Online Appendix
Social Clubs and Social Networks

Chaim Fershtman and Dotan Persitz*

1 Introduction

This online appendix includes some extensions and discussions that were omitted from the main text for the sake of conciseness and readability. We assume that the reader of this appendix is well acquainted with the main text and therefore we do not repeat here the definitions and notations presented there. Section 2 shows formally that the Club Congestion model includes the seminal Connections model (Jackson and Wolinsky (1996)) as a special case where the size of the clubs is exogeneously restricted to be equal to two. Section 3 discusses further the notion of the DCV in the club congestion model. Specifically, we characterize the DCV of the exponential club congestion functions. We demonstrate its usefulness by analyzing the stability of the Empty environment for a wide family of club congestion functions. Section 4 discusses the existence of OCS m-Complete environments in the Club Congestion model. Section 5 attempts to explore further, using analytic and numerical methods, the condition for the existence of OCS m-Star environments in the club congestion model. Section 6 provides a numeric demonstration of the stability-efficiency Gap in the club congestion model. This online appendix also includes two appendices. Appendix A includes the proofs for the propositions, claims and lemmata stated in the main text while Appendix B includes the proofs for the results stated in the online appendix itself.

2 The Club Congestion Model and the Connections Model

Jackson and Wolinsky (1996) introduce the connections model in which the utility of Individual i in the unweighted network g is $u_i^{JW}(g) = \sum_{j \neq i} \delta^{d_{ij}} - n_i(g) \times c$ where d_{ij} is the geodesic distance between individuals i and j, $\delta \in (0,1)$ the depreciation factor, $c > 0$ is the direct connection cost

*Coller School of Management, Tel Aviv University, Tel Aviv, 69978, ISRAEL.

and $n_i(g)$ is the number of Individual i's direct neighbors. Network g is pairwise stable if no single individual gains by severing any of her links and no pair of unlinked individuals wishes to establish a link between themselves.

Denote by $PS(\delta, c, n)$ the set of pairwise stable networks in the connections model and denote by $OCS(h, c, n)$ the set of OCS environments in the club congestion model (with congestion function h). For every unweighted network $g =< N, E >$ the corresponding environment $G_g =< N, S, A >$ is such that for each link $\{i, j\} \in E$ there exists a club $s_{ij} \in S$ that includes only individuals i and j, and there are no other populated clubs ($S = \cup_{\{i,j\} \in E}\{s_{ij}\}$; $A = \cup_{\{i,j\} \in E}\big\{\{i, s_{ij}\}, \{j, s_{ij}\}\big\}$). Denote the set of all unweighted networks with n individuals by $\mathbb{G}_n$ and the set of all corresponding environments by $\mathscr{G}_{\mathbb{G}_n} \subseteq \mathscr{G}_n$.

Proposition 1. *The Connections model is a special case of the Club Congestion model. Specifically, let the congestion function be $h(2) = \delta$ and $\forall m > 2 : h(m) = 0$.*

(i) $g \in PS(\delta, c, n)$ *if and only if* $G_g \in OCS(h, c, n)$.

(ii) If $G \in \mathscr{G}_n \backslash \mathscr{G}_{\mathbb{G}_n}$ *then* $G \notin OCS(h, c, n)$.

The concept of pairwise stability is closely related to OCS. Both solution concepts imply that leaving a club of size two destroys the club and the formation of a new club of size two is an acceptable deviation. However, OCS also allows for the formation of bigger clubs, for leaving bigger clubs without destroying them and for deviations in which an individual can join an existing club. Naturally, when the discussion is limited to clubs of size two, the two concepts coincide. Letting $h(2) = \delta$ and $h(m) = 0$ for every $m > 2$ implies that there is no OCS environment with clubs of size larger than 2.

Obviously, similar reasoning works also for the efficiency analysis. Hence, it is easy to see that the case of $m = 2$ of Proposition 2 in the main text, using the congestion function $h(2) = \delta$ and $\forall m > 2 : h(m) = 0$, yields the efficiency result of the Connections model (Proposition 1 in Jackson and Wolinsky (1996)).

3 DCV and Elasticity

3.1 The DCV of the Exponential Congestion Function

Lemma 1 summarizes the club size that maximizes the DCV for various sets of parameters of the exponential congestion function. Some technical notations are required: Denote $b(\delta, n_a) = \frac{1}{n_a - 2}(\delta - (n_a - 1)\delta^{n_a - 1})$ and let $\delta^\star(n_a)$ be the unique $\delta \in (0, 1)$ such that $b(\delta, n_a) = \delta(1 - 2\delta)$ and let $\hat{\delta}(n_a)$ be the unique $\delta \in (0, 1)$ such that $b(\delta, n_a) = 0$.

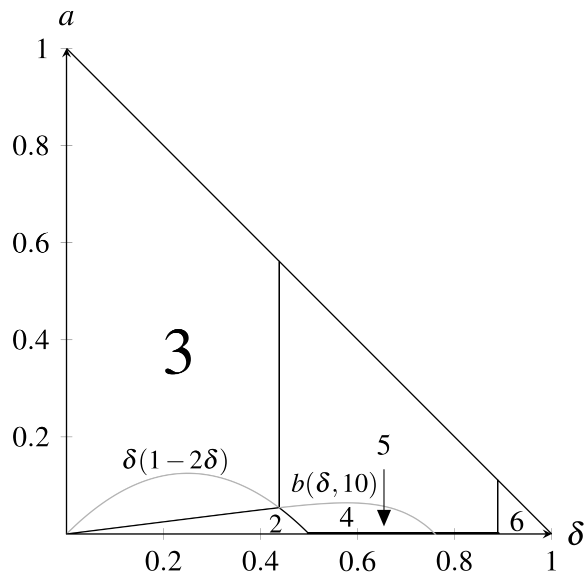

Figure 1: Lemma 1 for $n_a = 10$. Each number lies within the area characterized by the corresponding statement in the lemma.

Lemma 1. *Let $n_a \geq 4$ and let $h(m)$ be an exponential club congestion function.*

1. *The club size that maximizes the DCV weakly increases with a.*

2. *If $a \in [0, \min\{b(\delta, n_a), \delta(1-2\delta)\})$ then the DCV is maximized at $m = 2$.*

3. *If $\delta \in (0, \delta^\star(n_a))$ and $a \in (b(\delta, n_a), 1-\delta)$ then the DCV is maximized at $m = n_a$.*

4. *If $\delta \in (\delta^\star(n_a), \hat{\delta}(n_a))$ and $a \in (\max\{0, \delta(1-2\delta)\}, b(\delta, n_a))$ then the DCV is maximized at $m \in \{3, \ldots, n_a - 1\}$.*

5. *If $\delta \in [\frac{1}{2}, 1 - \frac{1}{n_a - 1}]$ and $a = 0$ then the DCV is maximized either at $m = \lfloor 1 - \frac{1}{\ln \delta} \rfloor$ or at $m = \lceil 1 - \frac{1}{\ln \delta} \rceil$.*

6. *If $\delta \in (1 - \frac{1}{n_a - 1}, 1)$ then the DCV is maximized at $m = n_a$.*

The case where $a = 0$ demonstrates the opposing effects of the club size on the DCV. When the number of individuals in the club increases, congestion increases (for every δ) but more direct links are formed in the club (the multiplicative effect). We use Lemma 2 in the main text to show that when $\delta < \frac{1}{2}$ the congestion effect is dominant and the DCV is maximized when the club is small (Part 2). When δ increases the effect of congestion weakens and increasingly larger clubs maximize the DCV (Parts 5 and 6).

When the non-congested component of the exponential congestion function is introduced it reinforces the multiplicative effect since the aggregate benefit from $a > 0$ increases with the size of the club. Therefore, the club size that maximizes the DCV weakly increases with a (Part 1). Part 2 shows that for relatively low values of δ and a, the congestion component is still dominant and the DCV is maximized by the smallest club. But, when the non-congestion component increases (and δ is still low) the DCV is maximized by the biggest club (Part 3).[1] Part 3 also makes use of the assertion in Part 1 to state that if the DCV is maximized by the biggest club for some a then it is maximized by the biggest club for any greater a (Part 6 uses the same assertion). If a is high enough (for $\delta > \frac{1}{2}$ its any value of a), a club of size two never maximizes the DCV since the multiplicative effect dominates the congestion component. Parts 4 and 5 show that for these values of δ, intermediate size clubs can maximize the DCV. Figure 1 demonstrates Lemma 1 for the case of $n_a = 10$.

3.2 The Stability of the Empty Environment

Proposition 2. *Let E_n be the Empty environment with $n_a \geq 4$ individuals.*

1. *Suppose $h(m)$ is the reciprocal congestion function. E_n is OCS if and only if $c \geq 1$.*

2. *Suppose $h(m)$ is an exponential congestion function.*

   (a) *Suppose $a \in [0, \min\{b(\delta, n_a), \delta(1-2\delta)\})$. E_n is OCS if and only if $c \geq a + \delta$.*

   (b) *Suppose that one of the following conditions hold:*

      i. *$\delta \in (0, \delta^\star(n_a))$ and $a \in (b(\delta, n_a), 1-\delta)$.*

      ii. *$\delta \in (1 - \frac{1}{n_a - 1}, 1)$.*

      E_n is OCS if and only if $c \geq (n_a - 1)(a + \delta^{n_a - 1})$.

   (c) *Suppose that $\delta \in [\frac{1}{2}, 1 - \frac{1}{n_a - 1}]$ and $a = 0$.*
   E_n is OCS if and only if $c \geq \max\{k_h(\lfloor 1 - \frac{1}{\ln \delta} \rfloor), k_h(\lceil 1 - \frac{1}{\ln \delta} \rceil)\}$.

The second part of Proposition 2 is a direct application of Lemma 1. Under the exponential congestion function, each club size provides its members with different payoffs. The minimal membership fee for which the Empty environment is OCS is determined by the most attractive deviation. In the case analyzed in Proposition 2.2a, the congestion component is dominant and therefore the most attractive deviation is to the smallest club. But, for the same δ, when the non-congestion component is high enough, the grand club becomes the most attractive deviation (Proposition 2.2(b)i).

One important implication of this discontinuity is on dynamic models where individuals join

[1] In the proof we use Lemma B.4 that shows that $k_h(m)$ has three parts - increasing, decreasing and increasing again. Therefore, to determine the club size that globally maximizes the DCV, the closest integer to the local maxima that separates the first two parts should be compared to the right-hand side limit, n_a.

the environment sequentially. Consider a dynamic model where the initial environment is the Empty environment, the clubs' rules follow the OCS rules and the membership fee is marginally high. Then, a tiny difference in the parameters of the congestion function or in the population size may lead to huge differences in the final environment's club composition.

4 Existence of OCS m-Complete Environments

The existence of a stable m-Complete environment is not guaranteed. It is possible that the lower bound may be higher than the upper bound. Claim 1 uses the exponential club congestion function with strong congestion (low δ) to demonstrate that even then m-Complete environments with large clubs may be OCS.

