Online Appendix

Uncovering the Effects of the Zero Lower Bound with an Endogenous Financial Wedge

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Contents

A Multiple Equilibria with (k_0, b_{-1}) as State Variables

One key reason why we solve the wealth-recursive equilibrium using (k_0, ω_0) as state variables, instead of a probably more straightforward choice (k_0, b_{-1}) is that, there can be multiple equilibria if we use the latter as state variables. Figure A.1 provides two such examples. The left panel in the figure plots the policy functions in ω_0 fixed a value of k_0 (1.85). In the top-left figure, b_{-1} is generated using equation (4). We see that b_{-1} is non-monotone in ω_0. Therefore, if we use (k_0, b_{-1}) as the state variables, then when b_{-1} lies between -1.806 and -1.805, there are two equilibria with binding ZLB and binding borrowing constraint, but with different values of output, labor supply, and markup. Similarly, the right panel in the figure shows that when b_{-1} lies between -1.90 and -1.89, there are two equilibria, one with binding ZLB and the other one with non-binding ZLB. For the same parameter values, Proposition 2 shows that there exists an unique equilibrium with (k_0, ω_0) as state variables.

Why are there multiple equilibria with (k_0, b_{-1}) as state variables? To understand this issue, we use the definition of wealth share (4) at $t = 0$:

$$\omega_0 = \frac{R_0^K k_0 + b_{-1}}{R_0^K k_0}.$$

When agents in the economy expect higher R_0^K, the entrepreneurs' wealth share is higher (b_{-1} is negative), which leads them to invest more. This implies higher aggregate demand and hence higher relative price of entrepreneurs' output, higher wage and higher level of labor supply from the households. All these factors support a higher value R_0^K because

$$R_0^K = 1 - \delta + P_0^e A_0^{1-\alpha} \left(\frac{L_0}{k_0} \right)^{1-\alpha},$$

making the expectation self-fulfilling. On the other hand, if agents expect lower R_0^K then the same factors imply a lower value of R_0^K. For this mechanism to generate multiple equilibria, we need investment to be significantly responsive to entrepreneurs' wealth share. This is more likely to be the case when the borrowing constraint binds as shown by strictly positive multipliers in Figure A.1. The figure also shows that, when two equilibria exists with the same value of (k_0, b_{-1}), one equilibrium features higher R_0^K, output, labor supply, and entrepreneurs' wholesale price than the other.

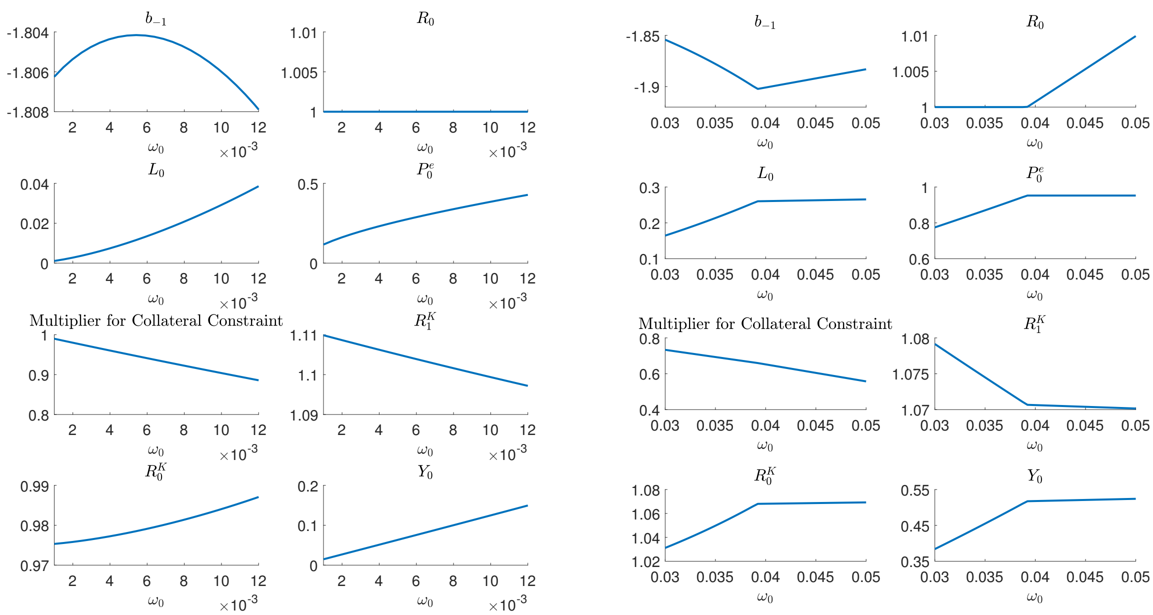


Figure A.1: Multiple Equilibria when using (k_0, b_{-1}) as States

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.9$ *and* $\epsilon = 21$. $k_0 = 1.85$.

B Two-Period Economy with Natural Borrowing Limit

In this appendix, we first provide the explicit solution for equilibrium in the last period $t = 1$ of the two-period economy. Then, we derive the expression for the AS-AD curves at $t = 0$ and their properties.

B.1 Last-Period Equilibrium

In the last period, there are no borrowing or lending, and thus we have $b_1 = 0$. The entrepreneurs makes no further investment either, i.e., $k_2 = 0$. We assume the markup takes its steady state value,

$$X_1 = X^*.$$

In the last period, the entrepreneurs and households consume all their wealth. From equations (3a),(3b), and (5a), (6a) at $t = 1$, we obtain

$$c_1 = R_1^K k_1 \omega_1, \tag{A.1a}$$

and

$$c_1' = R_1^K k_1 (1 - \omega_1) + \left(1 - \frac{\alpha}{X^*}\right) Y_1. \tag{A.1b}$$

Given ω_1 and k_1, for a labor supply L_1, we can solve w_1 and R_1^K from (6d) and (6e), and c_1 and c_1' from (A.1a) and (A.1b). Lastly, from (5b) at $t = 1$, we solve for L_1 from the following equation:

$$\frac{1-\alpha}{X^*} L_1^{-\alpha} - \left(1 - \frac{\alpha}{X^*}\omega_1\right) L_1^{1-\alpha} = (1 - \omega_1)(1 - \delta) \left(\frac{k_1}{A_1}\right)^{1-\alpha}. \tag{A.1c}$$

It follows that L_1 is decreasing in k_1 and increasing in ω_1.

B.2 Derivations of the AS-AD curves

By the result that $R_1^K = R_0$ and the expression for R_1^K in (6e), we obtain

$$R_1^K = 1 - \delta + \frac{\alpha}{X^*} A_1^{1-\alpha} \left(\frac{k_1}{L_1}\right)^{\alpha-1} = R_0.$$

So

$$\frac{k_1}{L_1} = \left[(R_0 - 1 + \delta) \frac{X^*}{\alpha} A_1^{\alpha-1}\right]^{\frac{1}{\alpha-1}}. \tag{A.2}$$

From the households' Euler equation and intra-temporal condition at $t = 1$, we obtain

$$c_0' = \frac{c_1'}{\beta R_0} = \frac{w_1}{\beta R_0}.$$

In addition

$$w_1 = \frac{1-\alpha}{X^*} A_1^{1-\alpha} \left(\frac{k_1}{L_1}\right)^{\alpha}.$$

Plugging the expression for k_1/L_1, (A.2), into this equation for w_1, we obtain the expression for c_0':

$$c_0' = \frac{1}{\beta R_0}\frac{1-\alpha}{X^*}A_1\left[(R_0 - 1 + \delta)\frac{X^*}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}. \tag{A.3}$$

From the households' intra-temporal condition at $t = 0$,

$$w_0 = c_0' = \frac{w_1}{\beta R_0}.$$

Therefore,

$$\frac{1-\alpha}{X_0}A_0^{1-\alpha}\left(\frac{k_0}{L_0}\right)^{\alpha} = \frac{1}{\beta R_0}\frac{1-\alpha}{X^*}A_1^{1-\alpha}\left(\frac{k_1}{L_1}\right)^{\alpha}$$

or equivalently,

$$\frac{L_0}{k_0} = \left(\beta R_0 \frac{X^*}{X_0}\frac{A_0^{1-\alpha}}{A_1^{1-\alpha}}\right)^{\frac{1}{\alpha}}\frac{L_1}{k_1} = \frac{1}{A_0}\left(\beta R_0 \frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}\left((R_0 - 1 + \delta)\frac{X^*}{\alpha}\right)^{\frac{1}{1-\alpha}}, \tag{A.4}$$

where the second inequality is obtained from (A.2). Plugging this expression for L_0/k_0 into the expression for R_0^K, (6e) at $t = 0$, we arrive at

$$R_0^K = 1 - \delta + \frac{A_0}{X_0}\alpha\left(\frac{L_0}{k_0}\right)^{1-\alpha} = 1 - \delta + \left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}(R_0 - 1 + \delta). \tag{A.5}$$

From the entrepreneurs' budget constraint

$$c_0 + \frac{1}{R_0}c_1 = R_0^K k_0 \omega_0,$$

and the Euler equation

$$c_1 = \gamma R_0 c_0,$$

we obtain

$$c_0 = \frac{1}{1+\gamma}R_0^K k_0 \omega_0.$$

Combining this with the last expression for R_0^K, we obtain

$$c_0 = \frac{1}{1+\gamma}R_0^K \omega_0 k_0, \tag{A.6}$$

where R_0^K given in (A.5) is the value of each unit of capital at time 0.

Lastly, the investment of the entrepreneurs is given by

$$I_0 = k_1 - (1-\delta)\, k_0. \tag{A.7}$$

From the feasibility constraint at $t = 1$,

$$\begin{aligned} c_1 + c_1' &= A_1^{1-\alpha} k_1 \left(\frac{L_1}{k_1}\right)^{1-\alpha} + (1-\delta) k_1 \\ &= k_1 \left(1 - \delta + (R_0 - 1 + \delta)\, \frac{X^*}{\alpha}\right). \end{aligned}$$

Therefore,

$$k_1 = \frac{c_1 + c_1'}{1 - \delta + \frac{X^*}{\alpha}\,(R_0 - 1 + \delta)}, \tag{A.8}$$

with

$$\begin{aligned} c_1 &= \gamma R_0 c_0, \\ c_1' &= \frac{1-\alpha}{X^*} A_1 \left((R_0 - 1 + \delta)\, \frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}, \end{aligned}$$

and the expressions for c_0 is given by (A.6).

For the aggregate supply equation, we first obtain the expression for $w_0 X_0$ from (6d):

$$w_0 X_0 = A_0^{1-\alpha} (1-\alpha) \left(\frac{k_0}{L_0}\right)^{\alpha}.$$

Then from the expression for L_0/k_0 in (A.4) and the expression for Y_0^{AS} as a function of $w_0 X_0$ from (9), we obtain:

$$Y_0^{AS} = \frac{X^*}{\alpha} \left(\beta R_0 \frac{A_0 X^*}{A_1 X_0}\right)^{\frac{1-\alpha}{\alpha}} (R_0 - 1 + \delta)\, k_0. \tag{A.9}$$

Lemma 1. *When the ZLB does not bind, the aggregate supply curve is upward slopping in R_0. There exists $\bar{R}_0$ such that aggregate demand curve is downward sloping when $R_0 \leq \bar{R}_0$.*

Proof. Given the ZLB is not binding, we can set $X_0 = X^*$ in the AS curve (A.9) and AD curve (8). Then it is easy to see that the AS curve is increasing in R_0. The AD curve becomes

$$Y_0^D = \left[\frac{1}{\beta R_0} + \frac{1}{1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)}\right]\frac{1-\alpha}{X^*}A_1\left[(R_0-1+\delta)\frac{X^*}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}$$
$$+\left(1+\frac{\gamma R_0}{1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)}\right)\frac{1}{1+\gamma}\omega_0\left[1-\delta+\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)\right]k_0-(1-\delta)k_0.$$

On the right-hand side, the first term is decreasing in R_0, and the second term is increasing in R_0. Taking derivative of Y_0^D with respect to R_0, we have

$$\frac{\partial Y_0^D}{\partial R_0} = -\left[\frac{1}{\beta R_0^2} + \frac{\frac{X^*}{\alpha}}{\left[1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)\right]^2}\right]\frac{1-\alpha}{X^*}A_1\left[(R_0-1+\delta)\frac{X^*}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}$$
$$-\left[\frac{1}{\beta R_0} + \frac{1}{1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)}\right]\frac{1-\alpha}{X^*}A_1\left[(R_0-1+\delta)\frac{X^*}{\alpha}\right]^{\frac{1}{\alpha-1}}\frac{X^*}{1-\alpha}$$
$$+\frac{\frac{\gamma}{1+\gamma}(1-\delta)\left(1-\frac{X^*}{\alpha}\right)}{\left[1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)\right]^2}\omega_0\left[1-\delta+\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)\right]k_0$$
$$+\left(1+\frac{\gamma R_0}{1-\delta+\frac{X^*}{\alpha}(R_0-1+\delta)}\right)\frac{1}{1+\gamma}\omega_0 k_0\left[\frac{1-\alpha}{\alpha}\beta\frac{A_0}{A_1}\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-2\alpha}{\alpha}}(R_0-1+\delta)+\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\right].$$

Since $\lim_{R_0\to 1-\delta}\frac{\partial Y_0^D}{\partial R_0} = -\infty$, and $\lim_{R_0\to\infty}\frac{\partial Y_0^D}{\partial R_0} = \infty$, and the expression of $\frac{\partial Y_0^D}{\partial R_0}$ is continuous, there exists an $\bar{R}_0$ such that the AD curve is downward sloping when $R_0 \le \bar{R}_0$. □

Lemma 2. *When the ZLB binds, both AS and AD curves, in Y_0 and P_0^e, are upward slopping.*

Proof. With binding ZLB, the AS curve can be expressed as

$$Y_0^S = (P_0^e)^{\frac{1-\alpha}{\alpha}}\left(\frac{1}{\beta X^*}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}}\frac{\delta X^*}{\alpha}k_0,$$

which is increasing in P_0^e. For the AD curve, inserting $R_0 = 1$ into (A.3) and using $c_1' = \beta R_0 c_0'$, we find that both c_0' and c_1' are independent of P_0^e. On the other hand, by (A.6) and $c_1 = \gamma R_0 c_0$, and (A.7), we find that c_0 and I_0 are increasing in P_0^e. Thus the AD curve is also increasing in P_0^e. □

B.3 Graphical Representations for Proposition 1 (Part 1)

When the ZLB does not bind, i.e., $R_0 > 1$, our specification of monetary policy, (2) implies $X_0 = X^*$. The AD and AS expressions, (8) and (A.9), can be represented by output Y_0 as functions of interest rate R_0. Lemma 1 in Appendix B.2 shows that the aggregate supply curve is upward slopping and the aggregate demand curve is downward slopping when R_0 is not too high.

Why is the AS curve upward sloping? In period 0, since both the TFP A_0 and capital stock k_0 are given, Y_0^S responds to R_0 through the labor supply. A higher R_0 reduces c_0' in two ways. First, it encourages saving and discourages consumption through the inter-temporal substitution effect. Second, with $R_0 = R_1^K$, a higher R_0 implies a lower capital-labor ratio in the last period $t = 1$ by (6e), and thus lower c_1' and w_1 by (5b) and (6d), which reduces c_0' through the income effect. Combining both effects, labor cost w_0 in the first period becomes cheaper by the labor supply equation (5b), which boosts the aggregate supply.

The intuition for why the AD curve is downward sloping is more involved since it is the summation of three variables: the households' consumption (c_0'), entrepreneurs' consumption (c_0), and investment (I_0). Investment demand I_0 is determined such that the capital stock at $t = 1$, $k_1 = (1-\delta)\,k_0 + I_0$, suffices to serve the consumption demand c_1 and c_1', with the marginal product of capital pinned down by R_0 in equilibrium. Combining this equation with the inter-temporal optimal choices $c_1 = \gamma R_0 c_0$ and $c_1' = \beta R_0 c_0'$, in Appendix B.2, we show that the AD curve can be written as

$$Y_0^{AD} = \left[1 + \frac{\beta R_0}{1-\delta+\frac{X^*}{\alpha}\left(R_0-1+\delta\right)}\right] c_0' + \left(1 + \frac{\gamma R_0}{1-\delta+\frac{X^*}{\alpha}\left(R_0-1+\delta\right)}\right) c_0 - (1-\delta)\,k_0. \tag{A.10}$$

On the right-hand side, the first component is associated with the households' consumption and is decreasing in R_0. As discussed earlier, a higher R_0 depresses c_0' and c_1', and thus depresses I_0 by reducing the demand in period 1. The second component is associated with the entrepreneurs' consumption and is increasing in R_0. This is because as labor cost w_0 becomes cheaper, the returns to capital and thus to the entrepreneurs' wealth become higher. However, we show that as long as the entrepreneur wealth share ω_0 and interest rate R_0 are not too high (as guaranteed by the conditions given in Proposition 1), the change in the second component is dominated by that of the first one, leaving the AD curve downward sloping. This is the case we focus on here.

Part 1 of Proposition 1 shows that R_0 is decreasing in k_0 and increasing in ω_0, which also implies that the ZLB tends to bind when k_0 is high or when ω_0 is low. The intuition for these results can be analyzed by plotting the AS and AD curves in Figure A.2 (output on the x-axis and interest rate on the y-axis).[31]

[31] The standard representations of the AS-AD curves plot price-level against output. We cannot strictly follow these representations in this case because the wholesale price level is constant at $1/X^*$. We can do

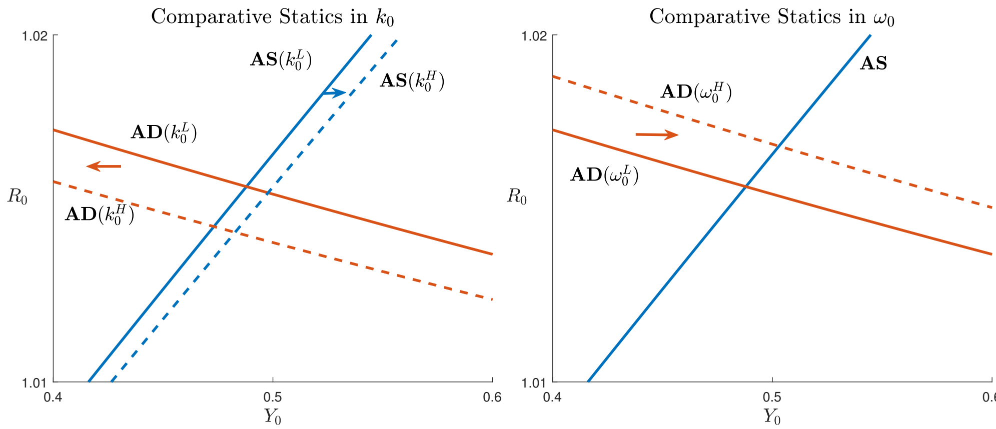


Figure A.2: AS-AD Curves when $R_0 > 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$ and $\epsilon = 21$. $k_0^L = 4$, $k_0^H = 4.1$; $\omega_0^L = 0.18$, $\omega_0^H = 0.2$. We set $k_0 = k_0^L$ and $\omega_0 = \omega_0^L$ as the baseline values.

Consider an exogenous increase in the initial capital, k_0. As implied by (A.9), given R_0, output increases in k_0, and the AS curve shifts to the right. This result is unsurprising since larger k_0 implies greater production capacity. The AD curve, on the contrary, shifts to the left. We show in Appendix B.2 that given R_0, c_0' does not depend on k_0. Although c_0 and k_1 are increasing in k_0, they increase by a smaller amount compared to the increase in $(1 - \delta) k_0$. Thus the aggregate demand is decreasing in k_0. As a result, R_0 is lower in equilibrium. This is illustrated in the left panel of Figure A.2.

For an exogenous increase in the entrepreneurs' wealth share, ω_0, the AS curve is not affected. For the AD curve, since the entrepreneurs now have more wealth, c_0 increases. In addition, the entrepreneurs also increase their consumption in period 1, c_1, which, in order to clear the good market in period 1, requires higher capital holding k_1 and thus higher investment. Given R_0, c_0' is does not depend on ω_0. Therefore, in sum, the AD curve shifts to the right. In equilibrium, both R_0 and output Y_0 are higher. This is illustrated in the right panel of Figure A.2.

C Proof of Proposition 1

Combining the expressions for c_0', c_0, and I_0 in (A.3), (A.6) and (A.7), the optimal intertemporal choices, $c_1 = \gamma R_0 c_0$ and $c_1' = \gamma R_0 c_0'$, we can reduce the whole system into one equation with one unknown: R_0 or X_0 depending on whether the ZLB is binding or not as follows:

so when the ZLB binds and the wholesale price level varies.

$$\begin{aligned}\frac{1}{k_0} =& \frac{(1-\delta)\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right)+\left(\frac{X^*}{\alpha}-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right]}{\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(R_0-1+\delta\right)\right]^{\frac{\alpha}{\alpha-1}}\left[1+\frac{X^*}{\alpha\beta}+\frac{1}{\beta R_0}\left(1-\delta\right)\left(1-\frac{X^*}{\alpha}\right)\right]} \\ &+\frac{\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(\frac{X_0}{\alpha}-\frac{1}{1+\gamma}\omega_0\right)+\left(\frac{X^*}{\alpha}\frac{X_0}{\alpha}-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right]\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}\left(R_0-1+\delta\right)}{\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(R_0-1+\delta\right)\right]^{\frac{\alpha}{\alpha-1}}\left[1+\frac{X^*}{\alpha\beta}+\frac{1}{\beta R_0}\left(1-\delta\right)\left(1-\frac{X^*}{\alpha}\right)\right]}.\end{aligned} \tag{A.11}$$

Lemma 3. *When the ZLB does not bind, R_0 is decreasing in k_0 and increasing in ω_0 if $\omega_0 \leq \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$.*

Proof. The monetary policy rule in equation (2) implies that, when $R_0 > 1$, $X_0 = X^*$. Inserting $X_0 = X^*$ into (A.11), we obtain

$$\begin{aligned}\frac{1}{k_0} =& \frac{(1-\delta)\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right)+\left(\frac{X^*}{\alpha}-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right]}{\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(R_0-1+\delta\right)\right]^{\frac{\alpha}{\alpha-1}}\left[1+\frac{X^*}{\alpha\beta}+\frac{1}{\beta R_0}\left(1-\delta\right)\left(1-\frac{X^*}{\alpha}\right)\right]} \\ &+\frac{\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(\frac{X^*}{\alpha}-\frac{1}{1+\gamma}\omega_0\right)+\left(\left(\frac{X^*}{\alpha}\right)^2-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right]\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}\left(R_0-1+\delta\right)}{\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(R_0-1+\delta\right)\right]^{\frac{\alpha}{\alpha-1}}\left[1+\frac{X^*}{\alpha\beta}+\frac{1}{\beta R_0}\left(1-\delta\right)\left(1-\frac{X^*}{\alpha}\right)\right]}.\end{aligned} \tag{A.12}$$

In the equation above, for the right-hand side, the denominator is decreasing in R_0, and the numerators are increasing in R_0 if $\omega_0 \leq \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$. As a result, its right-hand side is increasing in R_0. As $R_0 \to \infty$, the right-hand side goes to infinity. As $R_0 \to 1-\delta$, the right-hand side goes to zero. Thus there is a unique solution of R_0 between $[1-\delta, +\infty)$. As k_0 increases, R_0 decreases. When k_0 is large enough and hits the threshold $\hat{k}_0(\omega_0)$ as in Lemma 5, the ZLB is binding, and we switch to the other system with $R_0 = 1$ and X_0 being the unknowns.

To see how R_0 responds to ω_0, we can rewrite the equation above as follows:

$$\begin{aligned}&(1-\delta)\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right)+\left(\frac{X^*}{\alpha}-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right] \\ &+\left[(1-\delta)\left(1-\frac{X^*}{\alpha}\right)\left(\frac{X^*}{\alpha}-\frac{1}{1+\gamma}\omega_0\right)+\left(\left(\frac{X^*}{\alpha}\right)^2-\frac{\frac{X^*}{\alpha}+\gamma}{1+\gamma}\omega_0\right)R_0\right]\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}\left(R_0-1+\delta\right) \\ &-\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(R_0-1+\delta\right)\right]^{\frac{\alpha}{\alpha-1}}\left[1+\frac{X^*}{\alpha\beta}+\frac{1}{\beta R_0}\left(1-\delta\right)\left(1-\frac{X^*}{\alpha}\right)\right]\frac{1}{k_0} \\ =&0.\end{aligned}$$

Denote its left-hand side as $F(R_0, \omega_0)$. A careful examination shows that F is increasing in R_0 if $\omega_0 \leq \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$, and decreasing in ω_0. By the Implicit Function Theorem, $\partial R_0/\partial\omega_0 >$

0. $\square$

Lemma 4. *When the ZLB binds and $\alpha < \frac{X^*}{1+X^*}$, X_0 is increasing in k_0, and is decreasing in ω_0. Output Y_0 is decreasing in k_0, and increasing in ω_0.*

Proof. By the monetary policy rule (2), when $R_0 = 1$, $X_0 > X^*$. By setting $R_0 = 1$ in (A.11) and after some calculation, we have the following equation to pin down X_0:

$$\begin{aligned} &\frac{1-\alpha}{X^*} A_1 \left(\frac{\delta X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(1 + \frac{\delta X^*}{\alpha\beta} + \frac{1}{\beta}(1-\delta)\right) \frac{1}{k_0} \\ &= \left(1-\delta+\frac{\delta X^*}{\alpha}\right) \delta \left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}} \left(\frac{X_0}{\alpha}-1\right) \\ &+ \left[\left(1-\delta+\frac{\delta X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right) - \frac{\gamma}{1+\gamma}\omega_0\right] \left[(1-\delta) + \delta \left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}\right]. \end{aligned} \tag{A.13}$$

The left-hand side is independent of X_0. If $\alpha < \frac{X^*}{1+X^*}$, the right-hand side is decreasing in X_0. As k_0 increases, the left hand side decreases, so X_0 must increase. Since the right-hand side is decreasing in ω_0, when ω_0 increases, X_0 decreases.

For output, inserting the expression for Y_0 from the AS curve (A.9) into (A.13), we arrive at

$$\begin{aligned} &\frac{1-\alpha}{X^*} A_1 \left(\frac{\delta X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(1 + \frac{\delta X^*}{\alpha\beta} + \frac{1}{\beta}(1-\delta)\right) \\ &= \left[\left(1-\delta+\frac{\delta X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right) - \frac{\gamma}{1+\gamma}\omega_0\right] \left[(1-\delta)k_0 + Y_0\right] \\ &+ \left(1-\delta+\frac{\delta X^*}{\alpha}+\gamma\right) \frac{1}{1+\gamma}\omega_0 \left(1-\frac{\alpha}{X_0}\right) Y_0. \end{aligned} \tag{A.14}$$

Since the left-hand side of (A.14) is constant, and X_0 is increasing in k_0, with larger k_0, Y_0 must decrease to equate (A.14). Thus, Y_0 is decreasing in k_0 when the ZLB binds. To see how Y_0 responds to ω_0, we can insert the expressions of c_0', c_0 and I_0 from (A.3), (A.3) and (A.7) into the AD curve (8), and get

$$\begin{aligned} Y_0 = &\left(\frac{1}{\gamma} + \frac{1}{(1-\delta)\left(1-\frac{X^*}{\alpha}\right)+\frac{X^*}{\alpha}}\right) \frac{\gamma}{1+\gamma}\omega_0 \left[1-\delta+\delta\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}\right] k_0 - (1-\delta)k_0 \\ &+ \left(\frac{1}{\beta} + \frac{1}{(1-\delta)\left(1-\frac{X^*}{\alpha}\right)+\frac{X^*}{\alpha}}\right) \frac{1-\alpha}{X^*} A_1 \left[\frac{\delta X^*}{\alpha}\right]^{\frac{\alpha}{\alpha-1}}. \end{aligned}$$

As ω_0 increases, X_0 decreases, and we can see that Y_0 is increasing in ω_0. $\square$

Lemma 5. *Given an initial wealth distribution ω_0, there exists a cutoff value $\hat{k}_0(\omega_0)$ such that when $k_0 < \hat{k}_0(\omega_0)$, the ZLB is not binding; and when $k_0 > \hat{k}_0(\omega_0)$, the ZLB is binding. $\hat{k}_0(\omega_0)$*

is increasing in ω_0.

Proof. Inserting $R_0 = 1$ and $X_0 = X^*$ in equation (A.11), we obtain the expression for $\hat{k}_0(\omega_0)$ as

$$\hat{k}_0(\omega_0) = \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{\delta X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(1 + \frac{\delta X^*}{\alpha\beta} + \frac{1}{\beta}(1-\delta)\right)}{(1-\delta)\left(1-\delta+\frac{\delta X^*}{\alpha}\right)\left(1-\frac{1}{1+\gamma}\omega_0\right) - (1-\delta)\frac{\gamma}{1+\gamma}\omega_0 + \delta\left[\left(1-\delta+\frac{\delta X^*}{\alpha}\right)\left(\frac{X^*}{\alpha}-\frac{1}{1+\gamma}\omega_0\right) - \frac{\gamma}{1+\gamma}\omega_0\right]\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}}, \tag{A.15}$$

which is increasing in ω_0 and A_1.

We first show that when $k_0 < \hat{k}_0(\omega_0)$, the ZLB is not binding in any equilibrium. Otherwise, given a binding ZLB, X_0 is increasing in k_0 by Lemma 4, which implies $X_0 < X^*$ when $k_0 < \hat{k}_0(\omega_0)$. This contradicts the restriction $X_0 \geq X^*$. Thus, any equilibrium with $k_0 < \hat{k}_0(\omega_0)$ features non-binding ZLB. From equation (A.12), we can see that a unique equilibrium exists in this region without a binding ZLB.

Similarly, we can show that when $k_0 \geq \hat{k}_0(\omega_0)$, the ZLB is binding in any equilibrium. Otherwise, given $R_0 > 1$, R_0 is decreasing in k_0 by Lemma 3, which implies $R_0 < 1$ when $k_0 > \hat{k}_0(\omega_0)$. This contradicts the ZLB restriction. Thus the ZLB is binding when $k_0 \geq \hat{k}_0(\omega_0)$. □

Lemma 6. *When $m = 1$, given ω_0, there is an upper bound of initial capital $\bar{k}_0(\omega_0)$, such that an equilibrium does not exist when $k_0 > \bar{k}_0(\omega_0)$. $\bar{k}_0(\omega_0)$ is increasing in ω_0 and A_1.*

Proof. By setting $X_0 \to \infty$ in (A.13), we obtain the expression for $\bar{k}_0(\omega_0)$:

$$\bar{k}_0(\omega_0) = \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{\delta X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(1 + \frac{\delta X^*}{\alpha\beta} + \frac{1}{\beta}(1-\delta)\right)}{(1-\delta)\left(1-\delta+\frac{\delta X^*}{\alpha}\right) - \left(1-\delta+\frac{\delta X^*}{\alpha}+\gamma\right)\frac{1-\delta}{1+\gamma}\omega_0}. \tag{A.16}$$

By Lemma 4, X_0 increases with k_0. Since at $k_0 = \bar{k}_0(\omega_0)$, X_0 cannot be increased further, an equilibrium does not exist with binding ZLB when $k_0 > \bar{k}_0(\omega_0)$. Since by Lemma 5, $X_0 = X^*$, Lemma 4 implies that $\hat{k}_0(\omega_0) < \bar{k}_0(\omega_0)$. In Lemma 5, we also show that there is no equilibrium with $R_0 > 1$ when $k_0 > \hat{k}_0(\omega_0)$. Thus there is no equilibrium when $k_0 > \bar{k}_0(\omega_0)$. We can easily see from (A.16) that $\bar{k}_0(\omega_0)$ is increasing in ω_0 and A_1.

The intuition for this non-existence result is as follows – to simplify the discussion we assume exogenous labor supply. At the ZLB, the return to capital R_1^K is equal to 1. Therefore, by (6e) for $t = 1$, k_1 is bounded from above. Thus output and the consumption of households and entrepreneurs are all bounded from above by the market clearing condition (3a) in period 1. Then, from the Euler equations of the entrepreneurs and households,

c_0 and c_0' are also bounded from above. Therefore, if k_0 is sufficiently high,

$$(1-\delta)k_0 > c_0' + c_0 + k_1$$

violating (3a) in period 0 even if output falls to zero. □

Combining the results from Lemmas 3 to 6, we obtain a complete proof of Proposition 1.

D Proof of Proposition 2

The equilibrium in the last period, $t = 1$, is the same as for the natural borrowing limit and is provided in Appendix B.1. Here, we focus on equilibrium at $t = 0$.

Proposition 2. *Assume that $m < 1$, $\alpha < \frac{X^*}{1+X^*}$ and ω_0 is smaller than a threshold ϖ, which depends on model parameters and is given in Appendix D (equation (A.21a)). There is a cutoff value $\bar{\bar{k}}_0(\omega_0) > 0$ such that when $0 < k_0 \leq \bar{\bar{k}}_0(\omega_0)$, there exists a unique equilibrium and when $k_0 > \bar{\bar{k}}_0(\omega_0)$ there does not exist an equilibrium. In addition,*

1. if

$$\omega_0 < \Lambda_0(m, \gamma, \beta, \alpha, X^*),$$

(the expression for Λ_0 is given by (A.18f)) then $\bar{\bar{k}}_0(\omega_0) < \bar{k}_0(\omega_0)$, where $\bar{k}_0(\omega_0)$ is defined in Proposition 1. $\bar{\bar{k}}_0(\omega_0)$ is increasing in ω_0 and decreasing in m. If $\omega_0 > \Lambda_0$, $\bar{\bar{k}}_0(\omega_0) = \bar{k}_0(\omega_0)$;

2. the collateral constraint is binding if and only if $\omega_0 \leq \omega_0^{CC}(k_0)$, for some cutoff function $\omega_0^{CC}(k_0)$.

