Online Supplementary Appendix to Preference Conditions for Invertible Demand Functions

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Some additional technical results are presented herein that support and extend the analysis in the main paper. We also revisit the well-known results in Katzner (1970) and Hurwicz and Uzawa (1971) on sufficient conditions for an invertible demand function and homothetic preferences generating a non-invertible demand function, respectively, placing them in context as special cases of our analysis.

I. Preliminaries

In what follows, we maintain the notation introduced in the main text but extend the setting to the consumption set X being a convex subset of $\mathbb{R}^n_+$ with $\text{int}(X) \neq \emptyset$ - where $\text{int}(X)$ denotes the interior of X. That is, we allow also for consumption bundles on the boundary of $\mathbb{R}^n_+$.

To account for such bundles, we relax slightly the standard notions of strict convexity and strict monotonicity given in the main text. In particular, we will say that a continuous weak order $\succsim$ on X is *strictly convex* if, for all $(x, y) \in X \times \text{int}(X) \cup \text{int}(X) \times X$ and any $\alpha \in (0, 1)$, $x \succsim y$ implies $\alpha x + (1-\alpha)y \succ y$. Similarly, we will say that $\succsim$ is *strictly monotonic* if, for all $(x, y) \in X \times \text{int}(X) \cup \text{int}(X) \times X$, $x > y$ implies $x \succ y$. This slight weakening of the standard definitions in the main text refers to a restriction (of measure zero in $\mathbb{R}^n_+$) on the domain where the standard notions of strict convexity and strict monotonicity apply; namely, we consider these notions on X without requiring though that they hold also on the boundary of X. For instance, letting $X = \mathbb{R}^n_+$, our notions of strict convexity and strict monotonicity coincide with those in the main text for pairs of bundles when at least one bundle lies in $\mathbb{R}^n_{++}$; yet, they are not imposing any restrictions when both bundles lie on the boundary of $\mathbb{R}^n_+$. This allows our analysis to include preferences that are strictly convex and strictly monotonic on $\mathbb{R}^n_{++}$ but for which the boundary of $\mathbb{R}^n_+$ is an indifference set, a well-known example being the Cobb-Douglas preferences on $\mathbb{R}^n_+$ (see Section II below).

For the more general case where X is a convex subset of $\mathbb{R}^n_+$ (with non-empty interior) all of our results in the main text continue to hold, subject to trivial adjustments in the respective statements. Specifically, the expression $x \in X$ should be replaced everywhere by $x \in \text{int}(X)$ while the demand function should be written as $x : Y \to \text{int}(X)$. Furthermore, $\succsim$ being differentiable or weakly smooth of order 1 should apply only on $\text{int}(X)$. The latter should also be the domain on which the utility function $u : X \to \mathbb{R}$ is C^1 in Proposition 4.

The only non-trivial adaptation of our arguments concerns the first part of the proof for Proposition 1. For the more general case, we must complement Lemma II.1 in the main text with the following result. This provides now the desired contradiction for the argument in the proof of Lemma II.2 in the main text to remain valid.

LEMMA I.1: *Let $\succsim$ be a strictly convex, continuous weak order on X. For any $x \in int(X)$, $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x only if $x \in \max_{\succsim}\{z \in X : pz \leq px\}$.*

PROOF:

Let $p \in \mathbb{R}^n \setminus \{0\}$ support $\mathcal{U}_x$ at x. It suffices to show that $z \in X \setminus \{x\}$ and $pz \leq px$ implies $x \succsim z$. As this is obvious when $pz < px$, suppose that $pz = px$ and assume to the contrary that $z \succ x$. Define $z^\lambda = \lambda z + (1-\lambda)x$ for $\lambda \in (0,1)$ and observe that, since $||z^\lambda - x|| = \lambda||z - x||$, for any given $\varepsilon > 0$ we have $z^\lambda \in \mathcal{B}_\varepsilon(x)$ for sufficiently small λ. And as $x \in \text{int}(X)$, we have in fact $z^\lambda \in \mathcal{B}_\varepsilon(x) \subset \text{int}(X)$ for sufficiently small ε. However, by the strict convexity of $\succsim$, it must be $z^\lambda \succ x$ and thus $x \notin \max_{\succsim}\{z \in \text{int}(X) : pz \leq px\}$, a contradiction of Lemma II.1 in the main text. □

II. Onto Demand Functions with Full Price Domain

A special case of particular interest corresponds to the domain of the demand function being the entire orthant of strictly positive prices. When $X = \mathbb{R}^n_+$, we get that $Y = \mathbb{R}^n_{++}$ in Proposition 1 in the main text if and only if $\succsim$ is in addition *self-contained* in $\mathbb{R}^n_{++}$ in the sense that

$$\forall (z,x) \in \mathbb{R}^n_{++} \times X, z \sim x \ \Rightarrow\ x \in \mathbb{R}^n_{++} \tag{1}$$

PROPOSITION 1: *Let the onto demand function $x : Y \to \mathbb{R}^n_{++}$ for some $Y \subseteq \mathbb{R}^n_{++}$ be generated by the continuous weak order $\succsim$ on $X = \mathbb{R}^n_+$. Then $Y = \mathbb{R}^n_{++}$ if and only if $\succsim$ is self-contained in $\mathbb{R}^n_{++}$.*

The full price domain requires that the consumption domain includes the entire orthant of strictly positive bundles (see Lemma II.1 below). To obtain also a sufficient condition for the full price domain, we must extend X to the entire $\mathbb{R}^n_+$. In fact, the extension can be done in only one way: under the maintained assumption of strict monotonicity, $\succsim$ being self-contained in $\mathbb{R}^n_{++}$ means that the boundary $\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ (which we will denote also by $\text{bd}(\mathbb{R}^n_+)$ below) forms an indifference set (see Lemma II.2 below). The latter property implies in turn that the preference relation is indeed self-contained in $\mathbb{R}^n_{++}$. For this is the relevant consumption set if one assumes –as is often the case in the literature– that the main interest of the analysis is in strictly positive consumption bundles: extending the preference domain from $\mathbb{R}^n_{++}$ to $\mathbb{R}^n_+$ adds an indifference set which lies at the very bottom of the preference ranking over indifference sets, leaving the non-trivial part of the ranking unaffected. Put differently, restricting the preference domain to $\mathbb{R}^n_{++}$ instead of $\mathbb{R}^n_+$ is without loss of generality because no additional information about demand can be obtained by examining how preferences "behave" on the boundary $\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$.

The requirement that $X = \mathbb{R}^n_+$ while the preference relation is self-contained in $\mathbb{R}^n_{++}$ has appeared in the literature as Assumption 3.1-4 in Katzner (1970). It is satisfied, for example, by Cobb-Douglas and Leontieff preferences, but not by quasilinear preferences. Theorem 3.1-13 in Katzner (1970) states that a demand function $x : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is bijective if it is generated by a preference relation $\succsim$ on $\mathbb{R}^n_+$ which is self-contained in $\mathbb{R}^n_{++}$ and representable by a strictly concave and strictly increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that is twice continuously differentiable on $\mathbb{R}^n_{++}$. Proposition 1 above underlines the limits of the scope of Katzner's theorem: the price domain of the demand function under consideration must be the entire $\mathbb{R}^n_{++}$.