Claim 1. *Let $h(\cdot)$ be an exponential club congestion function where $\delta \in (0, \frac{1}{2})$ and $a > 0$. There exist two integers $\bar{m} \leq \tilde{m}$ such that, $\forall m : n_a > m > \bar{m}$ there exists a range of membership fees in which an m-complete environment is OCS. Moreover, there exists a range of membership fees in which every m-complete environment where $n > m > \tilde{m}$ is OCS.*

When congestion is strong, but there exists a non-congested part to the club congestion function, if the clubs are large enough ($m > \bar{m}$) the existence of a membership fee for which an m-Complete environment is OCS is guaranteed. The second part of Claim 1 shows that there even exists a range of membership fees for which multiple m-Complete environments are OCS (all those with $m > \tilde{m}$). This result implies non-monotonicity in the relationship between congestion and the size of clubs in stable environments: m-complete environments with intermediate size clubs are unstable while m-complete environments with either small clubs (wherein each individual maintains many high quality affiliations) or large clubs (wherein each individual maintains few low quality affiliations) are open clubwise stable.[2]

5 Existence of OCS m-Star Environments

5.1 One Analytic Result

Proposition 4 in the main text characterizes the membership fees for which an m-Star environment is OCS. However, it does not provide a condition for the existence of such membership fees since it does not guarantee that the upper bound is indeed greater than the lower bound. Claim 2 identifies one case in which existence is guaranteed.

[2]Consider the case where $h(m) = \frac{1}{32} + (\frac{1}{4})^{m-1}$. The All Paired environment is OCS in $[0, \frac{3}{16}]$, for $m \in \{3, \ldots, 9\}$ the m-complete is never OCS and for $m \geq 10$ every m-complete is OCS in [0.25,0.27].

Claim 2. *Let $n_a > m \geq 2$ and let $h(\cdot)$ be a club congestion function. Denote $\gamma \equiv \frac{n_a-1}{m-1}$ and $l_h = \min\{k \in \mathbb{Z} | h(k) \leq h^2(m)\}$. Suppose $\gamma < m$. If*

$$J_h(m) \geq \max\{\max_{\gamma \geq k \geq 2} FNS_h(k,m), \max_{m \geq k > \gamma} FNI_h(k,m,n_a), \max_{\min\{l_h,n_a\} > k > m} FNL_h(k,m,n_a)\}$$

then a range of membership fees for which an m-Star environment is OCS exists.

Claim 2 provides a sufficient condition for the existence of membership fees for which an m-Star environment is OCS in cases where the number of populated clubs is smaller than the size of the clubs (e.g. a 3-Star environment with 5 individuals). This condition is not vacant. Consider, for example, the case where $n_a = 13$ and $h(m) = 0.73 + 0.21^{m-1}$. We are guaranteed that a range of membership fees for which any 7-Star environment is OCS exists since it can be shown that for a peripheral individual joining the other club is more attractive than any deviation to form a new club.

5.2 Numerical Analysis

As noted above, Proposition 4 in the main text does not provide a condition for the existence of membership fees for which a given m-Star environment is OCS. Figure 2 demonstrates the application of Proposition 4 in the main text to the question of existence of such membership fees in the case of 13 individuals and an exponential club congestion function. In each of the six sub-figures, the shaded area presents the pairs of a (horizontal axis) and δ (vertical axis) for which the corresponding m-Star environment is OCS for some membership fees (since $a + \delta < 1$ only the lower left triangle is relevant). In addition, in each sub-figure we indicate, in terms of the size of the new club, the deviation that determines the envelop of the area where no membership fee exists for which the corresponding m-Star environment is OCS (restrictive intervals of the deviation are depicted as continuous while non-restrictive intervals are dotted).[3]

The upper leftmost sub-figure (the 2-Star environment) summarizes Claim 2(iv) in the main text. Claim 3(ii) in the main text could be recognized by the intersection of the shaded area with the Y-axis ($a = 0$) in the upper middle sub-figure (the 3-Star environment) and Claim 1(iii) in the main text could be recognized by the intersection of the non-shaded area with the Y-axis in the lower rightmost sub-figure (the Grand Club environment).

The main insight provided by Figure 2 is that holding δ constant, the effect of a on stability

[3] For each sub-figure (excluding the one for the Grand Club environment) we first calculated for each $k \in \{2,3,...,\min\{l_h,13\}\}$ and for 1000 values of $\delta \in (0,1)$ the set of as such that the upper bound is greater than the corresponding lower bound expression (using FNS_h, FNI_h or FNL_h). Claim 2 guarantees that the calculation of $J_h(m)$ is unnecessary. Next, we calculated the intersection of all the sets derived in the first stage and presented it by the shaded area. The curves were derived similarly to the first stage procedure, except that the upper bound was set to be equal to the lower bound expression. For the Grand Club environment we repeated the same procedure using the lower bound specified in Claim 1(i) in the main text.

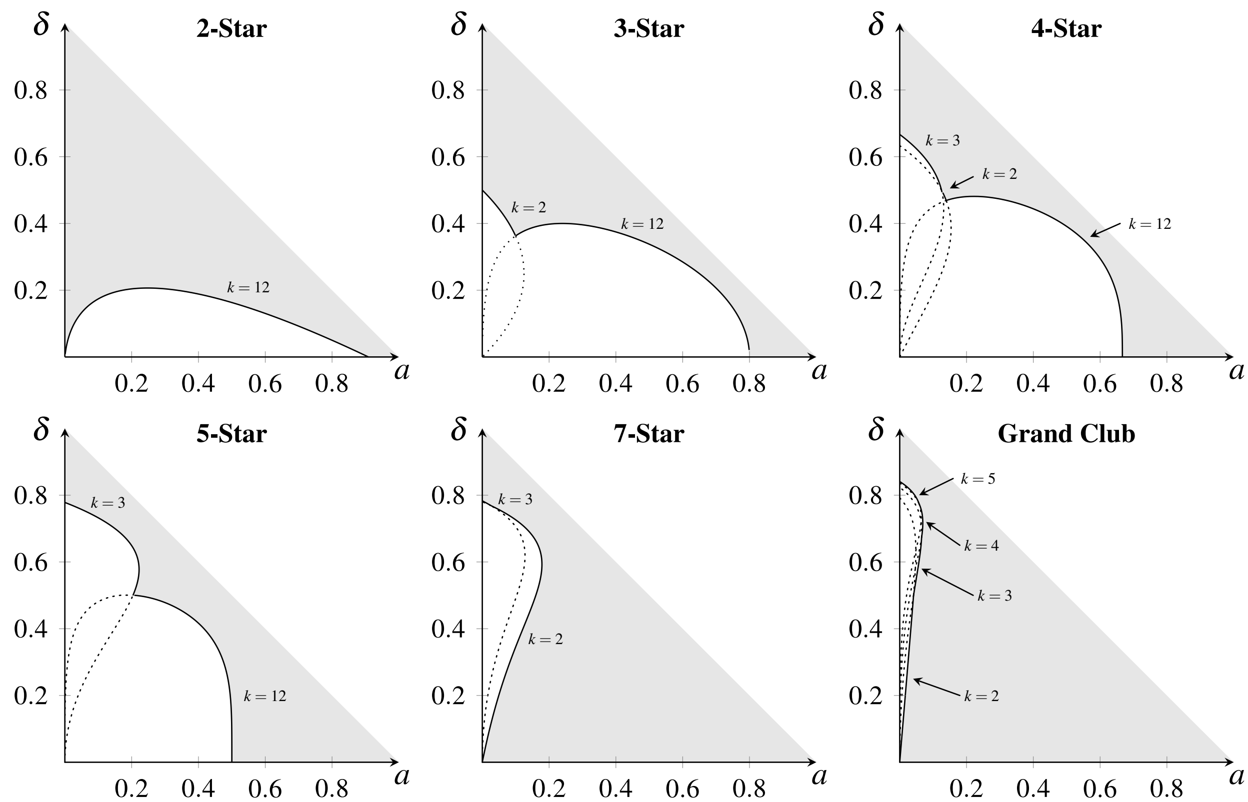

Figure 2: The existence of membership fees for which m-star environments are OCS when the club congestion function is exponential and $n_a = 13$. In each sub-figure, the shaded area presents the pairs of a (horizontal axis) and δ (vertical axis) for which the corresponding m-Star environment is OCS for some membership fees. The code and calculations can be found on GitHub: https://github.com/omri1348/Social-Clubs-and-Social-Networks/tree/master/code.

is non-monotonic. This reflects the complicated lower bound conditions in Proposition 4 in the main text, where the non-differentiable points denote changes in the effective lower bound. The cases of the 3-Star, 4-Star and 5-Star environments demonstrate the intuition very nicely. When the non-congested parameter is low, the effective bound is induced by a deviation of a small coalition since the effect of congestion is dominant and therefore should be minimized. However, when the non-congested parameter is high, the effective bound is a deviation of a large coalition, since congestion is relatively less important than the multiplicative effect introduced by a. Since the multiplicative effect strengthens with the size of the club, the most attractive deviation is to a club that includes all peripheral individuals. Note that the first consideration is missing from the sub-figure of the 2-Star environment since individuals in this environment suffer no congestion. Similarly, the second consideration is missing from the sub-figure of the Grand Club environment since the multiplicative effect is maximized (the reasoning is similar for the 7-Star environment).

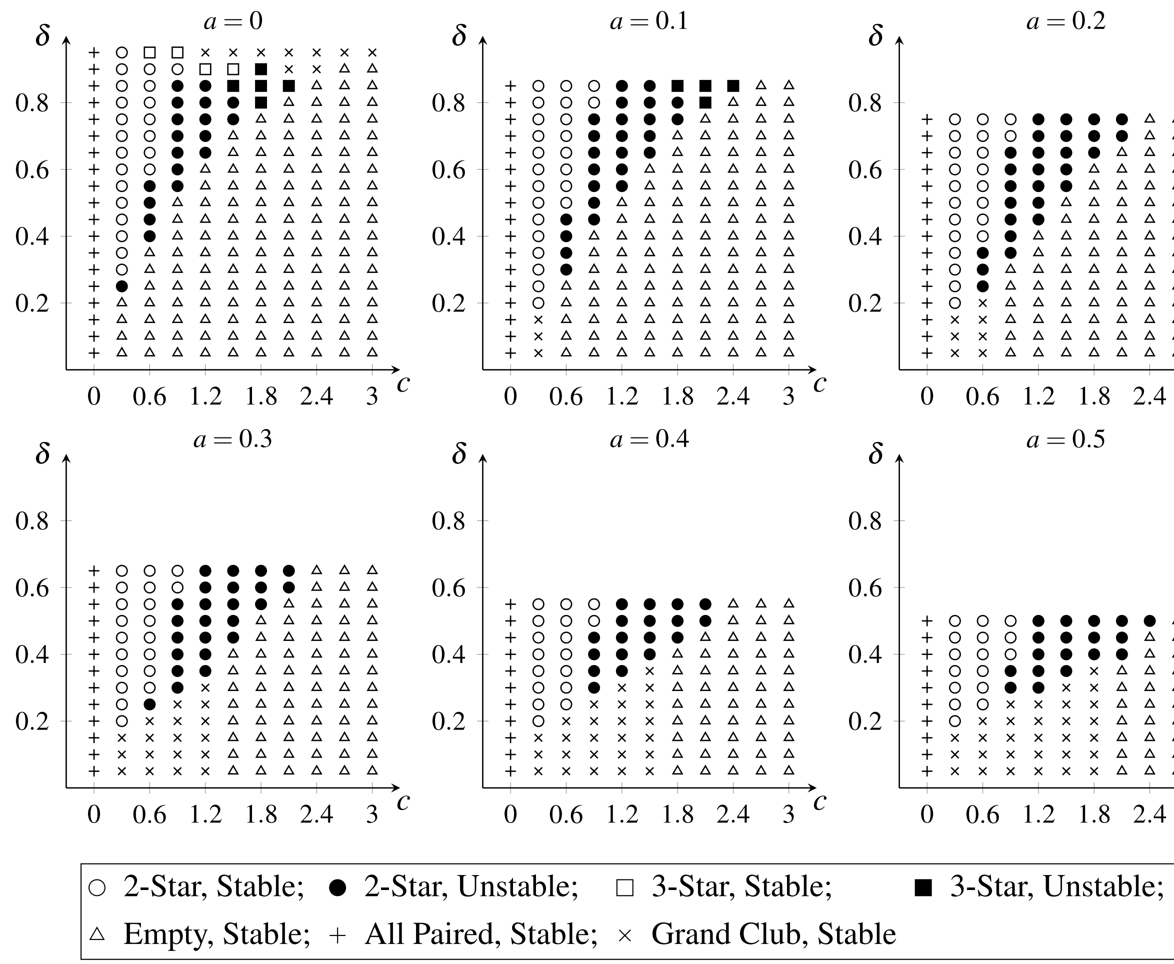

Figure 3: Strong efficiency analysis for 5 individuals and the exponential club congestion function. The code and calculations can be found on GitHub: https://github.com/omri1348/Social-Clubs-and-Social-Networks/tree/master/code.

