Multihoming and oligopolistic platform competition: online appendix

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This online appendix contains proofs of omitted results and details from the main paper.

B Further properties of the baseline demand function

We examine the continuity and differentiability of the complete demand function $Q_i\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right)$ derived in Section A.1 of the appendix of the main text.

Claim B.1 *For any $\hat{\mathbf{p}}$, $Q_i\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right)$ is continuous in p_i^b and p_i^s.*

Proof. Continuity with respect to p_i^b is obvious. To show continuity with respect to p_i^s, note from Lemma 3 that $\lim_{p_i^s \to \hat{p}^s-} \hat{v}_m = \hat{p}^s$ for $m = 2, ..., n$. Similarly, note from Lemma 1 that $\lim_{p_i^s \to \hat{p}^s+} \hat{v} = \hat{p}^s$. Thus,

$$\begin{aligned}
& \lim_{p_i^s \to \hat{p}^s-} Q_i\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right) \\
= \; & \left[1 - G\left(\hat{p}^s\right)\right] B_i^{(\mathbf{N})} + \sum_{m=0}^{n-2} \left[G\left(\hat{p}^s\right) - G\left(\hat{p}^s\right)\right] B_i^{(\mathbf{N}_{i,m})} \\
= \; & \left[1 - G\left(\hat{p}^s\right)\right] B_i^{(\mathbf{N})} \\
= \; & \lim_{p_i^s \to \hat{p}^s+} Q_i\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right),
\end{aligned}$$

so $Q_i\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right)$ is continuous for all p_i^b and p_i^s, which includes $\left(p_i^b, p_i^s\right) = \hat{p}$. ■

Claim B.2 *For any $\hat{\mathbf{p}}$,*

$$\lim_{p_i^s \to \hat{p}^s-} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right) \geq \lim_{p_i^s \to \hat{p}^s+} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right).$$

Equality holds if in addition (i) $n = 2$, or (ii) F, $F_0 \sim$Gumbel(μ).

Proof. Consider first $p_i^s \geq \hat{p}^s$. Then the right-hand side derivative is

$$\begin{aligned}
\lim_{p_i^s \to \hat{p}^s+} \frac{dQ_i}{dp_i^s}\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right) &= \lim_{p_i^s \to \hat{p}^s+} -\frac{d\hat{v}}{dp_i^s} B_i^{(\mathbf{N})} g\left(p_i^s\right) \\
&= \lim_{p_i^s \to \hat{p}^s+} \frac{-B_i^{(\mathbf{N})}}{\sum_{j \in \mathbf{N}, j \neq i}\left(B_j^{(\mathbf{N})} - B_j^{(\mathbf{N}_{-i})}\right) + B_i^{(\mathbf{N})}} B_i^{(\mathbf{N})} g\left(p_i^s\right) \\
&= \frac{-B_i^{(\mathbf{N})}}{\sum_{j \in \mathbf{N}, j \neq i}\left(B_j^{(\mathbf{N})} - B_j^{(\mathbf{N}_{-i})}\right) + B_i^{(\mathbf{N})}} B_i^{(\mathbf{N})} g\left(\hat{p}^s\right).
\end{aligned}$$

Evaluating the above at $p_i^b = \hat{p}^b$, all platforms become symmetry so that functions $B_j^{(\Theta)}$ are the same for any set Θ and any given $j \in \Theta$. So, for simplicity we denote any such generic term as $B^{(\Theta)}$. Hence we have

$$\lim_{p_i^s \to \hat{p}^s+} \frac{dQ_i}{dp_i^s}\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right) = \frac{-B^{(\mathbf{N})} B^{(\mathbf{N})}}{nB^{(\mathbf{N})} - (n-1) B^{(\mathbf{N}_{-i})}} g\left(\hat{p}^s\right). \tag{B.1}$$

When $p_i^s > \hat{p}^s$, the left hand side derivative is

$$
\begin{aligned}
& \lim_{p_i^s \to \hat{p}^s -} \frac{dQ_i}{dp_i^s}\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right) \\
= & \lim_{p_i^s \to \hat{p}^s -} \sum_{m=0}^{n-1} \left[\frac{d\hat{v}_{m+1}}{dp_i^s} g\left(\bar{v}_{m+1}\right) - \frac{d\hat{v}_m}{dp_i^s} g\left(\bar{v}_m\right)\right] B_i^{(\mathbf{N}_{i,m})} \\
= & \; g\left(\hat{p}^s\right)\left[\sum_{m=0}^{n-1}\left[\frac{d\hat{v}_{m+1}}{dp_i^s} - \frac{d\hat{v}_m}{dp_i^s}\right] B^{(\mathbf{N}_{i,m})}\right], \qquad \text{(B.2)}
\end{aligned}
$$

where $\frac{d\hat{v}_n}{dp_i^s} = 0$ because $\hat{v}_n \equiv \bar{v}$, $\frac{d\hat{v}_0}{dp_i^s} = 1$ since $\hat{v}_0 \equiv p_i^s$, while

$$
\begin{aligned}
\frac{d\hat{v}_m}{dp_i^s} &= \frac{B_i^{(\mathbf{N}_{i,m})} - B_i^{(\mathbf{N}_{i,m-1})}}{B_i^{(\mathbf{N}_{i,m})} - B_i^{(\mathbf{N}_{i,m-1})} + mB_j^{(\mathbf{N}_{i,m})} - (m-1) B_j^{(\mathbf{N}_{i,m-1})}} \text{ for } m = 1, ..., n-1 \\
\frac{d\hat{v}_m}{dp_i^s}|_{p_i^b = \hat{p}^b} &= \frac{B^{(\mathbf{N}_{i,m})} - B^{(\mathbf{N}_{i,m-1})}}{(m+1) B^{(\mathbf{N}_{i,m})} - mB^{(\mathbf{N}_{i,m-1})}},
\end{aligned}
$$

in which $B^{(\cdot)}$ is as denoted earlier due to symmetry. Hence, evaluating at $p_i^b = \hat{p}^b$, ($B.2$) can be expanded