LEMMA II.1: *Let the demand correspondence $x : Y \twoheadrightarrow X$ for some $Y \subseteq \mathbb{R}^n_{++}$ be generated by the strictly monotonic, continuous weak order $\succsim$ on $X \subseteq \mathbb{R}^n_+$. Then $Y = \mathbb{R}^n_{++}$ only if $\mathbb{R}^n_{++} \subseteq X$.*

PROOF:

We will show first that X_i is unbounded above. To establish this arguing ad absurdum, let there be $b > 0$ such that $x_i \leq b$ for all $x \in X$. Take also any $p \in Y$ and let $x \in x(p)$. As the strict monotonicity of $\succsim$ guarantees Walras' law, we have $1 = px \leq bp_i + p_{-i}x_{-i}$; equivalently, $p_i \geq (1 - p_{-i}x_{-i})/b$. Recall though that, by hypothesis, $Y = \mathbb{R}^n_{++}$. That is, $Y_{-i} = \mathbb{R}^{n-1}_{++}$ and we can take $|p_{-i}|$ to be arbitrarily small. For the last inequality above, therefore, to hold for all $p_{-i} \in Y_{-i}$, it cannot but be $p_i \geq 1/b$ for all $p \in Y$. But this is absurd given that $Y_i = \mathbb{R}_{++}$.
We will show next that $\inf_{x \in X} x_i = 0$. To establish this arguing again ad absurdum, let there be $a > 0$ such that $x_i \geq a$ for all $x \in X$. Taking again an arbitrary $p \in Y$ and letting $x = x(p)$, we now have $1 = px \geq ap_i + p_{-i}x_{-i}$; equivalently, $p_i \leq (1 - p_{-i}x_{-i})/a < 1/a$. Which is again absurd given that $Y_i = \mathbb{R}_{++}$.
Observe finally that, X being convex, so is X_i. It follows thus that X_i is path-connected; hence, connected. Clearly, X_i is an interval on $\mathbb{R}_{++}$ - see for instance Theorem 6.76 in Bowder (1996). And given the observations in the preceding two paragraphs, the claim follows. □

PROOF OF PROPOSITION 1We observe first that $\succsim$ is necessarily strictly convex and strictly monotonic (recall Proposition 1 in the main text).
"if". Take an arbitrary $p \in \mathbb{R}^n_{++}$. Being a continuous weak order on $X = \mathbb{R}^n_+$, $\succsim$ can be represented by a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. As a result, $B(p) := \{x \in X : px \leq 1\}$ being compact, the solution set of $\max_{z \in B(p)} u(z)$ is guaranteed to be non-empty; it is also a singleton by the strict convexity of $\succsim$. We will also make use of the following result.

LEMMA II.2: *Let $\succsim$ be a strictly monotonic, continuous weak order on $X = \mathbb{R}^n_+$. Then $\succsim$ is self-contained in $\mathbb{R}^n_{++}$ if and only if $\mathcal{I}_{\mathbf{0}} = bd(\mathbb{R}^n_+)$.*

PROOF:

That a strictly monotonic weak order on $X = \mathbb{R}^n_+$ satisfies (1) if $\mathcal{I}_{\mathbf{0}} = \text{bd}\left(\mathbb{R}^n_+\right)$ is obvious. For the "only if" part, observe first that, since $z \succ \mathbf{0}$ for any $z \in \mathbb{R}^n_{++}$, it must be $\mathcal{I}_0 \subseteq \text{bd}\left(\mathbb{R}^n_+\right)$. To show that also $\text{bd}\left(\mathbb{R}^n_+\right) \subseteq \mathcal{I}_{\mathbf{0}}$, suppose that there exist $x, x' \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$ such that $x \succ x'$ and let $x'' \in \mathbb{R}^n_{++}$ be given by $x''_i := x_i$ for $i \in \mathcal{N}^+_x$ and $x''_i := \varepsilon$ for $i \in \mathcal{N} \setminus \mathcal{N}^+_x$ and for some $\varepsilon > 0$. By the strict monotonicity of $\succsim$, we have that $x'' \succ x$. Define then the function $z : [0,1] \to \mathbb{R}^n_+$ by $z(\lambda) := \lambda x'' + (1-\lambda)x'$. Since $z(1) = x''$ while $x' = z(0)$, we have that $u(z(0)) < u(x) < u(z(1))$; the intermediate-value theorem ensures the existence of $\lambda_0 \in (0,1)$ such that $z(\lambda_0) \in \mathcal{I}_x$. As $z(\lambda_0) \in \mathbb{R}^n_{++}$, however, $\succsim$ cannot be self-contained in $\mathbb{R}^n_{++}$. We just established that $\succsim$ is self-contained in $\mathbb{R}^n_{++}$ only if $x \sim x'$ for any $x, x' \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$. That it must be also $0 \sim x$ for any $x \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$ follows from the continuity of $\succsim$. $\square$

Let now $x_0 := \max_{z \in B(p)} u(z)$. As the strict monotonicity of $\succsim$ guarantees Walras' law, we have that $1 = px_0$. Clearly, $x_0 \in X \setminus \{\mathbf{0}\}$ and thus $x_0 \succ \mathbf{0}$ (again by the strict monotonicity of $\succsim$). As though $X = \mathbb{R}^n_+$ while $\succsim$ is self-contained in $\mathbb{R}^n_{++}$, $x_0 \succ \mathbf{0}$ requires in fact that $x_0 \in \mathbb{R}^n_{++}$ (Lemma II.2). The latter being the image set of the onto function $x(\cdot)$, that $p \in Y$ follows from the very definition of Y.

"only if". To establish the contrapositive statement, we will make use of the following results.

LEMMA II.3: *Let $\succsim$ be a strictly monotonic, continuous weak order on $X = \mathbb{R}^n_+$. Then $\succsim$ is self-contained in $\mathbb{R}^n_{++}$ if it satisfies the following condition*[1]

A 1: For all $z \in \mathbb{R}^n_{++}$ and $x \in bd(\mathbb{R}^n_+) \setminus \{\mathbf{0}\}$, $z - x$ is an improvement direction at x.

PROOF:

We will establish the contrapositive statement. To this end, observe first that, as $z \succ \mathbf{0}$ for any $z \in \mathbb{R}^n_{++}$, it must be $\mathcal{I}_0 \subseteq \text{bd}\left(\mathbb{R}^n_+\right)$. By Lemma II.2, therefore, there exists $x \in \text{bd}\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$ with $x \succ \mathbf{0}$ if $\succsim$ is not self-contained in $\mathbb{R}^n_{++}$. Letting e denote the vector of ones in $\mathbb{R}^n$, define now the function $z : [0,1] \to \mathbb{R}^n_+$ by $z(\mu) = \mu e$. Since $\lim_{\mu \to 0} u(z(\mu)) = u(\mathbf{0}) < u(x)$, the continuity of $u(\cdot)$ ensures that $x \succ z(\mu_0)$ for small enough $\mu_0 > 0$. Define next the function $z^0 : [0,1] \to \mathbb{R}^n_+$ by $z^0(\lambda) = \lambda x + (1-\lambda) z(\mu_0)$. As $\lim_{\lambda \to 0} u\left(z^0(\lambda)\right) = u(z(\mu_0)) < u(x)$, the continuity of $u(\cdot)$ ensures now the existence of $\lambda_0 \in (0,1)$ such that $x \succ \lambda x + (1-\lambda) z(\mu_0) = x + (1-\lambda)(z(\mu_0) - x)$ for all $\lambda \in (0, \lambda_0)$. But this means that $z(\mu_0) - x$ is not an improvement direction at x. $\square$

LEMMA II.4: *Let $\succsim$ be a strictly monotonic and strictly convex, continuous weak order on $X = \mathbb{R}^n_+$. Let also $x \in bd\left(\mathbb{R}^n_+\right) \setminus \{\mathbf{0}\}$. There exists $z \in \mathbb{R}^n_{++}$ such that*

[1] We thank Phil Reny for suggesting condition A1 to us.