6 Demonstration of The Stability-Efficiency Gap

We used our Matlab code package (see Footnote 13 in the main text) to calculate the strongly efficient 5-individuals environment for various exponential congestion functions and membership fees.[4] Each shape in the graphs in Figure 3 represents the type of the strongly efficient environment and whether it is OCS. First, note that all the strongly efficient environments are either m-Complete (All Paired or Grand Club), m-Star (2-Star or 3-Star) or Empty. Second, the unstable strongly efficient environments are all m-Stars (2-Star or 3-Star).

[4] $a \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\delta \in \{0, 0.05, 0.1, \ldots, 0.95 - a\}$ for the exponential congestion function and $c \in \{0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3\}$ for the membership fee.

References

Berge, Claude (1989) *Hypergraphs: Combinatorics of Finite Sets*, Vol. 45 of North-Holland Mathematical Library: Elsevier Science Publishers B. V.

Jackson, Matthew O and Asher Wolinsky (1996) "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory*, Vol. 71, No. 1, pp. 44–74.

Appendix

A Proofs of Results from the Main Text

Proposition 1

Proof. Suppose $n_a - 1 > c > 0$. First, consider the case where G is a Minimally Connected environment of class $K(G) \geq c$. Since G is connected no individual can benefit by joining a club or by forming a new club. Also, by leaving a club, every individual loses connection to at least $K(G)$ individuals while gaining the membership fee. Since $K(G) \geq c$, the individual can not gain by leaving a club. Therefore, G is OCS.

Next, consider the case where G is not a Minimally Connected environment of class $K(G) \geq c$. If G is the Empty environment then consider the deviation where all the individuals form a new club. The benefit for each individual is $n_a - 1 - c > 0$. Hence, the Empty environment is not OCS. If G is a non-empty disconnected environment then there must exist a component H that contains $h > 1$ individuals. The maximal possible utility of an individual in H is $(h-1) - c$. If $c > h-1$ then every member of this component would like to leave any of her clubs. If $c \leq h-1$ then any individual that is not in H can improve if she joins one of H's clubs since she gets $h - c > 0$. Hence, no disconnected environment is OCS. If G is connected, but not minimally connected, there is an affiliation that can be removed while leaving the induced network connected. Denote this affiliation by $\{i,s\}$. Then, Individual i, by leaving Club s can improve his net utility by c. Hence, no connected, but not minimally connected, environment is OCS. Finally, suppose that G is a minimally connected network of class $K(G) < c$. Consider an affiliation $\{i^\star, s^\star\} \in \arg\min_{\{i,s\}\in A} n(C_{-i}(G - \{i,s\}))$. Individual $i^\star$ wishes to leave Club $s^\star$ since while losing the connection to $K(G)$ individuals she gains c, and $K(G) < c$. Hence, if G is not a Minimally Connected environment of class $K(G) \geq c$ then G is not OCS.

Since $n_a - 1 > c$, the maximal utility an individual can obtain is $u_i(G) = n_a - 1 - c$. In the Grand Club environment every individual achieves the maximal utility. Therefore, the Grand Club environment is SE. Moreover, any other environment is either disconnected or it contains at least one individual that maintains multiple affiliations. In both cases there is at least one individual with a utility lower than $n_a - 1 - c$. Therefore, the Grand Club environment is the unique SE environment.

Finally, suppose that $c > n_a - 1$. Every individual that maintains memberships in a non-empty environment, wishes to leave any of her clubs. Therefore, every non-empty environment is not OCS. Moreover, none of the individuals in those environments have positive utility and there is at least one with negative utility. However, in the Empty environment, no coalition of individuals is better off by establishing a new club. Therefore, the Empty environment is OCS. Since all the individuals have zero utility, it is also SE. Hence, the Empty environment is the unique OCS and SE environment. □

Lemma 1

Proof. By the definition of the club congestion function, the weight of each link is determined by a single club - the smallest club the two end individuals share. Since the benefit part of the individual's preferences depends only on the weights in the induced network, there may be at most $n_a - 1$ affiliations that determine the individual's benefits. Similarly, at most $\frac{n_a(n_a-1)}{2}$ clubs contribute to the benefits of any individual in the environment. However, all populated clubs contribute to the costs part of the preferences, since each affiliation is costly ($c > 0$). Hence, every G that includes an individual that maintains more than $n_a - 1$ affiliations is not OCS. Moreover, if G includes more than $\frac{n_a(n_a-1)}{2}$ populated clubs, there is at least one club that does not contribute to the benefits of any of its members. Therefore, each one of its members would wish to cancel this affiliation and G is not OCS. □

Lemma 2

Proof. By definition, $h(m)$ is inelastic if and only if $\forall m \in \{2, \ldots, n_a - 1\} : \eta_h(m) > -1$. $\eta_h(m) > -1$ if and only if $\frac{m \times h(m+1)}{h(m)} - m > -1$. Therefore, $h(m)$ is inelastic if and only if $k_h(m+1) = m \times h(m+1) > (m-1) \times h(m) = k_h(m)$. Hence, $h(m)$ is inelastic if and only if $k_h(m)$ is strictly increasing. Similar argument proves the case of elastic $h(m)$. □

Proposition 2

Proof. Throughout the proof we assume that $\frac{n_a-1}{m-1}$ and $\frac{n_a(n_a-1)}{m(m-1)}$ are integers. Consider first the maximal sum of utilities of a connected environment $G \in \mathscr{G}_n^m$ with at most $\frac{n_a(n_a-1)}{m(m-1)}$ clubs. By Proposition 2 in Berge (1989), the minimal number of clubs in a connected m-Uniform environment is $\frac{n_a-1}{m-1}$. Denote the number of clubs by $\frac{n_a(n_a-1)}{m(m-1)} \geq k \geq \frac{n_a-1}{m-1}$. The maximal total number of direct connections across all individuals in the environment is $km(m-1)$ and their value is $km(m-1)h(m)$. Since the induced network is connected, the number of indirect connections is $n_a(n_a-1) - km(m-1)$ and their maximal value is $[n_a(n_a-1) - km(m-1)]h^2(m)$. The total cost of membership in k clubs of size m is kmc. Thus, the maximal sum of utilities of a connected m-Uniform environment with k clubs is $km(m-1)h(m)+[n_a(n_a-1)-km(m-1)]h^2(m)-kmc$.

An m-Complete environment includes $\frac{n_a(n_a-1)}{m(m-1)}$ clubs (each club generates $\frac{m(m-1)}{2}$ links out of the $\frac{n_a(n_a-1)}{2}$ possible links and each link is generated exactly once). Therefore, the sum of utilities of an m-Complete environment is $n_a(n_a-1)h(m)-n_a\dfrac{n_a-1}{m-1}c$. The difference between these two expressions is $[km(m-1)-n_a(n_a-1)][h(m)-h^2(m)-\dfrac{c}{m-1}]$.

Obviously, there must be a k such that the maximal total sum of utilities of a connected m-Uniform environment is greater than the total sum of utilities of an m-Complete environment. Therefore, this difference must be non-negative for some k. If $c<(m-1)[h(m)-h^2(m)]$, $[km(m-1)-n_a(n_a-1)]$ must be non-negative, meaning that $k \geq \frac{n_a(n_a-1)}{m(m-1)}$. Since $\frac{n_a(n_a-1)}{m(m-1)} \geq k \geq \frac{n_a-1}{m-1}$ it must be that $k=\frac{n_a(n_a-1)}{m(m-1)}$ and the difference is zero. Also, if $c=(m-1)[h(m)-h^2(m)]$ the difference is zero. Therefore, the m-Complete environment achieves the maximal sum of utilities of the set of connected m-Uniform environments with at most $\frac{n_a(n_a-1)}{m(m-1)}$ clubs when $c \leq (m-1)[h(m)-h^2(m)]$.

Next, let G' be some m-Uniform environment with $k>\frac{n_a(n_a-1)}{m(m-1)}$ populated clubs. Each Individual $i \in \{1,\ldots,n_a\}$ in G' gets at most $(n_a-1)h(m)$ (in case she is directly connected to all other individuals). Therefore, the total benefits in G' are at most $n_a(n_a-1)h(m)$ while the total membership fees are $kmc > \frac{n_a(n_a-1)}{(m-1)}c$. Hence, the total sum of utilities of an m-Complete environment is weakly greater than the total sum of utilities of any m-Uniform environment with $k>\frac{n_a(n_a-1)}{m(m-1)}$ populated clubs. In particular, this means that the m-Complete environment achieves the maximal sum of utilities of the set of connected m-Uniform environments when $c \leq (m-1)[h(m)-h^2(m)]$.

This result implies that an environment that maximizes the sum of utilities from the set of non-empty m-Uniform environments when $c \leq (m-1)[h(m)-h^2(m)]$ is a collection of m-Complete components and isolated individuals. Note that the sum of utilities of an m-Complete component with n individuals ($n>1$) is $n(n-1)[h(m)-\frac{c}{m-1}]$. Since $[h(m)-\frac{c}{m-1}]$ is non-negative, the sum of utilities of an m-Complete component is a weakly increasing and weakly convex function of the number of individuals in the component. Therefore, if $c \leq (m-1)[h(m)-h^2(m)]$ an environment that achieves the maximum of the sum of utilities from the set of non-empty m-Uniform environments is the m-Complete environment. Also, when $c \leq (m-1)[h(m)-h^2(m)]$, the sum of utilities of the m-Complete environment is non-negative. Thus, when $c \in [0,(m-1)(h(m)-h^2(m)]$ the m-Complete environment maximizes the sum of utilities within the set of all m-Uniform environments.

An m-Star environment includes $\frac{n_a-1}{m-1}$ clubs. Therefore, the sum of utilities of an m-Star environment is

$$\frac{n_a-1}{m-1}m(m-1)h(m)+[n_a(n_a-1)-\frac{n_a-1}{m-1}m(m-1)]h^2(m)-\frac{n_a-1}{m-1}mc$$

The difference between the maximal sum of utilities of a connected m-Uniform environment with k clubs and an m-Star environment is

$$[k-\frac{n_a-1}{m-1}]m[(m-1)h(m)-(m-1)h^2(m)-c]$$

Again, there must be a k such that the maximal total sum of utilities of a connected m-Uniform environment is weakly greater than the total sum of utilities of an m-Star environment. Therefore, this difference must be non-negative for some k. If $c>(m-1)[h(m)-h^2(m)]$, $[k-\frac{n_a-1}{m-1}]m$ must be non-positive and since $k \geq \frac{n_a-1}{m-1}$ it must be that $k=\frac{n_a-1}{m-1}$ and the difference is zero. Also, if $c=(m-1)[h(m)-h^2(m)]$ the difference is zero. Therefore, the m-Star environment achieves the maximal sum of utilities of the set of connected m-Uniform environments when $c \geq (m-1)[h(m)-h^2(m)]$.

