),
\end{aligned}
$$

with boundary conditions $\mu^{g,h}(\tau) = \mu^{g,l}(\tau) = 0$. Matching implies

$$
\begin{aligned}
\mu^{p,b}\beta^{p-p}(t) &= \mu^{p,l}(t)\,\lambda^{p-p} \\
\mu^{p,b}\beta^{g-p}(t) &= \mu^{g,l}(t)\,\lambda^{g-p} \\
\mu^{g,b}\beta^{p-g}(t) &= \mu^{p,l}(t)\,\lambda^{p-g} \\
\mu^{g,b}\beta^{g,b-g}(t) &= \mu^{g,l}(t)\,\lambda^{g-g}.
\end{aligned}
$$

Define $\mu(t) = \left[\mu^{p,h}(t), \mu^{p,l}(t), \mu^{g,h}(t), \mu^{g,l}(t)\right]$. The boundary condition is $\mu(\tau) = [\mu^0, 0, 0, 0]$ and the system is $\dot{\mu}(t) = A\mu(t)$ where

$$
A = \begin{bmatrix}
\eta+\delta & -\lambda^{p-p} & 0 & -\lambda^{g-p} \\
-\eta & \delta + \lambda^{p-p} + \lambda^{p-g} & 0 & 0 \\
0 & -\lambda^{p-g} & \eta+\delta & -\lambda^{g-g} \\
0 & 0 & -\eta & \delta + \lambda^{g-p} + \lambda^{g-g}
\end{bmatrix}.
$$

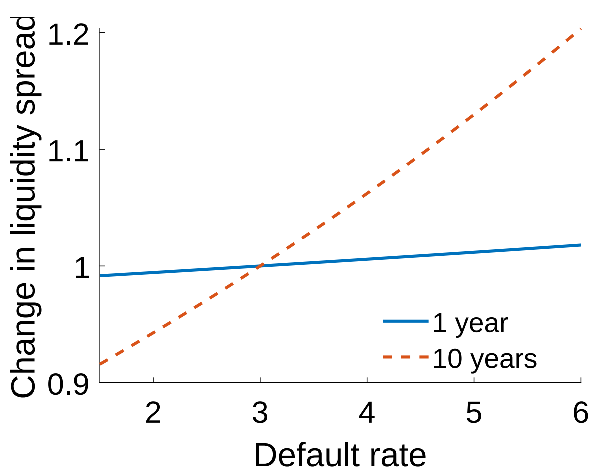
The solution of this system is standard. The only caveat is that we should pay attention to the real and complex eigenvalues of the matrix A.

B.2 Rollover and Default

Default affects credit spreads and investment decisions. Lemma 2 shows that the liquidity spread increases with the default intensity. Quantitatively, long-term rates react more than short-term rates to changes in default. Figure 9 shows how the default rate affects the liquidity spread at maturities of 1 and 10 years, relative to the benchmark of $\delta = 0.03$. The liquidity spread for long maturities reacts more to changes in the default rate. Hence, when δ increases, the yield curve shifts upward because of both default and liquidity. As a result, the firm chooses shorter-term projects (see rows four to six in Table 5).

Moreover, when there is no default risk and secondary markets are centralized, firms have no incentive to issue short-term debt regardless of the issuance cost. However, when default risk is positive, even if secondary markets are centralized, firms choose to rollover debt when the issuance cost is not too high. Hence, both default risk and trading frictions shape rollover choices.

Figure 9: **Liquidity-default interactions**



Note: The figure shows how the liquidity spread at 1 and 10 years changes with δ with respect to the benchmark of 0.03.

B.3 Segmented Markets

In the benchmark model, assets of different maturities are traded in a single secondary market. A potential concern could be that assets with short maturities, with small gains from trade, preclude the entry of buyers into the secondary market. To address this issue, we study an economy with secondary markets segmented by the time-to-maturity of assets. The main takeaway is that even though the market tightness for short-term bonds increases, the tightness for long-term assets remains similar to the case with only one market. Hence, the secondary market in the benchmark model is effectively a market for long-term assets. The intuition for this result is that, in equilibrium, the single market is not dominated by short-term assets, so there are always sufficient gains from trade.