$z - x$ is not an improvement direction at x only if there exists $p \in \mathbb{R}^n_{++}$ such that $x = \max_{\succsim} \{z \in X : pz \le px\}$.

PROOF:

Observe first that, $\succsim$ being convex and continuous, $\mathcal{U}_x$ is convex and closed. Suppose now that, for some $z \in \mathbb{R}^n_{++}$, $z - x$ is not an improvement direction at x. We must have then $x \succsim (1-\lambda)\, x + \lambda z \in X$ for arbitrarily small $\lambda > 0$. Given this and the strict monotonicity of $\succsim$, it is trivial to check that the convex and compact set

$$\mathcal{L}_{x,z} := \left\{ \widetilde{z} \in \mathbb{R}^n_{++} : \widetilde{z}_i \in \left\{ \begin{array}{ll} [x_i/2, x_i] & i \in \mathcal{N}^+_x \\ [z_i/3, z_i/2] & i \in \mathcal{N} \setminus \mathcal{N}^+_x \end{array} \right\} \right\}$$

gives $x \succ (1-\lambda)\, x + \lambda \widetilde{z}$ for any $\widetilde{z} \in \mathcal{L}_{x,z}$. That is, $(1-\lambda)\{x\} + \lambda \mathcal{L}_{x,z} \cap \mathcal{U}_x = \varnothing$ and by the separating hyperplane theorem (see for instance Theorem 1.F.2.2 in Mas-Colell (1985)) there exists $\widetilde{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\widetilde{p}\left((1-\lambda)\, x + \lambda \widetilde{z}\right) < \widetilde{p}\widetilde{x}$ for any $(\widetilde{z}, \widetilde{x}) \in \mathcal{L}_{x,z} \times \mathcal{U}_x$.

It must be in fact $\widetilde{p} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$. To see this arguing ad absurdum, suppose first that $\widetilde{p}_i < 0$ for some $i \in \mathcal{N}^+_x$. Let $z' \in \mathcal{L}_{x,z}$ be given by $z'_i := x_i/2$, $z'_j := x_j$ for $j \in \mathcal{N}^+_x \setminus \{i\}$ and $z'_j := z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$. Let also $x' \in \mathcal{U}_x$ be given by $x'_i := x_i$ for $i \in \mathcal{N}^+_x$ and $x'_j := \lambda z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$. It is trivial to verify that this leads to the absurdity $\widetilde{p}\left((1-\lambda)\, x + \lambda z'\right) > \widetilde{p}x'$. Suppose next that $\widetilde{p}_i < 0$ for some $i \in \mathcal{N} \setminus \mathcal{N}^+_x$. Let $z'' \in \mathcal{L}_{x,z}$ be given by $z''_i := z_i/3$, $z''_j := x_j$ for $j \in \mathcal{N}^+_x$ and $z'_j := z_j/2$ for $j \in \mathcal{N} \setminus (\mathcal{N}^+_x \cap \{i\})$. Let also $x'' \in \mathcal{U}_x$ be given by $x''_j := x_j$ for $j \in \mathcal{N}^+_x$ and $x''_j := \lambda z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$. This implies now the absurdity $\widetilde{p}\left((1-\lambda)\, x + \lambda z''\right) > \widetilde{p}x''$.

Take now any $i \in \mathcal{N}^+_x$. Let z' be defined as above, and observe that $(z' - x)_i = -x_i/2$, $(z' - x)_i = 0$ for $j \in \mathcal{N}^+_x \setminus \{i\}$ while $(z' - x)_j = (1-\lambda)\, z_j/2$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$. As $0 > \widetilde{p}\left((1-\lambda)\, x + \lambda z' - x\right) = \lambda \widetilde{p}\,(z' - x)$ reads also $\widetilde{p}_i x_i/2 > (1-\lambda) \sum_{j \in \mathcal{N} \setminus \mathcal{N}^+_x} \widetilde{p}_j z_j/2 \ge 0$, it must be $\widetilde{p}_i > 0$.

We have established the existence of $\widetilde{p} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ which supports $\mathcal{U}_x$ at x and has $\widetilde{p}_i > 0$ for any $i \in \mathcal{N}^+_x$. Define now $p \in \mathbb{R}^n_{++}$ by $p_i := \widetilde{p}_i$ for $i \in \mathcal{N}^+_x$ and $p_j := \widetilde{p}_j + \varepsilon$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$ and for some $\varepsilon > \max_{j \in \mathcal{N} \setminus \mathcal{N}^+_x} |\widetilde{p}_j|$. Since $p \gg \widetilde{p}$, we have $(p - \widetilde{p})\, z \ge 0$ for any $z \in X$. Moreover, as $x_j = 0$ for $j \in \mathcal{N} \setminus \mathcal{N}^+_x$, we also have $px = \widetilde{p}x$. And as $\widetilde{p}$ supports $\mathcal{U}_x$ at x, it follows that $pz \ge \widetilde{p}z \ge \widetilde{p}x = px$ for any $z \in \mathcal{U}_x$. Clearly, p also supports $\mathcal{U}_x$ at x. That $x = \max_{\succsim} \{z \in X : pz \le px\}$ follows now from Lemma II.4 in the main text. □

Suppose now that $\succsim$ is not self-contained in $\mathbb{R}^n_{++}$. By Lemma II.3, it does not satisfy condition A1 either. There exist thus $(z, x) \in \mathbb{R}^n_{++} \times \mathrm{bd}(\mathbb{R}^n_+) \setminus \{\mathbf{0}\}$ such that $z - x$ is not an improvement direction at x. But then, by the preceding lemma (and the strict convexity of $\succsim$), there exists also $p \in \mathbb{R}^n_{++}$ such that $x =$

$\max_{\succsim}\{z \in X : pz \le px\}$; equivalently, $x = \max_{\succsim}\{z \in X : \hat{p}z \le 1\}$ where $\hat{p} := p/px$.