This result implies that when $c \geq (m-1)[h(m)-h^2(m)]$ an environment that maximizes the sum of utilities from the set of non-empty m-Uniform environments is a collection of m-Star sub-environments and isolated individuals. In fact, an environment that maximizes the sum of utilities from the set of m-Uniform environments is a collection of m-Star sub-environments with non-negative sum of utilities and isolated individuals.

Suppose that C_1 and C_2 are two m-Star environments with $n_1>1$ and $n_2>1$ individuals, respectively. Let b_1 and b_2 be the central individuals of C_1 and C_2, respectively. Consider a new environment C that includes the clubs of C_1 and the clubs of C_2 where b_2 is replaced by b_1. Thus, C is an m-Star environment with n_1+n_2-1 individuals with an additional isolated individual. The utility of the central individual in C is the sum of utilities of b_1 and b_2 in C_1 and C_2, respectively. The utility of all other individuals improves due to the additional free indirect connections. Thus, uniting two m-stars into one bigger m-Star environment (and an isolate) always increases the sum of utilities. That is, when $c \geq (m-1)[h(m)-h^2(m)]$, an environment that achieves the maximal sum of utilities from the set of non-empty m-Uniform environments is an m-Star environment and some isolated individuals. Thus, if $c \geq (m-1)[h(m)-h^2(m)]$, an environment that achieves the maximal sum of utilities from the set of non-empty m-Uniform environments is the m-Star environment.

To complete the proof notice that the m-Star environment achieves the maximal sum of utilities from the set of all m-Uniform environments if and only if it has a non-negative sum of utilities. If it has non-positive sum of utilities, the Empty environment achieves the maximal the sum of utilities from the set of m-Uniform environments. The sum of utilities of the m-Star environment with n_a individuals is non-negative if $(m-1)h(m)+\dfrac{(n_a-m)(m-1)}{m}h^2(m) \geq c$.

Thus, when $c \in [(m-1)(h(m)-h^2(m)),(m-1)h(m)+\frac{(n_a-m)(m-1)}{m}h^2(m)]$ the m-Star environment achieves the maximal sum of utilities from the set of m-Uniform environments. When

$c \geq (m-1)h(m) + \frac{(n_a-m)(m-1)}{m}h^2(m)$ the Empty environment achieves the maximal the sum of utilities from the set of m-Uniform environments. $\square$

Proposition 3

Proof. Throughout the proof we assume that $\frac{n_a-1}{m-1}$ and $\frac{n_a(n_a-1)}{m(m-1)}$ are integers. Let G be an m-Complete environment. The utility of Individual i from Environment G is (denote $\gamma \equiv \frac{n_a-1}{m-1}$): $u_i(G) = (n_a-1)h(m) - \gamma c$.

To calculate the utility of Individual i from aborting any one of her affiliations, suppose that Individual i leaves Club s which she shares with Individual i'. In addition, suppose she shares the Club s' with Individual i''. By the definition of an m-Complete environment, $S_{G-\{i,s\}}(i) \cap S_{G-\{i,s\}}(i') = \emptyset$. Thus, the new shortest path between these two individuals must be indirect. Again, by the definition of m-Complete environments the populated clubs in $G-\{i,s\}$ are of size m except Club s which is of size $m-1$. Therefore, the new shortest path is of length 2 and its weight must be either $h(m-1)h(m)$ or $h^2(m)$ (any path of length of more than 2 has a lower or equal weight than $h^2(m)$). Now, let us show that in $G-\{i,s\}$ there is no shortest path between Individual i and Individual i' of the weight $h(m-1)h(m)$. Suppose such a path exists. Then, there is an Individual j who shares a club with Individual i (denoted by t) and also shares Club s with Individual i'. Thus, in G, Individual j shared s also with Individual i which implies, however, $S_G(i) \cap S_G(j) = \{s,t\}$ and the m-completeness of G is violated. However, a shortest path of weight $h^2(m)$ between Individual i and Individual i' in $G-\{i,s\}$ does exist. Recall that Individual i shares Club s' with individual i'' and note that Individual i'' is not a member of Club s (otherwise individuals i and i'' share two clubs in G) and that Individual i' is not a member of Club s' (otherwise individuals i and i' share two clubs in G). Hence, by the definition of an m-complete environment, $\exists s'' \in S \backslash \{s,s'\} : \{i',i''\} \subseteq N_G(s'')$. Thus, Individual i has a link of weight $h(m)$ with Individual i'' (Club s') and Individual i'' has a link of weight $h(m)$ with Individual i' (Club s''). Therefore, there is a path of weight $h^2(m)$ between Individual i and Individual i' in Environment $G-\{i,s\}$. Thus, the utility of Individual i from Environment $G-\{i,s\}$ is $u_i(G-\{i,s\}) = (n_a-m)h(m) + (m-1)h^2(m) - (\gamma-1)c$ and $u_i(G-\{i,s\}) - u_i(G) = (m-1)h^2(m) - (m-1)h(m) + c$. Individual i does not wish to leave any of her clubs if and only if $u_i(G-\{i,s\}) \leq u_i(G)$, meaning that she does not wish to leave any of her clubs if and only if $(m-1)[h(m)-h^2(m)] \geq c$. Thus, $(m-1)[h(m)-h^2(m)] \geq c$ guarantees that the "No Leaving" condition holds.

Next, let us calculate the utility of Individual i from joining an existing Club s. Since G is m-complete, $\forall i' \in N_G(s) : |S_{G+\{i,s\}}(i) \cap S_{G+\{i,s\}}(i')| = 2$. Moreover, since $\forall i' \in N_G(s) : w(i,i',G) = h(m)$, $n_{G+\{i,s\}}(s) = m+1$ and $h(m) \geq h(m+1)$, Individual i does not improve any of her shortest paths by joining Club s. However, she pays c as membership fee. Therefore, $c \geq 0$ guarantees that the "No Joining" condition holds.

Next, let us calculate the utility of Individual i from the formation of a new club by the group K ($i \in K, K \subseteq N$) and let $|K| = k$. Note that $c \geq 0$ guarantees the "No New Club Formation" condition for the case of $k \geq m$ due to similar considerations to those used in the case of the "No Joining" condition above. For $m > k \geq 2$, the utility of Individual i from the Environment $G+K$ is $u_i(G+K) = (n_a - k)h(m) + (k-1)h(k) - (\gamma+1)c$ (indirect connections cannot improve on direct connections). Since $u_i(G+K) - u_i(G) = (k-1)h(k) - (k-1)h(m) - c$ we get that if $c \geq (k-1)[h(k) - h(m)]$ then $u_i(G+K) \leq u_i(G)$. Thus, Individual i refuses to establish a new club as part of Group K if and only if $c \geq (k-1)[h(k) - h(m)]$. However, in order to ensure that Individual i refuses to establish a new club with any subset of individuals it must be that this condition holds $\forall k \in \{2, \ldots, m-1\}$. Therefore, $c \geq \max\limits_{k \in \{2,\ldots,m-1\}} (k-1)[h(k) - h(m)]$ guarantees that the "No New Club Formation" condition holds for $k < m$. Since $\forall k \in \{2, \ldots, m\} : (k-1)[h(k) - h(m)] \geq 0$, this condition also ensures that the "No Joining" condition holds.

Next, denote $k^\star = \min\{\arg\max_{k \in \{2,\ldots,m-1\}} (k-1)[h(k) - h(m)]\}$. Note that the condition above can be rewritten as $\max_{k \in \{2,\ldots,m-1\}} [k_h(k) - (k-1)h(m)]$. Let $k' \in \{\hat{k}+1, \ldots, n_a\}$. By the definition of $\hat{k}$, we get $k_h(\hat{k}) \geq k_h(k')$. In addition, since given an m-Complete environment $h(m)$ is fixed, we get $(\hat{k}-1)h(m) \leq (k'-1)h(m)$. Hence, for every $k' \in \{\hat{k}+1, \ldots, n_a\}$, we have $k_h(\hat{k}) - (\hat{k}-1)h(m) \geq k_h(k') - (k'-1)h(m)$. Therefore, $k^\star \leq \hat{k}$. This implies that the "No New Club Formation" and the "No Joining" conditions hold if $c \geq \max\limits_{k \in \{2,\ldots,\min\{m-1,\hat{k}\}\}} (k-1)[h(k) - h(m)]$.

$\square$

Claim 1

Lemma A.1. *Let $h(\cdot)$ be an exponential club congestion function where $\delta \in (0, \frac{1}{2})$. Then,* $\max_{k \in \{2,\ldots,m-1\}} (k-1)[h(k) - h(m)] = h(2) - h(m)$.

Proof. Note that

$$\forall l \in \{2, \ldots, m-2\}, \forall k \in \{0, \ldots, m-l-1\} : \frac{h(l+k) - h(l+k+1)}{h(l) - h(l+1)} = \delta^k$$

Thus,

$$\forall l \in \{2, \ldots, m-2\} : \sum_{k=0}^{m-l-1} \delta^k = \sum_{k=0}^{m-l-1} \frac{h(l+k) - h(l+k+1)}{h(l) - h(l+1)} = \frac{h(l) - h(m)}{h(l) - h(l+1)}$$

Since $\delta \in (0, \frac{1}{2})$, we get

$$\forall l \in \{2, \ldots, m-2\} : \frac{h(l) - h(m)}{h(l) - h(l+1)} = \frac{1 - \delta^{m-l}}{1 - \delta} < \frac{1}{1 - \delta} < 2$$

Therefore, $\forall l \in \{2, \ldots, m-2\} : h(l) - h(m) < 2[h(l) - h(l+1)]$ and since $h(\cdot)$ is non-increasing $\forall l \in \{2, \ldots, m-2\} : l[h(l+1) - h(m)] < (l-1)[h(l) - h(m)]$ which immediately implies the required: $\max_{k \in \{2,\ldots,m-1\}} (k-1)[h(k) - h(m)] = h(2) - h(m)$. $\square$

Proof of Claim 1

Proof. For the first part G includes one club that consists of all the individuals in the environment. Note that the considerations stated in the proof of Proposition 3 of the main text for the lower bound hold also here. Thus, if $c \geq \max\limits_{k \in \{2,\ldots,\min\{n_a-1,\hat{k}\}\}} (k-1)[h(k) - h(n_a)]$ the "No Joining" and the "No New Club Formation" conditions hold. However, the "No Leaving" condition is different since if an individual decides to leave the club her utility is zero. The utility of the individuals from G is $u_i(G) = (n_a - 1)h(n_a) - c$. Therefore, an individual will not leave the club as long as $(n_a - 1)h(n_a) \geq c$. Thus, $(n_a - 1)h(n_a) \geq c$ guarantees that the "No Leaving" condition holds. Hence, the Grand Club environment is OCS if and only if

$$c \in \Big[\max_{k \in \{2,\ldots,\min\{n_a-1,\hat{k}\}\}} (k-1)[h(k) - h(n_a)], (n_a - 1)h(n_a)\Big]$$

Therefore, there exists a range of membership fees in which the Grand Club environment is OCS if and only if $\max\limits_{k \in \{2,\ldots,\min\{n_a-1,\hat{k}\}\}} (k-1)[h(k) - h(n_a)] \leq (n_a - 1)h(n_a)$. Alternatively, such a range exists if and only if

$$\forall k \in \{2, \ldots, \min\{n_a - 1, \hat{k}\}\} : (k-1)[h(k) - h(n_a)] \leq (n_a - 1)h(n_a)$$

Such a range exists if and only if

$$\forall k \in \{2, \ldots, \min\{n_a - 1, \hat{k}\}\} : k_h(k) - (k-1)h(n_a) \leq k_h(n_a)$$

Equivalently, there exists a range of membership fees in which the Grand Club environment is OCS if and only if $\forall k \in \{2, \ldots, \min\{n_a - 1, \hat{k}\}\} : k_h(k) - \dfrac{k-1}{n_a - 1} k_h(n_a) \leq k_h(n_a)$. Thus, there exists a range of membership fees in which the Grand Club environment is OCS if and only if

$$\forall k \in \{2, \ldots, \min\{n_a - 1, \hat{k}\}\} : \frac{n_a - 1}{n_a + k - 2} \times k_h(k) \leq k_h(n_a)$$

Since for every $k \in \{2, \ldots, \min\{n_a - 1, \hat{k}\}\}$ we have $\frac{n_a-1}{n_a+k-2} < 1$, if the DCV is increasing then the inequality is satisfied. By Lemma 2 in the main text, if the club congestion function is inelastic the DCV is strictly increasing, and therefore if the club congestion function is inelastic, a range of membership fees in which the Grand Club environment is OCS exists.

