# Dominance and Competitive Bundling

## Online Appendix

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### *B1. Proof of Proposition 8*

From the example in section III.C, we see that under independent pricing the profit is $\pi^*_{A1}(\alpha_1) + \pi^*_{A2}(0) = 2(\frac{3+\alpha_1}{6})^2 + \frac{1}{2}$ for firm A, and $\pi^*_{B1}(\alpha_1) = 2(\frac{3-\alpha_1}{6})^2$ for firm $B1$, $\pi^*_{B2}(0) = \frac{1}{2}$ for firm $B2$.

In order to find the profit under bundling, we can rely on Proposition 7 and find that $\Pi^{**}_{2,A}(\alpha_1/2) = \frac{4F_2(y_2^{**}(\alpha_1/2))^2}{f_2(y_2^{**}(\alpha_1/2))}$, $\Pi^{**}_{2,B1}(\alpha_1/2) = \Pi^{**}_{2,B2}(\alpha_1/2) = \frac{4(1-F_2(y_2^{**}(\alpha_1/2)))^2}{f_2(y_2^{**}(\alpha_1/2))}$, and $y_2^{**}(\alpha_1/2)$ is the fixed point of the function $Y_2^{\alpha_1/2}(y) = \frac{1}{2} + \frac{1}{2}\frac{\alpha_1}{2} + \frac{2-3\left(1-2(1-y)^2\right)}{4(1-y)}$ in the interval $(\frac{1}{2}, 1)$, that is $y_2^{**}(\alpha_1/2) = \frac{1}{20}\alpha_1 + \frac{9}{10} - \frac{1}{20}\sqrt{\alpha_1^2 - 4\alpha_1 + 44}$. Numerical computations show the result. ■

### *B2. Mixed bundling in the baseline model*

Here, we consider the baseline model with $n = 2$ and study the case in which each firm is allowed to practice mixed bundling. This means that firm $i$ $(= A, B)$ chooses a price $P_i$ for the bundle of its own products and a price $p_i = p_{ij}$ for each single product $j = 1, 2$. Thus each consumer buys the bundle of a firm $i$ and pays $P_i$, or buys one object from each firm and pays $p_A + p_B$. The main result is that when $\alpha$ is sufficiently large, we find the same equilibrium outcome described by Proposition 2 under pure bundling because for firm $A$ a pure bundling strategy is superior to any alternative strategy when it has a large advantage over firm $B$. Moreover, we show that the same result holds when A competes with specialists B1 and B2.

Without loss of generality, we assume that $P_i \le 2p_i$ holds for $i = A, B$ and that each consumer willing to buy both products of $i$ buys $i$'s bundle. As a consequence, each consumer chooses one alternative among $AA$, $AB$, $BA$, $BB$, where for instance $AB$ means buying products $A1$ and $B2$. In order to describe the preferred alternative of each type of consumer, we introduce

$$s' \equiv \frac{1}{2} + \frac{\alpha + P_B - p_A - p_B}{2t} \text{ and } s'' \equiv \frac{1}{2} + \frac{\alpha + p_A + p_B - P_A}{2t}$$

where $s' \leq s''$ holds from $P_A \leq 2p_A$ and $P_B \leq 2p_B$.[1]

We find:

- Type $(s_1, s_2)$ buys $AA$ if and only if $s_1 \leq s''$, $s_2 \leq s''$, $s_1 + s_2 \leq s' + s''$.
- Type $(s_1, s_2)$ buys $AB$ if and only if $s_1 \leq s'$, $s_2 > s''$.
- Type $(s_1, s_2)$ buys $BA$ if and only if $s_1 > s''$, $s_2 \leq s'$.
- Type $(s_1, s_2)$ buys $BB$ if and only if $s_1 > s'$, $s_2 > s'$, $s_1 + s_2 > s' + s''$.