Suppose then that $Y = \mathbb{R}^n_{++}$. The function $x(\cdot)$ being onto, there exists $x(\hat{p}) \in \mathbb{R}^n_{++}$. And as obviously $x \ne x(\hat{p})$, we can define the function $\overline{x} : [0,1] \to \mathbb{R}^n_+$ by $\overline{x}(\lambda) := \lambda x + (1-\lambda) x(\hat{p})$. Since the strict monotonicity of $\succsim$ guarantees Walras' law, we have $\hat{p}x(\hat{p}) = 1 = \hat{p}x$; thus $\hat{p}\overline{x}(\lambda) = 1$ for all $\lambda \in (0,1)$. Moreover, as $\lim_{\lambda\to 0}\overline{x}(\lambda) = x(\hat{p})$, we also have $\overline{x}(\lambda) \in \mathbb{R}^n_{++}$ for small enough λ. A contradiction now obtains because, due to the strict convexity of $\succsim$, $x \sim x(\hat{p})$ means that $\overline{x}(\lambda) \succ x(\hat{p})$. ■

III. Supporting Results on Preference Gradients

In this section, we present some results that support our attempt in the main text to give an intuitive geometric inerpretation of preference differentiability via the notions of supporting hyperplanes and ordients.

LEMMA III.1: *Let $\succsim$ be a weak order on X and $x \in int(X)$. The collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x is a subset of the collection of decreasing ordients at x.*

PROOF:

Since $x \in \text{int}(X)$, for any $z \in X$, we have $x + \lambda(z-x) \in \text{int}(X) \subseteq X$ for sufficiently small $\lambda > 0$. Let now $p \in \mathbb{R}^n \setminus \{0\}$ support $\mathcal{U}_x$ at x and take $z \in H^-_{p,x}$. As $p(z-x) < 0$ is equivalent to $p(x + \lambda(z-x)) < px$, it must be $x \succ x + \lambda(z-x)$ for any $\lambda > 0$. That is, $z - x$ must be a worsening direction at x and p a decreasing ordient at x. □

LEMMA III.2: *Let $\succsim$ be a convex weak order on X and $x \in int(X)$. The collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x coincides with the collection of decreasing ordients at x.*

PROOF:

Since $x \in \text{int}(X)$, for any $z \in X$, we have $x + \lambda(z-x) \in \text{int}(X) \subseteq X$ for sufficiently small $\lambda > 0$. By Lemma III.1, moreover, it suffices to establish the collection of decreasing ordients at x as subset of the collection of $p \in \mathbb{R}^n \setminus \{0\}$ that support $\mathcal{U}_x$ at x. To this end, take $z \in \mathcal{U}_x$ and let $p \in \mathbb{R}^n \setminus \{0\}$ be a decreasing ordient at x. As $\succsim$ is convex, any $\lambda \in [0,1]$ gives $x + \lambda(z-x) = \lambda z + (1-\lambda)x \in \mathcal{U}_x$. Clearly, $z - x$ is not a worsening direction at x and thus we must have $p(z-x) \ge 0$. □

For the next result, we should point out that Lemma II.7 in the main text remains valid in the present more general case without any changes in the statement.

LEMMA III.3: *Let $\succsim$ be a strictly convex [resp. strictly convex and strictly monotonic] weak order on X and $x \in int(X)$. The collection of preference gradients at x is a subset* *of the collection of $p \in \mathbb{R}^n \setminus \{0\}$ [resp. $p \in \mathbb{R}^n_{++}$] that support $\mathcal{U}_x$ at x properly. As a result, the collection of preference gradients at x is a subset of the collection of ordients at x.*

PROOF:

Since $x \in \text{int}(X)$, for any $z \in X$, we have $x + \lambda(z - x) \in \text{int}(X)$ for sufficiently small $\lambda > 0$. The first part of the claim follows from Lemma II.7 in the main text [resp. Lemmas II.2 amd II.7 in the main text]. For the second part, observe that, in light also of Lemma III.2 above, a preference gradient at x is a decreasing ordient at x. The claim now follows from the fact that, by definition, a preference gradient at x is necessarily an increasing ordient at x. $\square$

LEMMA III.4: *Let $\succsim$ be a continuous weak order on $X \subseteq \mathbb{R}^n_+$ and suppose that $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at $x \in$ int (X). Then $v \in \mathbb{R}^n \setminus \{0\}$ is an improvement direction at x only if $pv > 0$.*

PROOF:

Since $x \in \text{int}(X)$, we have $\mathcal{B}_x(\epsilon_0) \subset \text{int}(X)$ for sufficiently small $\epsilon_0 > 0$. Suppose now that $p \in \mathbb{R}^n \setminus \{0\}$ supports $\mathcal{U}_x$ at x, and let $v \in \mathbb{R}^n \setminus \{0\}$ be an improvement direction at x. There exists then $\lambda^* > 0$ such that $x + \lambda v \succ x$ for all $\lambda \in (0, \lambda^*)$. Taking thus $\lambda_0 \in (0, \min\{\lambda^*, \epsilon_0/(2||v||)\})$, we have $x + \lambda_0 v \succ x$ while $x + \lambda_0 v \in \text{int}(X)$. Yet $x + \lambda_0 v \succ x$ can be only if $0 \leq p(x + \lambda_0 v - x) = \lambda_0 pv$.

It suffices therefore to rule out the case $pv = 0$. To establish this ad absurdum recall first that, $\succsim$ being complete and continuous, $\mathcal{U}_x \setminus \mathcal{I}_x$ is open. As a result, $x + \lambda_0 v \succ x$ necessitates that $z \succ x$ for any $z \in \mathcal{B}_{x+\lambda_0 v}(\epsilon_1)$ for sufficiently small $\epsilon_1 > 0$. More specifically, letting $\epsilon_2 \in (0, \min\{\epsilon_1, \epsilon_0/2\})$, we have $z \succ x$ for all $z \in \mathcal{B}_{x+\lambda_0 v}(\epsilon_2) \subset \text{int}(X)$. Taking then $z = x + \lambda v - \varepsilon p$ for some $\varepsilon \in (0, \epsilon_2/||p||)$ ensures that $x + \lambda v - \varepsilon p \in \text{int}(X)$ while $x + \lambda v - \varepsilon p \succ x$. And the latter relation implies in turn that $0 \leq p(x + \lambda v - \varepsilon p - x) = p(\lambda v - \varepsilon p) = \lambda pv - \varepsilon p^\intercal p = -\varepsilon p^\intercal p$, a contradiction. $\square$

LEMMA III.5: *Let $\succsim$ be a strictly convex and strictly monotonic, continuous weak order on $X \subseteq \mathbb{R}^n_+$ and $x \in$ int (X). The collection of preference gradients at x coincides with the collection of increasing ordients at x that support $\mathcal{U}_x$ at x properly. And either collection coincides also with the collection of ordients at x.*

PROOF:

By definition $p \in \mathbb{R}^n \setminus \{0\}$ is a preference gradient at x if it is an increasing ordient at x that satisfies "$v \in \mathbb{R}^n \setminus \{0\}$ is an improvement direction at x only if $pv > 0$." For the first part of the claim, that the collection of increasing ordients at x that support $\mathcal{U}_x$ at x properly is a subset of the collection of preference gradients at x follows from Lemma III.4. The opposite set inclusion is due to Lemma III.3.