Finally, by Lemma A.1, the lower bound of the range of membership fees in which the Grand Club environment is OCS becomes $h(2) - h(n_a) = \delta - \delta^{n_a - 1}$. When $a = 0$, the upper bound is $(n_a - 1)h(n_a) = (n_a - 1)\delta^{n_a - 1}$. Since $n_a \geq 4$ we can write $n_a \leq 2^{n_a - 2}$ or $\frac{1}{n_a} \geq \frac{1}{2^{n_a - 2}}$. Using $\delta \in (0, \frac{1}{2})$ we have $\frac{1}{n_a} > \delta^{n_a - 2}$ or $1 > n_a \delta^{n_a - 2}$ or $\delta > n_a \delta^{n_a - 1}$. Hence, $\delta - \delta^{n_a - 1} > (n_a - 1)\delta^{n_a - 1}$, that is the lower bound is greater than the upper bound. Thus, for $a = 0$, $\delta \in (0, \frac{1}{2})$ and $n_a \geq 4$ the Grand Club environment is never OCS. For the case of $a > 0$, a range of membership fees for which the Grand Club environment is OCS exists if and only if $\delta - \delta^{n_a - 1} \leq (n_a - 1)(a + \delta^{n_a - 1})$. That is, there exists a range of membership fees for which the Grand Club environment is OCS if and only if $(n_a - 1)a + n_a \delta^{n_a - 1} \geq \delta$. Let $\bar{n}_a = \frac{\delta}{a} + 1$. Then, $(\bar{n}_a - 1)a = \delta$, and therefore, for every $n_a > \bar{n}_a$ there exists a range of membership fees for which the Grand Club environment is OCS. □

Proposition 4

Proof. Let $G =< N, S, A >$ be an m-Star environment where $n_a > m \geq 2$ and denote the number of populated clubs in G by $\gamma \equiv \frac{n_a - 1}{m - 1}$ (we assume that γ is an integer). For simplicity, we refer to the central Individual as individual b and to the other individuals as individuals i, i', etc.

We begin with an upper bound on the range of membership fee where G is OCS. The upper bound is set by the membership fees above which individuals would wish to wave any of their affiliations. The utility of the central individual from Environment G is $u_b(G, h, c) = (n_a - 1)h(m) - \gamma c$. Consider Club s. Since all non-central club members have no other affiliations, no path exists between Individual b and these individuals once Individual b leaves Club s. Also, Individual b have no indirect connections through these individuals. Therefore, for every $\{b, s\} \in A$, $u_b(G - \{b, s\}, h, c) = (n_a - m)h(m) - (\gamma - 1)c$. Therefore, Individual b has no incentive to leave any of her affiliations if and only if $(m - 1)h(m) \geq c$. The utility of a non-central Individual i from Environment G is $u_i(G, h, c) = (m - 1)h(m) + (n_a - m)h^2(m) - c$. Consider Club s such that $\{i, s\} \in A$. The utility of Individual i after aborting her affiliation with Club s is zero. Therefore, Individual i has no incentive leave the club if and only if $(m - 1)h(m) + (n_a - m)h^2(m) \geq c$. Thus, no individual has an incentive to leave a club in G if and only if $(m - 1)h(m) \geq c$ or $k_h(m) \geq c$.

We continue with the lower bound on the range of membership fees where G is OCS. The lower bound is set by the membership fee below which individuals would wish to form new affiliations either by joining a new club or by forming a new club.

We begin by considering the benefits for a subset of individuals ($K \subseteq N$, $k = |K|$) from forming a new club (r). Denote by $K_s = N_{G+K}(s) \cap N_{G+K}(r)$ the set of individuals that share Club s in G and are affiliated with the new Club r and denote its magnitude by k_s. Denote the set of clubs represented in K by $Q = \{s \in S : k_s > 0\}$ and its magnitude by q.

We first consider the case where the new club is no larger than the existing clubs, $k \leq m$. Each individual in K gets a benefit from the direct connections with the other members in K. These connections replace either a link with a weight of $h(m)$ (if they share a club in G) or a path with a weight of $h^2(m)$ (if they do not share a club in G). Obviously, an improvement on an indirect connection is larger than an improvement on a direct connection ($h(k) - h^2(m) \geq h(k) - h(m)$). Therefore, non-central individuals get the same benefit as the central individual on links with individuals they already share a club with in G and higher benefit than the central individual on links with individuals they do not share a club with in G. Thus, in the case where $b \in K$, since the utility from environment $G+K$ to Individual b is $u_b(G+K,h,c) = (k-1)h(k) + (n_a - k)h(m) - (\gamma+1)c$ we get that $c \geq \max_{m \geq k \geq 2}(k-1)(h(k) - h(m))$ prevents the formation of K since it does not benefit individual b. For the case where $b \notin K$ we begin by considering the case where the size of the new club is not greater than the original club size and the original number of clubs ($m \geq k$ and $\gamma \geq k$). Note that on top of the improved direct connections that each individual in K gets, the partners with whom she did not share a club with in G supply her with improved indirect paths to the individuals in their original clubs that do not participate in K. These paths are better than the paths supplied in G by the central individual, since the new club is small ($k \leq m$). The utility from Environment $G+K$ for Individual i such that $\{i,s\} \in A$, $b \notin K$ and $i \in K$ is

$$\begin{aligned} u_i(G+K,h,c) =& (k-1)h(k) + (m-k_s)h(m) + ((q-1)(m-1) - (k-k_s))h(k)h(m) \\ &+ (\gamma - q)(m-1)h^2(m) - 2c \end{aligned}$$

For every Individual i and every m, $h(\cdot)$, c, q and k, $u_i(G+K,h,c)$ is maximized if $k_s = 1$.[5] Thus, the utility of Individual i from K is maximized if no other member in this club shares her original club in G. The utility of Individual i ($(i,s) \in A$) from $G+K$ when K includes no other individual from Club s is

$$\begin{aligned} u_i(G+K,h,c) =& (k-1)h(k) + (m-1)h(m) + ((q-1)(m-1) - (k-1))h(k)h(m) \\ &+ (\gamma - q)(m-1)h^2(m) - 2c \end{aligned}$$

[5] $\frac{\partial u_i(G+K,h,c)}{\partial k_s} = -h(m) + h(k)h(m) = -h(m)(1-h(k)) \leq 0$. Since q and k are held fixed, increasing k_s by 1 means that Individual j' of Club s' ($k_{s'} > 1$ since q is fixed) is replaced in K by a member j of Club s ($j \notin \{i,b\}$). The gain from this change is the improved path to Individual j ($h(k) - h(m)$) while the loss is the longer path to j' ($h(k)h(m) - h(k)$). Thus, the net benefit is $-h(m) + h(k)h(m)$.

In addition, to maximize $u_i(G+K,h,c)$ given m, $h(\cdot)$, c and k, q should be as high as possible.[6] Thus, the utility of Individual i from a new club where $q=k$ (no pair of individuals in the new club share a club in G) is

$$\begin{aligned} u_i(G+K,h,c) =&(k-1)h(k)+(m-1)h(m)+((k-1)(m-1)-(k-1))h(k)h(m) \\ &+(\gamma-k)(m-1)h^2(m)-2c \end{aligned}$$

Therefore, the non-central individuals have no incentive to form a new club of size $\min\{m,\gamma\} \geq k$ if

$$c \geq \max_{\min(\gamma,m)\geq k\geq 2}(k-1)[h(k)+(m-2)h(k)h(m)-(m-1)h^2(m)]$$

To complete the case of $k \leq m$, we consider the case of a new club of size $m \geq k > \gamma$ that does not include the central individual. Suppose $q<\gamma$. Then, there is a non-empty Club s in G such that $k_s = 0$. In this case, in $G+K$, $\forall i \in K$ there are some indirect paths with weight $h(k)h(m)$ and some indirect paths with weight $h^2(m)$. Alternatively, suppose $q=\gamma$. For every no non-empty club s in G, $k_s > 0$. Then, $\forall i \in K$ the direct links are the same as in the previous case, but all the indirect paths are of weight $h(k)h(m)$. Clearly, for each individual, the incentives to form a new club are weakly stronger when $q=\gamma$.

The utility from Environment $G+K$ to Individual i who participates in Club s and in Group K where $q=\gamma$ and $b \notin K$ is

$$u_i(G+K,h,c) = (k-1)h(k)+(m-k_s)h(m)+(n_a-m-(k-k_s))h(k)h(m)-2c$$

Given m, $h(\cdot)$ and c, $u_i(G+K,h,c)$ increases when k_s decreases (see Footnote 5). Thus, the most attractive K is the one that minimizes the maximal k_s (over all $s \in S$) where $q=\gamma$. In this new optimal club $\max_{s,s' \in S}|k_s-k_{s'}| \leq 1$ and the individuals that belong to the original clubs with the higher k_s have lower utility. Denote the optimal k_s by $\eta_k \equiv \lceil \frac{k}{\gamma} \rceil$. Then, the utility of $i \in K$ that belongs to the original Club $s \in \{s \in S | \forall s' \in S, k_s \geq k_{s'}\}$ is

$$u_i(G+K) = (k-1)h(k)+(m-\eta_k)h(m)+(n_a-m-(k-\eta_k))h(k)h(m)-2c$$

[6] $\frac{\partial u_i(G+K,h,c)}{\partial q} = (m-1)h(k)h(m)-(m-1)h^2(m) \geq 0$ (equality is achieved if and only if $k=m$). Since k is held fixed, increasing q by 1 means that Individual j' of Club s' ($k_{s'}>1$) is replaced in K by a member j of Club s that was not represented in K. The gain from this change is the paths to Individual j and her club members ($h(k)+(m-2)h(k)h(m)-(m-1)h^2(m)$) while the loss is the longer path to j' ($h(k)h(m)-h(k)$). Thus, the net benefit is $(m-1)(h(k)h(m)-h^2(m))$.