Let $S_{ii'}$ and $\mu_{ii'}$ denote, respectively, the set of types who choose $ii'$ and the measure of $S_{ii'}$ for $ii' = AA$, $AB$, $BA$, $BB$. Note that $\mu_{AB} = \mu_{BA}$, and moreover $\mu_{AB} > 0$ if $0 < s'$ and $s'' < 1$;[2] $\mu_{AB} = 0$ if $s' \leq 0$ and/or $s'' \geq 1$.[3] In either case, the firms' profits are given by

$$\pi_A = P_A\mu_{AA} + 2p_A\mu_{AB}; \qquad \pi_B = P_B\mu_{BB} + 2p_B\mu_{AB}.$$

Given a equilibrium $(p_A^*, P_A^*, p_B^*, P_B^*)$ with the corresponding measures, $\mu_{AA}^*, \mu_{AB}^*, \mu_{BB}^*$ for $S_{AA}, S_{AB}, S_{BB}$, we say that it is a *mixed bundling equilibrium* if $\mu_{AB}^* > 0$ and that it is a *pure bundling equilibrium* if $\mu_{AB}^* = 0$. It is almost immediate to see that a pure bundling equilibrium exists for any values of parameters as, for each firm, pure bundling is a best response to pure bundling.[4] The next proposition establishes that no mixed bundling equilibrium exists when the dominance of firm $A$ is sufficiently strong. In fact, this result also holds if firm A faces two specialist opponents B1 and B2, that is in each equilibrium firm A plays a pure bundling strategy, such that each consumer either buys firm A's bundle or products B1 and B2, at least as long as we consider symmetric equilibria such that $p_{A1} = p_{A2}$ and $p_{B1} = p_{B2}$. The reason is that when A faces two specialists such that $p_{B1} = p_{B2}$, A's pricing problem coincides with A's problem when A faces a generalist and $P_B = 2p_B$. Hence he has the same incentive to avoid mixed bundling strategies, as we describe immediately after the proposition.

PROPOSITION 9: *Consider the mixed bundling game with $n = 2$. Then both if firm A faces a generalist opponent or two specialists opponents, we have that*

*(i) there exists no mixed bundling equilibrium if $f(1) > 0$ and $\alpha \geq t + \frac{t}{f(1)}$;*

[1] Precisely, $s'$ is such that a consumer located at $(s_1, s_2) = (s', 1)$ (at $(s_1, s_2) = (1, s')$) is indifferent between the alternatives $BB$ and $AB$ (between the alternatives $BB$ and $BA$). Likewise, $s''$ is such that a consumer located at $(s_1, s_2) = (s'', 0)$ (at $(s_1, s_2) = (0, s'')$) is indifferent between the alternatives $AA$ and $BA$ (between the alternatives $AA$ and $AB$).

[2] The expressions for $\mu_{AA}, \mu_{AB}, \mu_{BB}$ are found in the proof of Proposition 9.

[3] Precisely, if $s' < 0$ then each type of consumer prefers $BB$ to $AB$ (and to $BA$). If $s'' > 1$, then each type of consumer prefers $AA$ to $AB$ (and to $BA$).

[4] Let $P_{2,A}^*, P_{2,B}^*$ be the equilibrium prices from Proposition 2. Under mixed bundling, $(p_A^*, P_{2,A}^*, p_B^*, P_{2,B}^*)$ is an equilibrium if $p_A^*$ and $p_B^*$ are large enough, as for firm $A\,(B)$ it is impossible to induce any type of consumer to choose $AB$ or $BA$ since $P_B = P_{2,B}^*$ and a large $p_B$ imply $s' < 0$ for any $p_A \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$ ($P_A = P_{2,A}^*$ and a large $p_A$ imply $s'' > 1$ for any $p_B \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$).

*(ii) when $f$ is the uniform density, there exists no mixed bundling equilibrium if $\alpha \geq \frac{9}{8}t$.*

Proposition 9(i) relies on proving that if $\alpha$ is sufficiently large and $(p_A, P_A, p_B, P_B)$ are such that $\mu_{AB} > 0$, then $s'' < 1$ and it is profitable for $A$ to reduce $P_A$. A small reduction in $P_A$ reduces $A$'s revenue from inframarginal consumers but attracts some marginal consumers. When $\alpha$ is large, the inequality $s'' < 1$ implies that $P_A$ is large. Hence, it follows that the revenue increase (which is proportional to the initial $P_A$) from the marginal consumers dominates the revenue decrease from inframarginal consumers (which is proportional to the reduction in $P_A$). This explains why it is profitable to reduce $P_A$ until $s''$ reaches the value of 1 to make $\mu_{AB} = 0$.[5]