Proof. We note that the proof for equilibrium uniqueness turns out to be rather challenging. For example, in the region of state space featuring an equilibrium with binding ZLB and binding collateral constraint, not only we need to establish equilibrium uniqueness with this binding pattern, we also need to rule out equilibria with all possible combinations of binding or non-binding ZLB and collateral constraint. What adds to the complexity is that the systems of equations determining equilibria with different binding patterns are fundamentally different. The non-existence result for $k_0 > \bar{\bar{k}}_0(\omega_0)$ is a novel result for this class of New Keynesian models. It is similar to the natural borrowing limit case analyzed in Subsection 2.3 and we provide more details in Appendix D.4. It points to the difficulties in solving the infinite-horizon version of these models which we and other researchers have encountered. In particular, if one solves the model using an iterative algorithm such as policy iterations, at some point the algorithm might not be able to find an equilibrium in certain region of the state space. One way to get around this issue of equilibrium non-existence is to add investment frictions to the model as in Subsection 2.5 and Section 3.

The detailed proof is provided in the remainder of this section and we proceed as follows.

(1) Describe the cutoff function $\omega_0^{CC}(k_0)$ of ω_0 for a binding collateral constraint, and the cutoff function $\bar{\bar{k}}_0(\omega_0)$ of k_0 for equilibrium existence.

(2) We show that when $\omega_0 \leq \omega_0^{CC}(k_0)$ and $k_0 < \bar{\bar{k}}_0(\omega_0)$, there exists a unique equilibrium with binding collateral constraint, while such an equilibrium does not exist when $\omega_0 > \omega_0^{CC}(k_0)$ or $k_0 \geq \bar{\bar{k}}_0(\omega_0)$

(3) We show that when $\omega_0 > \omega_0^{CC}(k_0)$ and $k_0 < \bar{\bar{k}}_0(\omega_0)$, there exists a unique equilibrium with non-binding collateral constraint, while such an equilibrium does not exist when $\omega_0 \leq \omega_0^{CC}(k_0)$ or $k_0 \geq \bar{\bar{k}}_0(\omega_0)$. □

D.1 Thresholds for Collateral Constraint and Equilibrium Existence

In this first step of the proof, we construct the threshold functions $\omega_0^{CC}(.)$ and $\bar{\bar{k}}_0(.)$ for binding collateral constraint and equilibrium existence. We use these thresholds later in Step 2 and Step 3 of the proof.

D.1.1 Cutoff Value for a Binding Collateral Constraint

The following two lemmas help us identify the cutoff value of a binding collateral constraint in the $\{k_0, \omega_0\}$ space.

Lemma 7. *With natural borrowing limit ($m = 1$), given $\alpha < \frac{X^*}{1+X^*}$ and $\omega_0 < \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$, in the unique equilibrium constructed in Proposition 1, the leverage ratio $-\frac{b_0}{R_1^K k_1}$ is decreasing in ω_0.*

Proof. From the entrepreneurs' budget and their optimal choice $c_0 = \frac{1}{1+\gamma} R_0^k \omega_0 k_0$, their leverage ratio is

$$-\frac{b_0}{R_1^k k_1} = 1 - \frac{\gamma}{1+\gamma}\frac{R_0^k \omega_0 k_0}{k_1}.$$

Showing that the leverage ratio is decreasing in ω_0 is equivalent to showing that $\frac{R_0^k \omega_0 k_0}{k_1}$ is increasing in ω_0.

First, consider the case in which ZLB is not binding, i.e. $R_0 > 1$. Using the expression for k_1 from (A.8) we obtain

$$\frac{R_0^k \omega_0 k_0}{k_1} = \frac{\omega_0 R_0^k k_0 \left[1 - \delta + \frac{X^*}{\alpha}(R_0 - 1 + \delta)\right]}{\frac{\gamma R_0}{1+\gamma}\omega_0 R_0^k k_0 + \frac{1-\alpha}{X^*} A_1 \left((R_0 - 1 + \delta)\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}$$
$$= \frac{1 - \delta + \frac{X^*}{\alpha}(R_0 - 1 + \delta)}{\frac{\gamma R_0}{1+\gamma} + \frac{\frac{1-\alpha}{X^*} A_1 \left((R_0-1+\delta)\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}}.$$

Differentiating the last line in ω_0, the derivative $\frac{d}{d\omega_0}\left[\frac{R_0^k \omega_0 k_0}{k_1}\right]$ has the same sign as

$$\frac{X^*}{\alpha}\frac{dR_0}{d\omega_0}\left[\frac{\gamma R_0}{1+\gamma} + \frac{\frac{1-\alpha}{X^*} A_1 \left((R_0 - 1 + \delta)\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}\right]$$
$$- \left[\frac{\gamma}{1+\gamma}\frac{dR_0}{d\omega_0} + \frac{d}{d\omega_0}\frac{\frac{1-\alpha}{X^*} A_1 \left((R_0 - 1 + \delta)\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}\right]\left[1 - \delta + \frac{X^*}{\alpha}(R_0 - 1 + \delta)\right]. \quad \text{(A.17)}$$

By Proposition 1, given $\omega_0 < \frac{X^* + X^*\gamma}{X^* + \alpha\gamma}$, R_0 increases with ω_0: $\frac{dR_0}{d\omega_0} > 0$. In addition, by setting $X_0 = X^*$ in (A.5), R_0^K becomes

$$R_0^K = 1 - \delta + \left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}(R_0 - 1 + \delta),$$

which is strictly increasing in R_0 and hence in ω_0. Therefore,

$$\frac{d}{d\omega_0}\frac{\frac{1-\alpha}{X^*} A_1 \left((R_0 - 1 + \delta)\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0} < 0.$$

To show that (A.17) is positive, we only need to show that

$$\frac{X^*}{\alpha}\frac{dR_0}{d\omega_0}\frac{\gamma R_0}{1+\gamma} > \frac{\gamma}{1+\gamma}\frac{dR_0}{d\omega_0}\left[1 - \delta + \frac{X^*}{\alpha}(R_0 - 1 + \delta)\right].$$

This inequality holds because $\frac{X^*}{\alpha} > 1$.

Now, consider the case in which ZLB is binding, i.e. $R_0 = 1$ and $X_0 > X^*$. Using the expression for k_1 from (A.8), we obtain

$$\frac{R_0^k \omega_0 k_0}{k_1} = \frac{\omega_0 R_0^k k_0 \left[1-\delta+\frac{X^*}{\alpha}\delta\right]}{\frac{\gamma}{1+\gamma}\omega_0 R_0^k k_0 + \frac{1-\alpha}{X^*}A_1\left(\delta\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}$$
$$= \frac{1-\delta+\frac{X^*}{\alpha}\delta}{\frac{\gamma}{1+\gamma}+\frac{\frac{1-\alpha}{X^*}A_1\left(\delta\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}}.$$

So leverage is decreasing in ω_0 if $\omega_0 R_0^k$ is increasing in ω_0. By setting $R_0 = 1$ in (A.5), R_0^K becomes

$$R_0^K = 1-\delta+\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\delta\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}.$$

Proposition 1 shows that when $\alpha < \frac{X^*}{1+X^*}$, X_0 is decreasing with ω_0. Therefore, R_0^K is increasing in ω_0. Hence, $\omega_0 R_0^K$ is increasing in ω_0 as desired. □

Lemma 8. *Given $m<1$, $\alpha < \frac{X^*}{1+X^*}$ and $\omega_0 < \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$, there is a cutoff value of wealth, $\omega_0^{CC}(k_0)$ such that in the unique equilibrium of the model with natural borrowing limit constructed in Proposition 1, the leverage ratio $-\frac{b_0}{R_1^k k_1} = m$ at $\omega_0 = \omega_0^{CC}(k_0)$.*

Proof. By Lemma 7, with natural borrowing limit, the leverage ratio $-\frac{b_0}{R_1^k k_1}$ is decreasing in ω_0. Thus we can derive the expression of $\omega_0^{CC}(k_0)$ by setting $-\frac{b_0}{R_1^k k_1} = m$.

At $\omega_0^{CC}(k_0)$, we have $\omega_1 = 1-m$ and $R_1^K = R_0$. After some calculations, given k_0, we obtain the following system of two equations with two unknowns, $\{\omega_0^{CC}, R_0\}$ or $\{\omega_0^{CC}, X_0\}$ depending on whether the ZLB is binding,

$$k_1 = \frac{\gamma}{1+\gamma}\frac{1-\delta+\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}(R_0-1+\delta)}{1-m}\omega_0^{CC}k_0, \tag{A.18a}$$

$$k_1 = \left[(1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right)+\left(\frac{X_0}{\alpha}-\frac{\omega_0}{1+\gamma}\right)\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)\right]k_0 \tag{A.18b}$$
$$-\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left[\frac{1}{\beta R_0}(R_0-1+\delta)^{\frac{\alpha}{\alpha-1}}\right],$$

in which k_1 is a decreasing function of R_0 :

$$k_1 = \frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}(R_0-1+\delta)^{\frac{\alpha}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{X^*}+\left(1-\frac{\alpha}{X^*}(1-m)\right)(R_0-1+\delta)}.$$

We consider the two cases, binding or non-binding ZLB separately.

Case 1: Non-binding ZLB

We first consider the case that the ZLB is not binding at $\omega_0^{CC}(k_0)$. So $X_0 = X^*$ in the two equations above. Given k_0, $\left\{R_0, \omega_0^{CC}\right\}$ are the two unknowns. After some calculations, we can express ω_0^{CC} and k_0 as functions of R_0:

$$\omega_0^{CC} = \frac{(1-\delta)+\frac{X^*}{\alpha}\left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)}{\left[\left(1+\frac{\frac{X^*}{\alpha}-(1-m)}{\beta}-\frac{\left(\frac{X^*}{\alpha}-1\right)(1-\delta)}{\beta R_0}\right)\frac{\gamma}{1+\gamma}\frac{1}{1-m}+\frac{1}{1+\gamma}\right]\left[1-\delta+\left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)\right]}, \tag{A.18c}$$

$$k_0 = \frac{1}{\beta R_0}\frac{X^*}{\alpha}\frac{\left[\frac{X^*}{\alpha}-1+m+\beta+\frac{\beta}{\gamma}(1-m)\right]R_0-\left(\frac{X^*}{\alpha}-1\right)(1-\delta)}{\left(\frac{X^*}{\alpha}-1+m\right)R_0-\left(\frac{X^*}{\alpha}-1\right)(1-\delta)}\frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}(R_0-1+\delta)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}R_0\right)^{\frac{1-\alpha}{\alpha}}(R_0-1+\delta)}.$$

Notice that R_0 is strictly decreasing in k_0. So we can write ω_0^{CC} as a function of k_0. By varying the value of R_0, we can trace out $\omega_0^{CC}(k_0)$. In particular, when k_0 equals to

$$\hat{k}_0^{CC} = \frac{\frac{\left(\frac{X^*}{\alpha}-1\right)\delta+m}{\beta}+1+\frac{1}{\gamma}(1-m)}{\left(\frac{X^*}{\alpha}-1+m\right)-\left(\frac{X^*}{\alpha}-1\right)(1-\delta)}\frac{\frac{1-\alpha}{\alpha}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\delta^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X^*}{\alpha}\delta\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}}, \tag{A.18d}$$

$R_0 = 1$ at $\omega_0 = \omega_0^{CC}\left(\hat{k}_0^{CC}\right)$. Thus out result here with non-binding ZLB only applies for $k_0 < \hat{k}_0^{CC}$.

Case 2: Binding ZLB

Now consider the case that the ZLB is binding at $\omega_0^{CC}(k_0)$. We set $R_0 = 1$ in equations (A.18a) and (A.18b). Given k_0, $\left\{X_0, \omega_0^{CC}\right\}$ are the two unknowns. After some calculations, we can express ω_0^{CC} and k_0 as functions of X_0:

$$\begin{aligned}\omega_0^{CC} &= \Lambda_0\frac{1-\delta+\delta\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\frac{X_0}{\alpha}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}}{1-\delta+\delta\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}}, \\ k_0 &= \Lambda_1\frac{\frac{1-\alpha}{X^*}A_1\left(\delta\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\delta\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\frac{X_0}{\alpha}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}},\end{aligned} \tag{A.18e}$$

in which Λ_0 and Λ_1 are constants:

$$\Lambda_0 = \frac{\frac{1+\gamma}{\gamma}(1-m)}{\frac{m-\delta+\frac{\delta X^*}{\alpha}}{\beta} + \frac{1+\gamma-m}{\gamma}}, \tag{A.18f}$$

$$\Lambda_1 = \frac{\frac{m-\delta}{\beta} + \frac{\delta X^*}{\alpha\beta} + 1 + \frac{1-m}{\gamma}}{m(1-\delta) + \left(\frac{X^*}{\alpha} - 1 + m\right)\delta}. \tag{A.18g}$$

Notice that X_0 is increasing in k_0. So we can write ω_0^{CC} as function of k_0. By varying the value of X_0, we can trace out $\omega_0^{CC}(k_0)$. In particular, when k_0 equals to $\hat{k}_0^{CC}$ in (A.18d), $X_0 = X^*$. Thus our result here with binding ZLB only applies for $k_0 > \hat{k}_0^{CC}$. Furthermore, when $k_0 = \Lambda_1 \frac{\frac{1-\alpha}{X^*} A_1 \left(\delta \frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta}$, $\omega_0^{CC}(k_0) = \Lambda_0$, and $X_0 \to +\infty$. Thus there is no solution for $\omega_0^{CC}(k_0)$ when $k_0 > \Lambda_1 \frac{\frac{1-\alpha}{X^*} A_1 \left(\delta \frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta}$. This happens because the $\{k_0, \omega_0\}$ lies in the region where no equilibrium exists as shown in Proposition 1.

To sum up, we have completed the calculation for $\omega_0^{CC}(k_0)$. When $k_0 \leq \hat{k}_0^{CC}$ in (A.18d), $\omega_0^{CC}(k_0)$ is given by (A.18c), and when $\hat{k}_0^{CC} < k_0 \leq \Lambda_1 \frac{\frac{1-\alpha}{X^*} A_1 \left(\delta \frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta}$, $\omega_0^{CC}(k_0)$ is given by (A.18e). When $k_0 > \Lambda_1 \frac{\frac{1-\alpha}{X^*} A_1 \left(\delta \frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta}$, $\omega_0^{CC}(k_0)$ is not defined. See the dashed line in Figure 3 for an example. □

D.1.2 Threshold for Equilibrium Existence

We define $\bar{\bar{k}}_0(\omega_0)$ as follows. When $\omega_0 \geq \Lambda_0$ given in equation (A.18f), we set $\bar{\bar{k}}_0(\omega_0) = \bar{k}_0(\omega_0)$ in equation (A.16), the cutoff value for equilibrium existence in the natural borrowing limit case. When $\omega_0 < \Lambda_0$, $\bar{\bar{k}}_0(\omega_0)$ is solved by the following two equations in which the marginal product of capital, $r_1^K = R_1^K - (1-\delta) \in (\delta, +\infty)$, is used as an auxiliary variable:

$$k_1 + \frac{1}{\beta}\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(r_1^K\right)^{\frac{\alpha}{\alpha-1}} = (1-\delta)\left(1 - \frac{\omega_0}{1+\gamma}\right)\bar{\bar{k}}_0, \tag{A.18h}$$

$$\left(1 - m(1-\delta) - m r_1^K\right) k_1 = \frac{\gamma}{1+\gamma}(1-\delta)\,\omega_0 \bar{\bar{k}}_0. \tag{A.18i}$$

In both equations, k_1 is a decreasing function of r_1^K as below:

$$k_1 = \frac{\frac{1-\alpha}{X^*} A_1 \left[\frac{X^*}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} \left[r_1^K\right]^{\frac{\alpha}{\alpha-1}}}{m(1-\delta) + \left(\frac{X^*}{\alpha} - 1 + m\right) r_1^K}. \tag{A.18j}$$

Equations (A.18h) and (A.18i) imply that r_1^K is a decreasing function of ω_0:

$$r_1^K = \frac{\left(\frac{1+\gamma}{\omega_0} - 1\right)\left(1 - m\left(1-\delta\right)\right) - \left(\gamma + \frac{\gamma}{\beta} m \left(1-\delta\right)\right)}{\left(\frac{1+\gamma}{\omega_0} - 1\right) m + \frac{\gamma}{\beta}\left(\frac{X^*}{\alpha} - 1 + m\right)}. \tag{A.18k}$$

Inserting this expression into (A.18h), we see that $\bar{\bar{k}}_0\left(\omega_0\right)$ is increasing, and thus r_1^K is decreasing in k_0 at $k_0 = \bar{\bar{k}}_0\left(\omega_0\right)$. In particular, at $\omega_0 = \Lambda_0$, the equation above implies $r_1^K = \delta$, and by (A.18i) we have $\bar{\bar{k}}_0\left(\omega_0\right) = \bar{k}_0\left(\omega_0\right)$ at $\omega_0 = \Lambda_0$. This shows that the function $\bar{\bar{k}}_0\left(\omega_0\right)$ is continuous at $\omega_0 = \Lambda_0$.

When $\omega_0 < \Lambda_0$, by the expression of $\bar{k}_0\left(\omega_0\right)$ in (A.16), we have

$$\frac{\left(1-\delta\right)\frac{\gamma}{1+\gamma}\omega_0}{\frac{\gamma}{\beta}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}\bar{k}_0\left(\omega_0\right) = \frac{1+\beta+\delta\left(\frac{X^*}{\alpha}-1\right)}{\left(\frac{1+\gamma}{\omega_0}-1\right)\left(1-\delta+\frac{\delta X^*}{\alpha}\right)-\gamma}\delta^{\frac{\alpha}{\alpha-1}}. \tag{A.18l}$$

Similarly, by (A.18i), (A.18j) and (A.18k), we have

$$\frac{\frac{\gamma}{1+\gamma}\left(1-\delta\right)\omega_0}{\frac{\gamma}{\beta}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}\bar{\bar{k}}_0\left(\omega_0\right) = \frac{1-m\left(1-\delta\right)+\frac{\beta m+m^2\left(1-\delta\right)}{\frac{X^*}{\alpha}-\left(1-m\right)}}{\left(\frac{1+\gamma}{\omega_0}-1\right)\left(1-m\left(1-\delta\right)+\frac{m^2\left(1-\delta\right)}{\frac{X^*}{\alpha}-\left(1-m\right)}\right)-\gamma}\left[r_1^K\right]^{\frac{\alpha}{\alpha-1}}, \tag{A.18m}$$

whose left-hand side is the same as that of (A.18l).

Given $\omega_0 < \Lambda_0$, we can show that the first term on the right-hand-side of (A.18l) is larger than that of (A.18m). Besides, from (A.18k), we have $r_1^K > \delta$ at $k_0 = \bar{\bar{k}}_0\left(\omega_0\right)$, and then $\delta^{\frac{\alpha}{\alpha-1}} > \left[r_1^K\right]^{\frac{\alpha}{\alpha-1}}$. Thus by comparing (A.18l) and (A.18m), we see that $\bar{\bar{k}}_0\left(\omega_0\right) < \bar{k}_0\left(\omega_0\right)$. Applying the Implicit Function Theorem to equations (A.18k) and (A.18m), we can show that $\bar{\bar{k}}_0\left(\omega_0\right)$ is decreasing in m when $\omega_0 < \Lambda_0$. By (A.18h) and (A.18k), it follows that $\bar{\bar{k}}_0\left(\omega_0\right)$ is increasing in A_1.

D.2 Equilibrium Properties with Binding Collateral Constraint

Having defined the thresholds $\omega_0^{CC}(.)$ and $\bar{\bar{k}}_0(.)$, now we proceed with Step 2 for the proof of Proposition 2. When the collateral constraint is binding, $b_0 = -mR_1^K k_1$, and by (4), $\omega_1 = 1 - m$. Using the equations in Appendix B.1, we can explicitly solve for the equilibrium at $t = 1$ given R_1^K. For example, by (6e) and (A.1c),

$$\frac{k_1}{L_1} = A_1 \left[\frac{X^*}{\alpha} \left(R_1^K - 1 + \delta \right) \right]^{\frac{1}{\alpha - 1}},$$

$$L_1 = \frac{\frac{1-\alpha}{X^*}}{\frac{m\alpha(1-\delta)}{X^*\left(R_1^K - 1 + \delta\right)} + 1 - \frac{\alpha}{X^*}(1-m)}.$$

Now, back to time $t = 0$. Denote the marginal product of capital as $r_t^K = R_t^K - (1-\delta)$. We can express the other variables as functions of $\{r_1^K, R_0\}$ or $\{r_1^K, X_0\}$ depending on whether the ZLB is binding. In particular,

$$k_1 = \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} \right)^{\frac{1}{\alpha-1}} \left(r_1^K \right)^{\frac{\alpha}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{X^*} + \left(1 - \frac{\alpha}{X^*}(1-m) \right) r_1^K}, \tag{A.19a}$$

which is decreasing in r_1^K.

By (5b) for period 0 and 1, (5c) and (6e), we have

$$r_0^K = \left(\beta R_0 \frac{A_0}{A_1} \right)^{\frac{1-\alpha}{\alpha}} \left(\frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} r_1^K. \tag{A.19b}$$

Equilibria with binding collateral constraint can be written as systems of two equations and two unknowns, $\{r_1^K, R_0\}$ or $\{r_1^K, X_0\}$:

$$\begin{aligned} k_1 = & \left[(1-\delta)\left(1 - \frac{\omega_0}{1+\gamma} \right) + \left(\frac{X_0}{\alpha} - \frac{\omega_0}{1+\gamma} \right) \left(\frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left(\frac{A_0}{A_1} \beta R_0 \right)^{\frac{1-\alpha}{\alpha}} r_1^K \right] k_0 \\ & - \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} \left[\frac{1}{\beta R_0} \left(r_1^K \right)^{\frac{\alpha}{\alpha-1}} \right], \end{aligned} \tag{A.19c}$$

and

$$\left(1 - \frac{m\left(1 - \delta + r_1^K\right)}{R_0} \right) k_1 = \frac{\gamma}{1+\gamma} \left(1 - \delta + r_0^K \right) \omega_0 k_0, \tag{A.19d}$$

in which k_1 and r_0^K are given in equations (A.19a) and (A.19b). Equation (A.19c) is derived by the feasibility condition (3a) in period 0, while equation (A.19d) is derived by applying $c_0 = \frac{1}{1+\gamma} R_0^K k_0 \omega_0$ and $b_0 = -m R_1^K k_1$ (binding collateral constraint) to the entrepreneurs' budget constraint in period 0.

A solution to the system of equations (A.19c) and (A.19d) corresponds to an equilibrium with binding collateral constraint if the multiplier μ_0 implied by (7) is positive, i.e., if

$$R_0 \leq R_1^K. \tag{A.19e}$$

In the next subsection, we characterize the properties of the solution to (A.19c) and (A.19d), depending on whether the ZLB is binding. We temporarily ignore the requirement (A.19e) and get back to it later in Subsection D.2.3.

D.2.1 Equilibrium with Non-binding ZLB and Binding Collateral Constraint

Lemma 9. *Assume that the collateral constraint is binding, and $\omega_0 < \frac{X^*}{\alpha}$. Given k_0, there is a cutoff value of ω_0, $\hat{\omega}_0(k_0)$ such that if $\omega_0 \geq \hat{\omega}_0(k_0)$, there exists a unique solution to* (A.19c) *and* (A.19d) *with non-binding ZLB, and R_0 is increasing in ω_0. If $\omega_0 < \hat{\omega}_0(k_0)$, there does not exist such a solution.*

Proof. **Step 1: Equilibrium Representation**

Setting $X_0 = X^*$, r_1^K can be expressed as functions of R_0 in both (A.19c) and (A.19d).

Equation (A.19c) becomes

$$\begin{aligned}
&\frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{\frac{m\alpha(1-\delta)}{X^*}+\left[1-\frac{\alpha}{X^*}(1-m)\right]r_1^K}+\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\frac{1}{\beta R_0}\\
&=\left[(1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right)+\left(\frac{X^*}{\alpha}-\frac{\omega_0}{1+\gamma}\right)\left(\frac{A_0}{A_1}\beta R_0\right)^{\frac{1-\alpha}{\alpha}}r_1^K\right]k_0\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned} \tag{A.20a}$$

in which r_1^K is a decreasing function of R_0. Denote this implicit function as $r_1^K = f_1(R_0)$. We can easily verify that $\lim_{R_0\to 0} f_1(R_0) \to +\infty$, and $\lim_{R_0\to+\infty} f_1(R_0) \to 0$.

We can write this equation in the form of

$$\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}} = -\psi_0^1(R_0)-\psi_1^1(R_0)r_1^K+\psi_2^1\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}}+\psi_3^1(R_0)\left(r_1^K\right)^{\frac{1}{1-\alpha}}+\psi_4^1(R_0)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}, \tag{A.20b}$$

where $\psi_0^1, \psi_1^1, \psi_2^1, \psi_3^1, \psi_4^1 > 0$. Denote its right-hand side as $F_1\left(r_1^K, R_0\right)$.

Equation (A.19d) becomes

$$1 = \frac{m\left(1-\delta+r_1^K\right)}{R_0}+\frac{\gamma}{1+\gamma}\omega_0 k_0\frac{\left[1-\delta+\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}r_1^K\right]\left[\frac{m\alpha(1-\delta)}{X^*}+\left(1-\frac{\alpha}{X^*}(1-m)\right)r_1^K\right]}{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{\alpha}{\alpha-1}}}, \tag{A.20c}$$

which can be similarly written as

$$\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}=\psi_0^2\left(R_0\right)+\psi_1^2\left(R_0\right)r_1^K+\psi_2^2\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}}+\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}}+\psi_4^2\left(R_0\right)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}, \tag{A.20d}$$

where $\psi_0^2, \psi_1^2, \psi_2^2, \psi_3^2, \psi_4^2 > 0$. Thus there exists a unique solution for r_1^K as a function of R_0. Denote this implicit function as $r_1^K = f_2\left(R_0\right)$. We can also easily verify that that $\lim_{R_0 \to m(1-\delta)} f_2\left(R_0\right) \to 0$, and as $\lim_{R_0 \to +\infty} f_2\left(R_0\right) \to 0$. Thus $f_2\left(R_0\right)$ is not monotone.

Step 2: Equilibrium Existence

First, we show that, given $\omega_0 < \frac{X^*}{\alpha}$, as $R_0 \to +\infty$, $f_2\left(R_0\right)$ is asymptotically higher than $f_1\left(R_0\right)$.

As $R_0 \to +\infty$, $f_1\left(R_0\right)$ and $f_2\left(R_0\right)$ both converge to zero. We can derive the following asymptotic behaviors as $R_0 \to +\infty$:

$$\left[f_1\left(R_0\right)\right]^{\frac{1}{1-\alpha}} \propto \frac{1}{\left(\frac{X^*}{\alpha}-\frac{\omega_0}{1+\gamma}\right)}\frac{\left(1-\alpha\right)A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}k_0 m\alpha\left(1-\delta\right)}R_0^{\frac{\alpha-1}{\alpha}},$$

$$\left[f_2\left(R_0\right)\right]^{\frac{1}{1-\alpha}} \propto \frac{\frac{1+\gamma}{\gamma}}{\omega_0}\frac{\left(1-\alpha\right)A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}k_0 m\alpha\left(1-\delta\right)}R_0^{\frac{\alpha-1}{\alpha}}.$$

If $\omega_0 < \frac{X^*}{\alpha}$, $f_2\left(R_0\right)$ is asymptotically higher than $f_1\left(R_0\right)$.

From the last two steps, we obtain $f_1(R_0) > f_2(R_0)$ at $R_0 = m(1-\delta)$ and $f_1(R_0) < f_2(R_0)$ when R_0 is sufficiently high. By the Intermediate Value Theorem, the two functions will cross at least once. This guarantees the existence of a solution $\left(R_0, r_1^K\right)$ for the two equations (A.20a) and (A.20c) without considering the ZLB.

Step 3: Equilibrium Uniqueness

We show at any intersection of f_1, f_2, i.e. $f_1\left(R_0\right) = f_2\left(R_0\right)$, the slope of f_2 must be steeper than the one for f_1, i.e. $f_1'\left(R_0\right) < f_2'\left(R_0\right)$.

By the Implicit Function Theorem,

$$f_1'\left(R_0\right) = -\frac{\partial F_1/\partial R_0}{\partial F_1/\partial r_1^K},$$

in which

$$\frac{\partial F_1}{\partial r_1^K} = -\psi_1^1\left(R_0\right) + \frac{\alpha}{1-\alpha}\psi_2^1\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}-1} + \frac{1}{1-\alpha}\psi_3^1\left(R_0\right)\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}} + \left(1+\frac{1}{1-\alpha}\right)\psi_4^1\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}},$$
$$\frac{\partial F_1}{\partial R_0} = -\frac{d}{dR_0}\psi_0^1\left(R_0\right) - \frac{d}{dR_0}\psi_1^1\left(R_0\right)r_1^K + \frac{d}{dR_0}\psi_3^1\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0}\psi_4^1\left(R_0\right)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}.$$

We can easily check that $\frac{\partial F_1}{\partial R_0} > 0$. Since r_1^K is a decreasing function of R_0, by the implicit function theorem, $\frac{\partial F_1}{\partial r_1^K} > 0$.

Similarly,

$$f_2'\left(R_0\right) = -\frac{\partial F_2/\partial R_0}{\partial F_2/\partial r_1^K},$$

in which

$$\frac{\partial F_2}{\partial r_1^K} = \psi_1^2\left(R_0\right) + \frac{\alpha}{1-\alpha}\psi_2^2\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}-1} + \frac{1}{1-\alpha}\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}} + \left(1+\frac{1}{1-\alpha}\right)\psi_4^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}},$$
$$\frac{\partial F_2}{\partial R_0} = \frac{d}{dR_0}\psi_0^2\left(R_0\right) + \frac{d}{dR_0}\psi_1^2\left(R_0\right)r_1^K + \frac{d}{dR_0}\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0}\psi_4^2\left(R_0\right)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}.$$

After lengthy algebras using these expressions, we find that $f_1'(R_0) < f_2'(R_0)$ is implied by

$$\begin{aligned}
&\left[\frac{d}{dR_0}\psi_3^1\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0}\psi_4^1\left(R_0\right)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}\right]\\
&\times\left[\frac{\alpha}{1-\alpha}\psi_2^2\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}-1} + \frac{1}{1-\alpha}\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}} + \left(1+\frac{1}{1-\alpha}\right)\psi_4^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}}\right]\\
>&\left[\frac{d}{dR_0}\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0}\psi_4^2\left(R_0\right)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}\right]\\
&\times\left[\frac{\alpha}{1-\alpha}\psi_2^1\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}-1} + \frac{1}{1-\alpha}\psi_3^1\left(R_0\right)\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}} + \left(1+\frac{1}{1-\alpha}\right)\psi_4^1\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}}\right].
\end{aligned}$$

Inserting the expressions of $\psi_1^1, \psi_2^1, \psi_3^1, \psi_4^1$ and $\psi_1^2, \psi_2^2, \psi_3^2, \psi_4^2$ above and after some calculations, we see that the inequality above holds.[32]

Combining the previous three steps together, we can see that with binding collateral constraint and non-binding ZLB, a solution to (A.20a) and (A.20c) exists and is unique (without checking whether the implied R_0 satisfies the ZLB).

Step 4: Cutoff of ω_0 for ZLB

An example of equations (A.20a) and (A.20c) are given in Figure A.3. In particular,

[32]The calculations are relatively straightforward since $F_1\left(r_1^K, R_0\right)$ and $F_2\left(r_1^K, R_0\right)$ take the same form, and many common terms cancel out.

by checking equations (A.20a) and (A.20c), we see that, as ω_0 increases both curves shift to the right, and the equilibrium R_0 increases. In other words, R_0 increases with ω_0. Thus we can identify the cutoff for binding ZLB, $\hat{\omega}_0(k_0)$, such that given k_0, $R_0 = 1$ at $\omega_0 = \hat{\omega}_0(k_0)$. The expression of $\hat{\omega}_0(k_0)$ can be solved implicitly by imposing $R_0 = 1$ in (A.20a) and (A.20c).

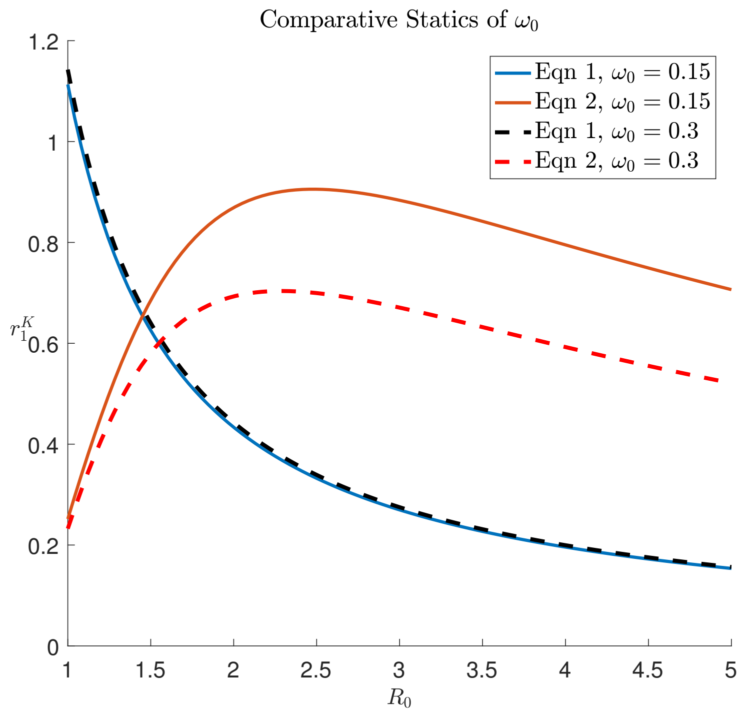


Figure A.3: Equilibria with Binding Collateral Constraint and no ZLB

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.8$ *and* $\epsilon = 21$. $k_0 = 0.1$.