Thus, the membership fee required to prevent the formation of a new club of size $\gamma < k \leq m$ is

$$c \geq \max_{m \geq k > \gamma} (k-1)h(k) - (\eta_k - 1)h(m) + (n_a - m - (k - \eta_k))h(k)h(m) - (n_a - m)h^2(m)$$

It is easy to see that the membership fee required to prevent the formation of a new club of size $k \leq m$ are higher when the central individual is not included in the group. Denote the membership fee required to prevent a deviation to a club of size k when $m \geq k$ and $\gamma \geq k$ by,

$$FNS_h(k,m) = (k-1)[h(k) + (m-2)h(k)h(m) - (m-1)h^2(m)]$$

and the membership fee required to prevent a deviation to a club of size k when $m \geq k$ and $k > \gamma$,

$$\begin{aligned} FNI_h(k,m,n_a) = (k-1)h(k) - (\eta_k - 1)h(m) + (n_a - m - (k - \eta_k))h(m)h(k) \\ - (n_a - m)h^2(m) \end{aligned}$$

Therefore, we can conclude that the minimal membership fee required to prevent the formation of a new club that is no larger than the existing clubs, $k \leq m$, depends on the relation between m and γ. If $m > \gamma$ then $c \geq \max\{\max_{\gamma \geq k \geq 2} FNS_h(k,m), \max_{m \geq k > \gamma} FNI_h(k,m,n_a)\}$ while if $\gamma \geq m$ then $c \geq \max_{m \geq k \geq 2} FNS_h(k,m)$. Note that when $k > m$ there are no gains to the members of the new club from shorter indirect paths. In addition, they have no gains from the members of the new club with whom they already share a club in G (therefore the central individual can never benefit from participating in clubs of size $k > m$). Moreover, if $h^2(m) \geq h(k)$ there are no gains also from the other members of the new club. Thus, no new club of size $k \geq l_h$ are formed where $l_h = \min\{k \in \mathbb{Z} | h(k) \leq h^2(m)\}$. However, the net gains for a non-central Individual i, that belongs to Club s in Environment G, from establishing a new club of size $\min\{l_h, n_a\} > k > m$ are $(k - k_s)(h(k) - h^2(m)) - c$. Since there is at least one individual in K for which $k_s \geq \eta_k$, she refuses to deviate if $c > (k - \eta_k)(h(k) - h^2(m))$. Denote the membership fee required to prevent a deviation to a club of size k when $\min\{l_h, n_a\} > k > m$ by $FNL_h(k,m,n_a) = (k - \eta_k)(h(k) - h^2(m))$. Thus, the minimal membership fee required to prevent the formation of a new club are

If $m > \gamma$

$$c \geq \max\{\max_{\gamma \geq k \geq 2} FNS_h(k,m,n_a), \max_{m \geq k > \gamma} FNI_h(k,m,n_a), \max_{\min\{l_h, n_a\} > k > m} FNL_h(k,m,n_a)\}$$

while if $\gamma \geq m$

$$c \geq \max\{\max_{m \geq k \geq 2} FNS_h(k,m,n_a), \max_{\min\{l_h,n_a\}>k>m} FNL_h(k,m,n_a)\}$$

Finally, we analyse the incentive to join a new club. Individual b is irrelevant since she is already present in all the populated clubs. A non-central Individual i who joins an existing Club s shortens her paths to the members of this club (excluding the center) while she pays the membership fee (and intensifies the congestion in Club s). The utility of Individual i from Environment $G+\{i,s\}$ (where $\{i,s\} \notin A$ and $n_G(s) \geq 2$) is

$$u_i(G+\{i,s\},h,c) = (m-1)h(m) + (m-1)h(m+1) + (n_a - 2m + 1)h^2(m) - 2c$$

Therefore, the net benefit for Individual i from joining an existing Club s is $(m-1)[h(m+1) - h^2(m)] - c$. Thus, no individual wishes to join a new club in G if and only if

$$c \geq (m-1)[h(m+1) - h^2(m)]$$

Denote $J_h(m) = (m-1)[h(m+1) - h^2(m)]$.

Note that $FNS_h(m,m) = (m-1)(h(m) - h^2(m))$ and therefore $FNS_h(m,m) \geq J_h(m)$. Thus, the lower bound on the membership are:

If $m > \gamma$

$$c \geq \max\{\max_{\gamma \geq k \geq 2} FNS_h(k,m), \max_{m \geq k > \gamma} FNI_h(k,m,n_a), \max_{\min\{l_h,n_a\}>k>m} FNL_h(k,m,n_a), J_h(m)\}$$

while if $\gamma \geq m$ then $c \geq \max\{\max_{m \geq k \geq 2} FNS_h(k,m), \max_{\min\{l_h,n_a\}>k>m} FNL_h(k,m,n_a)\}$. □

Claim 2

Proof. We assume $n_a > 2$. Since $m = 2$, the number of clubs is never smaller than m and therefore only the first part of Proposition 4 is relevant for the 2-Star environment.

Let us begin with Part (i). When $m = 2$ we get $k_h(2) = h(2)$ as the upper bound. For the lower bound only $FNS_h(2,2)$ and $FNL_h(k,2,n_a)$ for $k \in \{3,\ldots,\min\{l_h - 1, n_a - 1\}\}$ are relevant. $FNS_h(2,2) = h(2) - h^2(2)$. Since $m = 2$ and $k \leq n_a - 1$, we get $\eta_k = 1$. Therefore, $FNL_h(k,2,n_a) = (k-1)(h(k) - h^2(2))$. The 2-Star environment is therefore OCS if and only if $h(2) \geq c \geq \max_{k \in \{2,\ldots,\min\{l_h-1,n_a-1\}\}} (k-1)(h(k) - h^2(2))$.

Next, by Lemma 2, since the club congestion function is elastic then $k_h(\cdot)$ is strictly decreasing. Note that $(k-1)(h(k) - h^2(2)) = k_h(k) - (k-1)h^2(2)$. Thus, the first part decreases with k

while the second part increases with k, meaning that $(k-1)(h(k)-h^2(2))$ is maximized by $k=2$. Therefore the 2-Star environment is OCS if and only if $h(2) \geq c \geq h(2)-h^2(2)$.

The reciprocal club congestion function implies that $(k-1)(h(k)-h^2(2))$ equals $1-(k-1)h^2(2) = 1-(k-1) = 2-k$. Thus, using Part (i), if $h(\cdot)$ is the reciprocal club congestion function, the 2-star environment is OCS if and only if $c \in [0,1]$.

Finally, suppose that $h(\cdot)$ is an exponential club congestion function. A straight forward application of Part (i) suggests that the 2-Star environment is OCS if and only if

$$a+\delta \geq c \geq \max_{k \in \{2,\ldots,\min\{l_h-1,n_a-1\}\}} (k-1)((a+\delta^{k-1})-(a+\delta)^2)$$

It is easy to see that when $a=0$ the upper bound becomes δ. Since $a=0$ implies that $l_h=3$, the lower bound becomes $\delta-\delta^2$. Therefore, if $a=0$ then the 2-Star environment is OCS if and only if $c \in [\delta-\delta^2, \delta]$. □

Claim 3

Proof. Since $n_a \geq 9$ then $\gamma \geq 4 > 3$, so that only the first part of Proposition 4 is relevant.

First, let $h(m) = \frac{1}{m-1}$. Note that $l_h = 5$. Thus, the 3-Star environment is OCS if and only if

$$2h(3) \geq c \geq \max\{h(2)+h(2)h(3)-2h^2(3), 2[h(3)-h^2(3)], 3[h(4)-h^2(3)]\}$$

and using the functional form we get $1 \geq c \geq \max\{1, \frac{1}{2}, \frac{1}{4}\}$ and therefore the 3-Star environment is OCS if and only if $c=1$.

Next, let $h(m) = \delta^{m-1}$ for $\delta \in (0,1)$. Note that again $l_h = 5$. Thus, again the 3-star environment is OCS if and only if

$$2h(3) \geq c \geq \max\{h(2)+h(2)h(3)-2h^2(3), 2[h(3)-h^2(3)], 3[h(4)-h^2(3)]\}$$

and using the functional form the 3-Star environment is OCS if and only if

$$2\delta^2 \geq c \geq \max\{\delta+\delta^3-2\delta^4, 2\delta^2-2\delta^4, 3\delta^3-3\delta^4\}$$

Note that since $\delta \in (0,1)$ it must be that $\delta(1-\delta)^2 > 0$. Therefore, $\delta+\delta^3 > 2\delta^2$ and $\delta+\delta^3-2\delta^4 > 2\delta^2-2\delta^4$. Meaning that the 3-Star environment is OCS if and only if

$$2\delta^2 \geq c \geq \max\{\delta+\delta^3-2\delta^4, 3\delta^3-3\delta^4\}$$

Note that since $\delta \in (0,1)$ it must be that $\delta^2(2-\delta) < 1$.[7] Therefore, $2\delta^2 - \delta^3 < 1$ or $2\delta^3 - \delta^4 < \delta$ or $3\delta^3 - \delta^4 < \delta + \delta^3$ or $3\delta^3 - 3\delta^4 < \delta + \delta^3 - 2\delta^4$. Meaning that the 3-Star environment is OCS if and only if $2\delta^2 \geq c \geq \delta + \delta^3 - 2\delta^4$. Given that $\delta \in (0,1)$ then $(\delta^2+1)(2\delta-1) \geq 0$ if and only if $\delta \in [\frac{1}{2}, 1)$. Thus, $2\delta^3 - \delta^2 + 2\delta \geq 1$ if and only if $\delta \in [\frac{1}{2}, 1)$. And, $2\delta^4 - \delta^3 + 2\delta^2 \geq \delta$ if and only if $\delta \in [\frac{1}{2}, 1)$. Meaning that $2\delta^2 \geq \delta + \delta^3 - 2\delta^4$ if and only if $\delta \in [\frac{1}{2}, 1)$. Thus, since the 3-Star environment is OCS if and only if $c \in [\delta + \delta^3 - 2\delta^4, 2\delta^2]$, there is a range of membership fees for which it is OCS if and only if $\delta \geq \frac{1}{2}$. $\square$

B Proofs of Results from the Online Appendix

Proposition 1

Lemma B.1. *If $c > 0$ and the Club Congestion function is $h(2) = \delta$ and $\forall m > 2 : h(m) = 0$ then $\forall g \in \mathbb{G}_n, \forall i \in N : u_i(G_g) = u_i^{JW}(g)$.*

Proof. Note that for every un-weighted network $g =< N, E >$, the induced network of G_g denoted by $\bar{g} =< N, \bar{E}, W >$ is such that $\bar{E} = E$ and, by the choice of $h(\cdot)$, each link has a weight of δ since the clubs are all of size two.

Since all the weights in $\bar{g}$ are the same, the length of the shortest weighted path between individuals i and j in $\bar{g}$ is the same as the length of the shortest path between them in g. Therefore, the distance between individuals i and j in $\bar{g}$ equals $\delta^{d_{ij}}$ where d_{ij} is the geodesic distance between individuals i and j in $\bar{g}$ and therefore also in g. Hence, the benefits of the individuals in G_g equal their benefits in g.