$$
\begin{aligned}
& \frac{1}{g\left(\hat{p}^s\right)} \lim_{p_i^s \to \hat{p}^s -} \frac{dQ_i}{dp_i^s}\left(p_i^b, p_i^s; \hat{\mathbf{p}}\right) \\
= & -\frac{d\hat{v}_{n-1}}{dp_i^s} B^{(\mathbf{N})} + \sum_{m=1}^{n-2}\left[\frac{d\hat{v}_{m+1}}{dp_i^s} - \frac{d\hat{v}_m}{dp_i^s}\right] B^{(\mathbf{N}_{i,m})} + \left(\frac{d\hat{v}_1}{dp_i^s} - 1\right) B^{(\mathbf{N}_{i,0})} \\
= & \frac{-B^{(\mathbf{N}_{i,n-1})} B^{(\mathbf{N}_{i,n-1})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1) B^{(\mathbf{N}_{i,n-2})}} \\
& + \frac{B^{(\mathbf{N}_{i,n-2})} B^{(\mathbf{N}_{i,n-1})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1) B^{(\mathbf{N}_{i,n-2})}} \\
& + \sum_{m=1}^{n-2}\left[\frac{B^{(\mathbf{N}_{i,m+1})} - B^{(\mathbf{N}_{i,m})}}{(m+2) B^{(\mathbf{N}_{i,m+1})} - (m+1) B^{(\mathbf{N}_{i,m})}} - \frac{B^{(\mathbf{N}_{i,m})} - B^{(\mathbf{N}_{i,m-1})}}{(m+1) B^{(\mathbf{N}_{i,m})} - mB^{(\mathbf{N}_{i,m-1})}}\right] B^{(\mathbf{N}_{i,m})} \qquad \text{(B.3)} \\
& + \left(\frac{B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} - 1\right) B^{(\mathbf{N}_{i,0})}.
\end{aligned}
$$

By definition, proving differentiability at $\left(p_i^b, p_i^s\right) = \hat{p}$ requires us to show

$$
\lim_{p_i^s \to \hat{p}^s -} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right) = \lim_{p_i^s \to \hat{p}^s +} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right). \qquad \text{(B.4)}
$$

To prove this, we note that $\mathbf{N}_{i,n-1} = \mathbf{N}$ and that when all platforms are symmetry we have $\mathbf{N}_{i,n-2} = \mathbf{N}_{-i}$ (because both sets denote a set of $n-1$ symmetry platforms). Then, substituting for (B.1) we can rewrite (B.3) as

$$
\lim_{p_i^s \to \hat{p}^s -} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right) = \lim_{p_i^s \to \hat{p}^s +} \frac{dQ_i}{dp_i^s}\left(\hat{p}^b, p_i^s; \hat{\mathbf{p}}\right) + g\left(\hat{p}^s\right) \Phi\left(n\right),
$$

where $\Phi\left(n\right)$ is defined as the last three lines of (B.3), i.e.

$$
\begin{aligned}
\Phi\left(n\right) \equiv & \frac{B^{(\mathbf{N}_{i,n-2})} B^{(\mathbf{N}_{i,n-1})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1) B^{(\mathbf{N}_{i,n-2})}} \\
& + \sum_{m=1}^{n-2}\left[\frac{B^{(\mathbf{N}_{i,m+1})} - B^{(\mathbf{N}_{i,m})}}{(m+2) B^{(\mathbf{N}_{i,m+1})} - (m+1) B^{(\mathbf{N}_{i,m})}} - \frac{B^{(\mathbf{N}_{i,m})} - B^{(\mathbf{N}_{i,m-1})}}{(m+1) B^{(\mathbf{N}_{i,m})} - mB^{(\mathbf{N}_{i,m-1})}}\right] B^{(\mathbf{N}_{i,m})} \\
& + \left(\frac{B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} - 1\right) B^{(\mathbf{N}_{i,0})}.
\end{aligned}
$$

To conclude (B.4), it suffices to prove by induction that $\Phi\left(n\right) \geq 0$ for all $n \geq 2$. First, when $n = 2$ we have

$\mathbf{N}_{i,1} = \mathbf{N}$ so

$$\begin{aligned}\Phi(2) &= \frac{B^{(\mathbf{N}_{i,0})}B^{(\mathbf{N}_{i,1})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} + \left(\frac{B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} - 1\right) B^{(\mathbf{N}_{i,0})} \\ &= \frac{2B^{(\mathbf{N}_{i,0})}B^{(\mathbf{N}_{i,1})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} - \left[\frac{B^{(\mathbf{N}_{i,0})}B^{(\mathbf{N}_{i,0})}}{2B^{(\mathbf{N}_{i,1})} - B^{(\mathbf{N}_{i,0})}} + B^{(\mathbf{N}_{i,0})}\right] = 0.\end{aligned}$$

Note that this also proves the first part of the claim, that is, the case of $n = 2$. By the inductive hypothesis, suppose $\Phi(n-1) \geq 0$. For $n \geq 3$, if we expand one more term from the summation in $\Phi(n)$ and rearrange terms we get

$$\begin{aligned}\Phi(n) &= \frac{B^{(\mathbf{N}_{i,n-2})}B^{(\mathbf{N}_{i,n-1})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1)B^{(\mathbf{N}_{i,n-2})}} \\ &\quad + \left[\frac{B^{(\mathbf{N}_{i,n-1})} - B^{(\mathbf{N}_{i,n-2})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1)B^{(\mathbf{N}_{i,n-2})}} - \frac{B^{(\mathbf{N}_{i,n-2})} - B^{(\mathbf{N}_{i,n-3})}}{(n-1)B^{(\mathbf{N}_{i,n-2})} - (n-2)B^{(\mathbf{N}_{i,n-3})}}\right] B^{(\mathbf{N}_{i,n-2})} \\ &\quad + \sum_{m=1}^{n-3}\left[\frac{B^{(\mathbf{N}_{i,m+1})} - B^{(\mathbf{N}_{i,m})}}{(m+2)B^{(\mathbf{N}_{i,m+1})} - (m+1)B^{(\mathbf{N}_{i,m})}} - \frac{B^{(\mathbf{N}_{i,m})} - B^{(\mathbf{N}_{i,m-1})}}{(m+1)B^{(\mathbf{N}_{i,m})} - mB^{(\mathbf{N}_{i,m-1})}}\right] B^{(\mathbf{N}_{i,m})} \\ &\quad + \frac{B^{(\mathbf{N}_{i,n-2})}B^{(\mathbf{N})}}{nB^{(\mathbf{N})} - (n-1)B^{(\mathbf{N}_{i,n-2})}} \\ &= \frac{\left(2B^{(\mathbf{N}_{i,n-1})} - B^{(\mathbf{N}_{i,n-2})}\right)B^{(\mathbf{N}_{i,n-2})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1)B^{(\mathbf{N}_{i,n-2})}} - \frac{B^{(\mathbf{N}_{i,n-2})}B^{(\mathbf{N}_{i,n-2})}}{(n-1)B^{(\mathbf{N}_{i,n-2})} - (n-2)B^{(\mathbf{N}_{i,n-3})}} + \Phi(n-1)\,.\end{aligned} \tag{B.5}$$