Intuitively, in long-term markets, there are more gains from trade and therefore more entry of buyers. However, because there is Nash Bargaining over the gains from trade and buyers keep only a fraction $(1-\gamma)$ of the gains, the increase in the entry of buyers into long-term markets is not enough to compensate for the increase in the importance of the secondary markets for longer securities. As a result, the yield curve increases with maturity even with segmented markets. We describe the key features of the model and relegate to Appendix B.3.1 the full characterization of this extension.

Let τ be the initial maturity and consider the case in which secondary markets are segmented in N markets. Let $0=\tau_1<\cdots<\tau_{N+1}=\tau$, so each market $j=1,\ldots,N$ trades assets with time-to-maturity $t\in[\tau_j,\tau_{j+1}]$.

Matching and distribution of agents Let $\mu^j(y)=\left[\mu^{H,j}(y),\mu^{L,j}(y)\right]$ be the measure of high- and low-valuation agents holding an asset with time-to-maturity t in market j. We start with market N and solve for the distribution of agents backwards. The boundary condition is $\mu^N(\tau)=[\mu^0,0]$. Next, we iterate toward markets of shorter maturities with boundary conditions $\mu^j(\tau_{j+1})=\mu^{j+1}(\tau_{j+1})$ for $j=1,\ldots,N-1$. Lemma 4 characterizes the distribution of agents in each market.

Lemma 4. *The measure of agents for markets $j=1,\ldots,N$ is given by the following backward recursion:*

$$\begin{bmatrix}\mu^{H,N+1}(\tau)\\ \mu^{L,N+1}(\tau)\end{bmatrix}=\begin{bmatrix}\mu^0\\ 0\end{bmatrix}$$

and

$$\begin{aligned}\mu^{H,j}(y)=&\frac{\eta}{\eta+\lambda^j}\left[\frac{\lambda^j}{\eta}e^{\delta(y-\tau_{j+1})}\left(\mu^{H,j+1}(\tau_{j+1})+\mu^{L,j+1}(\tau_{j+1})\right)\right.\\&\left.-e^{\left(\eta+\lambda^j+\delta\right)(y-\tau_{j+1})}\left(-\mu^{H,j+1}(\tau_{j+1})+\lambda^j\mu^{L,j+1}(\tau_{j+1})\right)\right]\\ \mu^{L,j}(t)=&\frac{\eta}{\eta+\lambda^j}\left[e^{\delta(y-\tau_{j+1})}\left(\mu^{H,j+1}(\tau_{j+1})+\mu^{L,j+1}(\tau_{j+1})\right)\right.\\&\left.+e^{\left(\eta+\lambda^j+\delta\right)(y-\tau_{j+1})}\left(-\mu^{H,j+1}(\tau_{j+1})+\lambda^j\mu^{L,j+1}(\tau_{j+1})\right)\right],\end{aligned}$$

where λ^j is the selling intensity in market $j=1,\ldots,N$.

Valuations Let $D^j(y) = \left[D^{H,j}(y), D^{L,j}(y)\right]$ be the values for high- and low-valuation agents of holding an asset with time-to-maturity y in market $j = 1, \ldots, N$. To solve for the value of holding the asset, start with the first market, in which the boundary condition is that at maturity the value is equal to one, and then iterate forward, toward longer-term markets. The boundary condition for market $j = 1$ is $D^1(\tau_1) = [1, 1]$. Value matching for market $j = 2, \ldots, N$ implies $D^j(\tau_j) = D^{j-1}(\tau_j)$, and the Hamilton-Jacobi-Bellman equations are the same as in the benchmark model, Equations (9) and (10).