We can show that at $\omega_0 = \hat{\omega}_0(k_0)$, r_1^K is decreasing in k_0. In particular, there is a constant

$$\underline{\hat{k}}_0 = \frac{\left[\frac{1}{m(1-\delta)+\left(\frac{X^*}{\alpha}-(1-m)\right)\left(\frac{1}{m}-(1-\delta)\right)} + \frac{1}{\beta}\right] \frac{1-\alpha}{X^*} A_1 \left[\frac{X^*}{\alpha}\left(\frac{1}{m}-(1-\delta)\right)\right]^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{1}{m}-(1-\delta)\right)}, \qquad \text{(A.20e)}$$

such that when $k_0 = \underline{\hat{k}}_0$, $\hat{\omega}_0\left(\underline{\hat{k}}_0\right) = 0$, and $r_1^K = \frac{1}{m} - (1-\delta)$. As $k_0 \to +\infty$, $\lim_{k_0 \to +\infty} \hat{\omega}_0(k_0) = \frac{(1+\gamma)(1-m(1-\delta))}{1+\gamma+\left(\frac{\gamma}{\beta}-1\right)m(1-\delta)}$, and $\lim_{k_0\to+\infty} r_1^K[k_0, \hat{\omega}_0(k_0)] = 0$. See the blue solid line in Figure A.4 for an example of $\hat{\omega}_0(k_0)$.

□

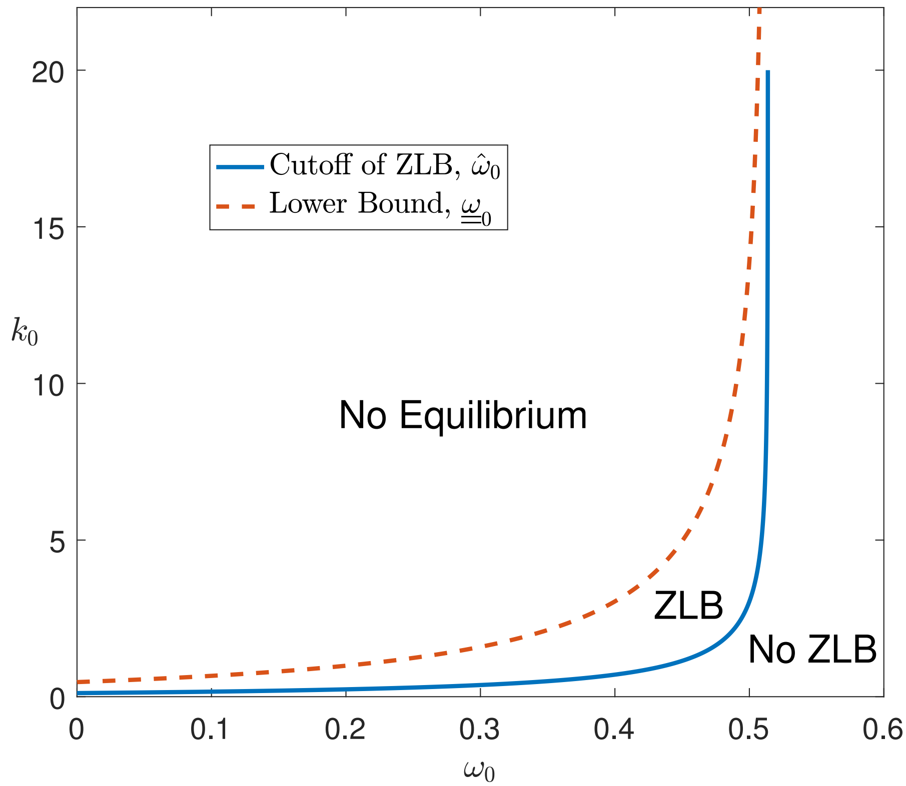


Figure A.4: Cut-offs with Binding Collateral Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.5$ and $\epsilon = 21$.

D.2.2 Equilibrium with Binding ZLB and Binding Collateral Constraint

Now we explore the situation when both the collateral constraint and the ZLB are binding.

Lemma 10. *Assume the collateral constraint is binding, and ω_0 is smaller than*

$$\varpi = \min\left\{(1+\gamma)\frac{1-\alpha}{\alpha}X^*, H\left(\gamma, \beta, \delta, m, \alpha, X^*\right)\right\}, \tag{A.21a}$$

where H is a function defined in (A.21g). Given k_0, there is a cutoff value of ω_0, $\underline{\underline{\omega}}_0\left(k_0\right) < \hat{\omega}_0\left(k_0\right)$ defined in Lemma 9 such that if $\underline{\underline{\omega}}_0\left(k_0\right) < \omega_0 < \hat{\omega}_0\left(k_0\right)$, there exists a unique solution to (A.19c) and (A.19d) with binding ZLB, and in this solution, X_0 is decreasing in ω_0. If $\omega_0 \geq \hat{\omega}_0\left(k_0\right)$ or $\omega_0 \leq \underline{\underline{\omega}}_0\left(k_0\right)$, there does not exist such an solution.

Proof. **Step 1: Equilibrium Representation**

In this case, we represent the system as functions of $\left\{r_1^K, r_0^K\right\}$. Setting $R_0 = 1$, and from

(A.19b) X_0 can be expressed as a function of $\left\{r_1^K, r_0^K\right\}$ as below:

$$X_0 = X^* \left(\beta \frac{A_0}{A_1}\right)^{1-\alpha} \left(\frac{r_1^K}{r_0^K}\right)^{\alpha}. \tag{A.21b}$$

The counterpart for the restriction $X_0 \geq X^*$ is

$$r_0^K \leq \left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} r_1^K. \tag{A.21c}$$

Equation (A.19c) becomes

$$\begin{aligned} & k_1 + \frac{1}{\beta}\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(r_1^K\right)^{\frac{\alpha}{\alpha-1}} \\ = & (1-\delta)\left(1 - \frac{\omega_0}{1+\gamma}\right) k_0 + \left(\frac{X^*}{\alpha}\left(\beta \frac{A_0}{A_1}\right)^{1-\alpha} \left(\frac{r_1^K}{r_0^K}\right)^{\alpha} - \frac{\omega_0}{1+\gamma}\right) r_0^K k_0, \end{aligned} \tag{A.21d}$$

If $\omega_0 < (1+\gamma)\frac{1-\alpha}{\alpha} X^*$, we can show that r_0^K is a decreasing function of r_1^K. Denote $r_0^K = g_1\left(r_1^K\right)$. There is an upper bound for r_1^K, denoted as $\hat{r}_1^K$, such that $g_1\left(\hat{r}_1^K\right) = 0$, and $X_0 \to \infty$ at $\hat{r}_1^K$.

Equation (A.19d) becomes

$$\left(1 - m(1-\delta) - m r_1^K\right) k_1 = \frac{\gamma}{1+\gamma} \omega_0 k_0 \left(1 - \delta + r_0^K\right) \tag{A.21e}$$

in which r_0^K is decreasing in r_1^K as well. Denote this function as $r_0^K = g_2\left(r_1^K\right)$. Similarly, in (A.21e), there is also an upper bound for r_1^K, denoted as $\tilde{r}_1^K$, such that $g_2\left(\tilde{r}_1^K\right) = 0$, and $X_0 \to \infty$ at the upper bound $\tilde{r}_1^K$.

Step 2: Equilibrium Existence

We show that given $\omega_0 < \hat{\omega}_0(k_0)$, defined in Lemma 9, and $\hat{r}_1^K > \tilde{r}_1^K$, there exists an equilibrium with both binding collateral constraint and ZLB.

The intuition of this result can be seen in Figure A.5. The black dashed line corresponds to $X_0 = X^*$ below which we have $X_0 \geq X^*$. When $\omega_0 < \hat{\omega}_0(k_0)$, by equations (A.20a) and (A.20c) in Subsection D.2.1 (also see Figure A.3), with $R_0 = 1$ and $X_0 = X^*$, r_1^K in (A.21d) is smaller than r_1^K in (A.21e). Correspondingly, in Figure A.5, Point A, the intersection of $g_1\left(r_1^K\right)$ and $X_0 = X^*$ lies to the lower left of Point B, the intersection of $g_2\left(r_1^K\right)$ and $X_0 = X^*$. In other words, at $r_1^K = r_{1,B}^K$, the value at point B, $g_1\left(r_{1,B}^K\right) < g_2\left(r_{1,B}^K\right)$. Now with $\hat{r}_1^K > \tilde{r}_1^K$, we see that $g_1\left(\tilde{r}_1^K\right) > g_2\left(\tilde{r}_1^K\right)$. Since both $g_1\left(r_1^K\right)$ and $g_2\left(r_1^K\right)$ are continuous, by the Intermediate Value Theorem, they intersect at least once at some $r_1^K \in \left[r_{1,B}^K, \tilde{r}_1^K\right]$ with $X_0 > X^*$. Thus there exists an equilibrium with both binding ZLB and collateral constraint in this range.

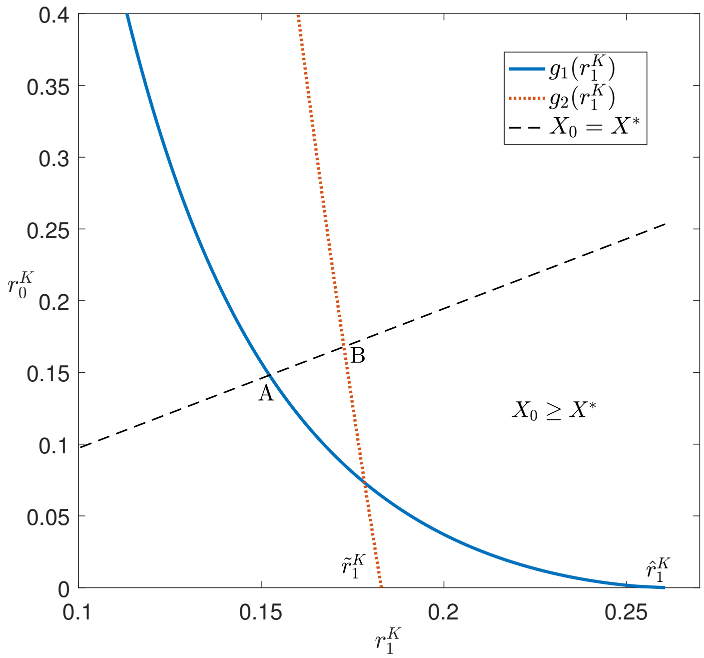


Figure A.5: Equilibria with Binding Collateral Constraint and ZLB

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.8$ and $\epsilon = 21$. $\omega_0 = 0.08$ and $k_0 = 1.27$.

Step 3: Equilibrium Uniqueness

When ω_0 is smaller than the value in equation (A.21g), the slope of $g_1\left(r_1^K\right)$ is higher than the slope of $g_2\left(r_1^K\right)$ when they intersect.

Using implicit function theorem, the derivatives of $g_1\left(r_1^K\right)$ and $g_2\left(r_1^K\right)$ are

$$\frac{\partial g_1}{\partial r_1^K}\left(r_1^K\right) = \frac{\frac{\partial k_1}{\partial r_1^K} - \frac{1}{\beta}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}} - X_0 k_0 \frac{r_0^K}{r_1^K}}{\left(\frac{1-\alpha}{\alpha}X_0 - \frac{\omega_0}{1+\gamma}\right)k_0},$$
$$\frac{\partial g_2}{\partial r_1^K}\left(r_1^K\right) = \frac{-mk_1 + \left(1 - m\left(1-\delta\right) - mr_1^K\right)\frac{\partial k_1}{\partial r_1^K}}{\frac{\gamma}{1+\gamma}\omega_0 k_0}.$$

We will show that given (A.21c) and $\delta \leq r_1^K \leq \frac{1-m(1-\delta)}{m}$,

$$\frac{\partial g_1}{\partial r_1^K}\left(r_1^K\right) > \frac{\partial g_2}{\partial r_1^K}\left(r_1^K\right)$$

holds at their intersection for ω_0 sufficiently small. The lower bound of r_1^K, δ is derived by the relation $R_1^K \geq R = 1$, while its upper bound, $\frac{1-m(1-\delta)}{m}$ is from equation (A.21e). Indeed, the inequality can be rewritten as

$$\begin{aligned}
&\left(\left(\left[-\frac{\partial k_1}{\partial r_1^K}\right] + \frac{1}{\beta}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}\right)\frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma}\left(mk_1 + \left(1-m\left(1-\delta\right)-mr_1^K\right)\left[-\frac{\partial k_1}{\partial r_1^K}\right]\right)\right)\omega_0 \\
&\leq \left(\frac{1-\alpha}{\alpha}\left(mk_1 + \left(1-m\left(1-\delta\right)-mr_1^K\right)\left[-\frac{\partial k_1}{\partial r_1^K}\right]\right) - \frac{\gamma}{1+\gamma}\omega_0 k_0 \frac{r_0^K}{r_1^K}\right)X_0,
\end{aligned} \tag{A.21f}$$

in which the expression of k_1 is from (A.19a).

After some calculations, we can show a stronger result as below:

$$\begin{aligned}
&\left(\left(\left[-\frac{\partial k_1}{\partial r_1^K}\right] + \frac{1}{\beta}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}\right)\frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma}\left(mk_1 + \left(1-m\left(1-\delta\right)-mr_1^K\right)\left[-\frac{\partial k_1}{\partial r_1^K}\right]\right)\right)\omega_0 \\
&\leq \frac{1-\alpha}{\alpha}\left(m + \frac{\left(1-\frac{\alpha}{X^*}\left(1-m\right)\right)\left(1-m\left(1-\delta\right)-mr_1^K\right)}{\frac{m\alpha(1-\delta)}{X^*} + \left(1-\frac{\alpha}{X^*}\left(1-m\right)\right)r_1^K}\right)k_1 X^*,
\end{aligned}$$

and this inequality holds if

$$\omega_0 < H(\gamma,\beta,\delta,m,\alpha,X^*) \equiv \min\left\{G\left(\delta\right), G\left(\frac{1-m(1-\delta)}{m}\right)\right\}, \tag{A.21g}$$

in which

$$\begin{aligned}
G\left(r_1^K\right) = &\frac{1-\alpha}{\alpha}\left(\frac{m^2\alpha\left(1-\delta\right)}{X^*} + \left(1-\frac{\alpha}{X^*}\left(1-m\right)\right)\left(1-m\left(1-\delta\right)\right)\right)X^*/ \\
&\{\left(1+\gamma+\left(\frac{\gamma}{\beta}-1\right)m\left(1-\delta\right)\right)\frac{1}{1+\gamma}\frac{\alpha}{1-\alpha}\frac{m\alpha\left(1-\delta\right)}{X^*}\frac{1}{r_1^K} \\
&+\left(\frac{\gamma}{\beta}\left(X^*-\alpha\left(1-m\right)\right)-\alpha m\right)\frac{1}{1-\alpha}\frac{1}{1+\gamma}\left(1-\frac{\alpha}{X^*}\left(1-m\right)\right)r_1^K \\
&+\frac{1}{1+\gamma}\frac{1}{1-\alpha}\left(1-\frac{\alpha}{X^*}\left(1-m\right)\right)\left(1+\gamma+\left(2\alpha\frac{\gamma}{\beta}-1\right)m\left(1-\delta\right)\right)+\frac{1}{1+\gamma}\frac{1-2\alpha}{1-\alpha}\frac{m^2\alpha\left(1-\delta\right)}{X^*}\}.
\end{aligned}$$

As a result, given k_0, $\omega_0 < \hat{\omega}_0\left(k_0\right)$ from Subsection D.2.1, and $\hat{r}_1^K > \tilde{r}_1^K$, an equilibrium with binding collateral constraint and ZLB exists and is unique. Otherwise, if there are multiple equilibria in this region, $g_1\left(r_1^K\right)$ and $g_2\left(r_1^K\right)$ cross for multiple times, and then one of these equilibria features $\frac{dg_1}{dr_1^K} \leq \frac{dg_2}{dr_1^K}$ which contradicts the slope comparison above.

Step 4: Equilibrium Non-Existence with Binding ZLB and Collateral Constraint

In Figure A.6, we show the comparative statics results for decreasing ω_0. We see that as ω_0 decreases, $g_1\left(r_1^K\right)$ shifts to the right, while $g_2\left(r_1^K\right)$ shifts to the left, making the equilibrium r_1^K higher and r_0^K lower. From (A.21b), X_0 also increases. Thus X_0 *decreases in* ω_0.

We also see that $\frac{\partial \hat{r}_1^K}{\partial \omega_0} > 0$ and $\frac{\partial \tilde{r}_1^K}{\partial \omega_0} < 0$. When ω_0 drops to the level $\underline{\underline{\omega}}_0(k_0)$ such that $\hat{r}_1^K = \tilde{r}_1^K$, at the intersection of $g_1\left(r_1^K\right)$and $g_2\left(r_1^K\right)$, $r_0^K = 0$, and X_0 goes to infinity.

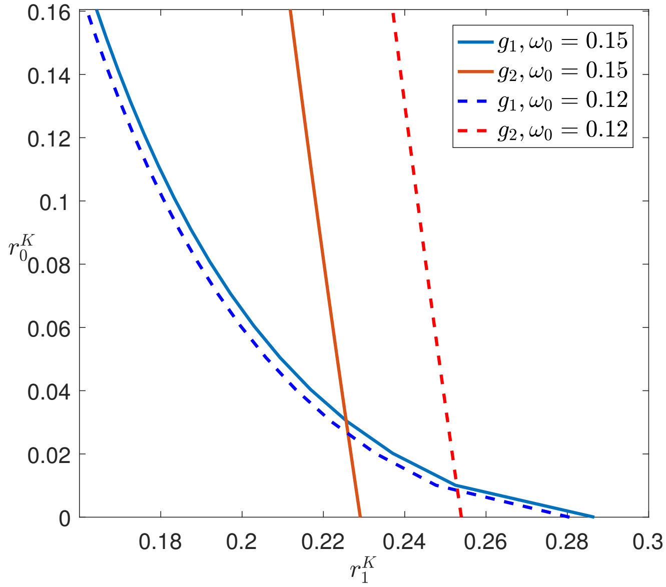


Figure A.6: Comparative Statics with Binding Collateral Constraint and ZLB

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.8$ and $\epsilon = 21$, and $k_0 = 1.27$.

Now we solve the cut-off value $\underline{\underline{\omega}}_0(k_0)$ by setting $\hat{r}_1^K = \tilde{r}_1^K$. Accordingly, by setting $r_0^K = 0$ in equations (A.21d) and (A.21e), we have

$$k_1 + \frac{1}{\beta}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\left(r_1^K\right)^{\frac{\alpha}{\alpha-1}} = (1-\delta)\left(1-\frac{\underline{\underline{\omega}}_0}{1+\gamma}\right)k_0,$$
$$\left(1-m(1-\delta)-mr_1^K\right)k_1 = \frac{\gamma}{1+\gamma}(1-\delta)\underline{\underline{\omega}}_0 k_0.$$

Notice that these two equations are exactly the same as equations (A.18h) and (A.18i) when we define $\bar{\bar{k}}_0(\omega_0)$. Thus $\underline{\underline{\omega}}_0(k_0)$ is the inverse function of $\bar{\bar{k}}_0(\omega_0)$ and is increasing in k_0.

We can show that $\forall k_0 > 0$, $\underline{\underline{\omega}}_0(k_0) < \hat{\omega}_0(k_0)$, the cutoff for binding ZLB in Lemma 9. First, when

$$k_0 = \underline{k}_0 = \frac{\frac{1-\alpha}{X^*}A_1\left[\frac{X^*}{\alpha}\left(\frac{1}{m}-(1-\delta)\right)\right]^{\frac{\alpha}{\alpha-1}}}{m(1-\delta)+\left(\frac{X^*}{\alpha}-(1-m)\right)\left(\frac{1}{m}-(1-\delta)\right)}, \tag{A.21h}$$

$\underline{\underline{\omega}}_0(\underline{k}_0) = 0$, and the implied $r_1^k = \frac{1}{m} - (1-\delta)$. Notice $\underline{k}_0 > \hat{\underline{k}}_0$ in equation (A.20e). As $k_0 \to +\infty$, $\lim_{k_0\to+\infty}\underline{\underline{\omega}}_0(k_0) = \frac{(1+\gamma)(1-m(1-\delta))}{1+\gamma+\left(\frac{\gamma}{\beta}-1\right)m(1-\delta)}$, and $\lim_{k_0\to+\infty} r_1^K\left[k_0, \underline{\underline{\omega}}_0(k_0)\right] = 0$.

Lastly, $\underline{\underline{\omega}}_0(k_0)$ and $\hat{\omega}_0(k_0)$ never cross. Otherwise, $r_0^K = 0$ at their intersection. To sum up, we have $\underline{\underline{\omega}}_0(k_0) < \hat{\omega}_0(k_0)$. See the red dashed line in Figure A.4 for an example of $\underline{\underline{\omega}}_0(k_0)$.

Now we show that, when $\omega_0 < \underline{\underline{\omega}}_0(k_0)$, there is no solution to (A.19c) and (A.19d) with binding collateral constraint and binding ZLB. We keep our definition of Point B as the intersection of $g_2(r_1^K)$ and $r_0^K = \left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} r_1^K$ in equation (A.21c). Since $\omega_0 < \underline{\underline{\omega}}_0(k_0) < \hat{\omega}_0(k_0)$, we have $g_1\left(r_{1,B}^K\right) < g_2\left(r_{1,B}^K\right)$ at $r_1^K = r_{1,B}^K$. We also have $g_1(\hat{r}_1^K) < g_2(\hat{r}_1^K)$. Then in the range $\left[r_{1,B}^K, \hat{r}_1^K\right]$, if $g_1(r_1^K)$ and $g_2(r_1^K)$ intersect, they would intersect at least twice, and at one of the intersections, the condition $\frac{dg_1}{dr_1^K} > \frac{dg_2}{dr_1^K}$ would be violated. Consequently, there is no solution with binding collateral constraint and ZLB in this region.

We can use similar argument to rule out any solution to (A.19c) and (A.19d) with binding collateral constraint and binding ZLB when $\omega_0 \geq \hat{\omega}_0(k_0)$. In this case, Point B lies to the lower-left of Point A in Figure A.5, and $\tilde{r}_1^K < \hat{r}_1^K$. Therefore, if $g_1(r_1^K)$ and $g_2(r_1^K)$ intersect, they would intersect at least twice. Then at one of the intersections, the condition $\frac{dg_1}{dr_1^K} > \frac{dg_2}{dr_1^K}$ would be violated. □

D.2.3 Equilibrium Existence and Uniqueness with Binding Collateral Constraint

In the previous subsections, we impose a binding collateral constraint and characterize the equilibrium properties without checking that whether $\mu_0 \geq 0$, or equivalently (A.19e). It is possible that $R_1^K < R_0$ and borrowing to the limit may not be the entrepreneurs' optimal choice. In this subsection, we check whether this is the case.

The following lemma is useful in proving the equilibrium existence and uniqueness with binding collateral constraint.

Lemma 11. *The derivative of the excess return $R_1^K - R_0$ is negative at $\omega_0 = \omega_0^{CC}(k_0)$ given in Lemma 8.*

Proof. It is equivalent to show that $r_1^K - R_0$ is decreasing in ω_0 at $\omega_0 = \omega_0^{CC}(k_0)$. Notice that since the collateral constraint is binding at $\omega_0 = \omega_0^{CC}(k_0)$, the equilibrium properties with binding collateral constraint from Lemmas 9 and 10 still apply at $\omega_0 = \omega_0^{CC}(k_0)$.

If the ZLB is binding at $\omega_0 = \omega_0^{CC}(k_0)$, $R_0 = 1$, and $R_1^K = 1 - \delta + r_1^K$ is decreasing in ω_0 as shown in Figure A.6, which implies a negative derivative of $R_1^K - R_0$ at $\omega_0 = \omega_0^{CC}(k_0)$.

If the ZLB is not binding, we write the system that determines r_1^K and R_0 as

$$F_1\left(r_1^K, R_0; \omega_0\right) = 0,$$
$$F_2\left(r_1^K, R_0; \omega_0\right) = 0,$$

where F_1 and F_2 are defined in equations (A.20b) and (A.20d). By the Implicit Function Theorem,

$$\begin{bmatrix} \frac{\partial F_1}{\partial r_1^K} & \frac{\partial F_1}{\partial R_0} \\ \frac{\partial F_2}{\partial r_1^K} & \frac{\partial F_2}{\partial R_0} \end{bmatrix} \begin{bmatrix} \frac{dr_1^K}{d\omega_0} \\ \frac{dR_0}{d\omega_0} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial \omega_0} \\ \frac{\partial F_2}{\partial \omega_0} \end{bmatrix},$$

we have

$$\begin{bmatrix} \frac{dr_1^K}{d\omega_0} \\ \frac{dR_0}{d\omega_0} \end{bmatrix} = -\frac{1}{\frac{\partial F_1}{\partial r_1^K}\frac{\partial F_2}{\partial R_0} - \frac{\partial F_1}{\partial R_0}\frac{\partial F_2}{\partial r_1^K}} \begin{bmatrix} \frac{\partial F_2}{\partial R_0} & -\frac{\partial F_1}{\partial R_0} \\ -\frac{\partial F_2}{\partial r_1^K} & \frac{\partial F_1}{\partial r_1^K} \end{bmatrix} \begin{bmatrix} \frac{\partial F_1}{\partial \omega_0} \\ \frac{\partial F_2}{\partial \omega_0} \end{bmatrix}.$$

In Subsection D.2.1, we have shown that

$$\frac{\partial F_1}{\partial r_1^K}\frac{\partial F_2}{\partial R_0} - \frac{\partial F_1}{\partial R_0}\frac{\partial F_2}{\partial r_1^K} < 0.$$

Thus to show $\frac{dr_1^K}{d\omega_0} - \frac{dR_0}{d\omega_0} < 0$, we need to show that

$$\frac{\partial F_2}{\partial R_0}\frac{\partial F_1}{\partial \omega_0} - \frac{\partial F_1}{\partial R_0}\frac{\partial F_2}{\partial \omega_0} + \frac{\partial F_2}{\partial r_1^K}\frac{\partial F_1}{\partial \omega_0} - \frac{\partial F_1}{\partial r_1^K}\frac{\partial F_2}{\partial \omega_0} < 0.$$

We can see that $\frac{\partial F_1}{\partial R_0} > 0$, $\frac{\partial F_1}{\partial r_1^K} > 0$, $\frac{\partial F_1}{\partial \omega_0} < 0$, $\frac{\partial F_2}{\partial r_1^K} > 0$, and $\frac{\partial F_2}{\partial \omega_0} > 0$. But the sign of $\frac{\partial F_2}{\partial R_0}$ is not clear. In particular, we have $\frac{\partial F_2}{\partial \omega_0} = -\gamma\frac{\partial F_1}{\partial \omega_0}$. Thus to show the inequality above, it is sufficient to show that

$$\begin{aligned} &\frac{\partial F_2}{\partial R_0}\frac{\partial F_1}{\partial \omega_0} + \gamma\frac{\partial F_1}{\partial R_0}\frac{\partial F_1}{\partial \omega_0} + \frac{\partial F_2}{\partial r_1^K}\frac{\partial F_1}{\partial \omega_0} + \gamma\frac{\partial F_1}{\partial r_1^K}\frac{\partial F_1}{\partial \omega_0} \\ =& \left(\frac{\partial F_2}{\partial R_0} + \gamma\frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} + \gamma\frac{\partial F_1}{\partial r_1^K}\right)\frac{\partial F_1}{\partial \omega_0} < 0. \end{aligned}$$

Since $\frac{\partial F_1}{\partial \omega_0} < 0$, it is then sufficient to show that

$$\frac{\partial F_2}{\partial R_0} + \gamma\frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} + \gamma\frac{\partial F_1}{\partial r_1^K} > 0.$$

Since $\frac{\partial F_1}{\partial r_1^K} > 0$, we can show a stronger result

$$\frac{\partial F_2}{\partial R_0} + \gamma \frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} > 0.$$

This expression can be written as

$$\begin{aligned}
\frac{\partial F_2}{\partial R_0} + \gamma \frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} =& \frac{d}{dR_0}\psi_0^2\left(R_0\right) - \gamma \frac{d}{dR_0}\psi_0^1\left(R_0\right) + \psi_1^2\left(R_0\right) \\
&+ \left[\frac{d}{dR_0}\psi_1^2\left(R_0\right) - \gamma \frac{d}{dR_0}\psi_1^1\left(R_0\right)\right] r_1^K \\
&+ \left[\frac{d}{dR_0}\psi_3^2\left(R_0\right) + \gamma \frac{d}{dR_0}\psi_3^1\left(R_0\right)\right] \left(r_1^K\right)^{\frac{1}{1-\alpha}} \\
&+ \left[\frac{d}{dR_0}\psi_4^2\left(R_0\right) + \gamma \frac{d}{dR_0}\psi_4^1\left(R_0\right)\right] \left(r_1^K\right)^{1+\frac{1}{1-\alpha}} \\
&+ \frac{\alpha}{1-\alpha}\psi_2^2\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}-1} + \frac{1}{1-\alpha}\psi_3^2\left(R_0\right)\left(r_1^K\right)^{\frac{\alpha}{1-\alpha}} \\
&+ \left(1+\frac{1}{1-\alpha}\right)\psi_4^2\left(R_0\right)\left(r_1^K\right)^{\frac{1}{1-\alpha}},
\end{aligned}$$

in which the expressions $\psi_0^1, \psi_1^1, \psi_3^1, \psi_4^1, \psi_0^2, \psi_1^2, \psi_3^2, \psi_4^2 > 0$ can be found in equations (A.20b) and (A.20d).

Since $\frac{d}{dR_0}\psi_3^1\left(R_0\right)$, $\frac{d}{dR_0}\psi_4^1\left(R_0\right)$, $\frac{d}{dR_0}\psi_3^2\left(R_0\right)$, $\frac{d}{dR_0}\psi_4^2\left(R_0\right) > 0$, it is then sufficient to show that

$$\frac{d}{dR_0}\psi_0^2\left(R_0\right) - \gamma \frac{d}{dR_0}\psi_0^1\left(R_0\right) + \psi_1^2\left(R_0\right) + \left[\frac{d}{dR_0}\psi_1^2\left(R_0\right) - \gamma \frac{d}{dR_0}\psi_1^1\left(R_0\right)\right] r_1^K > 0.$$

Inserting the expressions of $\psi_0^1\left(R_0\right)$, $\psi_1^1\left(R_0\right)$, $\psi_0^2\left(R_0\right)$, $\psi_1^2\left(R_0\right)$ into the expression above, and use the fact that $1-\delta+r_1^K = R_0$ at $\omega_0^{CC}\left(k_0\right)$, it remains to show that

$$\left[\frac{\gamma}{\beta}\left(1-\delta\right)m + \frac{\gamma}{\beta}\left(\frac{X^*}{\alpha} - \left(1-m\right)\right) r_1^K\right] \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{R_0^2} > 0,$$

which holds naturally. Thus $r_1^K - R_0$ is decreasing in ω_0 at $\omega_0^{CC}\left(k_0\right)$ when the ZLB is not binding. □

Lemma 12. *When $\omega_0 \leq \omega_0^{CC}\left(k_0\right)$ and $k_0 < \bar{\bar{k}}_0\left(\omega_0\right)$, there is a unique equilibrium with binding collateral constraint. When $\omega_0 > \omega_0^{CC}\left(k_0\right)$ or $k_0 \geq \bar{\bar{k}}_0\left(\omega_0\right)$, there is no such an equilibrium.*

Proof. In Lemma 10, we show $\bar{\bar{k}}_0\left(\omega_0\right)$ is the inverse function of $\underline{\underline{\omega}}_0\left(k_0\right)$. And the region with $k_0 < \bar{\bar{k}}_0(\omega_0)$ is equivalent to the region with $\omega_0 > \min\left\{0, \underline{\underline{\omega}}_0(k_0)\right\}$. When $\omega_0 \leq \min\left\{0, \underline{\underline{\omega}}_0(k_0)\right\}$, Lemma 10 shows that there is no equilibrium with binding collateral constraint and binding ZLB. Lemma 10 also shows that $\underline{\underline{\omega}}_0(k_0) < \hat{\omega}_0(k_0)$ in Lemma 9. By Lemma 9, there is also no equilibrium with binding collateral constraint and non-binding ZLB when $\omega_0 \leq \underline{\underline{\omega}}_0(k_0)$.

When $\omega_0 \in \left[\min\left\{0, \underline{\underline{\omega}}_0(k_0)\right\}, \omega_0^{CC}(k_0)\right]$, by Lemmas 9 and 10, we impose a binding collateral constraint and establish the existence of an equilibrium in this region. It remains to verify that $\mu_0 \geq 0$. In other words, borrowing up to the limit is indeed the entrepreneurs' optimal choice. Equivalently we need to show $R_1^K \geq R_0$. We prove this by contradiction.

Assume that in an equilibrium at some $k_0 = k_0^a$ and $\omega_0 = \omega_0^a \in \left[\min\left\{0, \underline{\underline{\omega}}_0(k_0)\right\}, \omega_0^{CC}(k_0)\right]$, the implied $\mu_0 < 0$. Since the equilibrium value of μ_0 is continuous in ω_0, as ω_0 decreases, the value of μ_0 also moves continuously. Now if $k_0^a \geq \underline{k}_0$ in (A.21h), we can decrease ω_0 to $\underline{\underline{\omega}}_0(k_0)$. Since the ZLB is binding at $\underline{\underline{\omega}}_0(k_0)$, and $r_1^k > \delta$ at $\underline{\underline{\omega}}_0(k_0)$, $\mu_0 > 0$ at $\underline{\underline{\omega}}_0(k_0)$. Then as ω_0 drops, the value of μ_0 switches from negative to positive, and it must be zero at a certain value of $\omega_0 \in \left[\underline{\underline{\omega}}_0(k_0), \omega_0^a\right]$. Call this value ω_0^b. Then at both $\left(k_0^a, \omega_0^b\right)$ and $\left(k_0^a, \omega_0^{CC}\left(k_0^a\right)\right)$, the leverage ratio is exactly m, and $R_1^k = R_0$. This violates the fact that $\omega_0^{CC}(k_0)$ is uniquely determined as in Lemma 13.