For the second part of the claim, again due to Lemma III.3, we only need to show that the collection of ordients at x is a subset of the collection of preference gradients at x. By definition though the former collection is a subset of the collection of decreasing ordients at x, which coincides with the collection of $p \in \mathbb{R}^n_{++}$ that support $\mathcal{U}_x$ at x properly (see Lemmas III.2 above as well as II.2 and II.4 in the main text). Hence, the collection of ordients at x is a subset of the collection of increasing ordients at x that support $\mathcal{U}_x$ at x properly. That the latter collection is a subset of the collection of preference gradients at x is due to the first part of the claim. □

IV. A Very Short Introduction on Local Subgradients

Recall that a set $A \subseteq \mathbb{R}^n$ is said to be locally convex if for every $x \in A$ there is $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \cap A$ is convex. Given a locally convex $A \subseteq \mathbb{R}^n$, a function $f : A \to \mathbb{R}$ is said to be locally convex if for every $x \in A$ there exists $\varepsilon > 0$ such that f is convex on $\mathcal{B}_\varepsilon(x) \cap A$.

Consider now an open and locally convex $A \subseteq \mathbb{R}^n$. Taking any $x_0 \in A$ and any $h \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we have that $x_0 + \lambda h \in \mathcal{B}_{\varepsilon_0}(x) \subset A$ for sufficiently small $\varepsilon_0 > 0$ and for any $\lambda \in (-\varepsilon_0/||h||, \varepsilon_0/||h||)$. If the function $f : S \to \mathbb{R}$ is locally convex, then the directional derivative of f at x_0 in any direction $h \in \mathbb{R}^n$ is well-defined (see Theorem 3.3.4 in Jahn (2007)) and given by

$$f'(x_0)(h) := \lim_{\lambda \searrow 0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}$$

Let next $\{e_1, \ldots, e_n\}$ be the orthonormal basis of $\mathbb{R}^n$. The ith partial derivative of f at x_0 is given by

$$\partial f(x_0)/\partial x_i := \lim_{\lambda \to 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda}$$

if this limit exists. And f is said to be partially differentiable at x_0 if $\partial f(x_0)/\partial x_i$ exists for all $i \in \{1, \ldots, n\}$. Observe also that

$$\begin{aligned}
\lim_{\lambda \searrow 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} &= f'(x_0)(e_i) \\
\lim_{\lambda \nearrow 0} \frac{f(x_0 + \lambda e_i) - f(x_0)}{\lambda} &= -\lim_{\lambda \searrow 0} \frac{f(x_0 - \lambda e_i) - f(x_0)}{\lambda} = -f'(x_0)(-e_i)
\end{aligned}$$

Clearly, the ith partial derivative of f at x_0 exists if and only if $f'(x_0)(e_i) = -f'(x_0)(-e_i)$. More precisely, f is partially differentiable at x_0 if and only if $f'(x_0)(e_i) = \partial f(x_0)/\partial x_i = -f'(x_0)(-e_i)$ for all $i \in \{1, \ldots, n\}$.

We will let also $\partial f(x_0)$ denote the *local subdifferential* of f at x_0. This is the set of vectors $q \in \mathbb{R}^n$ such that

$$f(x) \geq f(x_0) + q(x - x_0) \quad \forall x \in \mathcal{B}_{\varepsilon_0}(x_0)$$

Each $q \in \partial f(x_0)$ will be called a *local subgradient* of f at x_0. And f being locally convex, it is also continuous at x_0 (see Theorem 6.2.14 in de la Fuente (2000)). Which ensures in turn that $\partial f(x_0) \neq \varnothing$ (see Theorem 3.26 in Jahn (2007)), while

$$f'(x_0)(\lambda h) \max\{\lambda q h : q \in \partial f(x_0)\} \tag{2}$$

for any direction $h \in \mathbb{R}^n$ and any $\lambda \in (-\varepsilon_0/||h||, \varepsilon_0/||h||)$ - see Theorem 3.28 in Jahn (2007). In particular, taking $\lambda_0 \in (0, \varepsilon_0)$ we have

$$f'(x_0)(\lambda_0 e_i) = \lambda_0 \max\{q e_i : q \in \partial f(x_0)\}$$

and, thus, also

$$f'(x_0)(-\lambda_0 e_i) = \lambda_0 \min\{q e_i : q \in \partial f(x_0)\}$$

It is trivial then to check that f is partially differentiable at x_0 if and only if $\partial f(x_0)$ is a singleton; more precisely, if and only if $\partial f(x_0) = \{q\}$ for some $q \in \mathbb{R}^n \{\mathbf{0}\}$ while $\partial f(x_0)/\partial x_i = q_i$ for all $i \in \{1, \ldots, n\}$. In this case, (2) reads

$$f'(x_0)(h) = qh$$

while f is actually *differentiable* at x_0 - see Theorem 25.1 in Rockafellar (1970); that is, we have

$$0 = \lim_{x \to x_0} \frac{f(x) - f(x_0) - \nabla f(x_0)(x - x_0)}{||x - x_0||}$$

where $\nabla f(x_0) := (\partial f(x_0)/\partial x_1, \ldots, \partial f(x_0)/\partial x_n)$. And as differentiability implies partial differentiability, we conclude that f is in fact differentiable at x_0 if and only if $\partial f(x_0)$ is a singleton.

V. Supporting Results on Indifference-Projection Functions

Recall Lemma II.5 (and more importantly the corresponding proof) in the main text. It is noteworthy that our argument leads to $l_i(\cdot|x)$ being locally strictly convex because we are able to establish only that $\mathcal{I}_x^{-i}$ is locally convex at x_{-i}. Indeed, it turns out that $l_i(\cdot|x)$ is strictly convex (globally) on $\mathcal{I}_x^{-i}$ if the latter set is (globally) convex.

LEMMA V.1: *Let $\succsim$ be a strictly monotonic and strictly convex, continuous weak order on X. For $i \in \mathcal{N}$ and $x \in X$, suppose also that $\mathcal{I}_x^{-i}$ is convex. Then $l_i(\cdot|x)$ is strictly convex.*

PROOF:

$\succsim$ being a strictly monotonic and strictly convex, continuous weak order, it can be represented by a continuous, strictly monotonic and strictly quasi-concave utility function $u : X \to \mathbb{R}$. Take now any $z, y \in \mathcal{I}_x$ and any $\lambda \in (0,1)$. $\mathcal{I}_x^{-i}$ being convex, there exists $\widetilde{z}_i \in X_i$ such that $\widetilde{z}_i = l_i\left(\lambda z_{-i} + (1-\lambda) y_{-i} | x\right)$. To establish the claim arguing ad absurdum, observe that

$$\widetilde{z}_i = l_i\left(\lambda z_{-i} + (1-\lambda) y_{-i} | x\right) \geq \lambda l_i\left(z_{-i} | x\right) + (1-\lambda) l_i\left(y_{-i} | x\right) = \lambda z_i + (1-\lambda) y_i$$

implies the absurdity that

$$\begin{aligned} u(x) = u\left(\widetilde{z}_i, \lambda z_{-i} + (1-\lambda) y_{-i}\right) &\geq u\left(\lambda z_i + (1-\lambda) y_i, \lambda z_{-i} + (1-\lambda) y_{-i}\right) \\ &= u\left(\lambda z + (1-\lambda) y\right) > u(z) = u(x) \end{aligned}$$

the two inequalities, respectively, due to the strict monotonicity and strict quasi-concavity of $u(\cdot)$. □

The next result can be applied to verify that $\succsim$ is indeed strictly convex in our examples in the main text. This can be done of course also by showing directly that the respective utility functions are strictly quasi-concave. Nevertheless, the exercise is much easier if we deploy instead the strict convexity of the indifference-projection functions.