Free entry Free entry in each market implies that

$$c = (1 - \gamma) \int_{\tau_j}^{\tau_{j+1}} \beta^j(y) \left(D^{H,j}(y) - D^{L,j}(y)\right) dy,$$

where β^j is the intensity at which a buyer finds a seller in market $j = 1, \ldots, N$. Appendix B.3.1 provides analytical solutions for the value functions and the free-entry condition.

Results The first panel of Figure 10 shows the market tightness relative to the case of only one market when $N = 2$ and $N = 3$. With segmentation, markets for short-term assets are tighter (more sellers to buyers), as there are fewer gains from trade. However, for long-term bonds we find a tightness similar to the case of no segmentation. The second panel repeats the exercise under different degrees of segmentation ($N = 1, \ldots, 50$). Note that even with 50 different markets, the tightness for markets with maturity above four years is almost identical to the case of no segmentation.

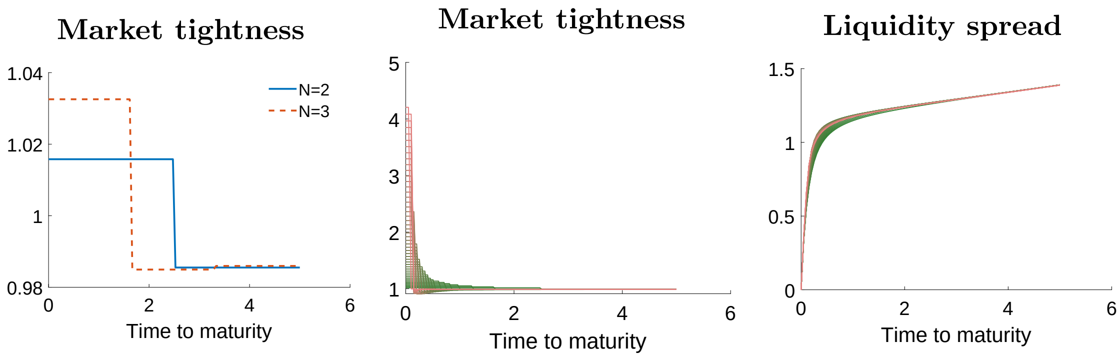
The third panel of Figure 10 shows the effects on the liquidity spread for different models with $N = 1$ to $N = 50$. As the market tightness after four years is identical in all these models, the implied liquidity spread is also the same. For short-term assets (maturities up to 4 years), there are some differences in the market tightness. However, they generate small variations in the yield curve. Therefore, we conclude that the secondary market in the benchmark model with $N = 1$ is effectively a market for long-term assets.

B.3.1 Proofs

Distributions of financiers

Proof of Lemma 4. Total assets are $\mu^j(t) = e^{(t-\tau)\delta}\mu^0$. The evolution for high- and low-valuation

Figure 10: **Segmented markets**



Note: The first (second) panel shows the market tightness relative to no segmentation for $N = 2, 3$ ($N = 2, \ldots, 50$). The third panel shows the liquidity spread for $N = 1, \ldots, 50$. In the second and third panels green lines are economies with larger N.

agents in market j are

$$-\dot{\mu}^{H,j}(t) = -(\eta+\delta)\,\mu^{H,j}(t) + \mu^{L,j}\beta^{j}(t)$$
$$-\dot{\mu}^{L,j}(t) = \eta\mu^{H,j}(t) - \left(\delta+\lambda^{j}\right)\mu^{L,j}(\tau_j).$$

Matching implies that $\mu^{B,j}\beta^{j}(t) = \mu^{L,j}(t)\,\lambda^{j}$. Hence, the system is

$$\dot{\mu}^{j}(t) = \begin{bmatrix} \eta+\delta & -\lambda^{j} \\ -\eta & \delta+\beta^{j} \end{bmatrix} \begin{bmatrix} \mu^{H,j}(t) \\ \mu^{L,j}(t) \end{bmatrix}$$

with eigenvalues δ and $\eta+\lambda^{j}+\delta$. Define V^{j} to be the matrix with the eigenvectors and R^{j} the diagonal matrix with the eigenvalues and $B^{j} = \left(V^{j}\right)^{-1}\mu^{j+1}(\tau_{j+1})$. Then

$$\mu^{H,j}(t) = \sum_{i=1}^{2} e^{R^{j}(i)(t-\tau_{j+1})}B^{j}(i)\,V^{j}(1,i)$$
$$\mu^{L,j}(t) = \sum_{i=1}^{2} e^{R^{j}(i)(t-\tau_{j+1})}B^{j}(i)\,V^{j}(2,i).$$