If $k_0^a < \underline{k}_0$ in (A.21h), as ω_0 drops from ω_0^a, it will hit $\omega_0 = 0$. If the ZLB is binding at $\{k_0^a, \omega_0 = 0\}$, then by equation (A.21e), $R_1^K = \frac{1}{m} > 1 = R_0$. If the ZLB is not binding at $\{k_0^a, \omega_0 = 0\}$, then by equation (A.20c), $R_1^K = \frac{R_0}{m} > R_0$. In either case, we have $\mu_0 > 0$ at $\{k_0^a, \omega_0 = 0\}$, and then we can find a value $\omega_0^b > 0$ at which $\mu_0 = 0$ and obtain a contradiction.

Next we show that when $\omega_0 > \omega_0^{CC}(k_0)$, the implied excess return $R_1^K - R_0 < 0$, and thus borrowing up to the limit is not the optimal choice for the entrepreneurs. We also show this by contradiction. Suppose that at some $k_0 = k_0^c$ and $\omega_0 = \omega_0^c > \omega_0^{CC}(k_0)$, the implied excess return $R_1^K - R_0 > 0$. Since the excess return moves continuously with ω_0, $R_1^K - R_0 = 0$ at $\omega_0^{CC}(k_0)$, and its derivative is negative at $\omega_0^{CC}(k_0)$ by Lemma 11, then there must exist a $\omega_0^d \in \left(\omega_0^{CC}(k_0), \omega_0^c\right)$ such that $R_1^K - R_0 = 0$ at $\omega_0 = \omega_0^d$, and the leverage ratio is m. But again this violates the fact $\omega_0^{CC}(k_0)$ is uniquely determined as in Lemma 13. Therefore, there is no equilibrium with binding collateral constraint given $\omega_0 > \omega_0^{CC}(k_0)$.

Lastly, when $k_0 \geq \bar{\bar{k}}_0(\omega_0)$, it must be that $\omega_0 \leq \underline{\underline{\omega}}_0(k_0)$. By Lemma 10, an equilibrium with binding collateral constraint does not exist. □

D.3 Equilibrium Properties with Non-binding Collateral Constraint

Now we proceed to Step 3 in the proof of Proposition 2.

Lemma 13. *Given $m < 1$, $\alpha < \frac{X^*}{1+X^*}$ and ω_0 is smaller than*

$$\min\left\{\frac{X^* + X^*\gamma}{X^* + \alpha\gamma}, (1+\gamma)\,\frac{1-\alpha}{\alpha}X^*, H(\gamma, \beta, \delta, m, \alpha, X^*)\right\},$$

when $\omega_0 \geq \omega_0^{CC}(k_0)$ and $k_0 < \bar{\bar{k}}_0(\omega_0)$ there exists a unique equilibrium with non-binding collateral constraint; when $\omega_0 < \omega_0^{CC}(k_0)$ or $k_0 \geq \bar{\bar{k}}_0(\omega_0)$, such an equilibrium does not exist.

Proof. Notice that an equilibrium with non-binding collateral constraint is equivalent to an equilibrium with natural borrowing limit defined in Proposition 1. Therefore, by Lemma 7, the leverage ratio $-\frac{b_0}{R_1^k k_1}$ is decreasing in ω_0. Recall that, the value of $\omega_0^{CC}(k_0)$ is determined by setting $-\frac{b_0}{R_1^k k_1} = m$. When $\omega_0 \geq \omega_0^{CC}(k_0)$, the leverage ratio is smaller than m, and the collateral constraint is not binding. Then an equilibrium exists and is unique given $k_0 < \bar{k}_0(\omega_0)$ by Proposition 1. In this region, $\bar{\bar{k}}_0(\omega_0) = \bar{k}_0(\omega_0)$ in equation (A.16) by construction.

When $\omega_0 < \omega_0^{CC}(k_0)$ and $k_0 \leq \bar{k}_0(\omega_0)$, assuming a non-binding collateral constraint, the implied leverage ratio would be larger than m by Lemma 7, which violates the collateral constraint. Thus there is no equilibrium with non-binding collateral constraint in that region.

When $k_0 \geq \bar{\bar{k}}_0(\omega_0)$, there are two cases, $\bar{\bar{k}}_0(\omega_0) \leq k_0 < \bar{k}_0(\omega_0)$ and $k_0 \geq \bar{k}_0(\omega_0)$. By Proposition 1, an equilibrium with non-binding collateral constraint does not exist in the latter case.

In the former case, we know both $\bar{k}_0(\omega_0)$ and $\bar{\bar{k}}_0(\omega_0)$ are increasing and $\bar{\bar{k}}_0(\omega_0) < \bar{k}_0(\omega_0)$ when $\omega_0 < \Lambda_0$ by Lemma 6 and by the construction of $\bar{\bar{k}}_0(\omega_0)$ in Subsection D.1.2. By Lemma 10, $\bar{\bar{k}}_0(.)$ is the inverse of $\underline{\underline{\omega}}_0(.)$. We now show that $\underline{\underline{\omega}}_0(k_0) < \omega_0^{CC}(k_0)$ for all $k_0 > 0$ where both functions are well-defined.

Indeed, if $k_0 \geq \hat{k}_0^{CC}$ defined in (A.18d), the ZLB is binding at $\omega_0^{CC}(k_0)$ by Lemma 8. Lemma 10 shows that X_0 is decreasing in ω_0 when both the collateral constraint and ZLB are binding. Since X_0 is finite and determined by (A.18e) at $\omega_0 = \omega_0^{CC}(k_0)$ and $X_0 = +\infty$ at $\omega_0 = \underline{\underline{\omega}}_0(k_0)$ defined in Lemma 10, it must be the case that $\underline{\underline{\omega}}_0(k_0) < \omega_0^{CC}(k_0)$.

If $k_0 < \hat{k}_0^{CC}$, the ZLB is not binding at $\omega_0^{CC}(k_0)$ by Lemma 8. Since Lemma 9 shows that R_0 is increasing in ω_0 with binding collateral constraint and non-binding ZLB, $\omega_0^{CC}(k_0) > \hat{\omega}_0(k_0)$, the cutoff value for a binding ZLB given by Lemma 9. Lemma 10 further shows $\underline{\underline{\omega}}_0(k_0) < \hat{\omega}_0(k_0)$. Hence, $\underline{\underline{\omega}}_0(k_0) < \omega_0^{CC}(k_0)$.

We have established that $\underline{\underline{\omega}}_0(k_0) < \omega_0^{CC}(k_0)$. Since $\underline{\underline{\omega}}_0(k_0)$ is the inverse function of $\bar{\bar{k}}_0(\omega_0)$, and when $\bar{\bar{k}}_0(\omega_0) \leq k_0 < \bar{k}_0(\omega_0)$, it must be the case that $\omega_0 \leq \underline{\underline{\omega}}_0(k_0)$, which implies $\omega_0 < \omega_0^{CC}(k_0)$. So for these k_0, an equilibrium with natural borrowing limit exists, but by Lemma 7, the implied leverage ratio would be larger than m which violates the collateral constraint. □

Combining the results from Lemmas 12, 13 and our definition for $\bar{\bar{k}}_0(\omega_0)$ in Subsection D.1.2, we complete the proof of Proposition 2.

D.4 Equilibrium Non-Existence

With tighter borrowing limit, Proposition 2 shows that there does not exist an equilibrium when capital k_0 is sufficiently high, similar to the natural borrowing limit case analyzed in Subsection 2.3. In Figure A.7, we plot the threshold for equilibrium existence, $\bar{\bar{k}}_0$, as a function of ω_0 under different borrowing limits m. The collateral constraint binds when ω_0 is low, and we see $\bar{\bar{k}}_0$ is lower when m is lower. The reason for equilibrium non-existence when capital goes beyond $\bar{\bar{k}}_0$ is insufficient demand. When the collateral constraint binds with smaller m, the entrepreneurs' consumption and investment are more constrained. Thus from the market clear condition (3a), the threshold for equilibrium nonexistence, $\bar{\bar{k}}_0$, is smaller. For large ω_0, the collateral constraint does not bind, so $\bar{\bar{k}}_0$ equals $\bar{k}_0$ independent of the value of m. In Appendix D.1.2, we also show that $\bar{\bar{k}}_0$ is increasing in A_1 and hence, equilibrium is less likely to exist when A_1 is low.

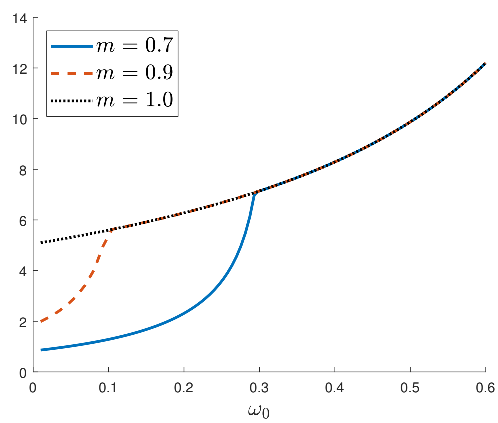


Figure A.7: Upper Bound of Capital

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$ and $\epsilon = 21$.

E Investment Friction and Endogenous Asset Price

In this appendix, we provide the formal statements of the equilibrium characterizations for 2-period economy with irreversible investment under the natural borrowing limit and tighter borrowing limits similar to Propositions 1 and 2. The next two appendices provide the proofs for these results.

E.1 Natural Borrowing Limit

Similar to Subsection 2.3, we focus on the case with natural borrowing limit first. The collateral constraint (1) does not bind in equilibrium, and $\mu_0 = 0$.

Equilibrium Characterizations The Proposition 3 below shows that an equilibrium always exists, is unique, and has intuitive properties. The state space $\{k_0, \omega_0\}$ can be partitioned into different regions with either binding or non binding ZLB and binding or non-binding investment irreversibility constraint. Figure A.8 shows what the different regions look like. The investment irreversibility constraint binds when k_0 is sufficiently high and the ZLB constraint binds when ω_0 is sufficiently low. Interestingly, unlike in the previous model without investment friction, the ZLB does not necessarily bind when k_0 is high. Indeed, the proposition below shows that for ω_0 sufficiently high, the ZLB does not bind for any $k_0 > 0$.

Proposition 3. *With the investment irreversibility constraint, $m = 1$, and ω_0 smaller than*

$$\min\left\{\frac{1}{\frac{\alpha}{X^*}+\frac{1}{\gamma}}, \frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\}\frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^*,$$

an equilibrium always exists and is unique.[33] *Besides, there is a threshold of k_0, $k_0^*(\omega_0)$ such that when $k_0 < k_0^*(\omega_0)$, the irreversibility constraint does not bind and $q_0 = 1$; and when $k_0 \geq k_0^*(\omega_0)$, the irreversibility constraint binds and q_0 is decreasing in k_0.*

In addition, denote ω_0^* as

$$\omega_0^* = \frac{\frac{X^*}{\alpha}\left[\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}} - \frac{1}{\beta}(1-\delta)\right]}{\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\left[(1-\delta)+\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\right]}. \tag{A.22}$$

[33] With our calibrated parameters, the value of the upper bound of ω_0 is 2.4, which is much larger than the typical values of ω_0.

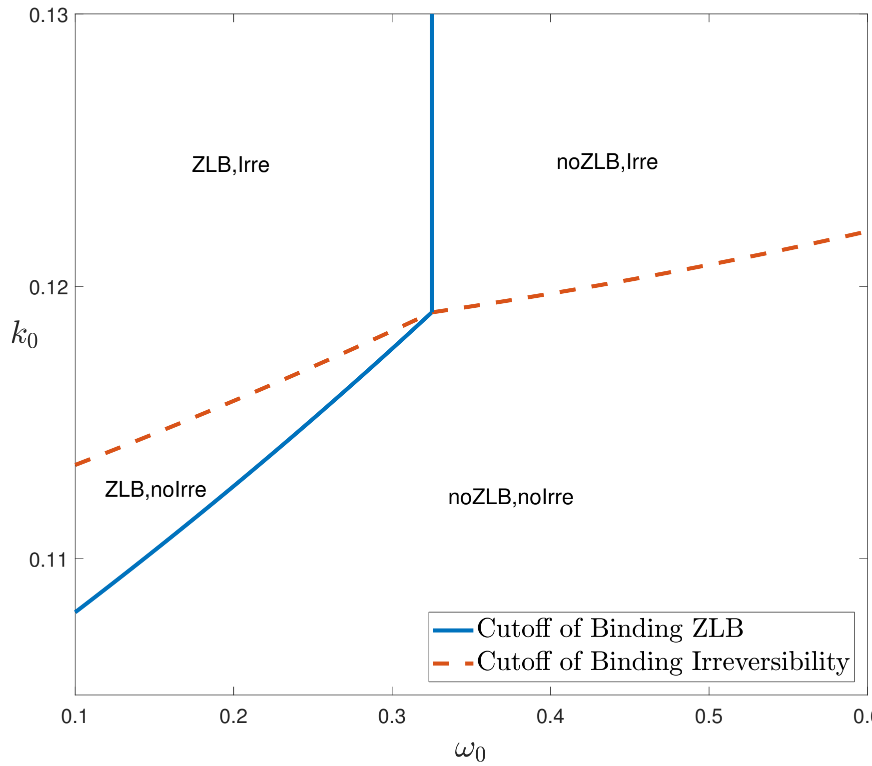


Figure A.8: Regions for Irreversibility and ZLB When $m = 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 0.95$, and $\epsilon = 21$.

1. If $\omega_0 > \omega_0^*$: the ZLB does not bind $\forall k_0 > 0$. R_0 is decreasing in k_0 when $k_0 < k_0^*(\omega_0)$, and is independent of k_0 when $k_0 \geq k_0^*(\omega_0)$.

2. If $\omega_0 \leq \omega_0^*$: there exists another threshold $\hat{k}_0(\omega_0)$ which is smaller than $k_0^*(\omega_0)$, such that:

(i) when $k_0 < \hat{k}_0(\omega_0)$, the ZLB does not bind and R_0 is decreasing in k_0;
(ii) when $k_0 \in \left[\hat{k}_0(\omega_0), k_0^*(\omega_0)\right]$, the ZLB binds, and X_0 is increasing in k_0;
(iii) and when $k_0 \geq k_0^*(\omega_0)$, ZLB binds and X_0 is independent of k_0.

Proof. *The proof is given in Appendix F. We also present the AS-AD representation of the equilibrium in Appendix F.5.*

□

One distinguishing feature of this model is that an equilibrium exists for any $k_0 > 0$. As discussed in Proposition 1, Part 3, in equilibrium, the rate of return on each unit of capital invested at time 0 is bounded from below by the ZLB and by (6e), which puts an upper bound on aggregate capital supply at time 1 if there is no capital adjustment cost. However, with investment irreversibility, k_1 can be large and R_1^K in (A.23b) can be very low. The entrepreneurs are still happy to hold units of capital at time 0 because the price

q_0 endogenous drops yielding high return to holding capital. More formally, we show that as k_0 goes to infinity, either R_0 converges to 1 or a constant strictly greater than 1, and

$$q_0 = \phi\left(\omega_0\right) k_0^{\alpha-1}$$

for some explicit function ϕ. Therefore $\lim_{k_0 \to \infty} q_0 = 0$, warranting that the return to each unit of capital is exactly R_0, as implied by (11).

E.2 Tighter Borrowing Limit

As in Subsection 2.4, the state-space $\{k_0, \omega_0\}$ can be partitioned into regions depending on whether each of the three constraints - the collateral constraint, ZLB, and investment irreversibility constraint - binds. Figure A.9 shows the different regions for a particular set of parameters. The ZLB and the investment irreversibility constraint tend to bind when k_0 is high and when ω_0 is low, and the collateral constraint tends to bind when k_0 is low and when ω_0 is low.

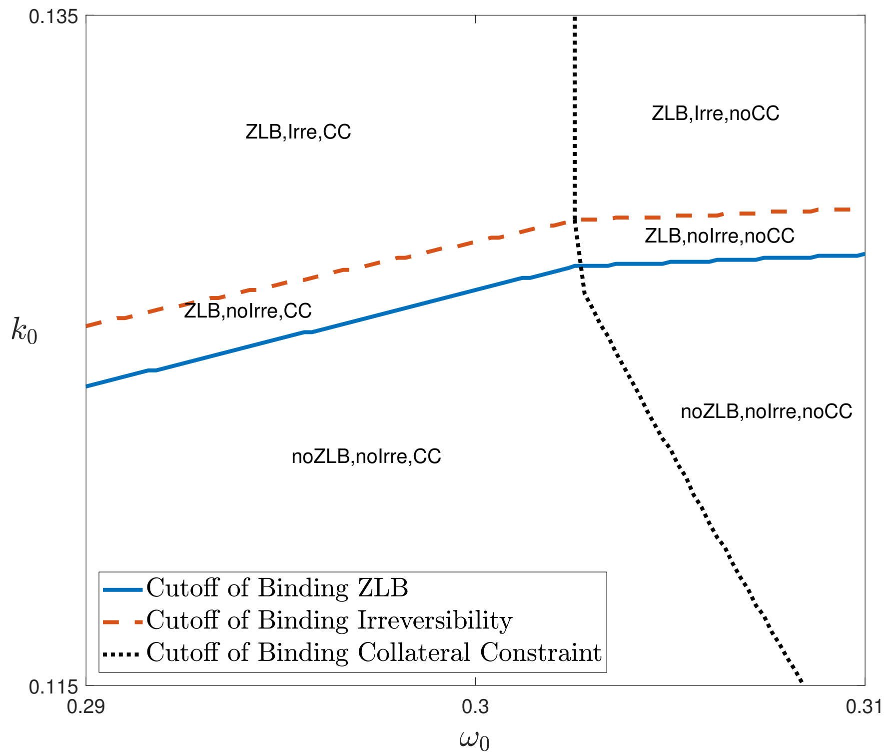


Figure A.9: Regions for Irreversibility, ZLB and Collateral Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 0.99$, $m = 0.7$ *and* $\epsilon = 21$.

The following proposition provides a complete characterization of equilibrium at time 0.

Proposition 4. *With $m < 1$ and the investment irreversibility constraint, given $\{k_0, \omega_0\}$, an equilibrium always exists and is unique. In addition, there is a threshold value for k_0, $k_0^{**}(\omega_0)$, such that*

(1) if $k_0 \geq k_0^{**}(\omega_0)$, the investment irreversibility constraint binds. As k_0 increases, q_0 decreases, but R_0, X_0 and the multiplier for the collateral constraint, μ_0 remain constant. In this region, the collateral constraint is binding if and only if $\omega_0 \leq \omega_{0,Irr}^{CC}$, a constant whose value depends on the parameters.

(2) if $k_0 < k_0^{**}(\omega_0)$, the investment irreversibility constraint does not bind. There is a cutoff value of wealth, $\omega_0^{CC}(k_0)$ such that the collateral constraint is binding if and only if $\omega_0 \leq \omega_0^{CC}(k_0)$.

Proof. The proof is given in Appendix G. We also present the AS-AD representation of the equilibrium in Appendix G.6.

□

Similar to the case with natural borrowing limit, we can also show that as k_0 goes to infinity, either R_0 converges to 1 or a constant strictly greater than 1, and

$$q_0 = \phi^{CC}(\omega_0)k_0^{\alpha-1}$$

for some explicit function ϕ^{CC}. So $\lim_{k_0\to\infty} q_0 = 0$, inducing the entrepreneurs to hold on to their old units of capital because the return to each unit of capital is higher than R_0, as implied by (11).

Policy Functions In the six left panels of Figure A.10, we plot several variables as functions of k_0 fixing ω_0 for $m = 1$ and $m = 0.7$. The shapes of the policy functions look similar under these two values of m. The interest rate R_0 is decreasing in k_0, and when the ZLB binds, markup X_0 increases from its steady state value X^*. As k_0 increases above some threshold the irreversibility constraint binds and X_0, μ_0 become constant. Capital price is decreasing in k_0. The bottom right panel shows that when m decreases from 1 to 0.7, output decreases (weakly), but the magnitude of the decrease is only significant when both collateral constraint and ZLB bind (for k_0 greater than 0.09). Similarly, the six right panels of Figure A.10 show several variables as functions of ω_0 fixing k_0 for $m = 1$ and $m = 0.7$. All variables, except for the multiplier on the collateral constraint, are increasing in ω_0. In addition, output decreases significantly when m decreases from 1 to 0.7 if both collateral constraint and ZLB bind (for ω_0 less than 0.2).

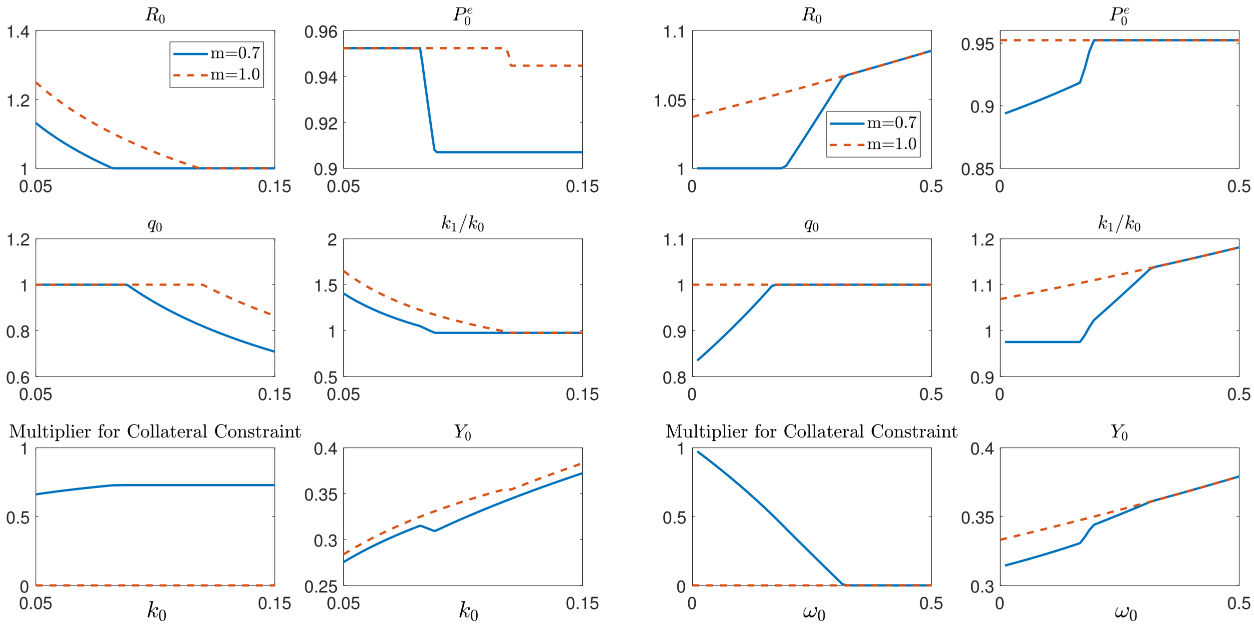


Figure A.10: Policy Functions with Irreversibility Constraint, $m = 1$

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 0.99$, and $\epsilon = 21$. In the left panel, $\omega_0 = 0.1$. In the right panel, $k_0 = 0.1$.

F Proof of Proposition 3

To prove Proposition 3, we proceed as follows:

1. Describe the threshold for a binding irreversibility constraint, $k_0^*(\omega_0)$.

2. Show that an equilibrium with binding irreversibility constraint exists, and is unique, if and only if $k_0 \geq k_0^*(\omega_0)$.

3. Show that an equilibrium with non-binding irreversibility constraint exists, and is unique, if and only if $k_0 < k_0^*(\omega_0)$.

Combining the results from the previous three steps, we can prove Proposition 3.

F.1 Equilibrium Properties

We first describe some equilibrium results which are useful for the following proof of Proposition 3.

Last Period In the last period, period 1, there is no return of investment, the irreversibility constraint is binding, i.e., $k_2 = (1-\delta) k_1$, and the capital price $q_1 = 0$. We still set the markup $X_1 = X^*$. The market clearing condition (3a) implies $c_1 + c_1' = Y_1$. Given $\{k_1, \omega_1\}$, we can solve for the equilibrium in closed-form. In particular,

$$L_1 = \frac{\frac{1-\alpha}{X^*}}{1 - \frac{\alpha}{X^*}\omega_1}, \tag{A.23a}$$

$$R_1^K = \frac{\alpha}{X^*}\left(\frac{\frac{1-\alpha}{X^*}A_1}{1 - \frac{\alpha}{X^*}\omega_1}\right)^{1-\alpha} k_1^{\alpha-1}. \tag{A.23b}$$

First Period In period 0, the gross return for holding capital k_1 is $\frac{R_1^K}{q_0}$. With natural borrowing limit, by the no-arbitrage condition,

$$R_0 = \frac{R_1^K}{q_0}. \tag{A.24a}$$

Given k_1, R_0 and q_0, in the last period, we can derive the follow expressions:

$$Y_1 = \frac{X^*}{\alpha} q_0 R_0 k_1, \tag{A.24b}$$

$$c_1' = \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} q_0 R_0\right)^{\frac{\alpha}{\alpha-1}},$$

$$c_1 = \frac{X^*}{\alpha} q_0 R_0 k_1 - \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} q_0 R_0\right)^{\frac{\alpha}{\alpha-1}}.$$

Using the optimal consumption choices $c_0 = \frac{c_1}{\gamma R_0}$ and $c_0' = \frac{c_1'}{\gamma R_0}$, we obtain the expressions for consumption at $t = 0$:

$$c_0 = \frac{1}{\gamma}\frac{X^*}{\alpha} q_0 k_1 - \frac{1}{\gamma R_0}\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} q_0 R_0\right)^{\frac{\alpha}{\alpha-1}}, \tag{A.24c}$$

$$c_0' = \frac{1}{\beta R_0}\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} q_0 R_0\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.24d}$$

The return to capital at $t = 0$ is

$$R_0^K = (1-\delta)\, q_0 + \left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}} q_0 R_0, \tag{A.24e}$$

and the aggregate output is

$$Y_0 = \frac{1}{\gamma}\frac{X^*}{\alpha}q_0 k_1 - \left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\frac{1}{R_0}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}q_0 R_0\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.24f}$$

By the market clearing condition of used capital, (10d), and the budget constraint of the entrepreneurs (10c) in period 0 and 1, we have

$$c_0 + \frac{c_1}{R_0} = \omega_0 R_0^K k_0.$$

Then the entrepreneurs' optimal consumption at $t = 0$ is

$$c_0 = \frac{1}{1+\gamma}\omega_0 R_0^K k_0.$$

We can use the following two equations to represent the equilibrium conditions with two unknowns: $\{k_1, R_0\}$, $\{q_0, R_0\}$, $\{k_1, X_0\}$ or $\{q_0, X_0\}$ depending on whether the ZLB or the irreversibility constraint is binding.[34] The first equation is derived from (A.24c) for c_0:

$$\frac{\gamma}{1+\gamma}\omega_0 R_0^K k_0 = \frac{X^*}{\alpha}q_0 k_1 - \frac{1}{R_0}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}q_0 R_0\right)^{\frac{\alpha}{\alpha-1}}, \tag{A.24g}$$

in which R_0^K is from equation(A.24e).

The second equation is derived by the feasibility condition at $t = 0$ and equations (A.24c), (A.24d) and (A.24f):

$$\begin{aligned} &\frac{1}{\gamma}\frac{X^*}{\alpha}q_0 k_1 - \left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\frac{1}{R_0}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}q_0 R_0\right)^{\frac{\alpha}{\alpha-1}} + k_1 \\ &= (1-\delta)\,k_0 + \frac{X^*}{\alpha}\left(\frac{1}{\beta R_0}\frac{A_1}{A_0}\frac{X_0}{X^*}\right)^{\frac{\alpha-1}{\alpha}} q_0 R_0 k_0. \end{aligned} \tag{A.24h}$$

[34]Or we can represent the system in a more rigorous way with 4 unknowns: $\{k_1, q_0, R_0, X_0\}$ with another two complementary conditions:

$$\begin{aligned} (1-q_0)\left[k_1 - (1-\delta)\,k_0\right] &= 0, \\ (R_0 - 1)\left(X_0 - X^*\right) &= 0. \end{aligned}$$

F.2 Threshold for Binding Irreversibility Constraint

Assume $\omega_0 < \frac{\frac{1-\alpha}{\alpha}X^*}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$. Here we construct the threshold of k_0 for a binding irreversibility constraint, $k_0^*\left(\omega_0\right)$. Intuitively, since $k_0^*\left(\omega_0\right)$ is the cutoff of binding irreversibility, we should have both $q_0 = 1$ and $k_1 = (1-\delta)\,k_0$ when $k_0 = k_0^*\left(\omega_0\right)$. Its expression also depends on whether the ZLB is binding at $k_0 = k_0^*\left(\omega_0\right)$. We give the expression of $k_0^*\left(\omega_0\right)$ here and verify it later.

In particular, we will show that, when $\omega_0 > \omega_0^*$ in equation (A.22), the ZLB is not binding at $k_0 = k_0^*\left(\omega_0\right)$, and its expression is

$$k_0^*\left(\omega_0\right) = \frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\frac{A_0}{A_1}\beta\left[\frac{(1-\delta)\left[\frac{1}{\beta}\frac{X^*}{\alpha}+\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\omega_0\right]}{\frac{X^*}{\alpha}-\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\omega_0}\right]^{\frac{\alpha}{\alpha-1}}}{(1-\delta)\left[\frac{\frac{X^*}{\alpha}-\omega_0}{\frac{X^*}{\alpha}-\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\omega_0}\right]}, \tag{A.25}$$

which is increasing in ω_0.

When $\omega_0 \leq \omega_0^*$, the ZLB is binding at $k_0 = k_0^*\left(\omega_0\right)$, and its expression is

$$k_0^*\left(\omega_0\right) = \frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\frac{X^*}{\alpha}(1-\delta)-\frac{\gamma}{1+\gamma}\omega_0\left[(1-\delta)+\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0^*(\omega_0)}{X^*}\right)^{-\frac{1}{\alpha}}\right]}, \tag{A.26}$$

in which $X_0^*\left(\omega_0\right)$ is given implicitly by the following equation:

$$\begin{aligned}&(1-\delta)\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}\omega_0+\frac{1}{\beta}(1-\delta)\frac{X^*}{\alpha} \\ &=\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0^*\left(\omega_0\right)}{\alpha}-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}\omega_0\right)\left[\frac{X^*}{X_0^*\left(\omega_0\right)}\right]^{\frac{1}{\alpha}}.\end{aligned} \tag{A.27}$$

When $\omega_0 < \frac{\frac{1-\alpha}{\alpha}X^*}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$, we can easily check that $X_0^*\left(\omega_0\right)$ is decreasing in ω_0, and then $k_0^*\left(\omega_0\right)$ is increasing in ω_0 when $\omega_0 \leq \omega_0^*$. We can also show that $k_0^*\left(\omega_0\right)$ is continuous at $\omega_0 = \omega_0^*$. See the red dashed line in Figure A.8 as one example of $k_0^*\left(\omega_0\right)$.

F.3 Region with binding irreversibility Constraint

Lemma 14. *With natural borrowing limit and* $\omega_0 < \frac{\frac{1-\alpha}{\alpha}X^*}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$*, an equilibrium with binding irreversibility constraint exists and is unique if and only if* $k_0 \geq k_0^*(\omega_0)$ *given in Subsection F.2. When* $k_0 \geq k_0^*(\omega_0)$*,* q_0 *is decreasing in* k_0*, and*

1. *if* $\omega_0 \leq \omega_0^*$ *in (A.22), ZLB is binding, and* $X_0 = X_0^*(\omega_0)$ *in equation (A.27) which is independent of* k_0*;*

2. *if* $\omega_0 > \omega_0^*$*, ZLB is not binding, and* $R_0 = R_0^*(\omega_0)$ *in equation (A.28b) which is also independent of* k_0*.*

Proof. When the irreversibility constraint is binding, $k_1 = (1-\delta)k_0$. Inserting it into equations (A.24g) and (A.24h) and after some calculation, we have the following equation with R_0 or X_0 being the only unknown:

$$\begin{aligned}&\frac{1}{\beta}(1-\delta)\frac{X^*}{\alpha}+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(1-\delta)\frac{\gamma}{1+\gamma}\omega_0 \\ =&\left[\frac{X_0}{\alpha}-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}\omega_0\right]\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}R_0.\end{aligned} \tag{A.28a}$$

Notice that the solution here does not depend on k_0.

Non-binding ZLB

If the ZLB is not binding, we set $X_0 = X^*$ in (A.28a), in which R_0 is the only unknown. The solution is $R_0 = R_0^*(\omega_0)$ as below, which is independent of k_0.

$$R_0^*(\omega_0) = \left(\frac{A_0}{A_1}\beta\right)^{\alpha-1}\left[\frac{(1-\delta)\left[\frac{1}{\beta}\frac{X^*}{\alpha}+\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\omega_0\right]}{\frac{X^*}{\alpha}-\frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\omega_0}\right]^{\alpha}. \tag{A.28b}$$

When $\omega_0 < \frac{\frac{X^*}{\alpha}}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$, $R_0^*(\omega_0)$ is increasing in ω_0, and $R_0^*(\omega_0) \geq 1$ if and only if $\omega_0 \geq \omega_0^*$ in (A.22).

The capital price q_0 is

$$q_0 = \frac{A_1^{1-\alpha}}{\frac{X^*}{\alpha}R_0^*(\omega_0)}\left[\frac{\frac{1}{\gamma}(1-\delta)-\left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}}\left[R_0^*(\omega_0)\right]^{\frac{1}{\alpha}}}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1-\alpha}{X^*}}\right]^{\alpha-1}k_0^{\alpha-1},$$

which is decreasing in k_0. We find in the equation above, $q_0 \leq 1$ if and only if $k_0 \geq k_0^*(\omega_0)$ in equation (A.25). As a result, there exists a unique equilibrium with binding irreversibility constraint and non-binding ZLB if and only if $\omega_0 > \omega_0^*$ and $k_0 \geq k_0^*(\omega_0)$.

Binding ZLB

Now consider the case when both the ZLB and the irreversibility constraint are binding. Then we set $R_0 = 1$ in (A.28a), and X_0 becomes the only unknown. After some calculation, equation (A.28a) becomes (A.27), in which the solution $X_0 = X_0^*(\omega_0)$ is independent of k_0. When $\omega_0 < \frac{\frac{1-\alpha}{\alpha}X^*}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$, $X_0^*(\omega_0)$ is decreasing in ω_0, and $X_0^*(\omega_0) \geq X^*$ if and only if $\omega_0 \leq \omega_0^*$.