In the case of the uniform distribution, the lower bound on $\alpha$ from Proposition 9(i) is $t + \frac{t}{f(1)} = 2t$, but Proposition 9(ii) relies on some particular features of the uniform distribution to establish that no mixed bundling equilibrium exists if $\alpha \geq \frac{9}{8}t$.[6] In order to see how this stronger result is obtained, fix $p_B, P_B$ arbitrarily and let $M_A$ denote the set of $(p_A, P_A)$ such that $\mu_{AB} > 0$. Whereas Proposition 9(i) is proved by showing that $\frac{\partial \pi_A}{\partial P_A}$ is negative at each $(p_A, P_A) \in M_A$ if $\alpha \geq t + \frac{t}{f(1)} = 2t$, for the uniform distribution we can show that if $\alpha \in [\frac{9}{8}t, 2t)$, there exists no $(p_A, P_A) \in M_A$ such that $\frac{\partial \pi_A}{\partial P_A} = 0$ and $\frac{\partial \pi_A}{\partial p_A} = 0$ are both satisfied; therefore no mixed bundling strategy is optimal for firm $A$ when $\alpha \in [\frac{9}{8}t, 2t)$.

It is interesting to notice that a well-established result in the literature is that mixed bundling reduces profits with respect to independent pricing, at least for symmetric firms: see Armstrong and Vickers (2010) and references therein.[7] Propositions 3(i) and 9(i), conversely, prove that if one firm's dominance over the other is strong enough, that is if $\alpha \geq t + \frac{t}{f(1)}$ and $\alpha > \bar{\alpha}$, then mixed bundling boils down to pure bundling, and each firm's profit is larger under mixed bundling than under independent pricing.

**Proof of Proposition 9 (i)**

In the case that $0 < s'$ and $s'' < 1$, each of the sets $S_{AA}, S_{AB}, S_{BB}$ has a

[5] Proposition 9(i) is linked to a result in Menicucci, Hurkens and Jeon (2015) (MHJ henceforth) about the optimality of pure bundling for a two-product monopolist. See Daskalakis, Deckelbaum and Tzamos (2017) for a similar resut in a monopoly context. In our duopoly setting, given $(p_B, P_B)$ chosen by firm $B$, the problem of maximizing $A$'s profit with respect to $(p_A, P_A)$ is equivalent to the problem of maximizing the profit of a two-product monopolist facing a consumer with suitably distributed valuations and such that the consumer enjoys a synergy of $2p_B - P_B \geq 0$ if she consumes both objects. Since MHJ do not allow for synergies, strictly speaking Proposition 9(i) is not a corollary of the results in MHJ.

[6] Numeric analysis suggests that (i) no mixed bundling NE exists as long as $\alpha \geq 0.72t$; (ii) when a mixed bundling NE exists, the firms' equilibrium profits are lower than under independent pricing.

[7] Armstrong and Vickers (2010) explain this result by referring to firms' incentives to compete fiercely for the consumers which choose to buy both products from the same firm. This is closely related to the strong demand elasticity effect we find when $\alpha = 0$, that is when the firms are symmetric.

positive measure as follows:

$$\mu_{AA} = F(s')F(s'') + \int_{s'}^{s''} F(s'+s''-s_1)f(s_1)ds_1 \quad (1a)$$
$$\mu_{AB} = F(s')[1-F(s'')]; \quad (1b)$$
$$\mu_{BB} = [1-F(s')][1-F(s'')] + \int_{s'}^{s''} [1-F(s'+s''-s_1)]f(s_1)ds_1. \quad (1c)$$

Therefore, given $\pi_A = P_A\mu_{AA} + 2p_A\mu_{AB}$, we find

$$\begin{aligned}\frac{\partial \pi}{\partial P_A} &= \mu_{AA} + P_A[2F(s')f(s'') + \int_{s'}^{s''} F(s'+s''-s_1)f(s_1)ds_1](-\frac{1}{2t}) - 2p_A F(s')f(s'')(-\frac{1}{2t}) \\ &= F(s')f(s'')\left[\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t}\right] + \int_{s'}^{s''} f(s_1)f(s'+s''-s_1)\left[\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)} - \frac{P_A}{2t}\right]ds_1\end{aligned}$$

and we prove that $\frac{\partial\pi}{\partial P_A} < 0$, given $s'' < 1$

- First, we prove that $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < 0$. Since $f$ is log-concave, it follows that $\frac{F}{f}$ is increasing and $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t}$ is decreasing in $P_A$. Since the inequality $s'' < 1$ is equivalent to $p_A + p_B - t + \alpha < P_A$, it follows that $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < \frac{1}{f(1)} + \frac{t - p_B - \alpha}{t}$, and the latter expression is negative given $\alpha \geq t + \frac{t}{f(1)}$.