By inductive hypothesis $\Phi(n-1) \geq 0$. Therefore, to prove $\Phi(n) \geq 0$, it remains to show

$$\frac{2B^{(\mathbf{N}_{i,n-1})} - B^{(\mathbf{N}_{i,n-2})}}{nB^{(\mathbf{N}_{i,n-1})} - (n-1)B^{(\mathbf{N}_{i,n-2})}} \geq \frac{B^{(\mathbf{N}_{i,n-2})}}{(n-1)B^{(\mathbf{N}_{i,n-2})} - (n-2)B^{(\mathbf{N}_{i,n-3})}}.$$

Rearranging the terms and cancelling out common coefficients, the inequality above is equivalent to

$$\begin{aligned}0 &\leq \frac{\left(B^{(\mathbf{N}_{i,n-2})} - B^{(\mathbf{N}_{i,n-1})}\right)}{B^{(\mathbf{N}_{i,n-1})}} - \frac{\left(B^{(\mathbf{N}_{i,n-3})} - B^{(\mathbf{N}_{i,n-2})}\right)}{B^{(\mathbf{N}_{i,n-3})}} \\ &\simeq \frac{\partial B^{(\mathbf{N}_{i,k})}}{\partial k}|_{k=n-1} - \frac{\partial B^{(\mathbf{N}_{i,k})}}{\partial k}|_{k=n-3}.\end{aligned} \tag{B.6}$$

We know $\frac{\partial B}{\partial k} \leq 0$, so we simply need to show that B is decreasing in k with a decreasing magnitude, i.e. B is convex in k. Recall that for $k \in \{0, ..., n-1\}$, we have

$$B^{(\mathbf{N}_{i,k})} = \int_{\underline{\epsilon}}^{\bar{\epsilon}-\hat{p}^b}\left[\frac{1 - F\left(\epsilon_0 + \hat{p}^b\right)^{k+1}}{k+1}\right] dF_0(\epsilon_0)\,.$$

Convexity of B in k then follows from the observation that $\frac{1-F\left(\epsilon_0+\hat{p}^b\right)^{k+1}}{k+1}$ is convex in k, and that convexity is preserved by integration when the integrand is always positive over the entire region of integration. So, $\Phi(n) \geq 0$ for all $n \geq 2$ as required. Finally, in the special case of Gumbel distribution, IIA property of logit-demand form implies that the right-hand side of (B.6) equals zero, so that $\Phi(n) = 0$ for all $n \geq 2$. ■

In order to determine the global quasi-concavity of the profit function in Section 3 for other distribution functions, we rely on numerical calculations. Details and codes of the numerical calculations are available from the authors upon request. Specifically, we focus on $n \in \{2, 3, 4\}$ and $c = 0.1$ and consider $F, F_0 \sim \mathit{Gumbel}(\mu)$ and $G \sim \mathit{Normal}(\mu_{norm}, \sigma)$ where the parameters are repeatedly picked in random from intervals: $\mu \in [1, 4]$, and $\mu_{norm} \in [-10, 10]$, $\sigma \in [1, 6]$. We also repeat the same exercise with (i) $F, F_0 \sim \mathit{Gumbel}(\mu)$ and $G \sim \mathit{Exponential}(\theta)$, $\theta \in [1/2, 2]$; (ii) $F, F_0 \sim \mathit{Exponential}(\theta_F)$ and $G \sim \mathit{Exponential}(\theta_G)$, $\theta_F, \theta_G \in [1/2, 2]$; and $F, F_0 \sim \mathit{Normal}(\mu_{norm}, \sigma)$ and $G \sim \mathit{Exponential}(\theta)$, $\mu_{norm} \in [1, 2]$, $\sigma \in [1, 2]$ and $\theta \in [1/2, 2]$.

In all the cases considered, the quasi-concavity assumption was satisfied, suggesting it does not require very special conditions to hold. Figure 7 below provides an example of the plot of platform profit function that shows quasi-concavity (assuming all other platforms are setting the equilibrium fees).

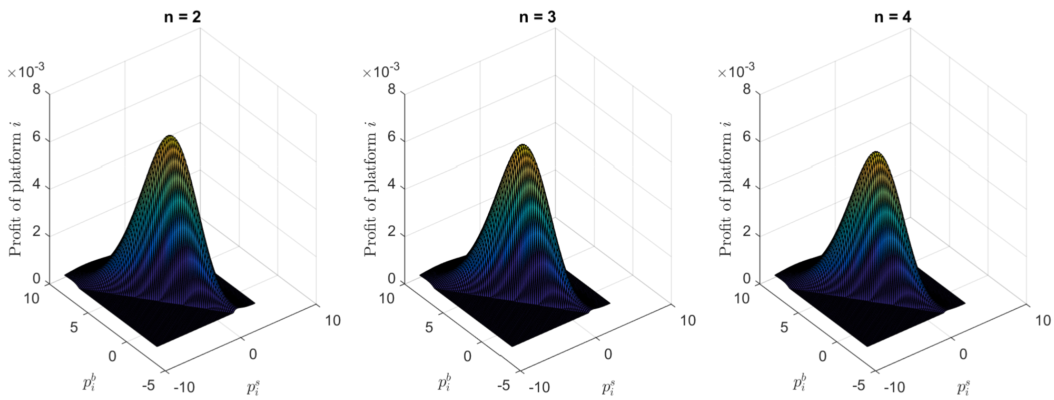

Figure 7: Profit function of firm i, assuming $c = 0.1$, F and $F_0 \sim Gumbel(1)$, and $G \sim Normal(-4, 2)$.

C Membership fee component

As stated in the main text, when buyers and sellers coordinate to not participating on any platform that charges strictly positive buyer or seller membership fees, then no platform has incentive to deviate from the equilibrium in Proposition 1 by charging positive membership fee component.

Meanwhile, given that all buyers are already multihoming in the equilibrium, no platform can profit from offering negative buyer membership fees. It remains to rule out deviations from the symmetric equilibrium with a negative seller fee (denoted as $P_i^s < 0$). We will focus on the case of logit buyer demand and show that, whenever a deviating platform i wants to set $P_i^s < 0$ to attract additional seller participation, it is instead better off setting $p_i^s < \hat{p}^s$ to attract the same amount of participation.