The capital price q_0 is

$$q_0 = \frac{A_1^{1-\alpha}}{\frac{X^*}{\alpha}}\left[\frac{(1-\delta) - \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X^*}\left((1-\delta) + \left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0^*(\omega_0)}{X^*}\right)^{-\frac{1}{\alpha}}\right)}{\frac{1-\alpha}{X^*}}\right]^{\alpha-1} k_0^{\alpha-1},$$

which is decreasing in k_0. We find in the equation above, $q_0 \leq 1$ if and only if $k_0 \geq k_0^*(\omega_0)$ in equation (A.26). As a result, there exists a unique equilibrium with binding irreversibility constraint and binding ZLB if and only if $\omega_0 \leq \omega_0^*$ and $k_0 \geq k_0^*(\omega_0)$.

To sum up, when $k_0 \geq k_0^*(\omega_0)$, there exists a unique equilibrium with binding irreversibility constraint. Otherwise, there does not exist such an equilibrium. □

F.4 Region with Non-binding Irreversibility Constraint

F.4.1 Region with Non-binding Irreversibility and Binding ZLB

Lemma 15. *With natural borrowing limit and ω_0 smaller than $\frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^* \min\left\{\frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}, \frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\}$, an equilibrium with non-binding irreversibility and binding ZLB exists and is unique if and only if $\omega_0 \leq \omega_0^*$, and $\hat{k}_0(\omega_0) \leq k_0 < k_0^*(\omega_0)$.[35] In addition, in this region, X_0 is increasing in k_0 and is decreasing in ω_0. $\frac{k_1}{k_0}$ is decreasing in k_0.*

Proof. When the ZLB is binding and the irreversibility constraint is not binding, using equations (A.24g) and (A.24h) and after some calculation, we have the following equation

[35] With our calibrated parameters, the value of this upper bound of ω_0 is 2.91. ω_0^* is given in (A.22), $k_0^*(\omega_0)$ is in (A.26) and $\hat{k}_0(\omega_0)$ is in (A.29b).

with X_0 being the only unknown:

$$(1-\delta)\left(1-\frac{1+\frac{\alpha}{X^*}\gamma}{1+\gamma}\omega_0\right)+\left(\frac{X_0}{\alpha}-\frac{1+\frac{\alpha}{X^*}\gamma}{1+\gamma}\omega_0\right)\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}} \tag{A.29a}$$
$$=\left(\frac{\alpha}{X^*}+\frac{1}{\beta}\right)\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\frac{1}{k_0}.$$

If $\omega_0<\frac{1-\alpha}{\alpha}X^*\frac{1+\gamma}{1+\frac{\alpha}{X^*}\gamma}$, by the implicit function theorem, X_0 is decreasing in ω_0 and increasing in k_0. Given ω_0, $X_0=X^*$ at $k_0=\hat{k}_0(\omega_0)$ given below

$$\hat{k}_0(\omega_0)=\frac{\left(\frac{\alpha}{X^*}+\frac{1}{\beta}\right)\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}-\left(\frac{\alpha}{X^*}+\frac{1}{\gamma}\right)\frac{\gamma}{1+\gamma}\omega_0\left[(1-\delta)+\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\right]}. \tag{A.29b}$$

Thus there is no equilibrium with binding ZLB and non-binding irreversibility constraint when $k_0<\hat{k}_0(\omega_0)$.

The ratio $\frac{k_1}{k_0}$ can be expressed as

$$\frac{k_1}{k_0}=\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\frac{1}{k_0}+\frac{\alpha}{X^*}\frac{\gamma}{1+\gamma}\omega_0\left[1-\delta+\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}\right], \tag{A.29c}$$

which is decreasing in k_0. In particular, $\frac{k_1}{k_0}=1-\delta$ when $k_0=k_0^*(\omega_0)$ in (A.29b). Thus there is no equilibrium with binding ZLB and non-binding irreversibility constraint when $k_0\geq k_0^*(\omega_0)$.

After some calculation, we find $\hat{k}_0(\omega_0)\leq k_0^*(\omega_0)$ if and only if $\omega_0\leq\omega_0^*$. When $\omega_0>\omega_0^*$, the set $\hat{k}_0(\omega_0)\leq k_0<k_0^*(\omega_0)$ is empty. When $\omega_0\leq\omega_0^*$ and $\hat{k}_0(\omega_0)\leq k_0<k_0^*(\omega_0)$, since X_0 is increasing in k_0, $X_0\in[X^*,X_0^*(\omega_0))$, in which $X_0^*(\omega_0)$ is given in equation (A.27). Thus we know an equilibrium with non-binding irreversibility and binding ZLB exists and is unique if and only if $\omega_0\leq\omega_0^*$ and $\hat{k}_0(\omega_0)\leq k_0<k_0^*(\omega_0)$. □

F.4.2 Region with Non-binding Irreversibility and Non-binding ZLB

Lemma 16. *With natural borrowing limit and* $\omega_0<\frac{\frac{1+\gamma}{\gamma}\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}\frac{X^*}{\alpha}$, *an equilibrium with non-binding irreversibility and non-binding ZLB exists if and only if* $\{k_0,\omega_0\}$ *lies in one of the following two regions:* $\{\omega_0>\omega_0^*,k_0<k_0^*(\omega_0)\}$ *or* $\left\{\omega_0\leq\omega_0^*,k_0<\hat{k}_0(\omega_0)\right\}$.[36] *In addition, in both regions,* R_0 *is decreasing in* k_0 *and increasing in* ω_0. *The ratio* $\frac{k_1}{k_0}$ *is decreasing in* k_0.

[36] ω_0^* is given in (A.22), $k_0^*(\omega_0)$ is in (A.25) and $\hat{k}_0(\omega_0)$ is in (A.29b).

Proof. When the ZLB is not binding, setting $X_0 = X^*$ in (A.24g) and (A.24h) and after some calculation, we get the following equation with R_0 as the only unknown:

$$\begin{aligned}
&\left[\frac{1}{\gamma} - \Pi_0\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\right]\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}R_0^{\frac{1}{\alpha-1}} \\
=&\left[(1-\delta)\left(\Pi_0 - \frac{1}{1+\gamma}\omega_0\right) + \left(\frac{X^*}{\alpha}\Pi_0 - \frac{1}{1+\gamma}\omega_0\right)\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}R_0^{\frac{1}{\alpha}}\right]k_0,
\end{aligned} \tag{A.30a}$$

in which Π_0 is a constant:

$$\Pi_0 = \frac{\frac{1}{\gamma}\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}. \tag{A.30b}$$

Given $\omega_0 < \frac{\frac{1+\gamma}{\gamma}\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}\frac{X^*}{\alpha}$, by the implicit function theorem, R_0 is decreasing in k_0 and increasing in ω_0. In particular, $R_0 = 1$ when $k_0 = \hat{k}_0(\omega_0)$ in (A.29b). This suggests that there is no equilibrium with non-binding irreversibility and non-binding ZLB when $k_0 \geq \hat{k}_0(\omega_0)$.

In addition, the ratio $\frac{k_1}{k_0}$ is

$$\frac{k_1}{k_0} = \Pi_1(1-\delta)\left(1 - \frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\omega_0\right) + \Pi_1\left(\frac{X^*}{\alpha} - \frac{\gamma}{1+\gamma}\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\omega_0\right)\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}R_0^{\frac{1}{\alpha}},$$

in which R_0 is derived from the previous equation, and Π_1 is a constant:

$$\Pi_1 = \frac{1}{\left(1+\frac{1}{\gamma}\frac{X^*}{\alpha}\right)\left(1 - \frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\right)}. \tag{A.30c}$$

We see that $\frac{k_1}{k_0}$ is decreasing in k_0. In particular, $\frac{k_1}{k_0} = 1-\delta$ when $k_0 = k_0^*(\omega_0)$ in (A.25). This suggests that there is no equilibrium with non-binding irreversibility and non-binding ZLB when $k_0 \geq k_0^*(\omega_0)$.

To sum up, for an equilibrium with non-binding irreversibility and non-binding ZLB to exist, we should have $k_0 < k_0^*(\omega_0)$ and $k_0 < \hat{k}_0(\omega_0)$. After some calculation, we find $\hat{k}_0(\omega_0) \leq k_0^*(\omega_0)$ if and only if $\omega_0 \leq \omega_0^*$. Thus the region for the existence of such an equilibrium is $\{\omega_0 > \omega_0^*, k_0 < k_0^*(\omega_0)\}$ and $\left\{\omega_0 \leq \omega_0^*, k_0 < \hat{k}_0(\omega_0)\right\}$. Given such an equilibrium exists, it is unique since we have a unique solution of R_0 from equation (A.30a). □

Combining the results of Lemmas 14, 15 and 16, we complete the proof for Proposition 3.

F.5 AS-AD Representation

If irreversibility constraint (10a) is not binding, $q_0 = 1$ from (10e) and locally, the equilibrium and the AS-AD curves are the same as the ones without irreversible investment (except for the last period equilibrium).[37] So in this section we focus on the case in which irreversibility constraint (10a) binds and, thus, $q_0 < 1$.

When the irreversibility constraint binds, $I_0 = 0$, and the aggregate demand is given by summing up c_0 in (A.24c) and c_0' in (A.24d) :

$$Y_0^{AD} = \frac{1}{\gamma}\frac{X^*}{\alpha}q_0\left(1-\delta\right)k_0 - \left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{1}{R_0}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}q_0R_0\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.31}$$

Using the production function, we derive the AS curve as follows:

$$Y_0^{AS} = \frac{X^*}{\alpha}\left(\frac{1}{\beta R_0}\frac{A_1}{A_0}\frac{X_0}{X^*}\right)^{\frac{\alpha-1}{\alpha}} q_0R_0k_0. \tag{A.32}$$

By Proposition 3, when the investment irreversibility constraint binds, both R_0 and X_0 are independent of k_0. As a result, we can plot the AS-AD curves as functions of q_0. Notice that both curves are increasing in q_0. For the AS curve in (A.32), a higher q_0 increases R_1^K by the non-arbitrage condition in (A.24a) and depresses wage at $t = 1$, w_1, which reduces the household's lifetime wealth. As a results, the household chooses to supply more labor in period 0, which increases output. For the AD curve, on the one hand, a higher q_0 increases R_1^K and depresses the households' consumption c_0'; on the other hand, the entrepreneurs enjoy higher c_0 since the households' higher labor supply increases the entrepreneurs' wealth. By (A.31), we see the net effect of q_0 on the aggregate demand is positive. We plot the AS-AD curves with binding ZLB and irreversibility constraint in Figure A.11. We see that as k_0 increases, Y_0 increases and q_0 decreases. As ω_0 increases, both Y_0 and q_0 increase.

G Proof of Proposition 4

To prove Proposition 4, we proceed in the following steps.

1. Describe the threshold of ω_0 for a binding collateral constraint, $\omega_0^{CC}(k_0)$, and the

[37] There is some difference though. In the benchmark model without investment irreversibility, $R_1^K = 1-\delta+\frac{\alpha}{X_1}\left(\frac{k_1}{A_1L_1}\right)^{\alpha-1}$; while with irreversibility, $R_1^K = \frac{\alpha}{X_1}\left(\frac{k_1}{A_1L_1}\right)^{\alpha-1}$ since capital price in the last period is 0. But this difference does not change the results qualitatively. We choose to omit this part here.

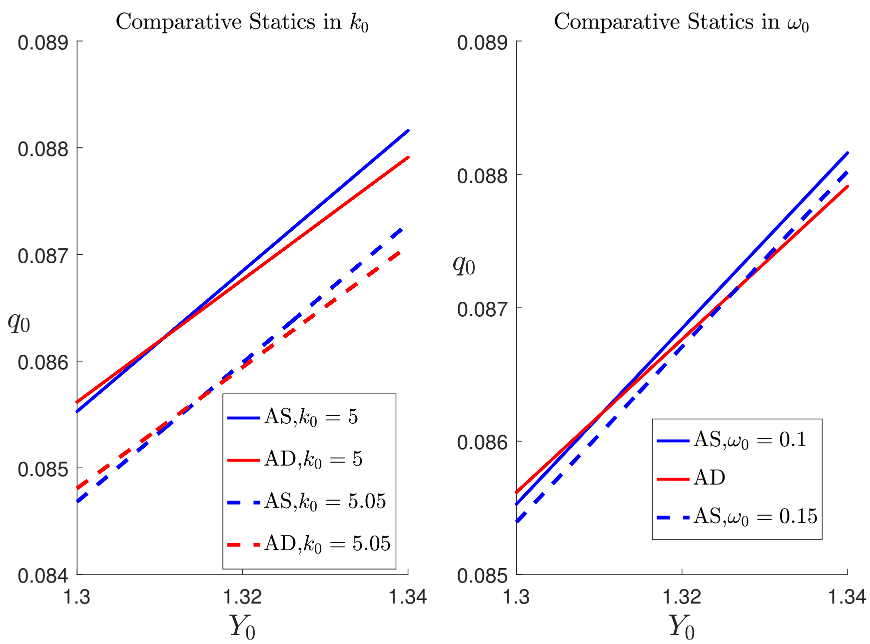


Figure A.11: AS-AD Curves with Binding ZLB and Irreversibility Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.9$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 0.99$ and $\epsilon = 21$. We choose $k_0 = 5$ and $\omega_0 = 0.1$ in the baseline case.

threshold of k_0 for a binding irreversibility constraint, $k_0^{**}(\omega_0)$.

2. Show an equilibrium with binding irreversibility constraint exists and is unique if and only if $k_0 \geq k_0^{**}(\omega_0)$.

3. Show an equilibrium with non-binding irreversibility constraint and non-binding collateral constraint exists and is unique if and only if $k_0 < k_0^{**}(\omega_0)$ and $\omega_0 > \omega_0^{CC}(k_0)$.

4. Show an equilibrium with non-binding irreversibility constraint and binding collateral constraint exists and is unique if and only if $k_0 < k_0^{**}(\omega_0)$ and $\omega_0 \leq \omega_0^{CC}(k_0)$.

G.1 Equilibrium Properties with Binding Collateral Constraint

The equilibrium in the last period is determined in Appendix F.1. In addition, when the collateral constraint binds at $t = 0$, by the definition of wealth share in equation (4), $\omega_1 = 1 - m$, and by equation (A.23a) the labor supply at $t = 1$ is constant and we denote it as L_1^{cc}:

$$L_1^{cc} = \frac{\frac{1-\alpha}{X^*}}{1-(1-m)\frac{\alpha}{X^*}}. \tag{A.33a}$$

The consumptions at $t=1$ are:

$$c_1 = (1-m)\frac{\alpha}{X^*}\left(A_1 L_1^{cc}\right)^{1-\alpha} k_1^{\alpha},$$
$$c_1' = \frac{1-\alpha}{X^*} A_1^{1-\alpha}\left(L_1^{cc}\right)^{-\alpha} k_1^{\alpha}.$$

In the first period,

$$c_0 = \frac{1}{1+\gamma}\omega_0 R_0^K k_0,$$
$$c_0' = \frac{c_1'}{\beta R_0},$$
$$Y_0 = \left(\beta\frac{A_0}{A_1}R_0\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{k_1}{A_1 L_1^{cc}}\right)^{\alpha-1} k_0,$$
$$R_0^K = (1-\delta)\, q_0 + \frac{\alpha}{X_0}\left(\beta R_0\frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{k_1}{A_1 L_1^{cc}}\right)^{\alpha-1}. \tag{A.33b}$$

G.2 Thresholds for Binding Collateral Constraint and Irreversibility

G.2.1 Threshold for Binding Collateral Constraint

We first show that in the model with natural borrowing limit in Subsection E.1, the leverage ratio $-\frac{b_1}{R_1^K k_1}$ is decreasing in ω_0. Now with $m<1$, if the collateral constraint is not binding, the equilibrium is the same as the natural borrowing limit model. Then there is a cutoff value $\omega_0^{CC}(k_0)$ such that $-\frac{b_1}{R_1^K k_1} = m$ at $\omega_0 = \omega_0^{CC}(k_0)$.

Lemma 17. *With $m=1$, the irreversibility constraint and*

$$\omega_0 \leq \frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^* \min\left\{\frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}, \frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\},$$

the leverage ratio $-\frac{b_0}{R_1^K k_1}$ is decreasing in ω_0.

Proof. When $m=1$, we have shown in Proposition 3 that the irreversibility constraint is binding if and only if $k_0 \geq k_0^*(\omega_0)$ given in Subsection F.2.

Case I: Non-binding Irreversibility Constraint

When $k_0 < k_0^*(\omega_0)$, the irreversibility constraint is not binding, and $R_0 = R_1^k$. From the entrepreneurs' budget and their optimal choice $c_0 = \frac{1}{1+\gamma} R_0^k \omega_0 k_0$, their leverage ratio is

$$-\frac{b_0}{R_1^k k_1} = 1 - \frac{\gamma}{1+\gamma} \frac{R_0^k \omega_0 k_0}{k_1}.$$

Showing that the leverage ratio is decreasing in ω_0 is equivalent to showing that $\frac{R_0^k \omega_0 k_0}{k_1}$ is increasing in ω_0.

First, consider the case in which ZLB is not binding, i.e. $R_0 > 1$. Substituting in the expression for k_1 using (A.24b), (A.24c) and (A.24d), we obtain

$$\frac{R_0^k \omega_0 k_0}{k_1} = \frac{\frac{X^*}{\alpha} R_0}{\frac{\gamma R_0}{1+\gamma} + \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} R_0\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}}.$$

Differentiating both sides of the equation above to ω_0, the derivative $\frac{d}{d\omega_0}\left[\frac{R_0^k \omega_0 k_0}{k_1}\right]$ has the same sign as the derivative of the right-hand side, which, after some calculation, is

$$\frac{X^*}{\alpha}\left[\frac{dR_0}{d\omega_0} \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} R_0\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0} - \frac{d}{d\omega_0}\left(\frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} R_0\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}\right) R_0\right]. \tag{A.34}$$

By Lemma 16, R_0 increases with ω_0: $\frac{dR_0}{d\omega_0} > 0$. In addition, by setting $X_0 = X^*$ and $q_0 = 1$ in (A.24e), R_0^K becomes

$$R_0^K = 1 - \delta + \left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} R_0,$$

which is strictly increasing in R_0 and hence in ω_0. Therefore,

$$\frac{d}{d\omega_0}\left(\frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} R_0\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}\right) < 0,$$

then expression (A.34) is positive.

Now, consider the case when $k_0 < k_0^*(\omega_0)$ and ZLB is binding, i.e. $R_0 = 1$ and $X_0 >$

X^*. Then we obtain

$$\frac{R_0^k \omega_0 k_0}{k_1} = \frac{\frac{X^*}{\alpha}}{\frac{\gamma}{1+\gamma} + \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\omega_0 R_0^k k_0}}.$$

By setting $R_0 = 1$ and $q_0 = 1$ in (A.24e), R_0^K becomes

$$R_0^K = 1 - \delta + \left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}.$$

Lemma 15 shows that when $\omega_0 < \frac{1-\alpha}{\alpha} X^* \frac{1+\gamma}{1+\frac{\alpha}{X^*}\gamma}$, X_0 is decreasing in ω_0. Therefore, R_0^K is increasing in ω_0. Hence, $\frac{R_0^k \omega_0 k_0}{k_1}$ is increasing in ω_0 as desired.

Case II: Binding Irreversibility Constraint

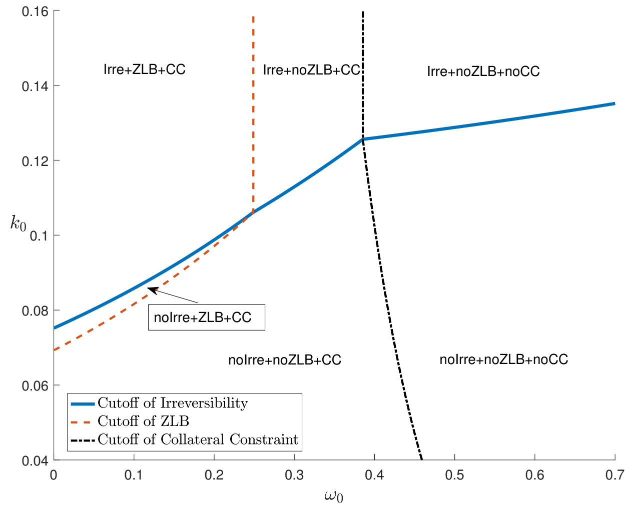
When $k_0 > k_0^* (\omega_0)$, the irreversibility constraint is binding, $q_0 < 1$, $k_1 = (1-\delta) k_0$, and $R_0 = \frac{R_1^k}{q_0}$. From the entrepreneurs' budget and their optimal choice $c_0 = \frac{1}{1+\gamma} R_0^k \omega_0 k_0$, their leverage ratio is

$$\begin{aligned} -\frac{b_0}{R_1^k k_1} &= 1 - \frac{\gamma}{1+\gamma} \frac{R_0^k \omega_0 k_0}{q_0 k_1} \\ &= 1 - \frac{\gamma}{1+\gamma} \frac{R_0^k \omega_0}{q_0 (1-\delta)}. \end{aligned}$$

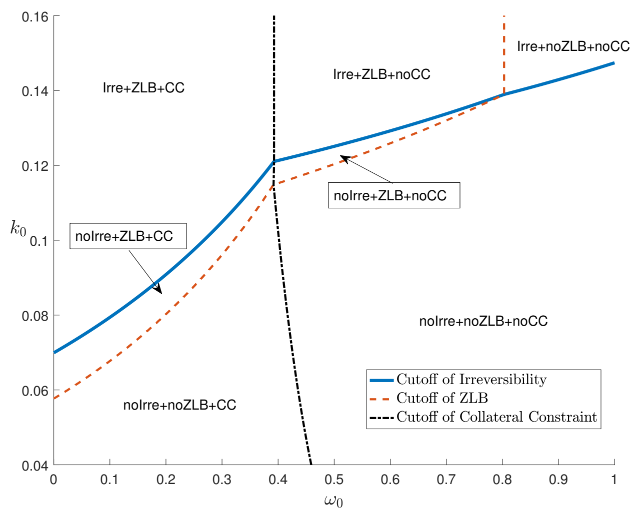
Replacing R_0^k by its expression in (A.24e), we have

$$-\frac{b_0}{R_1^k k_1} = 1 - \frac{\gamma}{1+\gamma} \omega_0 \left[1 + \frac{\left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}} R_0}{1-\delta} \right].$$

We show in Lemma 14 that given $\omega_0 < \frac{\frac{1-\alpha}{\alpha} X^*}{\left(\frac{1}{\gamma} - \frac{1}{\beta}\right) \frac{\gamma}{1+\gamma}}$, when $k_0 > k_0^* (\omega_0)$ and $\omega_0 \geq \omega_0^*$ in equation (A.22), $X_0 = X^*$ and $R_0 = R_0^* (\omega_0)$ in (A.28b) which is increasing in ω_0. Thus in the equation above, $-\frac{b_0}{R_1^k k_1}$ is decreasing in ω_0. When $k_0 > k_0^* (\omega_0)$ and $\omega_0 < \omega_0^*$, by Lemma 14, $R_0 = 1$ and $X_0 = X^* (\omega_0)$ in equation (A.27) which is decreasing in ω_0. Thus in the equation above, $-\frac{b_0}{R_1^k k_1}$ is also decreasing in ω_0. □



(a) Case I: $\mathscr{R}^{CC} > 1$



(b) Case II: $\mathscr{R}^{CC} < 1$

Figure A.12: Regions for Irreversibility, ZLB and Collateral Constraint

Note: Both figures are generated by setting $\beta = 0.99$, $\gamma = 0.6$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $m = 0.7$ and $\epsilon = 21$. In Case I, $A_1 = 1$; in Case II, $A_1 = 0.93$.

Lemma 18. *With $m = 1$, the irreversibility constraint and*

$$\omega_0 \leq \frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^* \min\left\{\frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}}, \frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\},$$

there is a threshold value of ω_0, $\omega_0^{CC}(k_0)$ such that the leverage ratio $-\frac{b_1}{R_1^K k_1} = m$ at $\omega_0 = \omega_0^{CC}(k_0)$. In particular, define a constant

$$\mathscr{R}^{CC} = \left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{1-\alpha}{\alpha}}(1-\delta)\left(\frac{1}{\gamma}(1-m)\frac{\alpha}{X^*}+\frac{1}{\beta}\left(1-\frac{\alpha}{X^*}(1-m)\right)\right). \tag{A.35a}$$

1. *If* $\mathscr{R}^{CC} \geq 1$, $\omega_0^{CC}(k_0) = \begin{cases} \omega_{0,noZLB}^{CC}(k_0), & k_0 \in \left[0, k_{0,Irr}^{CC}\right]; \\ \omega_{0,Irr}^{CC}, & k_0 \in \left(k_{0,Irr}^{CC}, +\infty\right). \end{cases}$
2. *If* $\mathscr{R}^{CC} < 1$, $\omega_0^{CC}(k_0) = \begin{cases} \omega_{0,noZLB}^{CC}(k_0), & k_0 \in \left[0, \widetilde{k}_0\right]; \\ \omega_{0,ZLB}^{CC}(k_0), & k_0 \in \left[\widetilde{k}_0, k_{0,Irr}^{CC}\right]; \\ \omega_{0,Irr}^{CC}, & k_0 \in \left(k_{0,Irr}^{CC}, +\infty\right). \end{cases}$

in which $\omega_{0,noZLB}^{CC}(k_0)$ is given in equation (A.35d), $\omega_{0,ZLB}^{CC}(k_0)$ is from (A.35g), $\widetilde{k}_0$ is a constant in (A.35e),

$$\omega_{0,Irr}^{CC} = \begin{cases} \frac{(1-m)\frac{X^*}{\alpha}\frac{1+\gamma}{\gamma}}{\frac{X^*}{\alpha}\left(1+\frac{1}{\beta}\right)+(1-m)\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}, & \text{if } \mathscr{R}^{CC} \geq 1; \\ \frac{\frac{1+\gamma}{\gamma}(1-m)(1-\delta)}{1-\delta+\frac{1}{\beta}\frac{A_1}{A_0}\left[\frac{1-\delta}{\beta}+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(1-m)(1-\delta)\frac{\alpha}{X^*}\right]^{\frac{1}{1-\alpha}}}, & \text{if } \mathscr{R}^{CC} < 1. \end{cases} \tag{A.35b}$$

and

$$k_{0,Irr}^{CC} = \begin{cases} \left[(1-\delta)\left(\frac{1-m}{\gamma}+\frac{1-\alpha}{\alpha\beta L_1^{cc}}\right)\right]^{\frac{\alpha}{\alpha-1}}\frac{\alpha\beta A_0 L_1^{cc}}{X^*(1-\delta)}, & \text{if } \mathscr{R}^{CC} \geq 1; \\ \frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\left(\frac{X^*}{\alpha}-1+m\right)(1-\delta)}, & \text{if } \mathscr{R}^{CC} < 1. \end{cases}$$

and L_1^{cc} is a constant from (A.33a). See the black dotted line in Figure A.12 for an example.

Proof. Proposition 3 shows that the irreversibility constraint is binding if and only if $k_0 \geq k_0^*(\omega_0)$. We first consider the case with non-binding irreversibility constraint.

Case I: Non-binding irreversibility constraint at $\omega_0^{CC}(k_0)$

Case 1.1: Non-binding irreversibility constraint and non-binding ZLB at $\omega_0^{CC}(k_0)$

Using the condition $-\frac{b_0}{R_1^k k_1} = m$ and (A.30a), we find the interest rate at ω_0^{CC} is

$$R_0^{CC} = (1-\delta)^{\alpha} \left(\frac{A_0}{A_1}\beta\right)^{\alpha-1} \left(\frac{\omega_0^{CC} - \Pi_2}{\frac{X^*}{\alpha}\Pi_2 - \omega_0^{CC}}\right)^{\alpha},$$

in which Π_2 is constant:

$$\Pi_2 = \frac{\Pi_0}{\frac{1}{1+\gamma} + \frac{\frac{X^*}{\alpha}-1+m}{\frac{1+\gamma}{\gamma}(1-m)} \left[\frac{1}{\gamma} - \Pi_0\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\right]}, \tag{A.35c}$$

and Π_0 is a constant defined in (A.30b). We see that R_0^{CC} is increasing in ω_0 when $\omega_0 \in \left(\Pi_2, \frac{X^*}{\alpha}\Pi_2\right)$. Inserting the expression of R_0^{CC} into (A.30a), we can derive the expression of k_0 as below:

$$k_0 = \left[\frac{1}{\gamma} - \Pi_0\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\right] \frac{\frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \left(R_0^{CC}\right)^{\frac{1}{\alpha-1}} \left(\frac{X^*}{\alpha}\Pi_2 - \omega_0^{CC}\right)}{(1-\delta)\left[\left(\frac{X^*}{\alpha} - 1\right)\Pi_0 - \Pi_2 \frac{\frac{X^*}{\alpha}-1}{1+\gamma}\right]\omega_0^{CC}}. \tag{A.35d}$$

Notice that k_0 is decreasing in ω_0^{CC}, and thus we denote its inverse function as $\omega_{0,noZLB}^{CC}(k_0)$, which is decreasing. At $k_0 = 0$, $\omega_0^{CC} = \frac{X^*}{\alpha}\Pi_2$, and $R_0^{CC} \to +\infty$. As k_0 increases, ω_0^{CC} and R_0^{CC} decrease. In particular, when k_0 reaches the cutoff value $\widetilde{k}_0$ as below:

$$\widetilde{k}_0 = \frac{\left(\frac{1}{\beta} + \frac{1+\frac{1-m}{\gamma}}{\frac{X^*}{\alpha}-1+m}\right) \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta + \frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}}, \tag{A.35e}$$

we have $R_0^{CC}\left(\widetilde{k}_0\right) = 1$, and

$$\omega_0^{CC}\left(\widetilde{k}_0\right) = \Pi_2 \left(1 + \frac{\left(\frac{X^*}{\alpha} - 1\right)}{1 + (1-\delta)\left(\beta\frac{A_0}{A_1}\right)^{\frac{\alpha-1}{\alpha}}}\right). \tag{A.35f}$$

Thus when $k_0 > \widetilde{k}_0$, our assumption that the irreversibility constraint and ZLB are both non-binding at $\omega_0^{CC}(k_0)$ does not hold.

Case 1.2: Non-binding irreversibility constraint and binding ZLB at $\omega_0^{CC}(k_0)$

Given k_0, we have two equations, $-\frac{b_0}{R_1^k k_1} = m$ and (A.29a) with two unknowns: $\{X_0, \omega_0^{CC}\}$.

After some calculations, we can express both ω_0^{CC} and k_0 as functions of X_0 as below:

$$\begin{aligned} k_0 &= \Pi_3 \frac{\left(\frac{\alpha}{X^*}+\frac{1}{\beta}\right)\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{1-\delta+\frac{X_0}{\alpha}\left(\frac{A_0}{A_1}\beta\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}}, \\ \omega_0^{CC} &= \Pi_2\left[1+\frac{\left(\frac{X_0}{\alpha}-1\right)\left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}}{(1-\delta)\left(\beta\frac{A_0}{A_1}\right)^{\frac{\alpha-1}{\alpha}}+\left(\frac{X_0}{X^*}\right)^{-\frac{1}{\alpha}}}\right], \end{aligned} \tag{A.35g}$$

in which Π_2 is defined in (A.29a), and Π_3 is a constant:

$$\Pi_3 = \frac{1+\frac{\left(\frac{X^*}{\alpha}-1+m\right)}{1-m}\left[1-\Pi_0\left(1-\frac{\gamma}{\beta}\right)\right]}{\frac{\left(\frac{X^*}{\alpha}-1+m\right)}{1-m}\left[1-\Pi_0\left(1-\frac{\gamma}{\beta}\right)\right]}.$$

Notice that in (A.35d), k_0 is increasing in X_0. Thus by varying the value of X_0 in the range $[X^*, +\infty)$, we can trace out $\omega_0^{CC}(k_0)$. In particular, when $X_0 = X^*$, the values of k_0 and ω_0^{CC} are the same as in equations (A.35e) and (A.35f), suggesting that $\omega_0^{CC}(k_0)$ is continuous at $k_0 = \widetilde{k}_0$.

Case II: Binding irreversibility constraint at $\omega_0^{CC}(k_0)$

We have proved in Lemma 17 that with binding irreversibility constraint, the leverage ratio is independent of k_0 and is decreasing in ω_0. Thus we only need to find the cutoff value of ω_0, $\omega_{0,Irr}^{CC}$ such that the leverage ratio $-\frac{b_0}{R_1^k k_1} = m$ at $\omega_0 = \omega_{0,Irr}^{CC}$. One question is, whether the ZLB is binding at $\omega_0 = \omega_{0,Irr}^{CC}$.

Case 2.1 Binding irreversibility constraint and non-binding ZLB at $\omega_0^{CC}(k_0)$

We find that when $\mathscr{R}^{CC} \geq 1$, the ZLB is not binding at $\omega_{0,Irr}^{CC}$. Using $k_1 = (1-\delta)k_0$, $R_0 = \frac{R_1^k}{q_0}$, the entrepreneurs' optimal choice $c_0 = \frac{1}{1+\gamma}R_0^k\omega_0 k_0$, and R_0^k in (A.24e), we have

$$\frac{\gamma}{1+\gamma}\omega_{0,Irr}^{CC}\left[1+\frac{\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left[R_0^*\left(\omega_{0,Irr}^{CC}\right)\right]^{\frac{1}{\alpha}}}{1-\delta}\right] = 1-m,$$

in which $R_0^*(\cdot)$ is given in (A.28b). Replacing $R_0^*\left(\omega_{0,Irr}^{CC}\right)$ by its expression, we have

$$\omega_{0,Irr}^{CC} = \frac{(1-m)\frac{X^*}{\alpha}\frac{1+\gamma}{\gamma}}{\frac{X^*}{\alpha}\left(1+\frac{1}{\beta}\right)+(1-m)\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}. \tag{A.35h}$$

We find that $R_0^*\left(\omega_{0,Irr}^{CC}\right) \geq 1$ if and only if $\mathscr{R}^{CC} \geq 1$.