Moreover, by construction, the number of direct links each individual maintains in g equals the number of her affiliations in $G_{\bar{g}}$. Therefore, the costs of the individuals in G_g equal their costs in g. Hence, $\forall i \in N : u_i(G_g) = u_i^{JW}(g)$. $\square$

Lemma B.2. *Let g be an un-weighted network and let $G_g =< N, S, A >$ be the corresponding environment. $\forall i, j \in N$ such that $\exists s \in S : \{\{i,s\},\{j,s\}\} \subseteq A$ then $u_i(G_{g-\{i,j\}}) = u_i(G_g - \{i,s\})$.*

Proof. By construction, the Environment $G_{g-\{i,j\}}$ includes the same clubs as G_g excluding Club s. Therefore its induced weighted network $\bar{g}_{-s}$ is identical to g excluding the link between individuals i and j. Denote the benefits of Individual i in $\bar{g}_{-s}$ by B. Then $u_i(G_{g-\{i,j\}}) = B - (s_{G_g}(i) - 1) \times c$.

Environment $G_g - \{i,s\}$ includes the same clubs as G_g, but the affiliation of Individual i in Club s is dropped. Since Club s is a singleton in $G_g - \{i,s\}$, it induces no links. Therefore, $\bar{g}_{is}$, the weighted network induced by $G_g - \{i,s\}$ is identical to $\bar{g}_{-s}$. Hence, the benefits of Individual i in $\bar{g}_{is}$ are B and $u_i(G_g - \{i,s\}) = B - (s_{G_g}(i) - 1) \times c$. Thus, we get $u_i(G_{g-\{i,j\}}) = u_i(G_g - \{i,s\})$. $\square$

[7] $\delta^2(2-\delta)$ has a local maximum at $\frac{4}{3}$, a local minimum at 0 and its value at $\delta = 1$ is 1.

Proof

Proof. We suppose that $g \in PS(\delta, c, n)$ and show that $G_g \in OCS(h, c, n)$. The "No Joining" condition holds since the utility from a club of size 3 is zero while the participation fee is positive. For the same reason, no coalition of size greater than two wishes to form a new club.

Next, consider two individuals, i and j, that do not share a club in G_g. Then, by construction, Individual i and Individual j are not linked in g. Since g is pairwise stable, if $u_i^{JW}(g) < u_i^{JW}(g + \{i,j\})$ then $u_j^{JW}(g) > u_j^{JW}(g + \{i,j\})$. By Lemma B.1, if $u_i(G_g) < u_i(G_{g+\{i,j\}})$ then $u_j(G_g) > u_j(G_{g+\{i,j\}})$. Denote by m_{ij} the coalition that includes only individuals i and j. Then, note that $G_{g+\{i,j\}}$ is identical to $G_g + m_{ij}$ since both denote the addition of Club s that includes individuals i and j to Environment G_g. Hence, if $u_i(G_g) < u_i(G_g + m_{ij})$ then $u_j(G_g) > u_j(G_g + m_{ij})$. Therefore, no coalition of size two wishes to form a new club and the "No New Club Formation" condition holds.

For the "No Leaving" condition, consider Individual i that participates, together with Individual j, in Club s in G_g. Then, by construction, Individual i and Individual j are linked in g. Since g is pairwise stable $u_i^{JW}(g) \geq u_i^{JW}(g - \{i,j\})$. By Lemma B.1, $u_i(G_g) \geq u_i(G_{g-\{i,j\}})$. By Lemma B.2, $u_i(G_g) \geq u_i(G_g - \{i,s\})$, meaning that this condition also holds. Therefore, $G_g \in OCS(h,c,n)$.

For the other direction, we suppose that $G_g \in OCS(h,c,n)$ and show that $g \in PS(\delta,c,n)$. First, consider Individual i that is linked with Individual j in g. By construction Individual i participates, together with Individual j, in Club s in G_g. Since G_g is OCS, $u_i(G_g) \geq u_i(G_g - \{i,s\})$. By Lemma B.2, $u_i(G_g) \geq u_i(G_{g-\{i,j\}})$. By Lemma B.1, $u_i^{JW}(g) \geq u_i^{JW}(g - \{i,j\})$, meaning that no individual wishes to discard an existing link. Next, consider two individuals, i and j, that are not linked in g. By construction individuals i and j do not share a club in G_g. Since G_g is OCS, if $u_i(G_g) < u_i(G_g + m_{ij})$ then $u_j(G_g) > u_j(G_g + m_{ij})$. But, as mentioned above, $G_{g+\{i,j\}}$ is identical to $G_g + m_{ij}$. Therefore, if $u_i(G_g) < u_i(G_{g+\{i,j\}})$ then $u_j(G_g) > u_j(G_{g+\{i,j\}})$. By Lemma B.1, if $u_i^{JW}(g) < u_i^{JW}(g + \{i,j\})$ then $u_j^{JW}(g) > u_j^{JW}(g + \{i,j\})$, meaning that no pair of individuals wishes to form a new link. Therefore, $g \in PS(\delta,c,n)$.

For the second part note that since we assume that $\mathscr{G}_n$ includes only environments with distinct clubs, every environment $G \in \mathscr{G}_n \backslash \mathscr{G}_{\bar{\mathbb{G}}n}$ includes at least one populated club of size greater than two. However, every individual that participates in a club of size greater than two wishes to leave the club since its benefits are zero (all induced links of such club are of weight zero) while the membership fee is positive. Therefore, $G \notin OCS(h,c,n)$. □

Lemma 1

Lemma B.3. *Let $h(m)$ be an exponential congestion function. Let $m > m'$ and suppose $k_h(m) > k_h(m')$ for a given parameter a. Then $k_h(m) > k_h(m')$ for every $\bar{a} \in [a, 1-\delta)$.*

Proof. For the given parameter a, $k_h(m) - k_h(m') > 0$. Therefore, $(m-1)(a + \delta^{m-1}) - (m'-1)(a+$

$\delta^{m'-1}) > 0$ or written differently, $(m-m')a+(m-1)\delta^{m-1}-(m'-1)\delta^{m'-1} > 0$. Now suppose a increases to $\bar{a}$. Since $m > m'$, $(m-m')\bar{a}+(m-1)\delta^{m-1}-(m'-1)\delta^{m'-1} > 0$ and therefore $(m-1)(\bar{a}+\delta^{m-1})-(m'-1)(\bar{a}+\delta^{m'-1}) > 0$. Hence, $k_h(m)-k_h(m') > 0$ given $\bar{a}$. □

Lemma B.4. *Let $n_a \geq 4$ and let $h(m)$ be an exponential club congestion function with $a > 0$. $k_h(m)$ has at most two extreme points, $\bar{m} < \bar{\bar{m}}$, where $\bar{m}$ is a local maximum and $\bar{\bar{m}}$ is a local minimum.*

Proof. Denote $g_h(m) = \frac{\eta_h(m)}{\eta_h(m+1)}$. To show that $g_h(m)$ is strictly increasing, it is helpful to rewrite it as

$$g_h(m) = \frac{\frac{\frac{h(m+1)-h(m)}{h(m)}}{\frac{1}{m}}}{\frac{\frac{h(m+2)-h(m+1)}{h(m+1)}}{\frac{1}{m+1}}}$$

Then,

$$g_h(m) = \frac{m}{m+1} \times \frac{h(m+1)}{h(m)} \times \frac{h(m+1)-h(m)}{h(m+2)-h(m+1)} = \frac{m}{m+1} \times \frac{h(m+1)}{h(m)} \times \frac{1}{\delta}.$$
$$g_h(m+1) = \frac{m+1}{m+2} \times \frac{h(m+2)}{h(m+1)} \times \frac{h(m+2)-h(m+1)}{h(m+3)-h(m+2)} = \frac{m+1}{m+2} \times \frac{h(m+2)}{h(m+1)} \times \frac{1}{\delta}.$$

Note that for every integer $m \geq 1$, $\delta \in (0,1)$ satisfies $\delta^{m-1}+\delta^{m+1} > 2\delta^m$. Therefore, $a^2+2a\delta^m+\delta^{2m} < a^2+a\delta^{m-1}+a\delta^{m+1}+\delta^{2m}$ which can be rewritten as $h^2(m+1) < h(m) \times h(m+2)$. Hence, $\forall m \in \{2,\dots,n_a-2\} : \frac{h(m+2)}{h(m+1)} > \frac{h(m+1)}{h(m)}$. Also, note that $\forall m \in \mathbb{N} : \frac{m+1}{m+2} > \frac{m}{m+1}$. Taken together, $\forall m \in \{2,\dots,n_a-2\} : g_h(m+1) > g_h(m)$, meaning $g_h(m)$ is strictly increasing.

Since $\eta_h(m) \leq 0$ and since $g_h(m)$ is strictly increasing, there exists $m^\star$ such that for every $m < m^\star$ the club-size elasticity $\eta_h(m)$ is decreasing ($g_h(m) < 1$) while for every $m > m^\star$, $\eta_h(m)$ is increasing ($g_h(m) > 1$). Thus, generally, $\eta_h(m)$ is unimodal with a single minimum at $m^\star$.

Thus, generally, $\eta_h(m)$ can be divided to four parts in the following order:

(i) $\eta_h(m) > -1$ and $\eta_h(m)$ is decreasing.

(ii) $\eta_h(m) < -1$ and $\eta_h(m)$ is decreasing.

(iii) $\eta_h(m) < -1$ and $\eta_h(m)$ is increasing.

(iv) $\eta_h(m) > -1$ and $\eta_h(m)$ is increasing.

Therefore, by Lemma 2 in the main text, $k_h(m)$ has at most three parts, the first increasing (corresponding to (i)), the second decreasing (corresponding to (ii) and (iii)) and the third increasing again (corresponding to (iv)). Hence, for $n_a \geq 4$, $k_h(m)$ has at most two extreme points, $\bar{m} < \bar{\bar{m}}$, where $\bar{m}$ is a local maximum and $\bar{\bar{m}}$ is a local minimum. □

Lemma B.5. *For every $n_a \geq 4$:*

1. $\hat{\delta}(n_a) > \frac{1}{2}$.
2. $\forall \delta \in (0, \hat{\delta}(n_a)) : b(\delta, n_a) > 0$.
3. $\arg\max_{\delta \in (0,1)} b(\delta, n_a) = \hat{\delta}^2(n_a)$.

Proof. First, note that $\hat{\delta}(n_a) = (\frac{1}{n_a-1})^{\frac{1}{n_a-2}}$ is the unique root of $b(\delta, n_a)$ that is real, positive and smaller than one. Next,

$$\frac{\partial \hat{\delta}(n_a)}{\partial n_a} = \frac{1}{n_a-2} \times (\frac{1}{n_a-1})^{\frac{1}{n_a-2}-1} \times \frac{-1}{(n_a-1)^2} + (\frac{1}{n_a-1})^{\frac{1}{n_a-2}} \times \ln \frac{1}{n_a-1} \times \frac{-1}{(n_a-2)^2}$$

Hence,

$$\frac{\partial \hat{\delta}(n_a)}{\partial n_a} = -\frac{1}{n_a-2} \times (\frac{1}{n_a-1})^{\frac{1}{n_a-2}} \times [\frac{1}{(n_a-1)} + \ln \frac{1}{n_a-1} \times \frac{1}{(n_a-2)}]$$

Since $n_a \geq 4$, $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$ if and only if $\frac{1}{(n_a-1)} + \ln \frac{1}{n_a-1} \times \frac{1}{(n_a-2)} < 0$. Hence, $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$ if and only if $\ln \frac{1}{n_a-1} < -1 + \frac{1}{n_a-1}$. Therefore, if $\ln \frac{1}{n_a-1} < -1$ then $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$. This means that if $n_a > e+1$ then $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$. Since $n_a \geq 4$ we showed that $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$.