LEMMA V.2: *Let $\succsim$ be a strictly monotonic, continuous weak order on X. Suppose also that there exist $i \in \mathcal{N}$ such that $\mathcal{I}_x^{-i} = X^{-i}$ for all $x \in X$. Then $l_i(\cdot|x)$ is strictly convex on X^{-i} if and only if $\succsim$ is strictly convex on X.*

PROOF:

For the "if" direction, by the preceding lemma, it suffices to show that X^{-i} is convex. But this is obvious: taking any $z, y \in X$ and any $\lambda \in (0,1)$, the fact that X is convex ensures that $\lambda z + (1-\lambda) y \in X$, which implies in turn that $\lambda z_{-i} + (1-\lambda) y_{-i} \in X^{-i}$.
For the "only if," letting $u : X \to \mathbb{R}$ represent $\succsim$ on X, it suffices to show that $u(\cdot)$ is strictly quasi-concave on X. To establish this by contradiction, take $x, y \in X$ such that $y \succsim x$ and suppose that $u(\lambda x + (1-\lambda) y) \leq u(x)$ for some $\lambda \in (0,1)$. Recall also that, $\succsim$ being strictly monotonic, $l_i(\cdot|x)$ is a well-defined function (recall Lemma 3.5 in the main text). And its domain being the entire X^{-i}, there exists $z_i \in X^i$ such that $z_i = l_i(y_{-i}|x)$. Observe now that, since $u(z_i, y_{-i}) = u(x) \leq u(y)$, the strict monotonicity of $\succsim$ necessitates also that $z_i \leq y_i$ and thus

$$\begin{aligned} u\left(\lambda x_i + (1-\lambda) z_i, \lambda x_{-i} + (1-\lambda) y_{-i}\right) &\leq u\left(\lambda x_i + (1-\lambda) y_i, \lambda x_{-i} + (1-\lambda) y_{-i}\right) \\ &= u\left(\lambda x + (1-\lambda) y\right) \leq u(x) \end{aligned} \tag{3}$$

Notice next that, X^{-i} being convex, there exists also $z_i' \in X^i$ such that

$$z_i' = l_i\left(\lambda x_{-i} + (1-\lambda)\, y_{-i}|x\right) < \lambda l_i\left(x_{-i}|x\right) + (1-\lambda)\, l_i\left(y_i|x\right) = \lambda x_i + (1-\lambda)\, z_i$$

the inequality due to the strict-convexity of $l_i\left(\cdot|x\right)$. Which implies in turn that

$$(4)\quad u\left(\lambda x_i + (1-\lambda)\, z_i, \lambda x_{-i} + (1-\lambda)\, y_{-i}\right) > u\left(z_i', \lambda x_{-i} + (1-\lambda)\, y_{-i}\right) = u\left(x\right)$$

again by the strict monotonicity of $\succsim$. From (3) and (4) we obtain the desired contradiction. $\square$

LEMMA V.3: *Let $\succsim$ be a strictly monotonic and strictly convex, continuous weak order $\succsim$ on X. Suppose also that it is represented by the utility function $u : X \to \mathbb{R}$. For any $x \in X$ any $z \in \mathcal{I}_x \cap int\left(X\right)$ and any $i \in \mathcal{N}$, the following statements hold.*
(i). If $\succsim$ is differentiable at z, there exist $\varepsilon_0 > 0$ such that $\mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0}\left(z\right) \subset int\left(X\right)$ and a family of functions $\left\{\mu_n : \left(-\varepsilon_0, \varepsilon_0\right) \to \left(-\varepsilon_0, \varepsilon_0\right)\right\}_{n \in \mathcal{N}}$ with $\lim_{\varepsilon\to 0} \mu_n\left(\varepsilon\right) = 0$ for all n such that for any $j \in \mathcal{N} \setminus \{i\}$ we have

$$\frac{\partial l_i\left(z_{-i}|x\right)}{\partial z_j} = -\lim_{\varepsilon \to 0} \frac{\left[u\left(z_j + \mu_j\left(\varepsilon\right), z_{-j}\right) - u\left(z\right)\right] / \mu_j\left(\varepsilon\right)}{\left[u\left(z_j + \mu_j\left(\varepsilon\right), z_{-j}\right) - u\left(z_i - \mu_i\left(\varepsilon\right), z_j + \mu_j\left(\varepsilon\right), z_{-(i,j)}\right)\right] / \mu_i\left(\varepsilon\right)}$$

(ii). If $u\left(\cdot\right)$ is C^1 at z then $l_i\left(\cdot|x\right)$ is C^1 at z_{-i}; specifically, for any $j \in \mathcal{N} \setminus \{i\}$ we have

$$\frac{\partial l_i\left(z_{-i}|x\right)}{\partial z_j} = -\left(\frac{\partial u\left(z\right)}{\partial z_i}\right)^{-1} \frac{\partial u\left(z\right)}{\partial z_j}$$

PROOF:

Take arbitrary $x \in X$, $z \in \mathcal{I}_x \cap \mathrm{int}\left(X\right)$ and $(i,j) \in \mathcal{N} \times \mathcal{N} \setminus \{i\}$. Choose also $\varepsilon_0 > 0$ such that $l_i\left(\cdot|x\right)$ is strictly convex on $\mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0}\left(z\right) \subset \mathrm{int}\left(X\right)$. For $\varepsilon \in \left(-\varepsilon_0, 0\right) \cup \left(0, \varepsilon_0\right)$ and $\mu \in [0,1]$ let $z\left(\varepsilon, \mu\right) \in \mathcal{B}_{\varepsilon_0}\left(z\right)$ be given by $z\left(\varepsilon, \mu\right)_i := z_i - \left(1-\mu\right)\varepsilon$, $z\left(\varepsilon, \mu\right)_j := z_j + \mu\varepsilon$ and $z\left(\varepsilon, \mu\right)_k := z_k$ for $k \in \mathcal{N} \setminus \{i, j\}$. Since

$$\varepsilon\left[u\left(z\left(\varepsilon, 0\right)\right) - u\left(z\right)\right] < 0 < \varepsilon\left[u\left(z\left(\varepsilon, 1\right)\right) - u\left(z\right)\right]$$

while $u\left(\cdot\right)$ is continuous, the intermediate-value theorem establishes the existence of $\mu_\varepsilon \in \left(0,1\right)$ such that $z\left(\varepsilon, \mu_\varepsilon\right) \in \mathcal{I}_x \cap \mathcal{B}_{\varepsilon_0}\left(z\right)$. By the strict monotonicity of $\succsim$, moreover, the mapping $\varepsilon \mapsto \mu\left(\varepsilon\right) = \mu_\varepsilon$ is a function. For all