The value of $k_{0,Irr}^{CC}$ can be derived by $k_0^*\left(\omega_{0,Irr}^{CC}\right)$ from equation (A.25):

$$k_{0,Irr}^{CC} = \left[(1-\delta)\left(\frac{1-m}{\gamma} + \frac{1-\alpha}{\alpha\beta L_1^{cc}}\right)\right]^{\frac{\alpha}{\alpha-1}} \frac{\alpha\beta A_0 L_1^{cc}}{X^*(1-\delta)}. \tag{A.35i}$$

When $k_0 \geq k_{0,Irr}^{CC}$, and $\omega_0 = \omega_{0,Irr}^{CC}$, the leverage ratio equals to m.

Case 2.2 Binding irreversibility constraint and binding ZLB at $\omega_0^{CC}(k_0)$

If $\mathscr{R}^{CC} < 1$, then the ZLB is binding at $\omega_{0,Irr}^{CC}$. Using $k_1 = (1-\delta)k_0$, $R_0 = \frac{R_1^k}{q_0}$, the entrepreneurs' optimal choice $c_0 = \frac{1}{1+\gamma}R_0^k\omega_0 k_0$, and R_0^k in (A.24e), we have

$$\frac{\gamma}{1+\gamma}\omega_{0,Irr}^{CC}\left[1 + \frac{\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{X_0^*\left(\omega_{0,Irr}^{CC}\right)}{X^*}\right)^{-\frac{1}{\alpha}}}{1-\delta}\right] = 1-m,$$

in which $X_0^*(\cdot)$ is given in (A.29a). After some calculations, we can pin down the value of $X_0^*\left(\omega_{0,Irr}^{CC}\right)$ as below:

$$X_0^*\left(\omega_{0,Irr}^{CC}\right) = \beta\frac{A_0}{A_1}X^*\left[\frac{1-\delta}{\beta} + \left(\frac{1}{\gamma} - \frac{1}{\beta}\right)(1-m)(1-\delta)\frac{\alpha}{X^*}\right]^{\frac{\alpha}{\alpha-1}}.$$

We find that $X_0^*\left(\omega_{0,Irr}^{CC}\right) > X^*$ if and only if $\mathscr{R}^{CC} < 1$. Inserting its expression into the expression for leverage, we have

$$\omega_{0,Irr}^{CC} = \frac{\frac{1+\gamma}{\gamma}(1-m)(1-\delta)}{1-\delta+\frac{1}{\beta}\frac{A_1}{A_0}\left[\frac{1-\delta}{\beta} + \left(\frac{1}{\gamma}-\frac{1}{\beta}\right)(1-m)(1-\delta)\frac{\alpha}{X^*}\right]^{\frac{1}{1-\alpha}}}, \tag{A.35j}$$

Again, the value of $k_{0,Irr}^{CC}$ can be derived by $k_0^*\left(\omega_{0,Irr}^{CC}\right)$ from equation (A.26):

$$k_{0,Irr}^{CC} = \frac{\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{\left(\frac{X^*}{\alpha}-1+m\right)(1-\delta)}. \tag{A.35k}$$

When $k_0 \geq k_{0,Irr}^{CC}$, and $\omega_0 = \omega_{0,Irr}^{CC}$, the leverage ratio equals to m.

Putting the Pieces Together

When $\mathscr{R}^{CC} \geq 1$, $k_{0,Irr}^{CC}$ in equation (A.35k) is smaller than $\widetilde{k}_0$ in (A.35e), thus as k_0 increases from 0, the curve $\omega_0^{CC}(k_0)$ intersects $k_0^*(\omega_0)$ in equation (A.25) before it reaches

$\widetilde{k}_0$. Then the ZLB never binds at $\omega_0^{CC}(k_0)$. When $k_0 \in \left[0, k_{0,Irr}^{CC}\right]$, $\omega_0^{CC}(k_0) = \omega_{0,noZLB}^{CC}(k_0)$ in equation (A.35d); when $k_0 > k_{0,Irr}^{CC}$ in (A.35i), $\omega_0^{CC}(k_0) = \omega_{0,Irr}^{CC}$ in equation (A.35h).

When $\mathscr{R}^{CC} < 1$, $k_{0,Irr}^{CC}$ in equation (A.35k) is larger than $\widetilde{k}_0$ in (A.35e). Thus when $k_0 \in \left[0, \widetilde{k}_0\right]$, $\omega_0^{CC}(k_0) = \omega_{0,noZLB}^{CC}(k_0)$ in equation (A.35d); when $k_0 \in \left[\widetilde{k}_0, k_{0,Irr}^{CC}\right]$, $\omega_0^{CC}(k_0) = \omega_{0,ZLB}^{CC}(k_0)$ in equation (A.35g); and when $k_0 > k_{0,Irr}^{CC}$ in (A.35k), $\omega_0^{CC}(k_0) = \omega_{0,Irr}^{CC}$ in equation (A.35j). □

G.2.2 Threshold for Binding Irreversibility Constraint

Assume $\omega_0 \leq \frac{(1+\gamma)\frac{1-\alpha}{\alpha}X^*}{1+\gamma\frac{1-\alpha}{\alpha}X^*}$.[38] Here we denote the threshold of k_0 for a binding irreversibility constraint as $k_0^{**}(\omega_0)$. As in the case for the natural borrowing limit in F.2, we give the expression of $k_0^{**}(\omega_0)$ first and verify it later. $k_0^{**}(\omega_0)$ is solved such that when $k_0 = k_0^{**}(\omega_0)$, $q_0 = 1$, and $k_1 = (1-\delta)k_0$. $k_0^{**}(\omega_0)$ also depends on whether the ZLB or the collateral constraint is binding or not at $k_0 = k_0^{**}(\omega_0)$. Eventually, we find its expression depends on the value of $\mathscr{R}^{CC}$ in equation (A.35a).

If $\mathscr{R}^{CC} \geq 1$,

$$k_0^{**}(\omega_0) = \begin{cases} k_0^*(\omega_0), & \omega_0 > \omega_{0,Irr}^{CC}; \\ \frac{A_1 L_1^{cc}}{1-\delta}\left[\frac{(1-\delta)m\frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X^*}\left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}}\left[R_0^{**}(\omega_0)\right]^{\frac{1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)R_0^{**}(\omega_0)}\right]^{\frac{1}{1-\alpha}}, & \omega_0 \in \left[\omega_0^{**}, \omega_{0,Irr}^{CC}\right]; \\ \frac{A_1 L_1^{cc}}{1-\delta}\left[\frac{(1-\delta)m\frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X_0^{**}(\omega_0)}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0^{**}(\omega_0)}{X^*}\right)^{\frac{\alpha-1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\right]^{\frac{1}{1-\alpha}}. & \omega_0 < \omega_0^{**}. \end{cases}$$

If $\mathscr{R}^{CC} < 1$, then

$$k_0^{**}(\omega_0) = \begin{cases} k_0^*(\omega_0), & \omega_0 > \omega_{0,Irr}^{CC}; \\ \frac{A_1 L_1^{cc}}{1-\delta}\left[\frac{(1-\delta)m\frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X_0^{**}(\omega_0)}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0^{**}(\omega_0)}{X^*}\right)^{\frac{\alpha-1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\right]^{\frac{1}{1-\alpha}}, & \omega_0 \in \left[0, \omega_{0,Irr}^{CC}\right]. \end{cases}$$

See the blue solid line in Figure A.12 for an example. In the expressions above, L_1^{cc} is given in (A.33a). $\omega_{0,Irr}^{CC}$ is given in equation (A.35b).

[38]With our calibrated parameters, this value is 1.32.

$R_0^{**}(\omega_0)$ is given as below:

$$R_0^{**}(\omega_0)=\left[\frac{\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}m\frac{\alpha}{X^*}+\frac{1}{\beta}\left(1-(1-m)\frac{\alpha}{X^*}\right)}{\frac{\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}}{1-\delta}\left(1-\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}\frac{\alpha}{X^*}\right)}\right]^{\alpha}. \tag{A.36}$$

When $\omega_0\le\frac{1+\gamma}{\gamma+\frac{\alpha}{X^*}}$, $R_0^{**}(\omega_0)$ is increasing in ω_0. $R_0^{**}(\omega_0)\ge 1$ if and only if $\omega_0\ge\omega_0^{**}$ as follows:

$$\omega_0^{**}=\frac{(1+\gamma)\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}-(1+\gamma)\frac{1-\delta}{\beta}\left(1-(1-m)\frac{\alpha}{X^*}\right)}{\left(\frac{\alpha}{X^*}+\gamma\right)\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}+\left(m+\frac{\gamma}{\beta}(1-m)\right)(1-\delta)\frac{\alpha}{X^*}-\frac{\gamma}{\beta}(1-\delta)}. \tag{A.37}$$

$X_0^{**}(\omega_0)$ is given implicitly by the equation below:

$$\begin{aligned}&\frac{\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}}{1-\delta}\left(1-\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}\frac{\alpha}{X_0^{**}(\omega_0)}\right)\left(\frac{X_0^{**}(\omega_0)}{X^*}\right)^{\frac{\alpha-1}{\alpha}}\\ =&\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}m\frac{\alpha}{X^*}+\frac{1}{\beta}\left(1-(1-m)\frac{\alpha}{X^*}\right).\end{aligned} \tag{A.38}$$

If $\omega_0<\frac{(1+\gamma)\frac{1-\alpha}{\alpha}X^*}{1+\gamma\frac{1-\alpha}{\alpha}X^*}$, $X_0^{**}(\omega_0)$ is decreasing in ω_0. At $\omega_0=\omega_0^{**}$, $X_0^{**}(\omega_0)=X^*$.

G.3 Region with Non-binding Collateral Constraint

Lemma 19. *With $m<1$, the irreversibility constraint and*

$$\omega_0\le\frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^*\min\left\{\frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}},\frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\},$$

there exists a unique equilibrium with non-binding collateral constraint if and only if $\omega_0>\omega_0^{CC}(k_0)$ given in Lemma 18.

Proof. Notice that an equilibrium with non-binding collateral constraint is equivalent to an equilibrium with natural borrowing limit analyzed in Proposition 3. By Lemma 17, given $\omega_0\le\frac{1+\gamma}{\gamma}\frac{1-\alpha}{\alpha}X^*\min\left\{\frac{\frac{X^*}{\alpha}}{1+\frac{1}{\gamma}\frac{X^*}{\alpha}},\frac{1}{\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}\right\}$, the leverage ratio $-\frac{b_1}{R_1^K k_1}$ is decreasing in

ω_0, and by Lemma 18, $\omega_0^{CC}(k_0)$ is defined such that $-\frac{b_1}{R_1^K k_1} = m$ at $\omega_0 = \omega_0^{CC}(k_0)$. When $\omega_0 > \omega_0^{CC}(k_0)$, the leverage ratio is smaller than m, and the collateral constraint is not binding. Then an equilibrium with non-binding collateral constraint exists and is unique by Proposition 3. When $\omega_0 \leq \omega_0^{CC}(k_0)$, assuming a non-binding collateral constraint, the implied leverage ratio would be larger than m17, which violates the collateral constraint. Thus there is no equilibrium with non-binding collateral constraint in that region.

By Proposition 3, we know that in this region, the irreversibility constraint is binding if and only if $k_0 \geq k_0^*(\omega_0)$ given in Subsection F.2. By Lemma 18, $k_0^*(\omega_0)$ and $\omega_0^{CC}(k_0)$ intersects at $\left\{k_{0,Irr}^{CC}, \omega_{0,Irr}^{CC}\right\}$. In addition, by the construction of $k_0^{**}(\omega_0)$ in Subsection G.2.2, $k_0^{**}(\omega_0) = k_0^*(\omega_0)$ when $\omega_0 > \omega_{0,Irr}^{CC}$. Thus we know that given $\omega_0 > \omega_0^{CC}(k_0)$, the irreversibility constraint is binding if and only if $k_0 \geq k_0^{**}(\omega_0)$. □

G.4 Region with Binding Collateral Constraint and Binding Irreversibility Constraint

Lemma 20. *With $m < 1$, the irreversibility constraint and $\omega_0 \leq \frac{(1+\gamma)\frac{1-\alpha}{\alpha}X^*}{1+\gamma\frac{1-\alpha}{\alpha}X^*}$, there exists a unique equilibrium with binding collateral constraint and binding irreversibility constraint if and only if $\omega_0 \leq \omega_{0,Irr}^{CC}$ in (A.35b) and $k_0 \geq k_0^{**}(\omega_0)$ given in Subsection G.2.2. In this equilibrium, q_0 is decreasing in k_0, and R_0, X_0 and the multiplier for the collateral constraint, μ_0 are all independent of k_0.*

Proof. Assuming the collateral constraint and the irreversibility constraint are binding. Then $k_1 = (1-\delta)k_0$ and $-\frac{b_1}{R_1^K k_1} = m$. We also have $\omega_1 = 1-m$ and $L_1 = L_1^{cc}$ as in (A.33a). In this case, we can express the system by two unknowns, (q_0, R_0) or (q_0, X_0) depending whether the ZLB is binding.

The first equation is derived by the entrepreneurs' consumption choice $c_0 = \frac{1}{1+\gamma}\omega_0 R_0^K k_0$, the expression of R_0^K in (A.33b) and their budget constraint (10c) as below:

$$k_0 = \frac{A_1 L_1^{cc}}{1-\delta}\left[\frac{(1-\delta)\frac{m}{R_0}\frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X_0}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0}{X^*}\right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}}}{q_0(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\right]^{\frac{1}{1-\alpha}}. \tag{A.39a}$$

The second equation is derived by the feasibility condition in period 0, $c_0 + c_0' = Y_0$ with expressions in Subsection G.1:

$$
\begin{aligned}
&\left(\beta \frac{A_0}{A_1} R_0 \frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\delta)\,k_0}{A_1 L_1^{cc}}\right)^{\alpha-1} k_0 \\
=&\frac{1}{1+\gamma}\omega_0 k_0 \left[q_0\,(1-\delta) + \frac{\alpha}{X_0}\left(\beta R_0 \frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\delta)\,k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}\right] \\
&+ \frac{1}{\beta R_0}\frac{1-\alpha}{X^*} A_1^{1-\alpha}\left(\frac{(1-\delta)\,k_0}{L_1^{cc}}\right)^{\alpha}.
\end{aligned}
\tag{A.39b}
$$

Combining these two equations together, we have one equation with one unknown, R_0 or X_0 depending on whether the ZLB is binding:

$$
\begin{aligned}
&\frac{1}{1-\delta}\left(1 - \frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0}\frac{\alpha}{X_0}\right)\left(\beta\frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}} R_0^{\frac{1}{\alpha}} \\
=&\frac{\frac{1}{1+\gamma}\omega_0}{1-\frac{\gamma}{1+\gamma}\omega_0} m \frac{\alpha}{X^*} + \frac{1}{\beta}\left(1-(1-m)\,\frac{\alpha}{X^*}\right).
\end{aligned}
\tag{A.39c}
$$

Notice that the solution of (A.39c) is independent of k_0.

From (A.39a), q_0 can be expressed as below:

$$
q_0 = \frac{(1-\delta)\,\frac{m}{R_0}\frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0\frac{\alpha}{X_0}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0}{X^*}\right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\left(\frac{(1-\delta)\,k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}. \tag{A.39d}
$$

Notice that q_0 is decreasing in k_0.

Next, by equation (11), the multiplier for the collateral constraint, μ_0 can be expressed as

$$
\begin{aligned}
\mu_0 &= \frac{1}{1-m}\left(\frac{1}{R_0} - \frac{q_0}{R_1^K}\right) \\
&= \frac{1}{1-m}\left(\frac{(1-\delta)\left(1-m-\frac{\gamma}{1+\gamma}\omega_0\right)\frac{1}{R_0} - \frac{\gamma}{1+\gamma}\omega_0\frac{X^*}{X_0}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0}{X^*}\right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\right).
\end{aligned}
\tag{A.39e}
$$

We find μ_0 is also independent of k_0. For our assumption of binding irreversibility constraint and collateral constraint to be valid, the two equations above should imply $q_0 \leq 1$ and $\mu_0 \geq 0$.

Case I: Non-binding ZLB

First, we assume that the ZLB is not binding and impose $X_0 = X^*$ in (A.39c). The solution of $R_0 = R_0^{**}(\omega_0)$ is given by equation (A.36), which is increasing in ω_0. $R_0^{**}(\omega_0) \geq 1$ if and only if $\omega_0 \geq \omega_0^{**}$ in (A.37).

The capital price becomes

$$q_0 = \frac{(1-\delta)\, m \frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0 \frac{\alpha}{X^*} \left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}} \left[R_0^{**}(\omega_0)\right]^{\frac{1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right) R_0^{**}(\omega_0)} \left(\frac{(1-\delta)\, k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}.$$

In particular, when $k_0 = k_{0,noZLB}^{**}(\omega_0)$ as below:

$$k_{0,noZLB}^{**}(\omega_0) = \frac{A_1 L_1^{cc}}{1-\delta}\left[\frac{(1-\delta)\, m \frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0 \frac{\alpha}{X^*} \left(\frac{1}{\beta}\frac{A_1}{A_0}\right)^{\frac{\alpha-1}{\alpha}} \left[R_0^{**}(\omega_0)\right]^{\frac{1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right) R_0^{**}(\omega_0)}\right]^{\frac{1}{1-\alpha}}, \tag{A.39f}$$

the implied $q_0 = 1$. Thus here we need $k_0 \geq k_{0,noZLB}^{**}(\omega_0)$. We also need to check whether μ_0 implied by equation (A.39e) is positive. We find that $\mu_0 \geq 0$ if and only if $\omega_0 \leq \omega_{0,Irr}^{CC}$ in equation (A.35b).

To sum up, for an equilibrium with non-binding ZLB, binding collateral constraint and irreversibility constraint to exist, we should have $\omega_0^{**} < \omega_0 \leq \omega_{0,Irr}^{CC}$ and $k_0 \geq k_{0,noZLB}^{**}(\omega_0)$.

Case II: Binding ZLB

Assuming the ZLB is binding and impose $R_0 = 1$ in (A.39c)with X_0 as the only unknown. Denote its solution as $X_0^{**}(\omega_0)$, which is given in equation (A.38). If $\omega_0 < \frac{(1+\gamma)\frac{1-\alpha}{\alpha}X^*}{1+\gamma\frac{1-\alpha}{\alpha}X^*}$, $X_0^{**}(\omega_0)$ is decreasing in ω_0. At $\omega_0 = \omega_0^{**}$, $X_0^{**}(\omega_0) = X^*$. Then for an equilibrium with binding ZLB to exist here, we need $\omega_0 \leq \omega_0^{**}$.

The capital price becomes

$$q_0 = \frac{(1-\delta)\, m \frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0 \frac{\alpha}{X_0^{**}(\omega_0)} \left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X_0^{**}(\omega_0)}{X^*}\right)^{\frac{\alpha-1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)} \left(\frac{(1-\delta)\, k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}.$$

In particular, when $k_0 = k^{**}_{0,ZLB}(\omega_0)$ as below:

$$k^{**}_{0,ZLB}(\omega_0) = \frac{A_1 L^{cc}_1}{1-\delta}\left[\frac{(1-\delta)\, m \frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma}\omega_0 \frac{\alpha}{X^{**}_0(\omega_0)}\left(\frac{1}{\beta}\frac{A_1}{A_0}\frac{X^{**}_0(\omega_0)}{X^*}\right)^{\frac{\alpha-1}{\alpha}}}{(1-\delta)\left(1-\frac{\gamma}{1+\gamma}\omega_0\right)}\right]^{\frac{1}{1-\alpha}}, \tag{A.39g}$$

the implied $q_0 = 1$. Thus here we need $k_0 \geq k^{**}_{0,ZLB}(\omega_0)$. We still need to check whether μ_0 implied by equation (A.39e) is positive. We find that $\mu_0 \geq 0$ if and only if $\omega_0 \leq \omega^{CC}_{0,Irr}$ in equation (A.35b).

To sum up, for an equilibrium with binding ZLB, collateral constraint and irreversibility constraint to exist, we should have $\omega_0 \leq \min\left\{\omega^{**}_0, \omega^{CC}_{0,Irr}\right\}$ and $k_0 \geq k^{**}_{0,ZLB}(\omega_0)$.

Putting the Two Pieces Together

We can verify that when $\mathscr{R}^{CC}$ in equation (A.35a) is larger than one, $\omega^{CC}_{0,Irr} \geq \omega^{**}_0$. Then the ZLB is binding when $\omega_0 < \omega^{**}_0$ and $k_0 \geq k^{**}_{0,ZLB}(\omega_0)$ in (A.39g), and not binding when $\omega_0 \in \left[\omega^{**}_0, \omega^{CC}_{0,Irr}\right]$ and $k_0 \geq k^{**}_{0,noZLB}(\omega_0)$ in (A.39f). By the construction of $k^{**}_0(\omega_0)$ in Subsection G.2.2, $k^{**}_0(\omega_0) = k^{**}_{0,ZLB}(\omega_0)$ when $\omega_0 < \omega^{**}_0$; and $k^{**}_0(\omega_0) = k^{**}_{0,noZLB}(\omega_0)$ when $\omega_0 \in \left[\omega^{**}_0, \omega^{CC}_{0,Irr}\right]$.

When $\mathscr{R}^{CC} < 1$, $\omega^{CC}_{0,Irr} < \omega^{**}_0$. Then the ZLB is always binding when $\omega_0 < \omega^{CC}_{0,Irr}$ and $k_0 \geq k^{**}_{0,ZLB}(\omega_0)$. By the construction of $k^{**}_0(\omega_0)$ in Subsection G.2.2, $k^{**}_0(\omega_0) = k^{**}_{0,ZLB}(\omega_0)$ when $\omega_0 \leq \omega^{CC}_{0,Irr}$.

To sum up, there exists a unique equilibrium with binding collateral constraint and binding irreversibility constraint if and only if $\omega_0 \leq \omega^{CC}_{0,Irr}$ and $k_0 \geq k^{**}_0(\omega_0)$. □

G.5 Regions with Binding Collateral Constraint and Non-binding Irreversibility Constraint

In this part, we assume the collateral constraint is binding and the irreversibility constraint is not binding. Then the equilibrium properties in this case is very similar to those in the simple two-period model in Subsection 2.4.

Here we have $q_0 = 1$, and $-\frac{b_1}{R^K_1 k_1} = m$. By the definition of the wealth share in (4), we also have $\omega_1 = 1 - m$ and $L_1 = L^{cc}_1$ as in (A.33a). In this case, we can express the system by two unknowns, $\left\{r^K_1, R_0\right\}$ or $\left\{r^K_1, X_0\right\}$ depending whether the ZLB is binding.

The first equation is derived from the feasibility condition (3a) at $t = 0$:

$$A_1 L_1^{cc} \left(\frac{X^*}{\alpha} r_1^K \right)^{\frac{1}{\alpha-1}} = \left[(1-\delta) \left(1 - \frac{\omega_0}{1+\gamma} \right) + \left(\frac{X_0}{\alpha} - \frac{\omega_0}{1+\gamma} \right) \left(\frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left(\beta \frac{A_0}{A_1} R_0 \right)^{\frac{1-\alpha}{\alpha}} r_1^K \right] k_0 \tag{A.40a}$$

$$- \frac{1}{\beta R_0} \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} r_1^K \right)^{\frac{\alpha}{\alpha-1}}.$$

The second equation is derived by the entrepreneurs' consumption choice $c_0 = \frac{1}{1+\gamma} \omega_0 R_0^K k_0$, the expression of R_0^K in (A.33b) and their budget constraint (10c) as below:

$$\left(1 - \frac{m r_1^K}{R_0} \right) A_1 L_1^{cc} \left(\frac{X^*}{\alpha} r_1^K \right)^{\frac{1}{\alpha-1}} = \frac{\gamma}{1+\gamma} \omega_0 k_0 \left(1 - \delta + \left(\frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left(\beta \frac{A_0}{A_1} R_0 \right)^{\frac{1-\alpha}{\alpha}} r_1^K \right). \tag{A.40b}$$

A solution to the system of equations (A.40a) and (A.40b) corresponds to an equilibrium with binding collateral constraint if $k_1 > (1-\delta) k_0$ and the multiplier μ_0 implied by (11) is positive, i.e., if

$$R_0 \leq R_1^K. \tag{A.40c}$$

In the next subsection, we characterize the properties of the solution to (A.40a) and (A.40b), depending on whether the ZLB is binding. We temporarily ignore the requirements (A.40c) and $k_1 > (1-\delta) k_0$, and will verify whether they hold or not later.

G.5.1 Equilibrium with Non-binding ZLB and Binding Collateral Constraint

Lemma 21. *With $\omega_0 < \frac{X^*}{\alpha}$, there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB if and only if $k_0 < k_0^{**}(\omega_0)$ given in Subsection G.2.2, $\omega_0 \leq \omega_0^{CC}(k_0)$ in Lemma 18 and ω_0 is larger than a cutoff value, $\hat{\omega}_0(k_0)$. In this region, R_0 is increasing in ω_0, and $\frac{k_1}{k_0}$ is decreasing in k_0.*

Proof. **Step 1: Equilibrium Existence**

Assuming that the collateral constraint is binding, the irreversibility constraint is non-binding, and ZLB is non-binding. Setting $X_0 = X^*$, r_1^K can be expressed as functions of R_0 in both (A.40a) and (A.40b).

Equation (A.19c) becomes

$$A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}} + \frac{1}{\beta R_0} \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} r_1^K$$
$$= \left[(1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right) + \left(\frac{X^*}{\alpha} - \frac{\omega_0}{1+\gamma}\right)\left(\beta \frac{A_0}{A_1} R_0\right)^{\frac{1-\alpha}{\alpha}} r_1^K\right] k_0 \left(r_1^K\right)^{\frac{1}{1-\alpha}}. \tag{A.41a}$$

in which r_1^K is a decreasing function of R_0. Denote this implicit function as $r_1^K = h_1(R_0)$. We easily verify that $\lim_{R_0 \to 0} h_1(R_0) \to +\infty$, and $\lim_{R_0 \to +\infty} h_1(R_0) \to 0$.

We can write the equation above in the form of

$$A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}} = -\phi_1^1(R_0) r_1^K + \phi_2^1 \left(r_1^K\right)^{\frac{1}{1-\alpha}} + \phi_3^1(R_0)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}, \tag{A.41b}$$

where ϕ_1^1, ϕ_2^1, ϕ_3^1>0. Denote its right-hand side as $H_1\left(r_1^K, R_0\right)$.

Equation (A.40b) becomes

$$1 = \frac{m r_1^K}{R_0} + \frac{\gamma}{1+\gamma} \omega_0 k_0 \frac{1-\delta+\left(\beta R_0 \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} r_1^K}{A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}} \left(r_1^K\right)^{\frac{1}{\alpha-1}}}, \tag{A.41c}$$

which can be similarly written as

$$A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}} = \phi_1^2(R_0) r_1^K + \phi_2^2(R_0)\left(r_1^K\right)^{\frac{1}{1-\alpha}} + \phi_3^2(R_0)\left(r_1^K\right)^{1+\frac{1}{1-\alpha}}, \tag{A.41d}$$

where ϕ_1^2, ϕ_2^2, ϕ_3^2>0. Denote its right-hand side as $H_2\left(r_1^K, R_0\right)$. Thus there exists a unique solution for r_1^K as a function of R_0. Denote this implicit function as $r_1^K = h_2(R_0)$. We can also easily verify that that $\lim_{R_0 \to 0} h_2(R_0) \to 0$, and as $\lim_{R_0 \to +\infty} h_2(R_0) \to 0$. Thus $h_2(R_0)$ is not monotone.

We show that, given $\omega_0 < \frac{X^*}{\alpha}$, as $R_0 \to +\infty$, $h_2(R_0)$ is asymptotically higher than $h_1(R_0)$.

As $R_0 \to +\infty$, $h_1(R_0)$ and $h_2(R_0)$ both converge to zero. We can derive the following asymptotic behaviors as $R_0 \to +\infty$:

$$[h_1(R_0)]^{1+\frac{1}{1-\alpha}} \propto \frac{1}{\left(\frac{X^*}{\alpha} - \frac{\omega_0}{1+\gamma}\right)} \frac{A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{\left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} k_0} R_0^{\frac{\alpha-1}{\alpha}},$$

$$[h_2(R_0)]^{1+\frac{1}{1-\alpha}} \propto \frac{\frac{1+\gamma}{\gamma}}{\omega_0} \frac{A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}}{\left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} k_0} R_0^{\frac{\alpha-1}{\alpha}}.$$

If $\omega_0 < \frac{X^*}{\alpha}$, $h_2(R_0)$ is asymptotically higher than $h_1(R_0)$.

Then we obtain $h_1(R_0) > h_2(R_0)$ at $R_0 = 0$ and $h_1(R_0) < h_2(R_0)$ when R_0 is sufficiently high. By the Intermediate Value Theorem, the two functions will cross at least once. This guarantees the existence of a solution $\left(R_0, r_1^K\right)$ for the two equations (A.41a) and (A.41c).

Step 2: Equilibrium Uniqueness

We show at any intersection of h_1 and h_2, i.e. $h_1(R_0) = h_2(R_0)$, the slope of h_2 must be steeper than the one for h_1, i.e. $h_1'(R_0) < h_2'(R_0)$.

Since the proof for this statement is the same as Step 3 of Subsection D.2.1, we choose to omit this part. Combining the previous two steps together, we see that assuming a binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB, a solution to (A.41a) and (A.41c) exists and is unique. (without checking whether the implied $R_0 \geq 1$.)

Step 3: Comparative Statics

In equation (A.41a), we see that fixing r_1^K, R_0 is increasing in ω_0. In (A.41c), we see that fixing R_0, r_1^K is decreasing in ω_0. Since the slope of $h_2(R_0)$ is steeper than the one for $h_2(R_0)$, as ω_0 increases both curves shift to the right, and the equilibrium R_0 increases. Thus R_0 *is increasing in* ω_0.

To see how the ratio $\frac{k_1}{k_0}$ responds to k_0, define $\rho_k = \frac{k_1}{k_0}$, and equations (A.41a) and (A.41c) can be respectively written as below:

$$\rho_k^{1-\alpha} \left[\rho_k - (1-\delta)\left(1 - \frac{\omega_0}{1+\gamma}\right)\right] k_0^{1-\alpha} + \frac{1}{\beta R_0} \frac{1-\alpha}{X^*} A_1 \left(A_1 L_1^{cc}\right)^{-\alpha} \rho_k$$
$$= \left(\frac{X^*}{\alpha} - \frac{\omega_0}{1+\gamma}\right) \left(\beta \frac{A_0}{A_1} R_0\right)^{\frac{1-\alpha}{\alpha}} \frac{\alpha}{X^*} \left(A_1 L_1^{cc}\right)^{1-\alpha},$$

and

$$\rho_k^{\alpha}\left[\rho_k^{1-\alpha}-\frac{m}{R_0}\frac{\alpha}{X^*}\left(A_1L_1^{cc}\right)^{1-\alpha}k_0^{\alpha-1}\right]$$
$$=(1-\delta)\frac{\gamma}{1+\gamma}\omega_0+\frac{\gamma}{1+\gamma}\omega_0\left(\beta R_0\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\frac{\alpha}{X^*}\left(A_1L_1^{cc}\right)^{1-\alpha}\rho_k^{\alpha-1}k_0^{\alpha-1}.$$

If we fix R_0, we see that ρ_k is decreasing in k_0 in both equations. As a result, $\frac{k_1}{k_0}$ *is decreasing in* k_0.

Step 4: Checking the Assumptions of Binding Collateral Constraint and Non-binding Irreversibility

Since the ratio $\frac{k_1}{k_0}$ is decreasing in k_0, we find at $k_0 = k_0^{**}(\omega_0)$ given in Subsection G.2.2, $\rho_k = 1-\delta$ in the two equations above. Thus there is no equilibrium with binding collateral constraint, Non-binding Irreversibility and non-binding ZLB when $k_0 > k_0^{**}(\omega_0)$. Otherwise, the restriction $\frac{k_1}{k_0} \geq 1-\delta$ would be violated.

We can also check the assumption of a binding collateral constraint. Similar to Lemma 11, here we can show that the derivative $R_1^K - R_0$ is negative at $\omega_0 = \omega_0^{CC}(k_0)$ given in Lemma 18. In addition, using the similar argument in Lemma 12, we can show the collateral constraint is violated if $\omega_0 > \omega_0^{CC}(k_0)$, while it is satisfied when $\omega_0 \leq \omega_0^{CC}(k_0)$.

Step 5: Cutoff of ω_0 for ZLB

It remains to check whether the assumption of non-binding ZLB holds. Since R_0 is decreasing in ω_0, we can identify the cutoff for binding ZLB, $\hat{\omega}_0(k_0)$, such that given k_0, $R_0 = 1$ at $\omega_0 = \hat{\omega}_0(k_0)$. The expression of $\hat{\omega}_0(k_0)$ can be solved implicitly by imposing $R_0 = 1$ in (A.41a) and (A.41c). To be specific, imposing $R_0 = 1$ in (A.41a), we have:

$$\hat{\omega}_0 = \frac{\left(1-\delta+\frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}r_1^K\right)k_0 - A_1L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}-\frac{1}{\beta}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}r_1^K\right)^{\frac{\alpha}{\alpha-1}}}{\frac{1}{1+\gamma}\left[1-\delta+\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}r_1^K\right]k_0}.$$

and imposing $R_0 = 1$ in (A.41c), we have:

$$\hat{\omega}_0 = \frac{\left(1-mr_1^K\right)A_1L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}}{\frac{\gamma}{1+\gamma}k_0\left[1-\delta+\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}r_1^K\right]}.$$

Combining both equations and after some calculation, we have

$$\frac{1+\gamma}{\gamma} A_1 L_1^{cc} \left(\frac{X^*}{\alpha} r_1^K\right)^{\frac{1}{\alpha-1}} + \left(\frac{1}{\beta} - \frac{\frac{m}{\gamma}}{\frac{X^*}{\alpha} - 1 + m}\right) \frac{1-\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha} r_1^K\right)^{\frac{\alpha}{\alpha-1}}$$
$$= \left(1 - \delta + \frac{X^*}{\alpha} \left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} r_1^K\right) k_0, \tag{A.41e}$$

in which r_1^K is decreasing in k_0. For a given value of k_0, inserting the solved r_1^K from the equation above into either one of the expression for $\hat{\omega}_0$, we get the expression for $\hat{\omega}_0(k_0)$.