For $j = 1, \ldots, N-1$ we have that

$$\begin{aligned}
\mu^{H,j}(t) =& \frac{\eta}{\eta+\lambda^j}\left[\frac{\lambda^j}{\eta} e^{\delta(t-\tau_{j+1})}\left(\mu^{H,j+1}(\tau_{j+1}) + \mu^{L,j+1}(\tau_{j+1})\right)\right] \\
&- \frac{\eta}{\eta+\lambda^j}\left[-e^{\left(\eta+\lambda^j+\delta\right)(t-\tau_{j+1})}\left(-\mu^{H,j+1}(\tau_{j+1}) + \lambda^j \mu^{L,j+1}(\tau_{j+1})\right)\right] \\
\mu^{L,j}(t) =& \frac{\eta}{\eta+\lambda^j}\left[e^{\delta(t-\tau_{j+1})}\left(\mu^{H,j+1}(\tau_{j+1}) + \mu^{L,j+1}(\tau_{j+1})\right)\right] \\
&+ \frac{\eta}{\eta+\lambda^j}\left[+e^{\left(\eta+\lambda^j+\delta\right)(t-\tau_{j+1})}\left(-\mu^{H,j+1}(\tau_{j+1}) + \lambda^j \mu^{L,j+1}(\tau_{j+1})\right)\right]
\end{aligned}$$

□

Value functions Let $Z^j(t) = D^{H,j}(t) - D^{L,j}(t)$ and $c_j = \rho + \delta + \eta + \gamma\lambda^j$, then $c_j Z^j(t) = h - \dot{Z}^j(t)$. The solution is $Z^j(t) = A^{Z,j} e^{-c_j t} + B^{Z,j}$ with $B^{Z,j} = \frac{h}{c_j}$, and the boundary condition pins down $A^{Z,j}$.

For $j = 1$ the boundary condition is $D^{H,1}(\tau_1) = D^{L,1}(\tau_1) = 1$ so $A^{Z,1} = -\frac{h}{c_1}$.

For $j = 2, \ldots, N$ we have that $Z^j(\tau_j) = Z^{j-1}(\tau_j)$, which implies $A^{Z,j} = e^{c_j \tau_j}\left(A^{Z,j-1} e^{-c_{j-1}\tau_j} + \frac{h}{c_{j-1}} - \frac{h}{c_j}\right)$.

Next, we can solve for the value of high- and low-valuation agents using Z^j and the initial conditions. For high-valuation agents

$$(\rho+\delta) D^{H,j}(t) = -\dot{D}^{H,j}(t) - \eta\left(A^{Z,j} e^{-c_j t} + \frac{h}{c_j}\right).$$

The solution is $D^{H,j}(t) = A^{H,j} + B^{H,j} e^{-\left(\rho+\lambda^D\right)t} + C^{H,j} e^{-c_j t}$, with $A^{H,j} = -\frac{\eta h}{(\rho+\delta)c_j}$, $C^{H,j} = \frac{\eta A^{Z,j}}{\eta+\gamma\lambda^j}$, and the boundary condition pins down $B^{H,j}$.

For $j = 1$, we have that $D^{H,1}(\tau_1) = 1$ and $\tau_1 = 0$, so $B^{U,1} = 1 - A^{U,j} - C^{U,j}$. For $j = 2, \ldots, N$, we have that $D^{H,j}(\tau_j) = D^{H,j-1}(\tau_j)$ so

$$B^{H,j} = e^{(\rho+\delta)\tau_j}\left(A^{H,j-1} - A^{H,j} + B^{H,j-1} e^{-(\rho+\delta)\tau_j} + C^{H,j-1} e^{-c_{j-1}\tau_j} - C^{H,j} e^{-c_j \tau_j}\right),$$

which defines a recursion in $B^{H,j}$.