- Now we prove that $\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)} - \frac{P_A}{2t} < 0$ for each $s_1 \in [s', s'']$. Since $f$ is log-concave, it follows that $\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)}$ is decreasing in $s_1$, and at $s_1 = s'$ we obtain the value $\frac{F(s'')}{f(s'')} - \frac{P_A}{2t}$, which is negative since it is smaller than $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < 0$, given $2p_A \geq P_A$.

**Proof of Proposition 9(ii)**

Given $b_1 \equiv P_B - p_B + \alpha$, $b_2 \equiv p_B + \alpha \geq b_1$, we say that firm $A$ plays a *pure bundling strategy* if and only if $p_A \geq b_1 + t$ and/or $P_A \leq b_2 - t + p_A$ because $\mu_{AB} = 0$ in either of these cases.[8] Given $b_1, b_2$, we define $M_A$ as the set of $(p_A, P_A)$ such that $\mu_{AB} > 0$, that is

$$M_A = \{(p_A, P_A) : p_A < b_1 + t, \quad b_2 - t + p_A < P_A \leq 2p_A\}.$$

We say that $A$ plays a *mixed bundling strategy* if $(p_A, P_A) \in M_A$. Notice that $M_A$ is non-empty if and only if $b_1 > -t$ and $b_2 < 2t + b_1$: see Figure 3 of this online appendix.

Using (1), for each $(p_A, P_A) \in M_A$ we have

$$\pi_A = \frac{1}{8t^2}\left(\begin{array}{c} P_A^3 + 4p_A^3 - 2(b_1+b_2+2t)P_A^2 - 6p_A^2P_A - 4(b_1-b_2+2t)p_A^2 + 8(b_1+t)P_Ap_A \\ +(2t^2+4tb_2+b_2^2+2b_1b_2-b_1^2)P_A - 4(b_2-t)(t+b_1)p_A \end{array}\right)$$

[8]Precisely, $s' \leq 0$ if and only if $p_A \geq b_1 + t$; $s'' \geq 1$ if and only if $P_A \leq b_2 - t + p_A$.

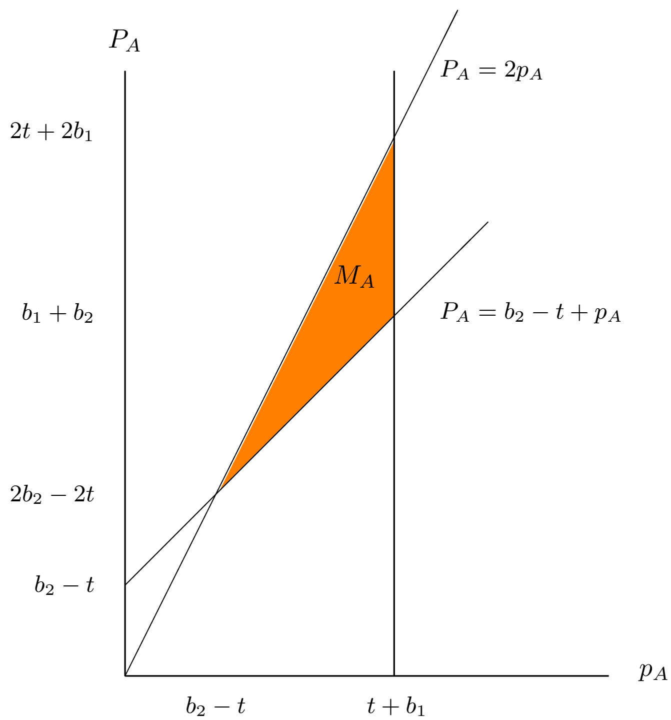


Figure 3: Mixed bundling strategies for firm A.