Since r_1^K is decreasing in k_0 in (A.41e), when $k_0 = \widetilde{k}_0$ in equation (A.35e), $r_1^K = 1$ in equation (A.41e). Thus when $k_0 > \widetilde{k}_0$, our assumption of a binding collateral constraint at $\hat{\omega}_0(k_0)$ does not hold anymore.

To see how the ratio $\rho_k = \frac{k_1}{k_0}$ changes along $\hat{\omega}_0(k_0)$, we replace r_1^K by ρ_k in the equation above, which becomes

$$k_0^{\alpha-1} = \frac{\left[\frac{1+\gamma}{\gamma}\rho_k - (1-\delta)\right] (\rho_k)^{1-\alpha}}{\left(A_1 L_1^{cc}\right)^{1-\alpha} \left[\left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} - \left(\frac{1}{\beta} - \frac{\alpha}{X^*}\left(\frac{1-m}{\beta} + \frac{m}{\gamma}\right)\right)\rho_k\right]}.$$

Check carefully this equation, we see that ρ_k is decreasing in k_0 along $\hat{\omega}_0(k_0)$. In particular, when $k_0 = \hat{k}_0^{**}$ as below:

$$\hat{k}_0^{**} = \frac{A_1 L_1^{cc}}{1-\delta} \left[\frac{\gamma}{1-\delta}\left(\beta \frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} - \gamma\left(\frac{1}{\beta} - \frac{\alpha}{X^*}\left(\frac{1-m}{\beta} + \frac{m}{\gamma}\right)\right)\right]^{\frac{1}{1-\alpha}}, \tag{A.41f}$$

$\hat{\omega}_0\left(\hat{k}_0^{**}\right) = \omega_0^{**}$ in equation (A.37). Thus suggests that when $k_0 > \hat{k}_0^{**}$, our assumption of a non-binding irreversibility constraint at $\hat{\omega}_0(k_0)$ does not hold anymore.

When $\mathscr{R}^{CC}$ in equation (A.35a) is larger than one, $\hat{k}_0^{**} < \widetilde{k}_0$, and as k_0 increases, the curve $\hat{\omega}_0(k_0)$ will cross $k_0^{**}(\omega_0)$ defined in Subsection G.2.2 at $k_0 = \hat{k}_0^{**}$. When $\mathscr{R}^{CC} < 1$, $\hat{k}_0^{**} > \widetilde{k}_0$, and as k_0 increases, the curve $\hat{\omega}_0(k_0)$ will cross $\omega_0^{CC}(k_0)$ given in Lemma 18 at $k_0 = \widetilde{k}_0$.

To sum up, when $\omega_0 < \frac{X^*}{\alpha}$, there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB if and only if $k_0 < k_0^{**}(\omega_0)$, $\omega_0 \leq \omega_0^{CC}(k_0)$ and $\omega_0 > \hat{\omega}_0(k_0)$. $\square$

G.5.2 Equilibrium with Binding ZLB and Binding Collateral Constraint

Lemma 22. *Assume ω_0 is smaller than*[39]

$$\min\left\{(1+\gamma)\frac{1-\alpha}{\alpha}X^*, \Xi\left(\alpha, m, X^*, \gamma, \beta\right)\right\},$$

in which Ξ is a function defined in (A.42f), there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and binding ZLB if and only if $k_0 < k_0^{**}\left(\omega_0\right)$ given in Subsection G.2.2, $\omega_0 \leq \omega_0^{CC}\left(k_0\right)$ in Lemma 18 and $\omega_0 \leq \hat{\omega}_0\left(k_0\right)$ given implicitly by (A.41e). In this region, X_0 is decreasing in ω_0, and $\frac{k_1}{k_0}$ is decreasing in k_0.

Proof. As in the proof of Lemma 21, we first assuming that the collateral constraint is binding, ZLB is binding, and the irreversibility constraint is non-binding. With these assumptions, we can represent the equilibrium by two equations. We will come back later to check whether the assumptions are valid.

Step 1: Equilibrium Representation

In this case, we represent the system as functions of $\left\{r_1^K, r_0^K\right\}$. X_0 can be expressed as a function of $\left\{r_1^K, r_0^K\right\}$ as below:

$$X_0 = X^*\left(\beta\frac{A_0}{A_1}\right)^{1-\alpha}\left(\frac{r_1^K}{r_0^K}\right)^{\alpha}. \tag{A.42a}$$

The counterpart for the restriction $X_0 \geq X^*$ is

$$r_0^K \leq \left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}} r_1^K,$$

and the irreversibility constraint sets a lower bound for r_1^K:

$$r_1^K \geq \frac{\alpha}{X^*}\left(\frac{(1-\delta)\,k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}. \tag{A.42b}$$

Equation (A.40a) becomes

$$A_1 L_1^{cc}\left(\frac{X^*}{\alpha}r_1^K\right)^{\frac{1}{\alpha-1}} + \frac{1}{\beta}\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}r_1^K\right)^{\frac{\alpha}{\alpha-1}}$$
$$= (1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right)k_0 + \left(\frac{X^*}{\alpha}\left(\beta\frac{A_0}{A_1}\right)^{1-\alpha}\left(\frac{r_1^K}{r_0^K}\right)^{\alpha} - \frac{\omega_0}{1+\gamma}\right)r_0^K k_0, \tag{A.42c}$$

[39] With our calibrated parameters, the value is 1.39.

If $\omega_0 < (1+\gamma)\frac{1-\alpha}{\alpha}X^*$, we can show that r_0^K is decreasing in r_1^K. Denote this implicit function as $r_0^K = u_1\left(r_1^K\right)$. Using implicit function theorem, we see that as k_0 increases, $u_1\left(r_1^K\right)$ shifts to the left, i.e., holding r_0^K unchanged, r_1^K is decreasing in k_0.

Equation (A.40b) becomes

$$\left(1 - mr_1^K\right) A_1 L_1^{cc} \left(\frac{X^*}{\alpha} r_1^K\right)^{\frac{1}{\alpha-1}} = \frac{\gamma}{1+\gamma}\omega_0 k_0 \left(1 - \delta + r_0^K\right) \tag{A.42d}$$

in which r_0^K is decreasing in r_1^K as well. Denote this implicit function as $r_0^K = u_2\left(r_1^K\right)$. Using implicit function theorem, we can show that as k_0 increases, $u_2\left(r_1^K\right)$ shifts to the left, i.e., holding r_0^K unchanged, r_1^K is increasing in k_0.

Step 2: Equilibrium Existence

We show that if $\omega_0 \leq \hat{\omega}_0(k_0)$, defined implicitly in equation (A.41e), and given $r_1^K \leq \frac{\alpha}{X^*}\left(\frac{(1-\delta)k_0}{A_1 L_1^{cc}}\right)^{\alpha-1}$ due to the irreversibility constraint, there exists a unique solution $\left\{r_1^K, r_0^K\right\}$ to equations (A.42c) and (A.42d).

The intuition of this result can be seen in Figure A.13. The black dashed line corresponds to $X_0 = X^*$ below which we have $X_0 \geq X^*$. When $\omega_0 < \hat{\omega}_0(k_0)$, by equations (A.42c) and (A.42d), with $R_0 = 1$ and $X_0 = X^*$, r_1^K in (A.42c) is smaller than r_1^K in (A.42d). Correspondingly, in Figure A.13, Point A, the intersection of $u_1\left(r_1^K\right)$ and $X_0 = X^*$ lies to the lower left of Point B, the intersection of $u_2\left(r_1^K\right)$ and $X_0 = X^*$. In other words, given $r_1^K = r_{1,B}^K$, the value at point B, $u_1\left(r_{1,B}^K\right) < u_2\left(r_{1,B}^K\right)$.

Denote the horizontal intercept of $u_1\left(r_1^K\right)$ as $\hat{r}_a^K$, and the horizontal intercept of $u_2\left(r_1^K\right)$ as $\hat{r}_b^K$. The question is whether $\hat{r}_a^K > \hat{r}_b^K$. Setting $r_0^K = 0$ in (A.41a) and (A.41c) and applying the implicit function theorem, we have $\frac{\partial \hat{r}_a^K}{\partial k_0} < 0$ and $\frac{\partial \hat{r}_b^K}{\partial k_0} > 0$. Thus given ω_0, there is a cutoff value of $\bar{k}_0^{Irr}$ such that $\hat{r}_a^K \geq \hat{r}_b^K$ if and only if $k_0 \leq \bar{k}_0^{Irr}$. From (A.42c) and (A.42d), we can show that $\bar{k}_0^{Irr}$ can be solved by the following equation:

$$2A_1 L_1^{cc}\left(\frac{X^*}{\alpha}\hat{r}_1^K\right)^{\frac{1}{\alpha-1}} + \left(\frac{1}{\beta} - \frac{m}{\frac{X^*}{\alpha} - 1 + m}\right)\frac{1-\alpha}{X^*}A_1\left(\frac{X^*}{\alpha}\hat{r}_1^K\right)^{\frac{\alpha}{\alpha-1}} = (1-\delta)\,\bar{k}_0^{Irr},$$

in which L_1^{cc} is from equation (A.33a). However, the restriction on r_1^K, (A.42b), is violated if $k_0 \geq \bar{k}_0^{Irr}$. Thus, for the current case, we must have $\hat{r}_a^K > \hat{r}_b^K$.

Now with $\hat{r}_a^K > \hat{r}_b^K$, we see that $u_1\left(\hat{r}_b^K\right) > u_2\left(\hat{r}_b^K\right)$. Since both $u_1\left(r_1^K\right)$ and $u_2\left(r_1^K\right)$ are continuous, they should intersect at least once when $r_1^K \in \left[r_{1,B}^K, \tilde{r}_1^K\right]$ with $X_0 > X^*$. Thus there exists at least one solution to equations (A.42c) and (A.42d).

Step 3: Equilibrium Uniqueness

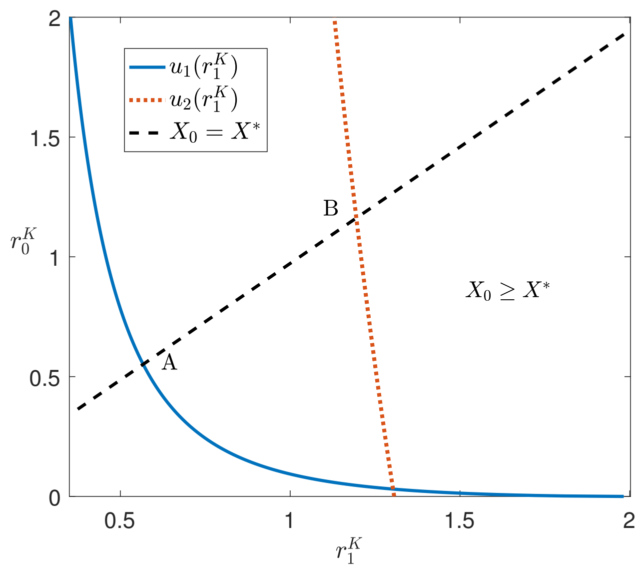


Figure A.13: Equilibria with Binding Collateral Constraint, ZLB and Non-binding Irreversibility

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.7$ and $\epsilon = 21$. $\omega_0 = 0.05$ and $k_0 = 0.3$.

When $\omega_0 \leq \hat{\omega}_0(k_0)$ and $u_1\left(r_1^K\right) = u_2\left(r_1^K\right)$, the slope of $u_1\left(r_1^K\right)$ is higher than the slope of $u_2\left(r_1^K\right)$ when they intersect.

Using implicit function theorem, the derivatives of $u_1\left(r_1^K\right)$ and $u_2\left(r_1^K\right)$ are

$$\frac{\partial u_1}{\partial r_1^K}\left(r_1^K\right) = -\frac{\frac{1}{1-\alpha}A_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{2-\alpha}{\alpha-1}} + \frac{1}{\beta}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}} + X_0\frac{r_0^K}{r_1^K}k_0}{\left(\frac{1-\alpha}{\alpha}X_0 - \frac{\omega_0}{1+\gamma}\right)k_0},$$

$$\frac{\partial u_2}{\partial r_1^K}\left(r_1^K\right) = -\frac{\frac{1}{1-\alpha}A_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{2-\alpha}{\alpha-1}} + \frac{\alpha}{1-\alpha}mA_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}}{\frac{\gamma}{1+\gamma}\omega_0 k_0}.$$

We will show that given $1 \leq r_1^K \leq \frac{1}{m}$,

$$\frac{\partial u_1}{\partial r_1^K}\left(r_1^K\right) > \frac{\partial u_2}{\partial r_1^K}\left(r_1^K\right).$$

Indeed, the inequality can be rewritten as

$$\left(\frac{1}{1-\alpha}A_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{2-\alpha}{\alpha-1}} + \frac{1}{\beta}A_1\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}\right)\frac{\gamma}{1+\gamma}\omega_0 + \frac{\gamma}{1+\gamma}\omega_0 X_0\frac{r_0^K}{r_1^K}k_0$$
$$\leq \left(\frac{1}{1-\alpha}A_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{2-\alpha}{\alpha-1}} + \frac{\alpha}{1-\alpha}mA_1 L_1^{cc}\left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(r_1^K\right)^{\frac{1}{\alpha-1}}\right)\left(\frac{1-\alpha}{\alpha}X_0 - \frac{\omega_0}{1+\gamma}\right).$$

After some calculations, we can show a stronger result as below:

$$\left(\frac{1}{1-\alpha}+\frac{1}{1+\gamma}\left(\frac{\gamma}{\beta}\frac{1}{L_1^{cc}}+\frac{\alpha}{1-\alpha}m\right)r_1^K\right)\omega_0 \\ \leq\left(\frac{1-\alpha}{\alpha}+2mr_1^K\right)X_0. \tag{A.42e}$$

and this inequality holds if [40]

$$\omega_0<\Xi\left(\alpha,m,X^*,\gamma,\beta\right)=\min\left\{V\left(1\right),V\left(\frac{1}{m}\right)\right\}, \tag{A.42f}$$

in which

$$V\left(r_1^K\right)=\frac{\left(\frac{1-\alpha}{\alpha}+2mr_1^K\right)X^*}{\frac{1}{1-\alpha}+\frac{1}{1+\gamma}\left(\frac{\gamma}{\beta}\frac{1-(1-m)\frac{\alpha}{X^*}}{\frac{1-\alpha}{X^*}}+\frac{\alpha}{1-\alpha}m\right)r_1^K}. \tag{A.42g}$$

As a result, given k_0, $\omega_0\leq\hat{\omega}_0\left(k_0\right)$ in (A.41e), a binding collateral constraint and a non-binding irreversibility constraint, an equilibrium with binding ZLB exists and is unique. Otherwise, if there are multiple equilibria in this region, $u_1\left(r_1^K\right)$ and $u_2\left(r_1^K\right)$ cross for multiple times, and then one of these equilibria features $\frac{du_1}{dr_1^K}\leq\frac{du_2}{dr_1^K}$ which contradicts the slope comparison above.

Step 4: Comparative Statics

By checking equations (A.42c) and (A.42d) carefully, we see that as ω_0 increases, $u_1\left(r_1^K\right)$ shifts to the left, while $u_2\left(r_1^K\right)$ shifts to the right, making the equilibrium r_1^K lower and r_0^K higher. From (A.42a), X_0 is also lower. Thus X_0 *is decreasing in* ω_0.

To see how the ratio $\rho_k=\frac{k_1}{k_0}$ changes with k_0, we express (A.42c) and (A.42d) as functions of $\{\rho_k,X_0\}$:

$$\rho_k^{1-\alpha}\left[\rho_k-(1-\delta)\left(1-\frac{\omega_0}{1+\gamma}\right)\right]k_0^{1-\alpha}+\frac{1}{\beta}\frac{1-\alpha}{X^*}A_1\left(A_1L_1^{cc}\right)^{-\alpha}\rho_k \\ =\left(\frac{X_0}{X^*}-\frac{\omega_0}{1+\gamma}\frac{\alpha}{X^*}\right)\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\left(A_1L_1^{cc}\right)^{1-\alpha},$$

[40]With our calibrated parameters, the value of this upper bound is 1.39.

and

$$\rho_k^{\alpha}\left[\rho_k^{1-\alpha}-m\frac{\alpha}{X^*}\left(A_1L_1^{cc}\right)^{1-\alpha}k_0^{\alpha-1}\right]$$
$$=(1-\delta)\frac{\gamma}{1+\gamma}\omega_0+\frac{\gamma}{1+\gamma}\omega_0\left(\frac{X^*}{X_0}\right)^{\frac{1}{\alpha}}\left(\beta\frac{A_0}{A_1}\right)^{\frac{1-\alpha}{\alpha}}\frac{\alpha}{X^*}\left(A_1L_1^{cc}\right)^{1-\alpha}\rho_k^{\alpha-1}k_0^{\alpha-1}.$$

In both functions, given ω_0 and fixing X_0, ρ_k is decreasing in k_0. Thus in equilibrium, $\frac{k_1}{k_0}$ *is decreasing in* k_0.

Lastly, since r_1^K is decreasing in ω_0, the excess return $R_1^K - R_0$ is also decreasing in ω_0 since $R_1^K = r_1^K$ and $R_0 = 1$.

Step 5: Checking the Assumptions of Binding Collateral Constraint and Non-binding Irreversibility

Since X_0 is decreasing at ω_0, and at $\omega_0 = \hat{\omega}_0(k_0)$ given in (A.41e), $X_0 = X^*$, the assumption of a binding ZLB is violated if and only if $\omega_0 > \hat{\omega}_0(k_0)$. In addition, since $\frac{k_1}{k_0}$ is decreasing in k_0 and we can show that at $k_0 = k_0^{**}(\omega_0)$ given in Subsection G.2.2, $\frac{k_1}{k_0} = 1-\delta$, the assumption of a non-binding irreversibility constraint is violated if and only if $k_0 \geq k_0^{**}(\omega_0)$. Lastly, since $R_1^K - R_0$ is decreasing in ω_0 and at $\omega_0 = \omega_0^{CC}(k_0)$ given in Lemma 18, $R_1^K - R_0 = 0$, the assumption of a binding collateral constraint is violated if and only if $\omega_0 > \omega_0^{CC}(k_0)$.

To sum up, when ω_0 is smaller than

$$\min\left\{(1+\gamma)\frac{1-\alpha}{\alpha}X^*, \Xi(\alpha, m, X^*, \gamma, \beta)\right\},$$

there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and binding ZLB if and only if $k_0 < k_0^{**}(\omega_0)$, $\omega_0 \leq \omega_0^{CC}(k_0)$ and $\omega_0 \leq \hat{\omega}_0(k_0)$. □

G.6 AS-AD Representation

If the collateral constraint is not binding, the equilibrium properties and the AS-AD curves are the same as the model with natural borrowing limit analyzed in Appendix F.5. On the other hand, if the irreversibility constraint is not binding, the equilibrium properties are similar to the simple two-period model in Subsection 2.4. So here we focus on the AS-AD curves with both binding collateral constraint and binding irreversibility constraint.

When the collateral constraint is binding, labor supply at $t = 1$ is constant and given by L_1^{cc} in equation (A.33a). Together with the labor-leisure choice of the households at $t = 0, 1$, the households' Euler equation for bond holding, as well as $k_1 = (1-\delta)\,k_0$ given a binding irreversibility constraint, the AS curve can be written as

$$Y_0^{AS} = \left(\beta R_0 \frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\delta)\,k_0}{A_0 L_1^{cc}}\right)^{\alpha-1} k_0. \tag{A.43a}$$

As $k_1 = (1-\delta)\,k_0$, the AD curve is given by summing up c_0 and c_0' in Subsection G.1:

$$\begin{aligned} Y_0^{AD} =& \frac{1}{1+\gamma}\omega_0 k_0 \left[q_0\,(1-\delta) + \frac{\alpha}{X_0}\left(\beta R_0 \frac{A_0}{A_1}\frac{X^*}{X_0}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{(1-\delta)\,k_0}{A_0 L_1^{cc}}\right)^{\alpha-1}\right] \\ &+ \frac{1}{\beta R_0}\frac{1-\alpha}{X^*}A_1^{1-\alpha}\left(\frac{(1-\delta)\,k_0}{L_1^{cc}}\right)^{\alpha}. \end{aligned} \tag{A.43b}$$

When the irreversibility constraint is binding, Proposition 4 shows that both R_0 and X_0 remain independent to k_0 and are functions of ω_0 only. Thus given $\{k_0, \omega_0\}$, we can express the AS-AD curves as functions of q_0 while replacing R_0 and X_0 by their equilibrium values.

As an example, the AS-AD curves are plotted in Figure A.14 when all three constraints: ZLB, collateral constraint and the irreversibility constraint are all binding. The AS curve is inelastic to q_0, and the AD curve is positively sloped. As k_0 increases, both AS and AD curves shift to the right leading Y_0 to increase. The effect on q_0 might be ambiguous but by equation (A.39d), q_0 decreases. Similarly, as ω_0 increases, both AS and AD curves shift to the right leading Y_0 to increase. By equation (A.39d), q_0 increases.

H More Details from the Quantitative Model

In Appendix H.1, we present the full quantitative model. Appendix H.2 describes our global solution method and Appendix H.3 provides the computed policy functions from the benchmark quantitative model. Appendix H.5 carries out analyses of numerical errors from our global solution.

H.1 Complete Setup

Section 3.1 describes the essential ingredients of the quantitative model. We now describe the remaining setup of the model and refer to the common components shared with the two-period model when necessary. Time is discrete, starts from 0 and goes to infinity.

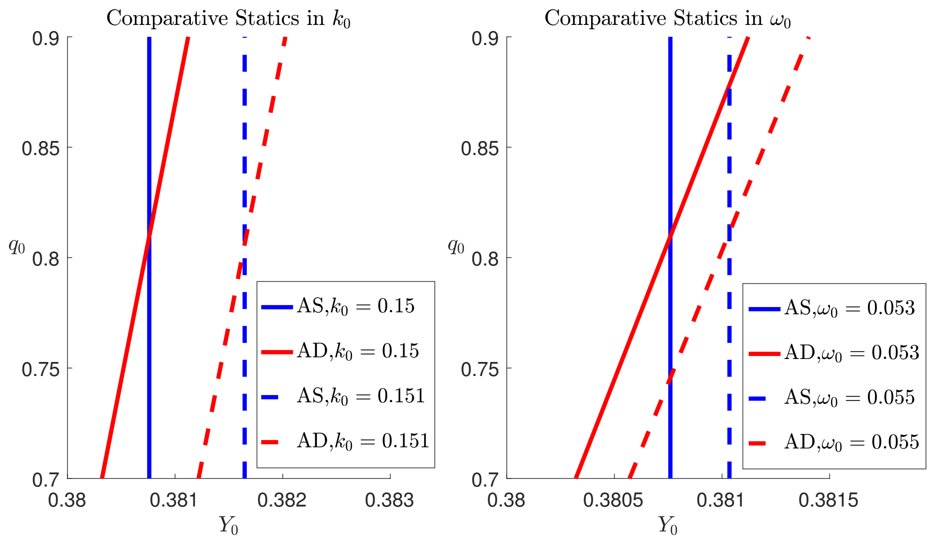


Figure A.14: AS-AD Curves with Binding ZLB, Collateral Constraint and Irreversibility Constraint

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.9$, and $\epsilon = 21$. We choose $k_0 = 0.15$ and $\omega_0 = 0.053$ in the baseline case.

The aggregate shocks consist of a productivity shock and a credit shock, as specified in Section 3.1.

The households The representative households supply labor endogenously and make saving and borrowing decisions to maximize the expected lifetime utility

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t[\log c_t' - \frac{1}{\eta}\left(L_t'\right)^{\eta}]$$

where c_t' is the consumption and L_t' is the labor supply. $\beta > 0$ is the households' common discount factor. The households are subject to the following sequential budget constraint

$$P_t c_t' + \frac{B_t'}{R_t} \leq B_{t-1}' + P_t w_t L_t' + P_t \int_0^1 \Xi_t(z)dz,$$

taking P_t, R_t, w_t and $\int_0^1 \Xi_t(z)dz$ as given, where B_{t-1}' is the nominal bond accumulated in the previous period, R_t is the nominal interest rate, w_t is the market real wage, and $\int_0^1 \Xi_t(z)dz$ is the profits transferred from intermediate good retailers in real terms.

The entrepreneurs The representative entrepreneurs maximize the following expected utility

$$\mathbb{E}\sum_{t=0}^{\infty}\gamma^t \log c_t$$

subject to the sequential budget constraint

$$P_t c_t + P_t q_t^{K'} k_{t+1} + \frac{B_t}{R_t} \leq B_{t-1} + P_t q_t^K k_t + P_t^e Y_t^e - P_t w_t L_t,$$

production technology

$$Y_t^e = k_t^\alpha (A_t L_t)^{1-\alpha},$$

and the collateral constraint

$$\min_{\{m_{t+1}, g_{t+1}, \chi_{t+1}^m\}} \left[m_t P_{t+1} [q_{t+1}^K + r_{t+1}^K] k_{t+1} + B_t \right] \geq 0,$$

taking $P_t, q_t^{K'}, q_t^K, R_t, \pi_t, P_t^e, w_t$ as given, where $q_t^{K'}$ and q_t^K are market prices for new capital and existing capital, respectively. P_t^e is the price of intermediate goods.

Equilibrium The problem of the final-good producers is the same as the one in the two-period model. The retailers' problem, capital producing firm's problem and monetary policy rules are specified in section 3.1. We define the real value of debt $b_t = \frac{B_t}{P_t}$ and the markup charged by retailers $X_t = P_t / P_t^e$. Then the budget constraints of the entrepreneurs and the households can be written in real variables.

We adopt the standard notation of uncertainty. Time is discrete and runs from 0 to infinity. In each period, an aggregate shock $s_t = (g_t, m_t, \chi_t^m)$ is realized. s_t follows a finite-state Markov chain described in Subsection 3.1. Let $s^t = (s_0, s_1, \ldots, s_t)$ denote the history of realizations of shocks until date t. Assume A_{-1}, $g_{-1}, m_{-1}, \chi_{-1}^m$ are given. To simplify notations, for each variable x, we use x_t as a shortcut for $x_t\left(s^t\right)$.

Definition 3. A competitive equilibrium is sequences, which depend on time t and the history of shocks s^t, of inflation and markup $\{\pi_t, X_t\}_{t,s^t}$, prices $\left\{w_t, r_t^K, q_t^K, q_t^{K'}, R_t\right\}_{t,s^t}$, retailer real profits $\left\{\int_0^1 \Xi_t(z)dz\right\}_{t,s^t}$ and allocations $\{c_t, c_t', k_{t+1}, Y_t, Y_t^e, b_t, b_t', L_t, L_t'\}_{t,s^t}$ such that given the initial conditions $k_0 > 0, b_{-1}$ and $b'_{-1} = -b_{-1}$:

(i) The allocations solve the entrepreneurs and the households' decision problems.

(ii) Markets for labor, bond, intermediate good and final good clear:

$$\begin{aligned} L_t &= L_t', \\ b_t + b_t' &= 0, \\ Y_t &= Y_t^e, \end{aligned}$$

$$c_t + c_t' + \Omega\left(k_t, k_{t+1}\right) + \theta\phi\left(\pi_t\right) Y_t = Y_t,$$

in which $\Omega\left(k_t, k_{t+1}\right)$ and $\theta\phi\left(\pi_t\right)Y_t$ are the capital adjustment cost and price adjustment cost, respectively, as specified in Section 3.1.

(iii) Retailers' profits satisfy equation (16). Capital prices satisfy equations (17).

(iv) The New-Keynesian Phillips Curve (13) holds. Taylor rule (18) holds.

As in the two-period model in Section 2, we focus on sequential competitive equilibria with the wealth share of the entrepreneurs as an endogenous state variable. Their wealth share is defined as

$$\omega_t = \frac{\left(r_t^K + q_t^K\right)k_t + \frac{b_{t-1}}{1+\pi_t}}{\left(r_t^K + q_t^K\right)k_t}. \tag{A.44}$$

Correspondingly, the households' wealth share is

$$\omega_t' = \frac{\frac{b_{t-1}'}{1+\pi_t}}{\left(r_t^K + q_t^K\right)k_t}.$$

From the bond market clearing condition, $\omega_t' = 1 - \omega_t$ in any competitive equilibrium.

Definition 4. A wealth-recursive equilibrium in the infinite-horizon economy is a sequential competitive equilibrium in which allocations $\{c_t, c_t', k_{t+1}, Y_t, b_t, L_t\}$, prices $\left\{w_t, r_t^K, q_t^K, q_t^{K'}, R_t\right\}$ and inflation and markup $\{\pi_t, X_t\}$ are functions of $\{k_t, \omega_t, s_t, A_t\}$.

H.2 Global Solution Method

To calculate wealth-recursive equilibria, we de-trend the retailers' real profits $\left\{\int_0^1 \Xi_t(z)dz\right\}$ and allocations $\{c_t, c_t', k_{t+1}, Y_t, b_t\}$ by the aggregate TFP shock A_t and remove A_t from the list of exogenous state variables. With some abuse of notations, we use the same symbols here to denote their corresponding de-trended values.

Given $\{k_t, \omega_t, s_t\}$, we have $7+S$ unknown variables: c_t, c_t', k_{t+1}, b_t, μ_t, π_t, X_t, $\{\omega_{t+1}\left(s_{t+1}\right)\}_{s_{t+1}}$, in which S is the number of states in period $t+1$, and $\omega_{t+1}\left(s_{t+1}\right)$ is the wealth share in period $t+1$ when the state in the next period is s_{t+1}.[41] We use the following system with $7+S$ equations to pin down the values of the variables:

1. Feasibility constraint:

$$c_t + c_t' + \Omega\left(k_t, k_{t+1}\right) + \theta\phi\left(\pi_t\right)Y_t = Y_t, \tag{A.45a}$$

[41] From equation (A.44), given $\{k_{t+1}, b_t\}$, ω_{t+1} is endogenous to state s_{t+1} since $\{r_{t+1}^K, q_{t+1}^K, \pi_{t+1}\}$ are affected by s_{t+1}.

2. FOC for households' bond holding:

$$-1+\beta\mathbb{E}_t\Big[\frac{1}{1+g_{t+1}}\frac{R_t}{1+\pi_{t+1}}\frac{c'_t}{c'_{t+1}}\Big]=0, \tag{A.45b}$$

3. Entrepreneurs' budget:

$$c_t+q_t^{K'}k_{t+1}+\frac{b_t}{R_t}=\frac{\left(r_t^K+q_t^K\right)k_t}{1+g_t}\omega_t, \tag{A.45c}$$

4. FOC for entrepreneurs' capital holding:

$$-1+\frac{\kappa_t\mu_t}{q_t^{K'}}+\gamma\mathbb{E}_t\left[\frac{1}{1+g_{t+1}}\left(\frac{r_{t+1}^K+q_{t+1}^K}{q_t^{K'}}\right)\frac{c_t}{c_{t+1}}\right]=0, \tag{A.45d}$$

5. Complementary-slackness condition for the collateral constraint:

$$\mu_t\Big[b_t+\kappa_t k_{t+1}\Big]=0, \tag{A.45e}$$

with $\mu_t\geq 0$,

6. F.O.C. for entrepreneurs' bond holding:

$$-1+R_t\mu_t+\gamma\mathbb{E}_t\Big[\frac{1}{1+g_{t+1}}\frac{R_t}{1+\pi_{t+1}}\frac{c_t}{c_{t+1}}\Big]=0, \tag{A.45f}$$

7. New-Keynesian Phillips curve:

$$(1+\pi_t)\phi'\left(\pi_t\right)=\frac{\varepsilon}{\theta}\Big(\frac{1}{X_t}-\frac{\varepsilon-1}{\varepsilon}\Big)+\beta\mathbb{E}_t\Big[\frac{1}{1+g_{t+1}}\frac{c'_t}{c'_{t+1}}(1+\pi_{t+1})\phi'\left(\pi_{t+1}\right)\frac{Y_{t+1}}{Y_t}\Big], \tag{A.45g}$$

8. Consistency condition:

$$\omega_{t+1}=\frac{\left(r_{t+1}^K+q_{t+1}^K\right)k_{t+1}+\frac{b_t}{1+\pi_{t+1}}}{\left(r_{t+1}^K+q_{t+1}^K\right)k_{t+1}},\quad\forall s_{t+1}, \tag{A.45h}$$

in which r_{t+1}^K, q_{t+1}^K and π_{t+1} are functions of $\{k_{t+1},\omega_{t+1},s_{t+1}\}$.