Note that $\hat{\delta}(4) = (\frac{1}{3})^{\frac{1}{2}} \approx 0.577$. Hence $\hat{\delta}(4) > \frac{1}{2}$. Since $\frac{\partial \hat{\delta}(n_a)}{\partial n_a} > 0$ we get that $\hat{\delta}(n_a) > \frac{1}{2}$ for every $n_a \geq 4$.

For every $n_a \geq 4$, $b(0, n_a) = 0$ and $b(\hat{\delta}(n_a), n_a) = 0$ and there is no other $\delta \in [0, \hat{\delta}(n_a)]$ such that $b(\delta, n_a) = 0$. Since $b(\delta, n_a)$ is continuous and its derivative with respect to δ at $\delta = 0$ is positive when $n_a \geq 4$ ($\frac{\partial b(\delta, n_a)}{\partial \delta}(0, n_a) = \frac{1}{n_a-2} > 0$), we infer that $\forall \delta \in (0, \hat{\delta}(n_a)) : b(\delta, n_a) > 0$ when $n_a \geq 4$.

Finally,

$$\frac{\partial b(\delta, n_a)}{\partial \delta} = \frac{1}{n_a-2} - \frac{(n_a-1)^2}{n_a-2} \times \delta^{n_a-2}$$

Thus, for a given n_a, the maximum of $b(\delta, n_a)$ is achieved at $\delta = (\frac{1}{n_a-1})^{\frac{2}{n_a-2}} = \hat{\delta}^2(n_a)$. □

Proof

Proof. First, by Lemma B.3, if $m > m'$ and $k_h(m) > k_h(m')$ for a given parameter a then $k_h(m) > k_h(m')$ for every $\bar{a} \in [a, 1-\delta)$. Hence, if $m^\star$ is the club size that maximizes the DCV for a, then for every $\bar{a} \in [a, 1-\delta)$ the DCV is maximized by $m \geq m^\star$. Hence, the club size that maximizes the DCV weakly increases with a (Part 1).

Second, we show that if $a \in [0, \min\{b(\delta, n_a), \delta(1-2\delta)\})$ then the DCV is maximized at $m = 2$. We begin by considering the case of $h(m) = a + \delta^{m-1}$ where $\delta \in (0,1)$, $a \in (0, \min\{b(\delta, n_a), \delta(1-2\delta)\})$ and $a + \delta \in (0,1)$. By Lemma B.4, $k_h(m)$ has at most two extreme points, $\bar{m} < \bar{\bar{m}}$, where $\bar{m}$ is a local maximum and $\bar{\bar{m}}$ is a local minimum. If $k_h(2) > k_h(3)$, rewritten as $a < \delta(1-2\delta)$, then $m = 2$ must be the local integer maximum of $k_h(m)$. Therefore, in these cases the global integer maximum is either at $m = 2$ or at $m = n_a$. Hence, if also $k_h(2) > k_h(n_a)$, rewritten as $a < \frac{1}{n_a-2}\delta(1-(n_a-1)\delta^{n_a-2}) = b(n_a, \delta)$, then the global integer maximum is at $m = 2$. Thus, if $a \in (0, \min\{\delta(1-2\delta), b(n_a, \delta)\})$ then $k_h(m)$ is maximized at $m = 2$. Note that $\delta(1-2\delta) > 0$ if and only if $\delta \in (0, \frac{1}{2})$. Hence, it is left to be shown that if $a = 0$ and $\delta \in (0, \frac{1}{2})$ then $k_h(m)$ is maximized at $m = 2$. In this case $h(m) = \delta^{m-1}$ and therefore the club-size elasticity is $\eta_h(m) = m(\delta - 1)$. Thus, the congestion function is elastic if $\delta < \frac{1}{2}$ since then $\eta_h(m) < -1$ for every club size. By Lemma 2 in the main text, $k_h(m)$ is decreasing and therefore maximized at $m = 2$. Hence, if $a \in [0, \min\{\delta(1-2\delta), b(n_a, \delta)\})$ then the DCV is maximized at $m = 2$ (Part 2).

Recall that $\delta(1-2\delta)$ is positive if and only if $\delta \in (0, \frac{1}{2})$ and that its derivative with respect to δ at $\delta = 0$ is one ($\frac{\partial \delta(1-2\delta)}{\partial \delta}(\delta = 0) = 1$). Also recall that when $n_a \geq 4$ by Lemma B.5, $b(n_a, \delta)$ is positive when $\delta \in (0, \hat{\delta}(n_a))$ where $\hat{\delta}(n_a) > \frac{1}{2}$ and its derivative with respect to δ at $\delta = 0$ is $\frac{1}{n_a-2} < 1$. Hence, these two function cross for some $\delta \in (0, \frac{1}{2})$ and since both are single peaked at this region, we denote it by $\delta^\star$ ($b(\delta^\star, n_a) = \delta^\star(1-2\delta^\star)$). Therefore, there is a unique $\delta^\star(n_a) \in (0, \frac{1}{2})$ such that $\forall \delta \in (0, \delta^\star) : \delta(1-2\delta) > b(n_a, \delta)$ and $\forall \delta \in (\delta^\star, \frac{1}{2}) : \delta(1-2\delta) < b(n_a, \delta)$. Consider the case where $\delta \in (0, \delta^\star)$ and $a \in (b(\delta, n_a), \delta(1-2\delta))$. In this range, $m = 2$ must be the local integer maximum of $k_h(m)$ (since $a < \delta(1-2\delta)$). However, the global maximum is $m = n_a$ since $a > b(\delta, n_a)$. Thus, for $\delta \in (0, \delta^\star)$ and $a \in (b(\delta, n_a), \delta(1-2\delta))$ the DCV is maximized at $m = n_a$. However, by Lemma B.3, by increasing a the club size that maximizes the DCV cannot decrease. Since n_a is the maximal size, then for $\delta \in (0, \delta^\star)$ and $a \in (b(\delta, n_a), 1-\delta)$ the DCV is maximized at $m = n_a$ (Part 3).

Next, consider the case where $\delta \in (\delta^\star, \hat{\delta})$ and $a \in (\max\{0, \delta(1-2\delta)\}, b(\delta, n_a))$. In this range, $m = 2$ is not the local integer maximum of $k_h(m)$ (since $a > \delta(1-2\delta)$). But, $k_h(2) > k_h(n_a)$ since $a < b(\delta, n_a)$. Therefore, the DCV is not maximized by $m = 2$ and it is not maximized by $m = n_a$. Therefore, the DCV is maximized at $m \in \{3, \ldots, n_a - 1\}$ (Part 4).

Next, consider the case where $\delta \in [\frac{1}{2}, 1 - \frac{1}{n_a-1}]$ and $a = 0$. In this case the congestion function reduces to $h(m) = \delta^{m-1}$ where $\delta \in [\frac{1}{2}, 1 - \frac{1}{n_a-1}]$. As a continuous function $k_h(m) = (m-1)\delta^{m-1}$ is single peaked and achieves its maximum at $m^\star = 1 - \frac{1}{\ln \delta}$. Therefore, the highest values achieved by integers are either in $\lfloor 1 - \frac{1}{\ln \delta} \rfloor$ or $\lceil 1 - \frac{1}{\ln \delta} \rceil$. Hence, when $a = 0$ and the club-size elasticity is indeterminate ($1 - \frac{1}{n_a-1} \geq \delta \geq \frac{1}{2}$) the DCV is maximized either at $m = \lfloor 1 - \frac{1}{\ln \delta} \rfloor$ or at $m = \lceil 1 - \frac{1}{\ln \delta} \rceil$ (Part 5).

Finally, consider the case where $\delta \in (1 - \frac{1}{n_a-1}, 1)$ and $a = 0$. Then the club congestion function becomes $h(m) = \delta^{m-1}$ where $\delta \in (1 - \frac{1}{n_a - 1}, 1)$. The club-size elasticity is $\eta_h(m) = m(\delta - 1)$ and the congestion function is inelastic since for $\delta > 1 - \frac{1}{n_a-1}$ we get $\eta_h(m) > -1$ for every club size. By Lemma 2 in the main text, $k_h(m)$ is increasing and therefore maximized at $m = n_a$. In addition, by Lemma B.3, by increasing a the club size that maximizes the DCV cannot decrease. Since n_a is the maximal size, then for $\delta \in (1 - \frac{1}{n_a-1}, 1)$ and every legitimate value of a the DCV is maximized at $m = n_a$ (Part 6). □

Proposition 2

Proof. The case of the reciprocal club congestion function is based on the DCV being a constant function that equals to 1. The case of the exponential club congestion function is based on Lemma 1. The first case results from Part 2 and from $k_h(2) = a + \delta$. The second case is an implication of parts 3 and 6 (recall that $k_h(n_a) = (n_a - 1)(a + \delta^{n_a - 1}))$. The final case results from Part 5. □

Claim 1

Proof. Let $h(\cdot)$ be an exponential club congestion function where $\delta \in (0, \frac{1}{2})$ and $a > 0$. By Proposition 3 in the main text and Lemma A.1 in the appendix of the main text, for every $n_a > m$, there exists a range of membership fees where the m-complete environment is OCS if $(m-1)[(a + \delta^{m-1}) - (a + \delta^{m-1})^2] \geq \delta - \delta^{m-1}$. Note that the right-hand-side of the inequality is bounded from above by $\delta < \frac{1}{2}$ and the left-hand-side of the inequality can be written as

$$(m-1)(a-a^2) + (1-2a)(m-1)\delta^{m-1} - (m-1)\delta^{2(m-1)}$$

Then, $(1-2a)(m-1)\delta^{m-1}$ and $(m-1)\delta^{2(m-1)}$ go to zero when m goes to infinity, while, since $a \in (0,1)$, $(m-1)(a-a^2)$ goes to infinity when m goes to infinity. Thus, the left-hand-side of the inequality is not bounded. Moreover, since the left-hand-side of the inequality is monotonic from some club size (depends on δ and a) there exists $\bar{m}$ such that $\forall m : m > \bar{m}$ the inequality holds. Thus, $\forall m : n_a > m > \bar{m}$ there exists a range of c in which the m-complete environment is OCS.

For similar reasons there exists an integer $\tilde{m}$ such that $\forall m : m > \tilde{m}$ the upper bound is higher than δ (since δ is greater than the right-hand-side for every $m \geq 2$, $\tilde{m} \geq \bar{m}$). Let $\bar{c} = (\tilde{m}-1)[(a + \delta^{\tilde{m}-1}) - (a + \delta^{\tilde{m}-1})^2]$. Thus, in the membership fees range $(\delta, \bar{c})$, every m-complete environment where $m \geq \tilde{m}$ is OCS. □

Claim 2

Proof. By Proposition 4 in the main text, $J_h(m) = (m-1)[h(m+1) - h^2(m)]$. Since club congestion functions are assumed to be non-increasing, we get $h(m) \geq h(m+1) - h^2(m)$. Therefore,

$(m-1)h(m) \geq (m-1)[h(m+1) - h^2(m)]$. Hence, $K_h(m) \geq J_h(m)$. If

$$J_h(m) \geq \max\{\max_{\gamma \geq k \geq 2} FNS_h(k,m), \max_{m \geq k > \gamma} FNI_h(k,m,n_a), \max_{\min\{l_h,n_a\} > k > m} FNL_h(k,m,n_a)\}$$

then by the second part of Proposition 4 in the main text, a range of membership fees for which the m-Star environment is OCS is guaranteed. $\square$