and

$$\frac{\partial \pi_A}{\partial p_A} = \frac{1}{8t^2}\left(12p_A^2 - 4\left(3P_A + 4t - 2b_2 + 2b_1\right)p_A + 8\left(b_1 + t\right)P_A - 4(b_2 - t)(t + b_1)\right)$$

$$\frac{\partial \pi_A}{\partial P_A} = \frac{1}{8t^2}\left(3P_A^2 - 4\left(2t + b_1 + b_2\right)P_A - 6p_A^2 + 8\left(b_1 + t\right)p_A + 2t^2 + 4tb_2 + b_2^2 + 2b_2b_1 - b_1^2\right).$$

Since $\alpha \geq \frac{9}{8}t$ implies $b_1 > \frac{9}{8}t$, we consider the following set $\mathcal{B}$ of possible values for $(b_1, b_2)$: $\mathcal{B} = \{(b_1, b_2) : \frac{9}{8}t < b_1 \leq b_2 < 2t + b_1\}$. We prove that for each $(b_1, b_2) \in \mathcal{B}$ it is never a best reply for firm A to play $(p_A, P_A)$ in $M_A$, that is the best reply of firm A is a pure bundling strategy. The proof is organized in three steps. In Step 1 we prove that for firm A playing independent pricing (that is, $P_A = 2p_A$) in $M_A$ is suboptimal. A mixed bundling strategy for firm A can thus be optimal only if it lies in the interior of $M_A$, which implies that the first (and second) order conditions must be satisfied. However, in Step 2 we show that if $(p_A, P_A) \in M_A$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$, then $P_A$ must be larger than a suitable $\bar{P}_A$, while in Step 3 we show that $\frac{\partial \pi_A}{\partial P_A} = 0$ implies that $P_A$ must be smaller than $\bar{P}_A$. Hence, it must be optimal for firm A to play a pure bundling strategy whenever $b_2 \geq \frac{9}{8}t$.

**Step 1** Suppose that $(b_1, b_2) \in \mathcal{B}$. Playing $(p_A, P_A) \in M_A$ such that $P_A = 2p_A$ is not a best reply for firm A because either $\frac{\partial \pi_A}{\partial p_A} > 0$ and/or $\frac{\partial \pi_A}{\partial P_A} < 0$.
We start by evaluating $\frac{\partial \pi_A}{\partial p_A}$ and $\frac{\partial \pi_A}{\partial P_A}$ at $P_A = 2p_A$ and we find

$$\begin{aligned}
\frac{\partial \pi_A}{\partial p_A} &= \frac{1}{t^2}\left(-\frac{3}{2}p_A^2 + (b_2+b_1)p_A - \frac{1}{2}\left(b_2 - t\right)\left(t + b_1\right)\right) \equiv z(p_A),\\
\frac{\partial \pi_A}{\partial P_A} &= \frac{1}{t^2}\left(\frac{3}{4}p_A^2 - (t+b_2)p_A + \frac{1}{8}\left(2b_2b_1 + b_2^2 + 4tb_2 + 2t^2 - b_1^2\right)\right) \equiv Z(p_A).
\end{aligned}$$

Notice that if $(p_A, P_A) \in M_A$, then $p_A \in (b_2 - t, b_1 + t)$. Let $p_A^*$ denote the larger solution to $z(p_A) = 0$, that is $p_A^* = \frac{1}{3}(b_1 + b_2 + \sqrt{(b_2-t)^2 + (b_1+t)(2t + b_1 - b_2)})$, and $b_2 - t < p_A^* < b_1 + t$ since $z(b_2 - t) = \frac{1}{2t^2}(b_2 - t)(b_1 - b_2 + 2t) > 0$ and $z(b_1 + t) = -\frac{1}{2t^2}(b_1 + t)(b_1 - b_2 + 2t) < 0$ in $\mathcal{B}$. In fact, from $z(b_2 - t) > 0 = z(p_A^*)$ we infer that $z(p_A) > 0$ for $p_A \in (b_2 - t, p_A^*)$. This implies that $(p_A, P_A)$ such that $P_A = 2p_A$ and $p_A \in (b_2 - t, p_A^*)$ is not a best reply for A since it is profitable to increase $p_A$.
For $p_A \in [p_A^*, b_1 + t)$ we prove that $Z(p_A) < 0$. This implies that $(p_A, P_A)$ such that $P_A = 2p_A$ and $p_A \in [p_A^*, b_1 + t)$ is not a best reply for A since it is profitable to reduce $P_A$. We find $Z(b_1 + t) = -\frac{1}{8t^2}(b_2 - b_1)(2t + b_1 - b_2 + 2t + 4b_1) \leq 0$ in $\mathcal{B}$ and