Free entry The free-entry condition in each market is

$$c_j = (1-\gamma)\int_{\tau_j}^{\tau_{j+1}} \beta^j(t)\left(D^{H,j}(t) - D^{L,j}(t)\right) dt,$$

where $\beta^j(t) = A\left(\theta^j\right)^\alpha \frac{\mu^{C,j}(t)}{\mu^{C,j}}$, and both the measures and value functions are the sum of exponential functions. Hence, it is easy to solve for the integrals on the free-entry condition in each market.

B.4 Bank Loans and other securities

The benchmark model assumes that firms borrow from corporate bond markets, but similar trading frictions apply to other sources of external finance such as bank loans, venture capital, or private equity funds. The main reason to study corporate bonds is that these assets are already well studied in the literature about trading frictions and we can quantify the effects (e.g., He and Milbradt, 2014). However, for these alternative sources of finance there are also frictional secondary markets. For example, in 2006 the U.S. secondary loan market reached a volume of more than 200 billion (see Drucker and Puri, 2008; Altman et al., 2010).

We can interpret the financial sector of the model as bank lending in which the primary market represents the origination of the loan and the secondary market is a market for loans across banks. A moment commonly used to compare intermediation costs across countries is the bank's net interest margin, which measures the difference between interest income and payments to lenders using bank balance-sheet data from Bankscope (the data is available at The World Bank Global Financial Development Database). For example, Greenwood et al. (2013) attribute all of the spreads to the intermediation costs related to acquiring information about borrowers. Through the lens of this paper, however, the net interest margin can also reflect the illiquidity cost that banks will charge at origination of the loan, taking into account that they might need to later sell the loan in a frictional secondary market. In the data, net interest margins are about 300 basis points higher, while the maturity of loans is 3.5 years shorter in emerging countries than in advanced ones.[34] According to this data, trading frictions also seems to be higher in emerging economies, implying higher interest rates for longer-term loans and inducing firms to borrow and invest at shorter maturities.

A wider interpretation is that the life-cycle of cash-flows matters for liquidity spreads. For example, consider two stocks with different cash-flow structures. On the one-hand imagine a "growth" company that is not paying dividends in the short-run. On the other-hand, consider a "mature" firm that pays a smooth and roughly constant dividend stream to shareholders. The growth stock looks like a longer-term bond relative to the mature stock. By dividend payments the investor is getting out of the position without trading in the secondary market. There are two differences between the growth and mature stock when an investor receives a shock and becomes low valuation. First, due to previous dividend payments, the mature investor has less exposure to the security. Second, as future dividends arrive, the mature investor is getting out of the financial position without trading in the secondary market. Hence, the growth stock will have a higher liquidity spread than the mature stock. This example shows that the life-cycle of

[34]While we cannot decompose the net interest margin into the which fraction correspond to agency frictions and which fractions to liquidity components, we conjecture that an important fraction of it can be attributed to liquidity considerations. We left open the decomposition for future research.

cash-flows matters for liquidity spreads and similar results would arise in a model with equity finance as long as there are trading frictions in the secondary market.

C Additional Empirical Results

C.1 Non-linear effects on credit spreads

In this appendix we extend the main empirical specification to allow for non-linear effects on maturity. We split the maturity difference in five groups (less than 3 years, 3 to 6 years, 6 to 9 years, 9 to 12 years, and more than 12 years) and estimate fixed effects at the group level as:

$$s_{i,t,m_j} - s_{i,t,m_1} = \gamma_t + \sum_{g=1}^{5} \beta_g \mathcal{I}\left((m_j - m_1) \in G_g\right) + \epsilon_{i,t,m_1,m_j}.$$

Table 7 shows the maturities for each group and the estimated coefficient. The last column shows that the marginal effect of an additional year of maturity is about 13 basis points for the first group but it reduces to about 4 basis points for the last group. This results are consistent with the concavity results of the theory shown in Proposition 1.