Starting from the symmetric equilibrium, when $P_i^s < 0$, all sellers with $v \geq \hat{p}^s$ will continue to join all platforms while sellers with $v \in \left[\frac{P_i^s}{B_i^{(\{i\})}} + \hat{p}^s, \hat{p}^s\right]$ will singlehome on platform i. We reframe the platform's problem as choosing the marginal participating seller type

$$t_i^s \equiv \frac{P_i^s}{B_i^{(\{i\})}} + \hat{p}^s \leq \hat{p}^s.$$

Then, platform i's profit when it charges a negative seller membership fee is

$$\tilde{\Pi}_i\left(t_i^s\right) = \left(\hat{p}^s + \hat{p}^b - c\right)\left(\left(1 - G\left(\hat{p}^s\right)\right)B_i^{(\mathbf{N})} + \left(G\left(\hat{p}^s\right) - G\left(t_i^s\right)\right)B_i^{(\{i\})}\right) + \left(t_i^s - \hat{p}^s\right)\left(1 - G\left(t_i^s\right)\right)B_i^{(\{i\})}$$

where

$$\frac{d\tilde{\Pi}_i}{dt_i^s} = \left(1 - G\left(t_i^s\right)\right)B_i^{(\{i\})} - \left(t_i^s + p_i^b - c\right)g\left(t_i^s\right)B_i^{(\{i\})}.$$

Meanwhile, from Section A.6 of the appendix of the main text, we know that when platform i deviates by lowering its seller transaction fee $p_i^s < \hat{p}^s$, the marginal participating seller type is $t_i^s = p_i^s$ (and is singlehoming on platform i). Reframe the platform's problem as choosing the marginal participating seller type:

$$\Pi_i\left(t_i^s\right) = \left(t_i^s + \hat{p}^b - c\right)\left(\left(1 - G\left(\left(\hat{p}^s - t_i^s\right)\exp\left\{-p^b/\mu\right\} + \hat{p}^s\right)\right)B_i^{(\mathbf{N})} + \left(G\left(\left(\hat{p}^s - t_i^s\right)\exp\left\{-p_i^b/\mu\right\} + \hat{p}^s\right) - G(t_i^s)\right)B_i^{(\{i\})}\right).$$

Using $B_i^{(\mathbf{N})} < B_i^{(\{i\})}$, it is easily shown

$$\frac{d\Pi_i}{dt_i^s} < (1 - G(t_i^s))B_i^{(\{i\})} - \left(t_i^s + p_i^b - c\right) g\left(t_i^s\right) B_i^{(\{i\})}. \tag{C.1}$$

Consider a scenario where platform i wants to lower the participation threshold from $t_i^s \leq \hat{p}^s$ to $t_i^s + \Delta$, where $\Delta < 0$. Whenever the platform finds it profitable to do so through a deviation with a negative seller membership fee ($\frac{d\tilde{\Pi}_i}{dt_i^s}\Delta > 0$), inequality (C.1) immediately implies

$$\frac{d\Pi_i}{dt_i^s}\Delta > \frac{d\tilde{\Pi}_i}{dt_i^s}\Delta > 0,$$

that is, a deviation by lowering seller transaction fee is more profitable.

D Multihoming behaviors of buyers

This section corresponds to the equilibrium analysis of Section 5 in the main text.

Demand derivation. We first derive the demand functions facing each platform. Consider a deviating platform i that charges $\left(p_i^b, p_i^s\right) \neq \left(\hat{p}^b, \hat{p}^s\right)$. Note that the decisions of buyers are shown in the main text and so are omitted here. Consider $p_i^s \geq \hat{p}^s$. For a seller with type v, we write her total surplus from joining all platforms $j \neq i$ as

$$\begin{aligned} & (v - \hat{p}^s) \sum_{j \in \mathbf{N}_{-i}} \left((1-\lambda) B_j^{(\mathbf{N})} + \lambda B_j^{(\mathbf{N}_{-i})}\right) \\ = \; & (v - \hat{p}^s) \sum_{j \in \mathbf{N}_{-i}} \tilde{B}_j^{(\mathbf{N}_{-i})}, \end{aligned} \tag{D.1}$$

where

$$\tilde{B}_j^{(\Theta)} \equiv (1-\lambda) B_j^{(\mathbf{N})} + \lambda B_j^{(\Theta)} \tag{D.2}$$

can be thought of as a "composite" buyer quasi-demand. Likewise, if the seller joins all platforms including i, then her total surplus is

$$\begin{aligned} & (v - \hat{p}^s) \left[(1-\lambda) \sum_{j \in \mathbf{N}_{-i}} B_j^{(\mathbf{N})} + \lambda \sum_{j \in \mathbf{N}_{-i}} B_j^{(\mathbf{N})}\right] + (v - p_i^s) \left[(1-\lambda) B_i^{(\mathbf{N})} + \lambda B_i^{(\mathbf{N})}\right] \\ = \; & (v - \hat{p}^s) \sum_{j \in \mathbf{N}_{-i}} \tilde{B}_j^{(\mathbf{N})} + (v - p_i^s) \tilde{B}_i^{(\mathbf{N})}. \end{aligned} \tag{D.3}$$

Denote

$$\tilde{\sigma}_i \equiv 1 - \frac{\sum_{j \in \mathbf{N}_{-i}} (\tilde{B}_j^{(\mathbf{N})} - \tilde{B}_j^{(\mathbf{N}_{-i})})}{\tilde{B}_i^{(\mathbf{N})}}.$$

Comparing (D.1) and (D.3), we can pin down the threshold $\tilde{v}$ as in Lemma 1:

$$\tilde{v} = \frac{p_i^s}{\tilde{\sigma}_i} - \frac{1 - \tilde{\sigma}_i}{\tilde{\sigma}_i} \hat{p}^s.$$

Likewise, when $p_i^s < \hat{p}^s$, with similar calculations we can pin down thresholds $\tilde{v}_m$ for $m = 1, ..., n-1$ as in Lemma 3:

$$\tilde{v}_m \equiv \frac{p_i^s \left[\tilde{B}_i^{(\mathbf{N}_{i,m})} - \tilde{B}_i^{(\mathbf{N}_{i,m-1})}\right] + \hat{p}^s \left[m \tilde{B}_j^{(\mathbf{N}_{i,m})} - (m-1) \tilde{B}_j^{(\mathbf{N}_{i,m-1})}\right]}{\tilde{B}_i^{(\mathbf{N}_{i,m})} - \tilde{B}_i^{(\mathbf{N}_{i,m-1})} + m \tilde{B}_j^{(\mathbf{N}_{i,m})} - (m-1) \tilde{B}_j^{(\mathbf{N}_{i,m-1})}}.$$

We can further define $\tilde{v}_n \equiv \bar{v}$ and $\tilde{v}_0 \equiv p_i^s$. Then the formal characterization of seller participation decisions is the same as Lemma 3 after replacing the relevant thresholds with $\hat{v}_m$.