$$Z(p_A^*) = -\frac{(2t + b_2 - b_1)\left(b_2 + b_1 + 4\sqrt{(b_2-t)^2 + (b_1+t)(2t + b_1 - b_2)}\right) - 12t^2}{24t^2}$$

which now we prove to be negative in $\mathcal{B}$. Precisely, we define $\xi_1(b_1, b_2) \equiv (2t + b_2 - b_1)(b_2 + b_1 + 4\sqrt{(b_2-t)^2 + (b_1+t)(2t + b_1 - b_2)})$ and show that

$$\xi_1(b_1, b_2) > 12t^2 \qquad \text{for} \quad \text{any} \quad (b_1, b_2) \in \mathcal{B}. \tag{2}$$

To this purpose we prove below that $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$, and $\xi_1(b_1, b_1) = 4t(b_1 + 2\sqrt{b_1^2 + 3t^2}) > 12t^2$ for any $b_1 > t$ implies (2). Precisely, $\frac{\partial \xi_1}{\partial b_2} = 2b_2 + 2t + \frac{6b_1^2 + 8b_2^2 - 10b_2b_1 + 14b_1t - 10tb_2}{\sqrt{(b_2-t)^2 + (b_1+t)(2t + b_1 - b_2)}}$ and $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$ since $\xi_2(b_1, b_2) \equiv 6b_1^2 + 8b_2^2 - 10b_2b_1 + 14b_1t - 10tb_2 > 0$ in $\mathcal{B}$.[9] ■

**Step 2** Suppose that $(b_1, b_2) \in \mathcal{B}$. If $(p_A, P_A) \in M_A$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$, then $P_A \geq \bar{P}_A$, for a suitable $\bar{P}_A$.
For the equation $\frac{\partial \pi_A}{\partial p_A} = 0$ in the unknown $p_A$, there exists at least a real solution if and only if $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1+t)(b_2-t)})$ or $P_A \geq \frac{2}{3}(b_1 + b_2 + \sqrt{(b_1+t)(b_2-t)}) \equiv \bar{P}_A$. We now prove that if $(p_A, P_A)$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$ and $P_A \leq \frac{2}{3}(b_1 + b_2 - \sqrt{(b_1+t)(b_2-t)})$, then $(p_A, P_A) \notin M_A$; therefore $\frac{\partial \pi_A}{\partial p_A} = 0$ implies $P_A \geq \bar{P}_A$.

[9] Minimizing $\xi_2$ over the closure of $\mathcal{B}$ yields the minimum point $b_1 = t$, $b_2 = \frac{5}{4}t$, with $\xi_2(t, \frac{5}{4}t) = \frac{15}{2}t^2 > 0$.

First notice that $\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$ is smaller than $b_1+b_2$ and in fact it is sometimes smaller than $2b_2-2t$ for some $(b_1,b_2)\in\mathcal{B}$. If $\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})>2b_2-2t$, then the line $P_A=\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$ has a non-empty intersection with $M_A$, and we find that (i) at $p_A=P_A-b_2+t$ (i.e., along the south-east boundary of $M_A$) $\frac{\partial\pi_A}{\partial p_A}=\frac{1}{2}(b_2-t)(b_1+b_2-P_A)$, which is positive given $P_A\leq\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$; (ii) $\frac{\partial\pi_A}{\partial p_A}$ is decreasing with respect to $p_A$ for $p_A\leq\frac{1}{2}P_A+\frac{1}{3}(b_1-b_2)+\frac{2}{3}t$, and $P_A-b_2+t<\frac{1}{2}P_A+\frac{1}{3}(b_1-b_2)+\frac{2}{3}t$ given $P_A\leq\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$. Therefore $\frac{\partial\pi_A}{\partial p_A}>0$ for each $(p_A,P_A)\in M_A$ such that $P_A\leq\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$, and in fact for each $(p_A,P_A)\in M_A$ such that $P_A<\bar{P}_A$. ■