$\varepsilon \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0)$, therefore, we have

$$
\begin{aligned}
(5) \qquad & \frac{l_i\left(z_j + \varepsilon\mu(\varepsilon), z_{-(i,j)}|x\right) - l_i(z_{-i}|x)}{\varepsilon\mu(\varepsilon)} \\
= \quad & \frac{z_i - \varepsilon(1-\mu(\varepsilon)) - z_i}{\varepsilon\mu(\varepsilon)} \\
= \quad & -\frac{1-\mu(\varepsilon)}{\mu(\varepsilon)} \\
= \quad & -\frac{1-\mu(\varepsilon)}{\mu(\varepsilon)} \\
\times \quad & \frac{u(z_j + \varepsilon\mu(\varepsilon), z_{-j}) - u(z)}{u(z_j + \varepsilon\mu(\varepsilon), z_{-j}) - u\left(z_i - \varepsilon(1-\mu(\varepsilon)), z_j + \varepsilon\mu(\varepsilon), z_{-(i,j)}\right)}
\end{aligned}
$$

To establish now statement (i) of the claim, define the functions $\widetilde{\mu}, \mu_i, \mu_j : (-\varepsilon_0, \varepsilon_0) \to (-\varepsilon_0, \varepsilon_0)$ by $\widetilde{\mu}(0) := 0$ while $\widetilde{\mu}(\varepsilon) := \varepsilon\mu(\varepsilon)$ on $(-\varepsilon_0, 0) \cup (0, \varepsilon_0)$ as well as $\mu_i(\varepsilon) := \varepsilon - \widetilde{\mu}(\varepsilon)$ and $\mu_j(\varepsilon) := \widetilde{\mu}(\varepsilon)$. Then (5) gives

$$
\begin{aligned}
& \lim_{\varepsilon\to 0} \frac{l_i\left(z_j + \mu_j(\varepsilon), z_{-(i,j)}|x\right) - l_i(z_{-i}|x)}{\mu_j(\varepsilon)} \\
(6) \quad = \quad & -\lim_{\varepsilon\to 0} \frac{\left[u(z_j + \mu_j(\varepsilon), z_{-j}) - u(z)\right]/\mu_j(\varepsilon)}{\left[u(z_j + \mu_j(\varepsilon), z_{-j}) - u\left(z_i - \mu_i(\varepsilon), z_j + \mu_j(\varepsilon), z_{-(i,j)}\right)\right]/\mu_i(\varepsilon)}
\end{aligned}
$$

And as $\lim_{\varepsilon\to 0} \mu_j(\varepsilon) = 0$, it suffices to note that, $l_i(\cdot|x)$ being differentiable at z_{-i} (recall Proposition 2 in the main text), the limit on the left-hand side of (6) coincides with $\partial l_i(z_{-i}|x)/\partial z_j$.

To show next statement (ii), notice first that, $u(\cdot)$ being differentiable, the mean-value theorem ensures the existence of a function $\lambda : (-\varepsilon_0, \varepsilon_0) \to (0, 1)$ such that

$$
u(z_j + \mu_j(\varepsilon), z_{-j}) - u\left(z_i - \mu_i(\varepsilon), z_j + \mu_j(\varepsilon), z_{-(i,j)}\right) = \mu_i(\varepsilon) \frac{\partial u\left(z_j + \lambda(\varepsilon)\mu_j(\varepsilon), z_{-j}\right)}{\partial z_i}
$$

We can re-write therefore (5) as

$$\begin{aligned}\frac{l_i\left(z_j+\mu_j(\varepsilon),z_{-(i,j)}|x\right)-l_i(z_{-i}|x)}{\mu_j(\varepsilon)} &= -\left(\frac{\partial u\left(z_j+\lambda(\varepsilon)\mu_j(\varepsilon),z_{-j}\right)}{\partial z_i}\right)^{-1}\\ &\times \frac{u\left(z_j+\mu_j(\varepsilon),z_{-j}\right)-u(z)}{\mu_j(\varepsilon)}\end{aligned}$$

which gives

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{l_i\left(z_j+\mu_j(\varepsilon),z_{-(i,j)}|x\right)-l_i(z_{-i}|x)}{\mu_j(\varepsilon)}\\ &= -\lim_{\varepsilon\to 0}\left(\frac{\partial u\left(z_j+\lambda(\varepsilon)\mu_j(\varepsilon),z_{-j}\right)}{\partial z_i}\right)^{-1}\times\frac{u\left(z_j+\mu_j(\varepsilon),z_{-j}\right)-u(z)}{\mu_j(\varepsilon)}\\ &= -\left(\frac{\partial u(z)}{\partial z_i}\right)^{-1}\times\left(\frac{\partial u(z)}{\partial z_j}\right)\qquad(7)\end{aligned}$$

the last equality because $u(\cdot)$ is in fact C^1 at z. The latter property ensures that $l_i(\cdot|x)$ is also C^1 at z_{-i}, for it implies that $\succsim$ is weakly C^1 at z (recall Lemma 3.9 in the main text).

It remains only to show that the first limit in (7) coincides with $\partial l_i(z_{-i}|x)/\partial z_j$. To this end, observe first that $\mu(\cdot)$ is strictly increasing everywhere on its domain. To show this arguing ad absurdum, let $\varepsilon<\varepsilon'$ and suppose that $\mu(\varepsilon)\geq\mu(\varepsilon')$. Then $1-\mu(\varepsilon)\leq 1-\mu(\varepsilon')$ and, by the strict monotonicity of $\succsim$, it must be

$$l_j\left(z_i-\varepsilon'(1-\mu(\varepsilon)),z_{-(i,j)}|x\right)<l_j\left(z_i-\varepsilon'\left(1-\mu(\varepsilon')\right),z_{-(i,j)}|x\right)$$

Moreover, the function $l_j(\cdot|x)$ being strictly convex, $\varepsilon<\varepsilon'$ implies also that[2]

$$\frac{l_j\left(z_i-\varepsilon'(1-\mu(\varepsilon)),z_{-(i,j)}|x\right)}{\varepsilon'}>\frac{l_j\left(z_i-\varepsilon(1-\mu(\varepsilon)),z_{-(i,j)}|x\right)}{\varepsilon}$$

[2]Given $K\in\mathbb{N}\setminus\{0\}$ and a strictly convex function $f:S\to\mathbb{R}$ defined on an open and convex set $S\subseteq\mathbb{R}^K$, a vector $v\in\mathbb{R}^K$, and $\varepsilon\in\mathbb{R}\setminus\{0\}$, the ratio $[f(x+\varepsilon v)-f(x)]/\varepsilon$ is a strictly increasing function of ε (see Theorem 6.2.15 in de la Fuente (2000)). For the application of this result in the text, let $K=n-1$ and $v_i=1-\mu(\varepsilon)$ while $v_k=0$ for $k\in\mathcal{N}\setminus\{i,j\}$.