Table 7: Non-linear effects on credit spreads

Change in maturity				**Effect on spreads**		
Group	Minimum	Maximum	Mean	Coefficient	Standard Error	Marginal effect
1	0	3	1.8	22.8	1.29	13
2	3	6	4.6	26.2	1.47	6
3	6	9	7.7	48.5	1.65	6
4	9	12	10.5	29.4	2.26	3
5	12	30	18.6	65.3	2.32	4

C.2 Argentina

As an example, we repeat the estimates of the slope of liquidity spreads in Argentina to discipline the counterfactuals. We find that credit spreads are steeper in Argentina than in the US, with a slope of 50 basis points per year in Argentina, relative to about 5 in the US (first column of Table 8).[35] To control for default, we look at sovereign CDS and estimate how they change with maturity. We consider sovereign instead of corporate CDS because we only have data on

[35] Of course, the Argentinean market is much smaller than the US market, so when we restrict to firms issuing two bonds on the same day we end with a much smaller sample than in the US. Nevertheless, for the year 2017 we have 70 issuance of 15 firms generating 35 observations for the difference on credit spreads and maturity.

sovereign CDS for Argentina. Another advantage is that sovereign CDS are more liquid than corporate CDS, so the bias due to liquidity should be smaller. The second column of Table 8 shows that the estimated coefficient for sovereign CDS is 10 bps, about one-fifth of the total slope for corporate spreads. Interestingly, if we estimate the slope of sovereign CDS for the US we also find that they are about one-fifth of the slope of corporate spreads. We conclude that credit spreads are steeper in Argentina than in the US and that a large fraction of this slope can be attributed to liquidity considerations. For the quantitative evaluation it is useful to summarize the empirical results with the effect at the median. In Argentina the average maturity is 2.5 years, and when maturity increases from 1.5 to 3 years, credit spreads increase by 75 basis points, while CDS spreads increase by 14 basis points. We attribute the difference between corporate and sovereign CDS slope, 61 basis points, to the liquidity component.

Table 8: **Slope of credit spreads in Argentina**

	Corporate	Sovereign CDS
Maturity difference	50.04***	9.529***
	(7.377)	(0.104)
Observations	35	99
Number of firms	15	
R-squared	0.930	0.577
FE	Time	Time

*Note: Standard errors in parentheses; *, **, and *** denote statistical significance at the 10, 5, and 1 percent level, respectively.*

D Data sources

D.1 Credit spreads

We consider corporate debt issuances in the US on the Mergent Fixed Income Securities Database (FISD). We keep corporate bonds of domestic borrowers in local currency (i.e., US dollars) and with a fixed interest rate. We follow Gilchrist and Zakrajšek (2012) to define credit spreads that are not subject to the "duration mismatch" by constructing a synthetic risk-free security that mimics exactly the cash flows of the corresponding corporate debt instrument. We use the US Treasury yield curve estimated by Gürkaynak et al. (2007). Empirical results are similar to an alternative definition of spreads. For example, we find similar results when we define credit spreads as the difference in coupon rates between corporate and sovereign bonds

Table 9: **Summary Statistics**

	Mean	Median	SD
Corporate Bonds			
# of Bond Issuances per Firm/Month	6.69	3.00	7.46
Maturity at Issue (years)	6.95	5.00	6.45
Coupon Rate (pct.)	3.28	3.70	2.70
Nominal Effective Yield (pct.)	3.34	3.74	4.10
Nominal Effective Treasury Yield (pct.)	2.73	2.50	1.60
Credit Spread (bps.)	60	59	369
Firms			
Profitability	0.10	0.10	0.04
Average Maturity	4.82	4.38	1.95
Long Share	0.78	0.80	0.13

Note: Issuances: 994 issuers; 35,513 bonds of which 23,182 bonds are rated. Firms: there are 20,163 firms and 150,477 firm-year observations.

of similar maturity. For CDS we use Markit for 2000-2017.