Equilibrium. To derive the equilibrium fees, it suffices to focus on the seller participation profile after an upward deviation by platform i, that is, $p_i^s \geq \hat{p}^s$. Note

$$\begin{aligned}\Pi_i &= \left(p_i^b + p_i^s - c\right) Q_i \\ &= \left(p_i^b + p_i^s - c\right)\left(1 - G\left(\tilde{v}\right)\right) B_i^{(\mathbf{N})},\end{aligned}$$

where we use $\tilde{B}_i^{(\mathbf{N})} = B_i^{(\mathbf{N})}$ given (D.2). We assume that Π_i is quasi-concave in $\left(p_i^b, p_i^s\right)$ so that the equilibrium can be characterized by the usual first-order condition. We numerically verified that Π_i is quasi-concave for $\lambda \in \{0.1, 0.5, 0.9\}$ over all the distributional and parameter configurations considered in the baseline model. The details and codes of the numerical exercises are available from the authors upon request.

The demand derivatives, after imposing symmetry, can be calculated as follows:

$$\frac{dQ_i\left(\hat{\mathbf{p}};\hat{\mathbf{p}}\right)}{dp_i^b} = \left(1 - G\left(\hat{p}^s\right)\right)\frac{\partial B_i^{(\mathbf{N})}}{\partial p_i^b}\Big|_{\mathbf{p}=\hat{\mathbf{p}}} = -\frac{1}{X} Q_i\left(\hat{\mathbf{p}};\hat{\mathbf{p}}\right)$$

and

$$\begin{aligned}\frac{dQ_i\left(\hat{\mathbf{p}};\hat{\mathbf{p}}\right)}{dp_i^s} &= -\frac{d\tilde{v}}{dp_i^s} g\left(\hat{p}^s\right) B_i^{(\mathbf{N})}\Big|_{\mathbf{p}=\hat{\mathbf{p}}} \\ &= -\frac{1}{\sigma_\lambda}\frac{g\left(\hat{p}^s\right)}{1 - G\left(\hat{p}^s\right)} Q_i\left(\hat{\mathbf{p}};\hat{\mathbf{p}}\right).\end{aligned}$$

Then, the standard first-order conditions yield the equilibrium in (17).

The proofs of the propositions are provided in the Appendix A.4 and A.5.

D.1 Observable seller fee

We now extend our analysis in Section 5 of the main text to the case where buyers can observe the fees set on the seller side. We will focus on the polar cases where $\lambda \to 0$ and $\lambda \to 1$. We know that $\lambda \to 1$ corresponds to our baseline model. In what follows, we focus on the equilibrium for $\lambda \to 0$.

Denote the symmetric fee equilibrium with $\lambda \to 0$ as $\tilde{p} = \left(\tilde{p}^b, \tilde{p}^s\right)$. Consider a deviating platform i that sets $p_i = \left(p_i^b, p_i^s\right) \neq \tilde{p}$, while all the remaining platforms continue to set $\tilde{p}$. We first note that the seller's participation decision is the same as in the baseline model, such that the number of sellers joining platform i is $1 - G\left(p_i^s\right)$. Then, a given buyer's total utility from participating and being able to transact with sellers through platform i is

$$\begin{aligned}U_i^b &= \max\left\{\epsilon_i - p_i^b, \epsilon_0\right\}\left(1 - G\left(p_i^s\right)\right) + \epsilon_0 G\left(p_i^s\right) \\ &= \max\left\{\epsilon_i - p_i^b - \epsilon_0, 0\right\}\left(1 - G\left(p_i^s\right)\right) + \epsilon_0.\end{aligned}$$

A buyer joins platform i if and only if $U_i^b \geq \max_{j\in\mathbf{N}} U_j^b$, and then uses it for a transaction (with each seller) if $\epsilon_i - p_i^b > \epsilon_0$. Therefore, the total mass of buyers using platform i for transactions (or buyer quasi-demand) is

$$\begin{aligned}B_i &\equiv \Pr\left(\left(\epsilon_i - p_i^b - \epsilon_0\right)\left(1 - G\left(p_i^s\right)\right) \geq \max_{j\in\mathbf{N}}\left\{\left(\epsilon_j - \tilde{p}^b - \epsilon_0\right)\left(1 - G\left(\tilde{p}^s\right)\right), 0\right\}\right) \\ &= \int_{\underline{\epsilon}}^{\bar{\epsilon}}\int_{\underline{\epsilon}}^{\bar{\epsilon}} 1 - F\left(\max\left\{\epsilon - \tilde{p}^b - \epsilon_0, 0\right\}\frac{1 - G(\tilde{p}^s)}{1 - G(p_i^s)} + p_i^b + \epsilon_0\right) dF^{n-1}(\epsilon) dF_0(\epsilon_0).\end{aligned}$$

Notice that when buyers observe the seller fee, their quasi-demand for platform i is decreasing in p_i^s because a higher seller fee decreases seller participation, making it less attractive to buyers.