Putting these observations together, we get the contradiction that

$$\begin{aligned}
\mu(\varepsilon) = \frac{z_j + \varepsilon\mu(\varepsilon) - z_j}{\varepsilon} &= \frac{l_j\left(z_i - \varepsilon(1-\mu(\varepsilon)), z_{-(i,j)}|x\right) - l_j(z_{-j}|x)}{\varepsilon} \\
&< \frac{l_j\left(z_i - \varepsilon'(1-\mu(\varepsilon)), z_{-(i,j)}|x\right) - l_j(z_{-j}|x)}{\varepsilon'} \\
&< \frac{l_j\left(z_i - \varepsilon'(1-\mu(\varepsilon')), z_{-(i,j)}|x\right) - l_j(z_{-j}|x)}{\varepsilon'} = \mu(\varepsilon')
\end{aligned}$$

Clearly, $\mu(\cdot)$ being strictly increasing, $\widetilde{\mu}(\cdot)$ is injective. It is also continuous. Indeed, given any $\varepsilon^* \in (-\varepsilon_0, \varepsilon_0)$ and any $\delta > 0$, choosing $\varepsilon_1 \in (0, \delta/2)$ sufficiently small, the continuity of $l_j(\cdot|x)$ in conjunction with the fact that $\mu(\cdot)$ is strictly increasing ensure that

$$\begin{aligned}
\delta/2 &> |l_i\left(z_j + \varepsilon\mu(\varepsilon), z_{-(i,j)}|x\right) - l_i\left(z_j + \varepsilon^*\mu(\varepsilon^*), z_{-(i,j)}|x\right)| \\
&= |l_i\left(z_j + \varepsilon\mu(\varepsilon), z_{-(i,j)}|x\right) - l_i(z_{-i}|x) + l_i(z_{-i}|x) - l_i\left(z_j + \varepsilon^*\mu(\varepsilon^*), z_{-(i,j)}|x\right)| \\
&= |\varepsilon^*(1-\mu(\varepsilon^*)) - \varepsilon(1-\mu(\varepsilon))| \\
&= |\varepsilon\mu(\varepsilon) - \varepsilon^*\mu(\varepsilon^*) - (\varepsilon - \varepsilon^*)| \\
&> |\varepsilon\mu(\varepsilon) - \varepsilon^*\mu(\varepsilon^*)| - |\varepsilon - \varepsilon^*| \\
&> |\varepsilon\mu(\varepsilon) - \varepsilon^*\mu(\varepsilon^*)| - \varepsilon_1 \\
&> |\varepsilon\mu(\varepsilon) - \varepsilon^*\mu(\varepsilon^*)| - \delta/2
\end{aligned}$$

and thus $|\widetilde{\mu}(\varepsilon) - \widetilde{\mu}(\varepsilon^*)| < \delta$ everywhere on $(\varepsilon^* - \varepsilon_1, \varepsilon^* + \varepsilon_1)$.
Moreover, for any $k > 0$, $\widetilde{\mu} : (-\varepsilon_0/k, \varepsilon_0/k) \to (-\varepsilon_0\mu(\varepsilon_0/k)/k, \varepsilon_0\mu(\varepsilon_0/k)/k)$ is surjective. To see this, observe that there cannot be $\varepsilon' \in (0, \varepsilon_0\mu(\varepsilon_0/k)/k)$ [resp. $\varepsilon' \in (-\varepsilon_0\mu(-\varepsilon_0/k)/k, 0)$] such that $\varepsilon' < \widetilde{\mu}(\varepsilon)$ [resp. $\varepsilon' > \widetilde{\mu}(-\varepsilon)$] for all $\varepsilon \in (0, \varepsilon_0/k)$ because the right-hand side of the inequality vanishes as $\varepsilon \searrow 0$. Nor can there be $\varepsilon' \in (0, \varepsilon_0\mu(\varepsilon_0/k)/k)$ [resp. $\varepsilon' \in (-\varepsilon_0\mu(-\varepsilon_0/k)/k, 0)$] such that $\varepsilon' > \widetilde{\mu}(\varepsilon)$ [resp. $\varepsilon' < \widetilde{\mu}(-\varepsilon)$] for all $\varepsilon \in (0, \varepsilon_0/k)$ since $\varepsilon' < \varepsilon_0\mu(\varepsilon_0/k)/k = \widetilde{\mu}(\varepsilon_0/k)$ [resp. $\varepsilon' > -\varepsilon_0\mu(-\varepsilon_0/k)/k = \widetilde{\mu}(-\varepsilon_0/k)$] while $\widetilde{\mu}(\cdot)$ is continuous.
Hence, $\widetilde{\mu}(\cdot)$ is invertible. It maps a given (sufficiently small) neighbourhood around the origin onto a smaller neighbourhood around the origin - with the latter neighbourhood completely identified by the choice of the former neigh-

bourhood. It follows thus that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{l_i\left(z_j+\widetilde{\mu}\left(\varepsilon\right),z_{-(i,j)}|x\right)-l_i\left(z_{-i}|x\right)}{\widetilde{\mu}\left(\varepsilon\right)} &= \lim_{\varepsilon\to 0}\frac{l_i\left(z_j+\varepsilon,z_{-(i,j)}|x\right)-l_i\left(z_{-i}|x\right)}{\varepsilon}\\ &= \frac{\partial l_i\left(z_{-i}|x\right)}{\partial z_j}\end{aligned}$$

To complete the argument it suffices to recall the definition of $\mu_j\left(\cdot\right)$. $\square$

Our closing remark refers to Example 2 (and thus also to Proposition 4) in the main text. Our intuition applies for the more general formulation where $\succsim$ is represented by

$$u\left(x\right):=\begin{cases} f\left(x_1,x_2\right), & \text{if } \left(x_1,x_2\right)\in S:=\left\{x\in X: x_1\leq x_2\right\}\\ f\left(x_2,x_1\right), & \text{otherwise}\end{cases}$$

for some function $f: X\to\mathbb{R}$ that is strictly quasiconcave, strictly increasing and C^1 on $\operatorname{int}\left(X\right)\setminus S_0$ where $S_0:=\left\{x\in X: x_1=x_2\right\}$. For as long as $\partial f\left(x\right)/\partial x_1\neq\partial f\left(x\right)/\partial x_2$ $\succsim$ will not be differentiable anywhere on $\operatorname{int}\left(X\right)\cap S_0$. Indeed, by the preceding lemma, for any $\overline{x}\in\operatorname{int}\left(X\right)$ we have

$$l_2'\left(x_1|\overline{x}\right)=\begin{cases} -\frac{\partial f(x)/\partial x_1}{\partial f(x)/\partial x_2}, & \text{if } x\in\mathcal{I}_{\overline{x}}\cap S\setminus S_0\\ -\frac{\partial f(x)/\partial x_2}{\partial f(x)/\partial x_1}, & \text{if } x\in\mathcal{I}_{\overline{x}}\cap\left(X\setminus S\right)\end{cases}$$

Clearly, the quantity $\lim_{x_1\to\overline{x}_1} l_2'\left(x_1|\overline{x}\right)$ takes different values as we approach $\overline{x}\in\operatorname{int}\left(X\right)\cap S_0$ from within S as opposed to from outside. Example 2 in the main text has $f\left(x_1,x_2\right):=x_1^{1/3}x_2^{2/3}$. The example of homothetic preferences generating a non-invertible demand function in Hurwicz and Uzawa (1971) has $f\left(x_1,x_2\right):=x_2\phi\left(x_1/x_2\right)$ with $\phi:[0,1]\to\mathbb{R}$ given by $\phi\left(t\right):=3-\left(1-t\right)\left(2+\sqrt{1-t}\right)$.

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