Table 9 describes the data in the final sample. Our sample considers the set of firms that in a given period issues two or more bonds of different maturities. In the benchmark specification we define the period as a day and perform robustness exercises for definitions at the week and month level. As the length of the period increases (from day to week to month), there are more issuances within each group allowing us to also include firm-period fixed-effects. The reason for this sample selection is important for identification and is discussed in the main text. To ensure that our results are not driven by a small number of extreme observations, we trimmed the data at the top and bottom 1 percent. Our sample period is January 2000 to December 2017. There are 994 issuers and 35,513 bond issuances; 23,182 bonds are rated and the median rating from Moody's is A2. On average, a firm that is issuing bonds in a given month makes 6.69 different issuances; however, there is a large variation across firms. The average maturity is 6.95 years with an average credit spread of 60 basis points. Again, note the large dispersion in maturity and credit spreads across issuances.

D.2 High-quality corporate bonds

The corporate yield curve corresponds to the high-quality market (bonds rated above A), and it is available at `https://www.treasury.gov/resource-center/economic-policy/corp-bond-yield/`. Define the corporate yield curve as the monthly average for the year 2017. Tables 10 and 11 show the default rates, default credit losses, and the transition probabilities of credit ratings for high-quality issuers.

Table 10: **Default Credit Losses**

Rating	Default credit losses Average 1982-2014	Default credit losses Maximum 2008	Default rates Average 1920-2014	Default rates Maximum 2008
Aaa	0.00%	0.00%	0.000%	0.00%
Aa	0.03%	0.48%	0.061%	0.724%
A	0.03%	0.37%	0.096%	0.547%

Source: Moody's 2015.

Table 11: **Five-year Transitions (cumulative)**

	Aaa-A	Baa-B	Caa-C	Default
Aaa-A	88.70%	10.62%	0.15%	0.52%

D.3 Maturity and Profitability

We use Compustat data for 1976 to 2014. We follow the same cleaning as in Crouzet (2017). Firm-year observations are kept in the sample if (1) their 2-digit sic code is not between 60 and 69 (financials) or equal to 49 (utilities); (2) debt in current liabilities (dlc) and debt in long-term liabilities (dltt) are not missing and weakly positive; (3) book assets (at) is not missing and weakly greater than 1m$; (4) book leverage, the ratio of (dlc+dltt) to at, is between 0 and 1; (5) the variables ddi, for i = 2, ..., 5 (which capture the portion of long-term debt due in 2,...,5 years) are all non-missing and weakly positive; (6) their sum is weakly smaller than 1.01× dltt; and (7) operating income before depreciation (oibdp) is non-missing. There are no direct measures of average time to maturity of outstanding debt, but a proxy can be obtained from the data as $\frac{1}{dlc+dltt}\left(dlc + \sum_{i=1}^{5} dd_i \times i + dvlt \times x\right)$, where dvlt represents long-term debt due in more than five years and is defined as $dvlt = dltt - \sum_{i=1}^{5} dd_i$, and we set $x = 15$ as Crouzet (2017). Finally, we define the year-industry variables as the mean across firm-year observations of each sector but results are robust to consider the median instead. Table 9 describes the data. There are 20,163 firms and 150,477 firm-year observations with an average profitability and maturity of 10% and 4.82 years, respectively.

D.4 Credit spreads in Argentina

Consider all the active corporate bonds in August 2017 in the domestic market (MAE) and keep issuances in local currency, with 100% amortization, and interest rates as a spread on the Badlar rate (which is the reference short-term rate in Argentina). These are floating interest rate bonds with a fixed spread, so the credit spread is just the spread on the Badlar rate because non-arbitrage implies that agents can swap the variable Badlar rate for a fixed rate.