The total demand is $Q_i(\mathbf{p}_i; \hat{\mathbf{p}}) = (1 - G(p_i^s)) B_i$. The usual demand derivative terms can be calculated as

$$-\frac{Q_i(\hat{\mathbf{p}}; \hat{\mathbf{p}})}{dQ_i(\hat{\mathbf{p}}; \hat{\mathbf{p}})/dp_i^b} = X(\tilde{p}^b; n)$$

and

$$\begin{aligned}
-\frac{Q_i(\hat{\mathbf{p}}; \hat{\mathbf{p}})}{dQ_i(\hat{\mathbf{p}}; \hat{\mathbf{p}})/dp_i^s} &= \frac{1 - G(\tilde{p}^s) B_i}{g(\tilde{p}^s)\left(B_i + \int_{\underline{\epsilon}_0}^{\bar{\epsilon}_0} \int_{p^b+\epsilon_0}^{\bar{\epsilon}} f(\epsilon)\, dF^{n-1}(\epsilon)\, dF_0(\epsilon_0)\right)} \\
&= \frac{1 - G(\tilde{p}^s)}{g(\tilde{p}^s)} \delta(\tilde{p}^b; n),
\end{aligned}$$

where we have defined, for any arbitrarily given (symmetric) equilibrium buyer fee p^b,

$$\delta(p^b; n) \equiv \left(1 + \frac{\int_{\underline{\epsilon}_0}^{\bar{\epsilon}_0} \int_{p^b+\epsilon_0}^{\bar{\epsilon}} f(\epsilon)\, dF^{n-1}(\epsilon)\, dF_0(\epsilon_0)}{\int_{\underline{\epsilon}_0}^{\bar{\epsilon}_0} \int_{\underline{\epsilon}}^{\bar{\epsilon}} (1 - F(\max\{\epsilon, p^b + \epsilon_0\}))\, dF^{n-1}(\epsilon) dF_0(\epsilon_0)}\right)^{-1}. \tag{D.4}$$

Here, $\delta(p^b; n)$ is an inverse measure of how changes in seller participation affect the total demand for platform i. The first component in (D.4) reflects that, each additional seller participating increases the number of transactions that can be made by inframarginal buyers who have participated on platform i. The second component in (D.4) reflects that, each additional seller participating also makes platform i more attractive to buyers, expanding the number of buyers that participate on platform i.

Provided that the profit function $\Pi_i = \left(p_i^b + p_i^s - c\right) Q_i(\mathbf{p}_i; \hat{\mathbf{p}})$ is quasiconcave in $\mathbf{p}_i$, the usual first-order condition shows that a pure symmetric pricing equilibrium is characterized by all platforms setting $\tilde{\mathbf{p}} = (\tilde{p}^b, \tilde{p}^s)$ which solves

$$\tilde{p}^b + \tilde{p}^s - c = X(\tilde{p}^b; n) = \frac{1 - G(\tilde{p}^s)}{g(\tilde{p}^s)} \delta(\tilde{p}^b; n). \tag{D.5}$$

D.2 Comparative statics with observable seller fees

To derive further results, we focus on the special case in which F, $F_0 \sim$ Gumbel with scale parameter μ. In this case, it can be shown that

$$\delta(\tilde{p}^b; n) = \frac{1 + n \exp\left\{-\tilde{p}^b/\mu\right\}}{1 + \left(n + \frac{n-1}{\mu}\right) \exp\{-\tilde{p}^b/\mu\}},$$

which is decreasing in n and increasing in $\tilde{p}^b$. Meanwhile, the expressions for $X\left(p^b; n\right)$ and $\sigma\left(p^b; n\right)$ follow from (9) in Section A.6 of the appendix of the main text.

The following proposition corresponds to Proposition 3. We compare the equilibrium with $\lambda \to 1$ ($\hat{p}^b$ and $\hat{p}^s$) and the equilibrium with $\lambda \to 0$ ($\tilde{p}^b$ and $\tilde{p}^s$).

Proposition D.1 *(Effect of buyer multihoming) Suppose $\mu \geq 1$. A change from $\lambda \to 0$ to $\lambda \to 1$ decreases total fees ($\hat{p}^b + \hat{p}^s < \tilde{p}^b + \tilde{p}^s$), decreases the seller fee ($\hat{p}^s < \tilde{p}^s$), and increases the buyer fee ($\hat{p}^b > \tilde{p}^b$).*

Proof. First, it is straightforward to verify that $\mu \geq 1$ implies $n \exp\left\{-\tilde{p}^b/\mu\right\} \geq \frac{1}{\mu} - 1$, which implies

$$\sigma(\tilde{p}^b; n) = \frac{1}{1 + (n-1)\exp\{-\tilde{p}^b/\mu\}} < \frac{1 + n \exp\left\{-\tilde{p}^b/\mu\right\}}{1 + \left(n + \frac{n-1}{\mu}\right) \exp\{-\tilde{p}^b/\mu\}} = \delta(\tilde{p}^b; n).$$

From the respective equilibrium conditions, $\sigma(\tilde{p}^b;n) < \delta(\tilde{p}^b;n)$ immediately implies $\hat{p}^b + \hat{p}^s < \tilde{p}^b + \tilde{p}^s$. Next, define the function $P^s\left(p^b\right) = X(p^b;n) - p^b + c$, which is decreasing in p^b. Notice $\tilde{p}^s = P^s\left(\tilde{p}^b\right)$ and $\hat{p}^s = P^s\left(\hat{p}^b\right)$. Using this notation, $\tilde{p}^b$ is pinned down by

$$X(\tilde{p}^b;n)\frac{g\left(P^s\left(\tilde{p}^b\right)\right)}{1-G\left(P^s\left(\tilde{p}^b\right)\right)} - \delta(\tilde{p}^b;n) = 0.$$

We know $\sigma(\tilde{p}^b;n) < \delta(\tilde{p}^b;n)$, so

$$X(\tilde{p}^b;n)\frac{g\left(P^s\left(\tilde{p}^b\right)\right)}{1-G\left(P^s\left(\tilde{p}^b\right)\right)} - \sigma(\tilde{p}^b;n) > 0.$$

Notice the left-hand side of this expression is decreasing in p^b. To show $\tilde{p}^b < \hat{p}^b$, suppose by contradiction $\tilde{p}^b \geq \hat{p}^b$. Then it implies

$$X(\hat{p}^b;n)\frac{g\left(P^s\left(\hat{p}^b\right)\right)}{1-G\left(P^s\left(\hat{p}^b\right)\right)} - \sigma(\hat{p}^b;n) \geq X(\tilde{p}^b;n)\frac{g\left(P^s\left(\tilde{p}^b\right)\right)}{1-G\left(P^s\left(\tilde{p}^b\right)\right)} - \sigma(\tilde{p}^b;n) > 0,$$

contradicting the definition of $\hat{p}^b$. Therefore, we must have $\tilde{p}^b < \hat{p}^b$, which immediately implies $\hat{p}^s < \tilde{p}^s$. ∎

The next proposition corresponds to the second part of Proposition 4 in the main text (recall that the first part of the proposition is exactly Proposition 2).