**Step 3** Suppose that $(b_1,b_2)\in\mathcal{B}$ and that $b_2\geq\frac{9}{8}t$. If $(p_A,P_A)\in M_A$ is a best reply for firm A, then $P_A<\bar{P}_A$.
The equation $\frac{\partial\pi_A}{\partial P_A}=0$ is quadratic and convex in $P_A$. In order to satisfy the second order condition, the best reply for firm A must be such that $P_A$ is equal to the smaller solution of $\frac{\partial\pi_A}{\partial P_A}=0$. We now show that $\frac{\partial\pi_A}{\partial P_A}<0$ at $P_A=\bar{P}_A$, which implies that the smaller solution to $\frac{\partial\pi_A}{\partial P_A}=0$ is smaller than $\bar{P}_A$. We find

$$\frac{\partial\pi_A}{\partial P_A}=-\frac{3}{4t^2}p_A^2+\frac{b_1+t}{t^2}p_A+\frac{2b_2b_1-7b_1^2-b_2^2-20tb_1+2t^2-16t\sqrt{(b_2-t)(b_1+t)}}{24t^2}\equiv W(p_A)$$

and notice that $\bar{P}_A<b_1+b_2$; therefore $W$ is defined for $p_A\in(\frac{1}{2}\bar{P}_A,\bar{P}_A-b_2+t)$. We prove that $W(p_A)<0$ for each $p_A\in(\frac{1}{2}\bar{P}_A,\bar{P}_A-b_2+t)$, and to this purpose we notice that $W$ is maximized with respect to $p_A$ at

$$p_A=\begin{cases}\frac{2}{3}t+\frac{2}{3}b_1 & \text{if } b_2\leq\frac{3-\sqrt{5}}{2}b_1+\frac{5-\sqrt{5}}{2}t\\ \frac{1}{2}\bar{P}_A & \text{if } b_2>\frac{3-\sqrt{5}}{2}b_1+\frac{5-\sqrt{5}}{2}t\end{cases}$$

- If $b_2\leq\frac{3-\sqrt{5}}{2}b_1+\frac{5-\sqrt{5}}{2}t$, then $b_1\leq\sqrt{5}t$ in order to satisfy $b_1\leq b_2$, and $W(\frac{2}{3}t+\frac{2}{3}b_1)=\frac{1}{12t^2}(5t^2-2b_1t-\frac{1}{2}b_2^2+b_2b_1+\frac{1}{2}b_1^2-8t\sqrt{(b_1+t)(b_2-t)})\equiv\xi_3(b_1,b_2)$, which is decreasing in $b_2$ and $\xi_3(b_1,b_1)=\frac{1}{12t^2}(5t^2-2tb_1+b_1^2-8t\sqrt{b_1^2-t^2})$ is negative for $b_1\in[\frac{9}{8}t,\sqrt{5}t]$.

- If $b_2>\frac{3-\sqrt{5}}{2}b_1+\frac{5-\sqrt{5}}{2}t$, then we evaluate $W(\frac{1}{2}\bar{P}_A)=\frac{1}{24t^2}(4t^2-10tb_1+6tb_2-b_1^2-3b_2^2+4b_1b_2-4(2t-b_1+b_2)\sqrt{(b_1+t)(b_2-t)})$, and we prove it is negative. Precisely, we show that

  $$\xi_4(b_1,b_2)\equiv4(2t-b_1+b_2)\sqrt{(b_2-t)(b_1+t)}-4t^2+10tb_1-6tb_2+b_1^2+3b_2^2-4b_1b_2$$

  is positive, and from $b_1+t>b_2-t$ we obtain $\xi_4(b_1,b_2)>4(2t-b_1+b_2)(b_2-t)-4t^2+10tb_1-6tb_2+b_1^2+3b_2^2-4b_1b_2=b_1^2+7b_2^2-8b_1b_2-12t^2+14tb_1-2tb_2\equiv\xi_5(b_1,b_2)$. It is immediate that $\xi_5$ is increasing with respect to $b_2$, and $\xi_5(b_1,\frac{3-\sqrt{5}}{2}b_1+\frac{5-\sqrt{5}}{2}t)=-\frac{1}{2}(13\sqrt{5}-27)b_1^2+(61-23\sqrt{5})tb_1-\frac{1}{2}(33\sqrt{5}-71)t^2>0$ for $b_1\in(\frac{9}{8}t,\sqrt{5}t)$; $\xi_5(b_1,b_1)=12t(b_1-t)>0$. ■