The auxiliary variables in the equations above are given as follows:

$$\Omega\left(k_t, k_{t+1}\right) = k_{t+1} - \frac{1-\delta}{1+g_t} k_t + \frac{\xi}{2} \frac{\left(k_{t+1} - \frac{k_t}{1+g_t}\right)^2}{\frac{k_t}{1+g_t}},$$

$$\kappa_t = m_t \min_{s_{t+1}} \left\{ \left(r^K_{t+1} + q^K_{t+1}\right) \pi_{t+1} \right\},$$

$$q_t^{K'} = 1 + \xi \frac{k_{t+1} - \frac{k_t}{1+g_t}}{\frac{k_t}{1+g_t}},$$

$$q_t^K = (1-\delta) - \frac{\xi}{2}\left[1 - \left(\frac{(1+g_t)\, k_{t+1}}{k_t}\right)^2\right],$$

$$\phi\left(\pi_t\right) = \frac{\pi_t - \bar{\pi}}{\sqrt{\bar{\pi} - \underline{\pi}}} - 2\sqrt{\pi_t - \underline{\pi}} + 2\sqrt{\bar{\pi} - \underline{\pi}},$$

$$R_t = \max\left\{ \bar{R}\left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_Y}, 1 \right\},$$

$$L_t = \left[\frac{1-\alpha}{X_t} \frac{k_t^\alpha}{(1+g_t)\, c_t'}\right]^{\frac{1}{\alpha+\eta-1}},$$

$$r_t^K = \frac{\alpha}{X_t}\left[\frac{k_t}{(1+g_t)\, L_t}\right]^{\alpha-1},$$

$$Y_t = \left(\frac{k_t}{1+g_t}\right)^\alpha L_t^{1-\alpha}.$$

In addition, we can invert equation (A.44) to get b_{t-1} :

$$b_{t-1} = (\omega_t - 1)(1+\pi_t)\left(r_t^K + q_t^K\right) k_t.$$

We solve for the recursive equilibrium in this economy using the algorithm in Cao and Nie (2017) and Cao (2018). The original algorithm in Cao (2018) uses wealth share as endogenous state variables. Cao and Nie (2017) add labor choice as well as housing consumption decisions. In the current paper, we show that the original algorithm works similarly when we add imperfect price stickiness, capital and capital adjustment cost, and Taylor-rule based monetary policy with two occasional binding constraints, ZLB and the collateral constraint.[42] Here we present the details of this algorithm.

Our algorithm looks for a recursive equilibrium mapping from $\{k_t, \omega_t\}$ and the exogenous aggregate shock, $s_t = \{g_t, m_t, \chi_t^m\}$, to the allocations $\{c_t, c_t', k_{t+1}, Y_t, b_t, L_t\}$, prices

[42]We also solved another version of the infinite-horizon economy with a third occasional binding constraint: investment irreversibility as discussed in Section E.

$\left\{w_t, r_t^K, q_t^K, q_t^{K'}, R_t\right\}$, inflation and markup $\{\pi_t, X_t\}$, as well as the future financial wealth distributions, $\omega_{t+1}(s_{t+1})$, depending on the realization of future aggregate shocks, s_{t+1}. Indeed, given the mapping from $\{k_{t+1}, \omega_{t+1}, s_{t+1}\}$ to $\left\{r_{t+1}^K, q_{t+1}^K, \pi_{t+1}, c_{t+1}, c_{t+1}', Y_{t+1}\right\}$, for a given set of $\{k_t, \omega_t, s_t\}$, we can solve for the other variables using equations (A.45a) to (A.45h). In particular, we follow Cao (2018) in solving for ω_{t+1} simultaneously with other unknowns. The additional equations needed to solve for ω_{t+1} are equation (A.45h) applied to each of the future state s_{t+1} in which the mapping from $\{k_{t+1}, \omega_{t+1}, s_{t+1}\}$ to r_{t+1}^K, q_{t+1}^K and π_{t+1} are given by the mapping obtained in the previous iteration of the algorithm.

We solve for the recursive equilibrium using backward induction. The algorithm starts by solving for the equilibrium mapping for 1-period economy. Then given the mapping from $t = 0$ to $t = 1$ for T-period economy, we can solve for the mapping for $(T+1)$-period economy following the procedure described above. The algorithm converges when the mappings for $T-$period economy and $(T+1)$-period economy are sufficiently close to each other.[43]

[43] For the model without the credit shock, i.e., $m_t \equiv m$, based on the definition of ω_t in (A.44) and the borrowing constraint (15), we can easily see that the lower bound of ω_t is $1 - m$. With shocks to m_t, the expression of the lower bound of ω_t is unknown ex ante, and we show the model can also be solved using c_t as an endogenous state variable instead of ω_t.

H.3 Policy Functions and ZLB duration from the Quantitative Model

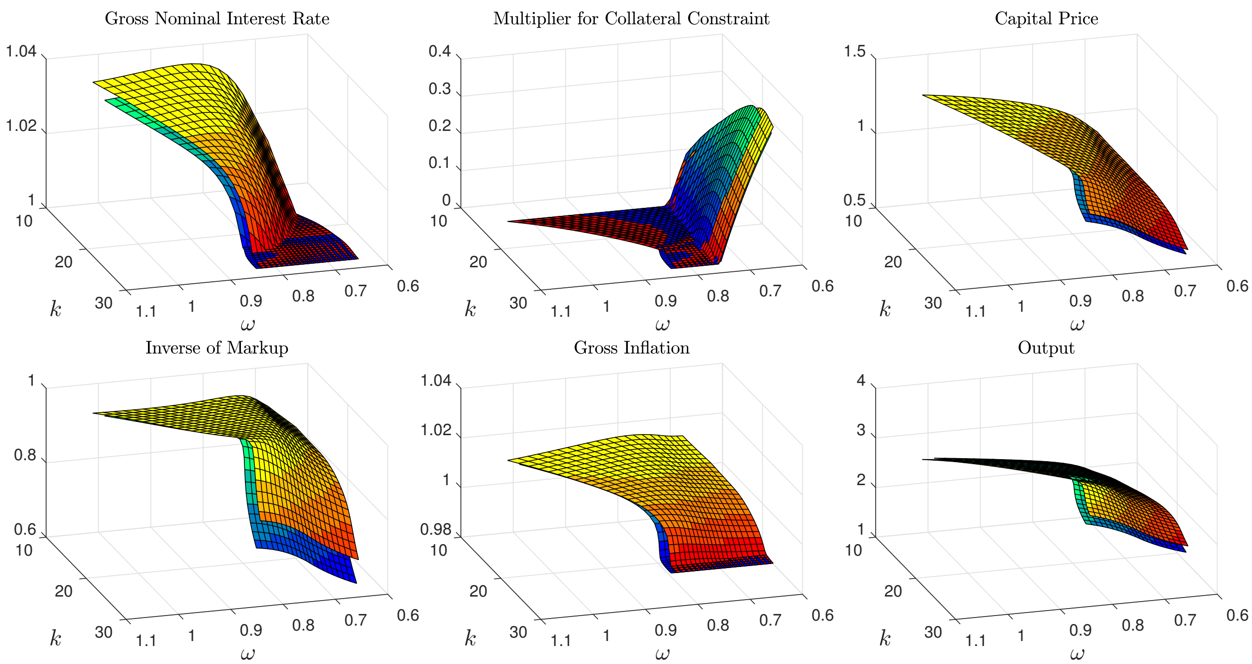


Figure A.15: Policy Functions over (k, ω) Space

Note: The policy functions are evaluated with productivity growth rate equal to its unconditional mean. Surfaces with warm colors correspond to m High ($m = 0.45$). Surfaces with cold colors correspond to m Low ($m = 0.1$).

The policy functions in the full quantitative model carry all the intuitions we have learned from the two-period model. Figure A.15 plots the policy functions for several key equilibrium variables over the endogenous states variables (k, ω). The policy functions are evaluated with productivity growth rate equal to its unconditional mean. The two surfaces correspond to policy functions with different levels of leverage constraint m (the warm-color surfaces correspond to $m = 0.45$ and the cold-colored ones correspond to $m = 0.1$). As shown in the figure, given the productivity, the ZLB tends to bind when capital stock is high or the entrepreneur wealth share is low. The collateral constraint tends to bind when the entrepreneur wealth share is low. The inverse of markup drops substantially when the ZLB binds, and even more so when the ZLB and the collateral constraint both bind. Both capital price and output are substantially lower in the regions where zero lower bound binds, and more so when both constraints bind.

What is new in the quantitative model with imperfect price stickiness compared to the two-period model is that now inflation is allowed to be different from one. Inflation changes in the same direction along with the inverse of markup over the state space since it is associated with the inverse of markup through the New-Keynesian Phillips Curve. It changes more smoothly due to the forward-looking nature of the Phillips Curve. Introducing imperfect price stickiness allows the movement of inflation to feedback into the collateral constraint, through a traditional Fisherian "nominal debt-deflation channel". However, from both the policy functions and the crisis episode we study in Section 3, the movement in inflation is usually small and is not likely to play a quantitatively important role in determining the severity of the crisis compared to other channels.

Figure A.16 and A.17 plot the regions for binding ZLB and collateral constraints, and the policy functions projecting onto either k or ω space. Both figures resemble their counterparts for the two-period model and we refer the reader to the main text for the analytical characterizations and the discussions.

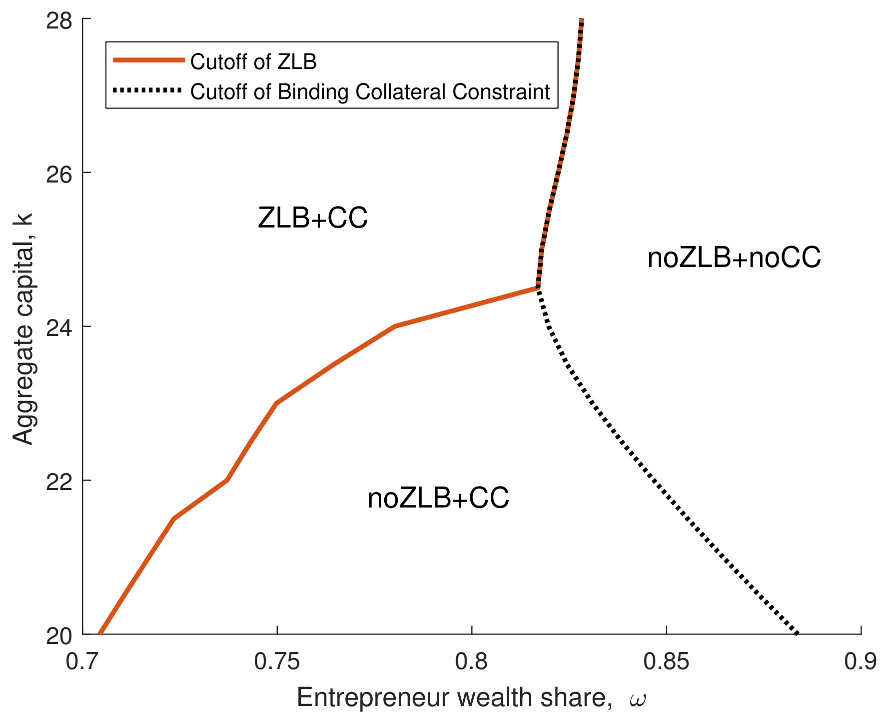


Figure A.16: Regions for ZLB and Binding Collateral Constraint

Note: The region is based on policy functions at $m = \bar{m}$ and $g = \bar{g}$.

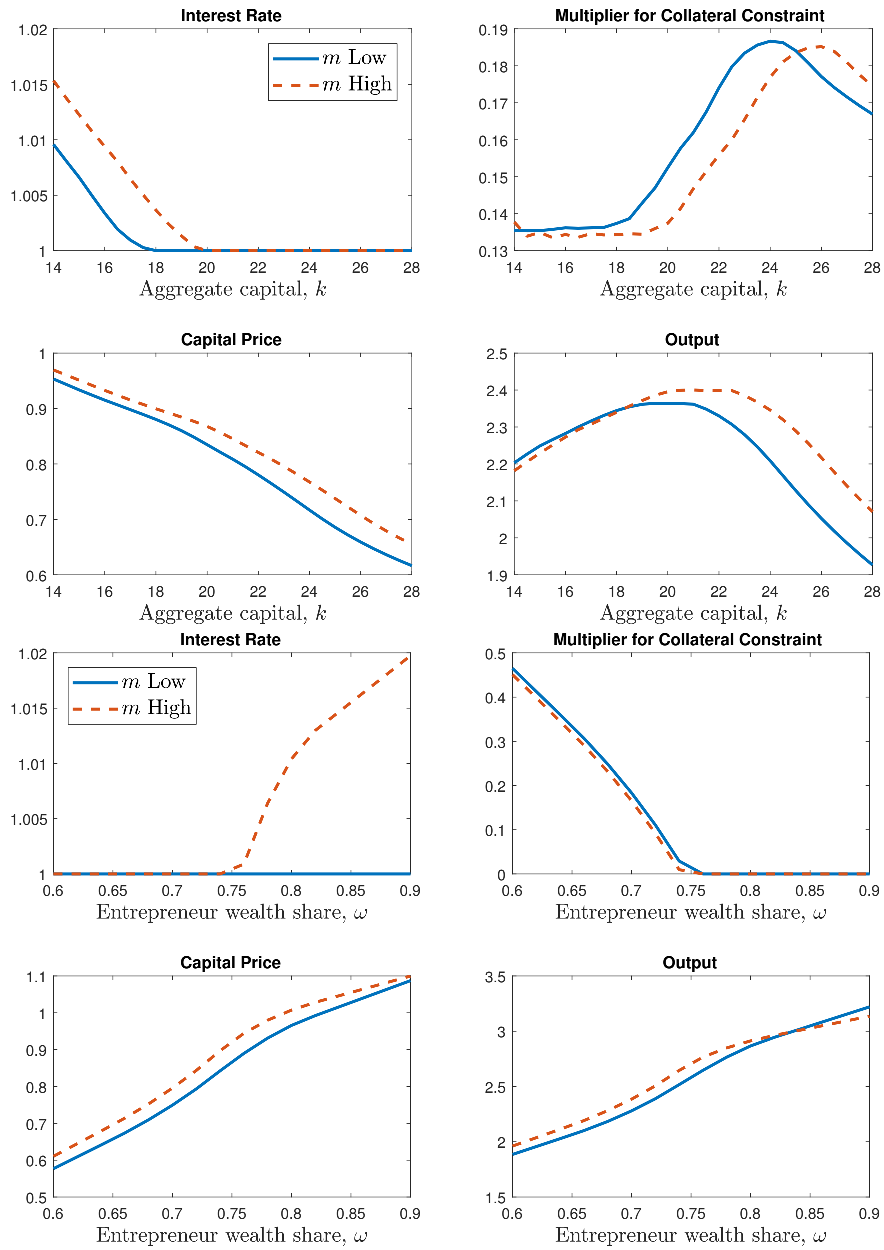


Figure A.17: Policy Functions Varying Capital (four upper panels) and Wealth Share (four lower panels)

Note: The policy functions are at $g = \bar{g}$. m High corresponds to $m = 0.45$. m Low corresponds to $m = 0.1$.

The average duration of a ZLB episode in the ergodic set is around 2 quarters. Nevertheless, the histogram of ZLB durations, shown in Figure A.18, exhibits a long right tail, and the model can produce lengthy ZLB episodes with the appropriate choice of realized shock series, albeit with low probability.

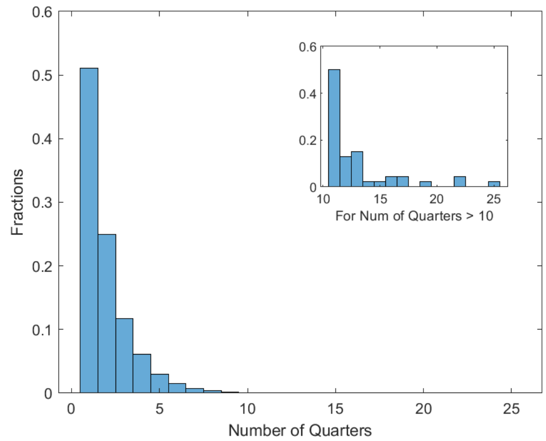


Figure A.18: Histogram of ZLB Durations in the Ergodic Set

Note: The histogram is based on 24 sample paths, each with 50,000 periods and with the first 5,000 periods dropped. The longest ZLB episode in the simulated sample lasts for 25 quarters.

H.4 Asset Prices in the Data and in the Model

In this appendix, we discuss the dynamics of asset prices implied by the quantitative model during the Great Recession and compare them to the data. As we described in Subsection 3.1, capital in the model stands in for a combination of housing and non-housing capital. Therefore, in the left panel of Figure A.19, we plot the model capital price against the price indices of both housing and stock market from the data. Stock prices dropped significantly more than housing prices but recovered more quickly. Overall, our model captures relatively well the timing and magnitude of the average dynamics of prices in these time series. The magnitude of the drops in the model is slightly smaller because we leave out other important factors influencing prices during the Great Recession, such as changes in risk premium and liquidity, and deteriorated balance sheet of the financial sector.

The right panel of Figure A.19 plots the model implied excess returns against the credit spread constructed by Gilchrist and Zakrajsek (2012) (GZ spread). The dynamics of excess return in the model tracks the overall timing of the rise and fall of the GZ spread. However, the magnitude of the rise of the excess return is significantly larger than that of the GZ spread. This is partly because the size of the credit shock in the model is calibrated to match the overall drop in bank loans and the excess returns correspond to excess returns on a broad range of assets, whereas the GZ spread measures the spread on bonds issued by publicly listed firms which have better access to external financing.

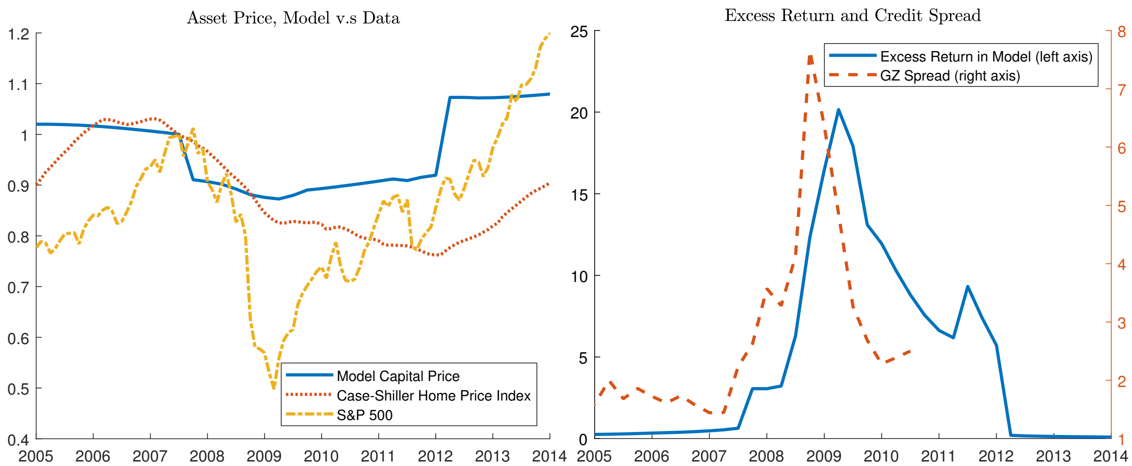


Figure A.19: Asset Price and Excess Return, Model versus Data

Note: The variables in the left panel are reported as ratios to their 2007Q3 values.

H.5 Numerical Error Analyses

For each state $x_t = (k_t, \omega_t, g_t, m_t, \chi_t^m)$, following Judd, Maliar, and Maliar (2011) and Guerrieri and Iacoviello (2015), we define the unit-free Euler equation errors for the bond and capital choices of the households and the entrepreneurs as below:

$$\begin{aligned}
\mathcal{E}^{b'}(x_t) &= -1 + \beta \mathbb{E}_t \left[\frac{1}{1+g_{t+1}} \frac{R_t}{1+\pi_{t+1}} \frac{c_t}{c_{t+1}} \right] \\
\mathcal{E}^{b}(x_t) &= -1 + R_t \mu_t + \gamma \mathbb{E}_t \left[\frac{1}{1+g_{t+1}} \frac{R_t}{1+\pi_{t+1}} \frac{c_t}{c_{t+1}} \right]. \\
\mathcal{E}^{k}(x_t) &= -1 + \frac{\kappa_t \mu_t}{q_t^{K'}} + \gamma \mathbb{E}_t \left[\frac{1}{1+g_{t+1}} \left(\frac{r_{t+1}^K + q_{t+1}^K}{q_t^{K'}} \right) \frac{c_t}{c_{t+1}} \right].
\end{aligned} \tag{A.46}$$

We evaluate the errors for a sample of 120000 x_t drawn from the model's ergodic set.[44]

Table A.1: Euler Equation Errors

	mean $\lvert\mathcal{E}^{b'}\rvert$	mean $\lvert\mathcal{E}^{b}\rvert$	mean $\lvert\mathcal{E}^{k}\rvert$
All Samples	2.9E-04	2.5E-05	4.1E-04
ZLB Binding	4.3E-04	4.1E-05	3.4E-04
ZLB & CC Binding	4.2E-04	4.0E-05	3.1E-04

[44]To draw samples from the ergodic set, we simulate 24 paths of 6000 periods. Notice by the ergodic theory, the long run distribution of samples across time of a single simulation path converges to the ergodic distribution; we choose multiple paths to utilize parallel computation. We drop the first 1000 periods and keep the remaining 5000 periods of the 24 paths, which give us 120000 observations in total.

Table A.1 reports the mean absolute errors across x_t in the full samples, the samples with binding ZLB, and the samples with both binding ZLB and collateral constraints. As shown, the mean absolute errors across all samples and subsamples are below $5E-4$. The accuracy is slightly lower for states with binding ZLB due to the nonlinear dynamics in these regions of the state space. The numerical errors are of similar magnitude as the errors from Guerrieri and Iacoviello's OccBin for their model without ZLB Guerrieri and Iacoviello (2015, Figure 4) and are lower than the OccBin errors for their model with ZLB Guerrieri and Iacoviello (2015, Figure 6).

H.6 Comparisons with Piecewise-linear Solutions

The global nonlinear solutions provide a full characterization of the economy in and out of normal times, and capture agents' precautionary motives facing severe although infrequent crises. An alternative approach, popularized by the toolbox OccBin (Guerrieri and Iacoviello, 2015), approximates the nonlinear solutions with piecewise linear functions. This section compares the OccBin solution with the global nonlinear solution, highlighting the non-linearity of the current model and the importance of capturing agents' precautionary motives in understanding the crisis dynamics.

To do so, we need to modify the benchmark model in several ways. First, the benchmark collateral constraint is specified as

$$m_t \cdot \min\left[P_{t+1}\left(q_{t+1}^K + r_{t+1}^K\right)k_{t+1} + B_t\right] \geq 0.$$

This constraint corresponds to a condition that the entrepreneurs will not default under any realization of future exogenous states, and hence equips the lenders with a strong precautionary motive. Since the local solution cannot handle the min operator, we modify the collateral constraint to

$$m_t \cdot \mathbb{E}_t\left[P_{t+1}\left(q_{t+1}^K + r_{t+1}^K\right)k_{t+1} + B_t\right] \geq 0,$$

where $\mathbb{E}_t$ is the expectation operator conditional on the current state. Second, in the benchmark model we specify the credit shock, m_t, to have innovations with asymmetric distributions, aiming at capturing the infrequent nature of financial crises. The piecewise linear solution method cannot handle this asymmetry, so we modify the process of the credit shock to an AR(1) process:[45]

$$m_{t+1} = (1-\rho^m)\bar{m} + \rho^m m_t + \varepsilon^m,$$

where ε^m satisfies the normal distribution with mean zero and standard deviation σ^m. We choose $\bar{m}$ to be the same as in the benchmark model, $\rho^m = 0.99$ and $\sigma^m = 0.01$, so the process is close to the one used in the benchmark model.[46] Third, after the two modifications above, we recalibrate the discount factor of the entrepreneurs, β, so that the average nominal interest rate is 5% in the ergodic set based on the nonlinear solution.[47]

[45]For the collateral constraint to be well-defined, we need to truncate m_t to be within $[0, 1]$, but in the simulated ergodic set, m_t never hits the upper of lower bounds with the chosen standard deviation of the shock.

[46]Notice for the event study interested in this subsection, the choice of σ^m matters for the global nonlinear solution but not for the piecewise linear solution. Setting $\sigma^m = 0.01$ allows us to discretize the innovation to be $\{-0.01, 0.01\}$ based on a two-point Gaussian quadrature, and together with a high ρ^m, brings the process of m_t close to the one in the benchmark model.

[47]This procedure is mainly to ensure comparability across models. The recalibrated $\beta = 0.9991$, close to the calibrated value 0.9993 in the benchmark model.

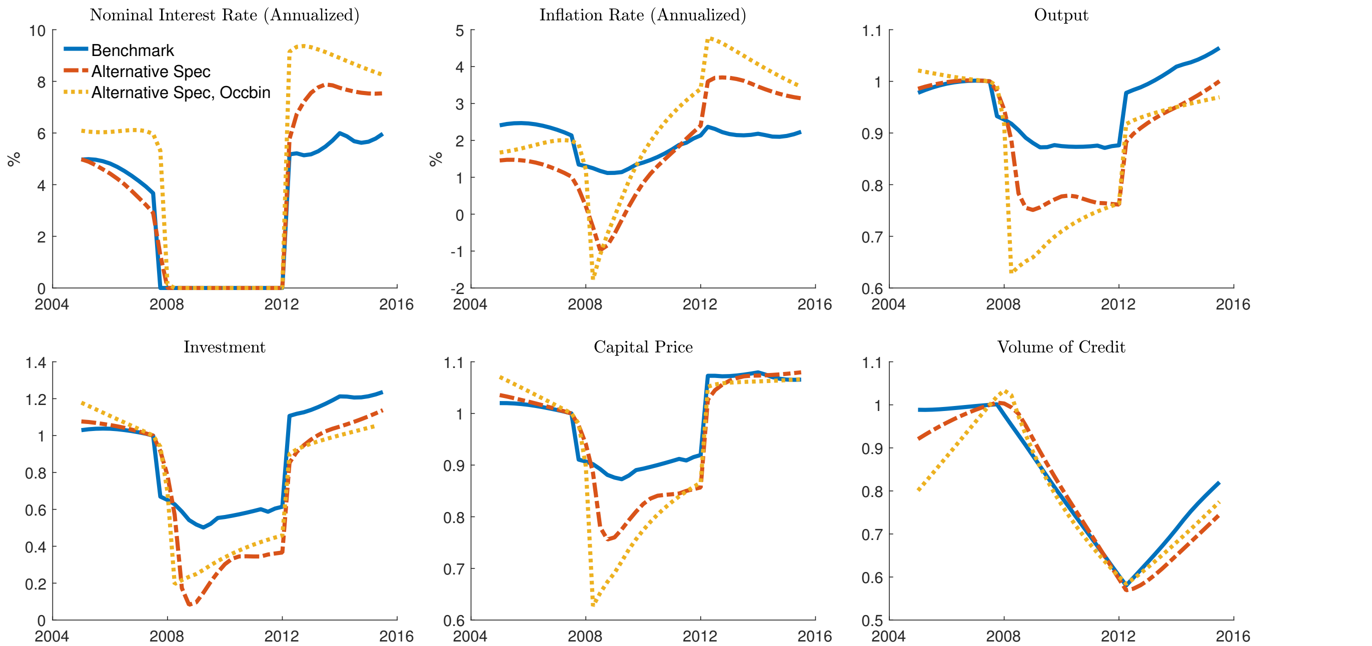


Figure A.20: The ZLB Episode, Global Nonlinear Solutions v.s. Piecewise Linear Solutions

Note: Output, investment, capital price, and the volume of credit are ratios to their 2007Q3 values. "Benchmark" corresponds to the benchmark result in Section 3.3. "Alternative Spec" corresponds to the alternative collateral constraint specification that associates the borrowing limit with the expected value of future capital. "Alternative Spec, OccBin" corresponds to the piecewise linear solution obtained using toolbox OccBin. The initial capital and deb levels are calibrated so that the nominal interest rate is 5% and the debt-to-asset ratio is 35%, based on the nonlinear solution for each specification.

Since the piecewise linear solution method is not designed to produce policy functions over a global domain such as those in Figure A.15, we focus on the comparison for the ZLB episode studied in Section 3.3. To do so, we compare the dynamics implied by different solution methods by starting from the same initial capital and debt levels,[48] and feeding in the same sequences of g_t and m_t as described in Section 3.3. Figure A.20 plots the aggregate dynamics in models with different specifications and solved with different solution methods. The solid lines correspond to the benchmark results in Section 3.3. The dash-dotted lines correspond to the results under the alternative collateral constraint specification, based on the global nonlinear solution. As shown, the effects of the negative productivity growth shock and credit shock are already much larger under the alternative specification than the benchmark.[49] This is because in the benchmark model

[48]The initial capital and debt levels are calibrated so that the nominal interest rate is 5% and the debt-to-asset ratio is 35% based on the nonlinear solution, the same targets as in the benchmark experiment in Section 3.3.

[49]One intermediate model specification between the benchmark and the current alternative model is to

the lenders assign full weight to the worst scenario when evaluating the collateral value of future capital, and thus are well prepared for the crisis state by allowing lower leverages ex ante. Whereas with the alternative specification, the lenders are less precautionary by allowing the entrepreneurs to borrow against the expected value of future capital, putting less weight on the rare but severe crisis state. Consistent with this intuition, the bottom right panel shows that the volume of credits grows much faster before the crisis hits under the alternative specification than under the benchmark specification (the volume of credit in 2007Q3 is normalized to 1). Similarly, the dotted lines correspond to the piecewise log-linear solution for the alternative specification, obtained using OccBin. As shown, although the piecewise log-linear solution captures the overall shape of the dynamics, it overstates the severity of the crisis even more than the global solution for the alternative specification: output and capital price drop by more than 40%, whereas both drops implied by the benchmark model are modest and well align with the data.

In summary, for the model to produce crises with magnitude in line with data, it is important to model agents taking precautionary measures against the rare but severe crisis state, and to capture the high nonlinearity of the model when the crisis hits. The global nonlinear solution is able to appropriately take into account both features.

I Representative Agent Model with Exogenous Wedges

The representative agent model shares all the ingredients with the full model, except that the households and entrepreneurs are combined into representative households, who solve the following problem:

$$\max_{c_t, L_t, k_{t+1}, B_t} \mathbb{E}_0 \left[\log c_t - \frac{1}{\eta} (L_t)^\eta \right]$$

$$s.t. \quad P_t c_t + P_t \frac{1}{1-\Delta_t^k} q_t^{K'} k_{t+1} + \frac{1}{1+\Delta_t^b} \frac{B_t}{R_t} \leq B_{t-1} + P_t (r_t^K + q_t^K) k_t + P_t w_t L_t + P_t \int_0^1 \Xi_t(z) dz,$$

where, to remind readers, P_t is the price level, R_t is the bond nominal interest rate, r_t^K is the real return on capital, $q_t^{K'}$ and q_t^K are the market prices for new and existing capital, w_t is the real wage, and $\int_0^1 \Xi_t(z)dz$ is the profits transferred from the retailers in real terms. The terms Δ_t^b and Δ_t^k are exogenous wedges and correspond to errors in the Euler equations

use the benchmark credit shock process and the alternative collateral constraint specification. The responses in this intermediate model are also significantly larger than those in the benchmark model, suggesting that it is the collateral constraint specification rather than the credit shock process that drives the main difference. Results from the intermediate model are available upon requests.

for bond and capital holdings of the representative households. The Euler equations with these wedges are given in (19).

Following a tradition in the literature (e.g., Smets and Wouters (2007), Coibion et al. (2012), Christiano et al. (2015), Gust et al. (2017)), we use this representative agent model to interpret the data generated from the full model. To do so, we treat Δ_t^b and Δ_t^k as exogenous consumption and financial wedges and back out them from the simulated time series data from the full model. We treat the nominal interest rate, R_t, expected inflation rate $\mathbb{E}_t[\pi_{t+1}]$, and the expected real return to capital $\mathbb{E}_t[\frac{r_{t+1}^K + q_{t+1}^K}{q_t^{K'}}]$ as observables[50] and, following Christiano et al. (2015), we set $\frac{\mathcal{M}_{t+1}}{\mathcal{M}_t} = \beta$ when constructing the two wedges.[51]

We allow households to correctly forecast future prices and allocations taking into account the effects of their expectation errors, and to understand that Δ_t^b and Δ_t^k are recurrent exogenous shocks. To do so, we estimate an AR(1) process for each of Δ_t^b and Δ_t^k:

$$\Delta_{t+1}^x = \mu^x + \rho^x \Delta_t^x + \varepsilon_t^x, \quad \varepsilon_t^x \sim Normal(0, \sigma_{\varepsilon^x}^2)$$

for $x = b, k$. In implementation, we draw a 50000-period time series from the full model, drop the first 10000 period observations and estimate the AR(1) processes with the remaining sample. Table A.2 reports the point estimates for the coefficients.

Table A.2: Estimated AR(1) Processes for Consumption and Financial Wedges

	μ	ρ	σ_ε
Δ^b	-0.0015	0.678	0.0032
Δ^k	-0.0068	0.470	0.0123

We embed the processes of the two wedges, as well as the productivity growth process that is the same as in the full model, to the representative agent model. This leaves four continuous state variables (three for exogenous shocks and one for capital). We solve the model using the global solution method described in Appendix H.2. We keep all parame-

[50] To estimate Δ_t^b, Christiano et al. (2015) use the federal funds rate for R_t and the core CPI-inflation forecasts from the Survey of Professional Forecasters for $\mathbb{E}_t \pi_{t+1}$. Then the consumption wedge is calculated from $1 + \Delta_t^b = (1 + \mathbb{E}_t \pi_{t+1})/(\beta R_t)$ by ignoring the covariance terms. They estimate Δ_t^k using the credit spread constructed in Gilchrist and Zakrajsek (2012). The nominal interest rate, expected inflation rate, and capital return in our model, which we treat as observables to the econometrician, contain the same set of information as the empirical counterparts used in Christiano et al. (2015).

[51] The choice of the deterministic discount factor affects the estimated mean of the consumption wedge and the financial wedge, but not the time variations. As long as the discount factor is recalibrated to match the same bond interest rate, which we do, the choice of the deterministic discount factor when backing out the wedges does not matter.

ters the same as in the full model, except for the households' discount factor β, which we recalibrate to match the same target as in the full model—an annualized average nominal bond interest rate of 5%.

We then take the representative agent model to analyze the ZLB episode. We set the initial capital stock (in 2015Q1) such that the annualized nominal bond interest rate is at 5%. For the "Benchmark" experiment presented in Figure 8, we construct the consumption and financial wedges from the ZLB episode in the full model that is described in Section 3.3. In the experiment labeled "No ZLB," we keep the consumption and financial wedges constructed from the full model. In the experiment labeled "Shutdown ZLB wedge," we construct the wedges from the full model with the ZLB constraint relaxed.