Proposition D.2 *(Increased platform competition) Suppose buyers observe seller fees. In the equilibrium characterized by (D.5), an increase in n (i.e. platform entry) decreases the total fee $\tilde{p}^s + \tilde{p}^b$. Furthermore, an increase in n decreases the buyer fee $\tilde{p}^b$ if $\exp\left\{-\tilde{p}^b/\mu\right\} > \frac{1}{\mu}$, and increases the seller fee $\tilde{p}^s$ if in addition $\exp\left\{-\tilde{p}^b/\mu\right\} > \Gamma$, where Γ is a threshold defined in (D.8) and Γ is decreasing in μ.*

Proof. Denote $M\left(\hat{p}^s\right) \equiv \frac{1-G(\hat{p}^s)}{g(\hat{p}^s)}$. Total differentiation on the equilibrium (D.5), in matrix form, gives

$$\begin{bmatrix} 1-X' & 1 \\ 1-M\frac{\partial\delta}{\partial\tilde{p}^b} & 1-\delta\frac{\partial M}{\partial\tilde{p}^s} \end{bmatrix}\begin{bmatrix} \frac{d\tilde{p}^b}{dn} \\ \frac{d\tilde{p}^s}{dn} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial n} \\ M\frac{\partial\delta}{\partial n} \end{bmatrix}. \tag{D.6}$$

Since $X' \leq 0$, $\frac{\partial M}{\partial\tilde{p}^s} < 0$, so accordingly the matrix in (D.6) has determinant

$$Det \equiv \underbrace{(1-X')\left(1-\delta\frac{\partial M}{\partial\tilde{p}^s}\right)}_{>1} - 1 + M\underbrace{\frac{\partial\delta}{\partial\tilde{p}^b}}_{>0} > 0.$$

By Cramer's rule,

$$\frac{d\tilde{p}^s}{dn} = \frac{1}{Det}\begin{vmatrix} 1-X' & \frac{\partial X}{\partial n} \\ 1-M\frac{\partial\delta}{\partial\tilde{p}^b} & M\frac{\partial\delta}{\partial n} \end{vmatrix} = \frac{1}{Det}\left[\left(M\frac{\partial\delta}{\partial n} - \frac{\partial X}{\partial n}\right) + M\left(\underbrace{\frac{\partial\delta}{\partial\tilde{p}^b}\frac{\partial X}{\partial n}}_{<0} - \underbrace{\frac{\partial\delta}{\partial n}X'}_{\geq 0}\right)\right];$$

$$\frac{d\tilde{p}^b}{dn} = \frac{1}{Det}\begin{vmatrix} \frac{\partial X}{\partial n} & 1 \\ M\frac{\partial\sigma}{\partial n} & 1-\sigma\frac{\partial M}{\partial\tilde{p}^s} \end{vmatrix} = \frac{1}{Det}\left[\left(\frac{\partial X}{\partial n} - M\frac{\partial\delta}{\partial n}\right) - \delta\underbrace{\frac{\partial M}{\partial\tilde{p}^s}\frac{\partial X}{\partial n}}_{\geq 0}\right].$$

Clearly, $\frac{d\tilde{p}^b}{dn}+\frac{d\tilde{p}^s}{dn}<0$. Next, after appropriate substitutions, we can arrive at

$$\begin{aligned}
& M\frac{\partial\delta}{\partial n}-\frac{\partial X}{\partial n} \\
= \quad & \frac{\mu\exp\left\{-\tilde{p}^b/\mu\right\}}{1+(n-1)\exp\left\{-\tilde{p}^b/\mu\right\}}\left(\frac{\exp\left\{-\tilde{p}^b/\mu\right\}}{1+(n-1)\exp\left\{-\tilde{p}^b/\mu\right\}}-\frac{1+\exp\left\{-\tilde{p}^b/\mu\right\}}{\mu+(\mu n+n-1)\exp\left\{-\tilde{p}^b/\mu\right\}}\right),
\end{aligned}$$

which is positive if and only if $\exp\left\{-\tilde{p}^b/\mu\right\}>\frac{1}{\mu}$. Therefore, $\frac{d\tilde{p}^b}{dn}<0$ if $\exp\left\{-\tilde{p}^b/\mu\right\}>\frac{1}{\mu}$ holds. Similarly, we can calculate

$$\begin{aligned}
& M\left(\frac{\partial\delta}{\partial\hat{p}^b}\frac{\partial X}{\partial n}-\frac{\partial\delta}{\partial n}\frac{\partial X}{\partial\hat{p}^b}\right) \\
= \quad & -\frac{\mu\exp\left\{-\tilde{p}^b/\mu\right\}^2\left[1+(n-1)\exp\left\{-\tilde{p}^b/\mu\right\}+\exp\left\{-\tilde{p}^b/\mu\right\}\right]}{\left[1+(n-1)\exp\left\{-\tilde{p}^b/\mu\right\}\right]^3\left[\mu+(n+\mu n-1)\exp\left\{-\tilde{p}^b/\mu\right\}\right]}.
\end{aligned}$$

Substituting these expressions and rearranging, we get $\frac{d\tilde{p}^s}{dn}>0$ if and only if

$$\frac{2}{\exp\left\{-\tilde{p}^b/\mu\right\}}+\frac{1}{n+1}\left(\frac{1}{\exp\left\{-\tilde{p}^b/\mu\right\}}+1\right)^2-\mu-(n-1)\left(\mu\exp\left\{-\tilde{p}^b/\mu\right\}-1\right)<0. \tag{D.7}$$

Notice that the left-hand side of (D.7) is decreasing in $\exp\left\{-\tilde{p}^b/\mu\right\}$. Therefore, an application of the intermediate value theorem implies that there exists a unique cutoff Γ, defined implicitly by

$$\frac{2}{\Gamma}+\frac{1}{n+1}\left(\frac{1}{\Gamma}+1\right)^2-\mu-(\mu\Gamma-1)(n-1)=0, \tag{D.8}$$

such that (D.7) holds if and only if $\exp\left\{-\tilde{p}^b/\mu\right\}>\Gamma$. Notice that Γ is decreasing in μ and n. ■

To the extent that $\exp\left\{-\tilde{p}^b/\mu\right\}$ is bounded below (e.g. when $\tilde{p}^b\leq 0$ so that $\exp\left\{-\tilde{p}^b/\mu\right\}\geq 1$), then the conditions in Proposition D.2 hold whenever μ is sufficiently large (i.e. platforms are sufficiently differentiated from buyers' perspective).

E Value of transactions and user heterogeneity

This section corresponds to Section 6 in the main text. Throughout, we denote

$$M\left(\hat{p}^s\right)\equiv\left(\frac{1-G(\frac{\hat{p}^s-\alpha^s}{\gamma^s})}{g(\frac{\hat{p}^s-\alpha^s}{\gamma^s})}\right)\gamma^s,$$

where $\frac{\partial M}{\partial\hat{p}^s}<0$ by log-concavity of $1-G$.

Proof of Proposition 5. By linearity, note that $\frac{\partial X}{\partial\alpha^b}=-\frac{\partial X}{\partial\hat{p}^b}=-\frac{1}{\gamma^b}X'\geq 0$ and $\frac{\partial\sigma}{\partial\alpha^b}=-\frac{\partial\sigma}{\partial\hat{p}^b}=-\frac{1}{\gamma^b}\sigma'<0$, where $X'\leq 0$ and $\sigma'>0$ are the derivatives of X and σ with respect to the first argument (Lemma 4 and Lemma 5). Total differentiation of the equilibrium (18), in matrix form, gives

$$\begin{bmatrix} 1-X' & 1 \\ 1-\frac{M}{\gamma^b}\sigma' & 1-\sigma\frac{\partial M}{\partial\hat{p}^s}\end{bmatrix}\begin{bmatrix}\frac{d\hat{p}^b}{d\alpha^b} \\ \frac{d\hat{p}^s}{d\alpha^b}\end{bmatrix}=\begin{bmatrix}-X' \\ -\frac{M}{\gamma^b}\sigma'\end{bmatrix}. \tag{E.1}$$

The matrix in (E.1) has a strictly positive determinant

$$Det=-\left(1-X'\right)\sigma\frac{\partial M}{\partial\hat{p}^s}+\frac{M}{\gamma^b}\sigma'-X'>0,$$

By Cramer's rule,

$$\begin{aligned}
\frac{d\hat{p}^s}{d\alpha^b} &= \frac{-1}{Det}\left[\frac{M}{\gamma^b}\sigma' - X'\right] = -\left(1 + \frac{(1-X')}{Det}\sigma\frac{\partial M}{\partial \hat{p}^s}\right) \in (-1,0) && \text{(E.2)}\\
\frac{d\hat{p}^b}{d\alpha^b} &= \frac{1}{Det}\left[\left(\frac{M}{\gamma^b}\sigma' - X'\right) + \sigma\frac{\partial M}{\partial \hat{p}^s}X'\right] = 1 + \frac{1}{Det}\sigma\frac{\partial M}{\partial \hat{p}^s} \in (0,1) && \text{(E.3)}\\
\frac{d\hat{p}^s}{d\alpha^b} + \frac{d\hat{p}^b}{d\alpha^b} &= \frac{1}{Det}\sigma\frac{\partial M}{\partial \hat{p}^s}X' \geq 0.
\end{aligned}$$

The results of $\frac{d\hat{p}^b}{d\alpha^s} < 0$, $\frac{d\hat{p}^s}{d\alpha^s} > 0$, and $\frac{d\hat{p}^s}{d\alpha^b} + \frac{d\hat{p}^b}{d\alpha^b} \geq 0$ can be proven similarly by applying Cramer's rule to the following matrix:

$$\begin{bmatrix} 1-X' & 1 \\ 1-\frac{M}{\gamma^b}\sigma' & 1-\sigma\frac{\partial M}{\partial \hat{p}^s} \end{bmatrix}\begin{bmatrix} \frac{d\hat{p}^b}{d\alpha^s} \\ \frac{d\hat{p}^s}{d\alpha^s} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial M}{\partial \hat{p}^s}\sigma \end{bmatrix},$$

where

$$\begin{aligned}
\frac{d\hat{p}^s}{d\alpha^s} &= \frac{1}{Det}\left[\frac{\partial M}{\partial \hat{p}^s}\sigma\right] < 0\\
\frac{d\hat{p}^b}{d\alpha^s} &= \frac{-1}{Det}(1-X')\frac{\partial M}{\partial \hat{p}^s}\sigma = \frac{-(1-X')\frac{\partial M}{\partial \hat{p}^s}\sigma}{-(1-X')\sigma\frac{\partial M}{\partial \hat{p}^s} + \frac{M}{\gamma^b}\sigma' - X'} \in (0,1).
\end{aligned}$$

Proof of Proposition 6. To prove the first result, we note from the proof of Proposition 5 that $\hat{p}^b - \alpha^b$ is strictly decreasing in α^b while $\hat{p}^s + \alpha^b$ is strictly increasing in α^b. Thus, if α^b is sufficiently small, we have $\hat{p}^b - \alpha^b \geq 0$ and $\hat{p}^s + \alpha^b \leq 0$. From equilibrium (18),

$$\frac{\partial X}{\partial \gamma^b} = -\left(\frac{\hat{p}^b - \alpha^b}{\gamma^{b2}}\right)X' \geq 0 \quad \text{and} \quad \frac{\partial \sigma}{\partial \gamma^b} = -\left(\frac{\hat{p}^b - \alpha^b}{\gamma^{b2}}\right)\sigma' \leq 0.$$

Total differentiation of the equilibrium (18), in matrix form, gives

$$\begin{bmatrix} 1-X' & 1 \\ 1-\frac{M}{\gamma^b}\sigma' & 1-\sigma\frac{\partial M}{\partial \hat{p}^s} \end{bmatrix}\begin{bmatrix} \frac{d\hat{p}^b}{d\gamma^b} \\ \frac{d\hat{p}^s}{d\gamma^b} \end{bmatrix} = \begin{bmatrix} X + \gamma^b\frac{\partial X}{\partial \gamma^b} \\ M\ \frac{\partial \sigma}{\partial \gamma^b} \end{bmatrix},$$

where the matrix on the left hand side is the same as in (E.1). By Cramer's rule,

$$\begin{aligned}
\frac{d\hat{p}^b}{d\gamma^b} &= \frac{1}{Det}\left[\left(X + \gamma^b\frac{\partial X}{\partial \gamma^b}\right)\left(1 - \sigma\frac{\partial M}{\partial \hat{p}^s}\right) - M\ \frac{\partial \sigma}{\partial \gamma^b}\right] \geq 0\\
\frac{d\hat{p}^s}{d\gamma^b} &= \frac{1}{Det}\left[(1-X')M\ \frac{\partial \sigma}{\partial \gamma^b} - \left(1 - \frac{M}{\gamma^b}\sigma'\right)\left(X + \gamma^b\frac{\partial X}{\partial \gamma^b}\right)\right]\\
&= \frac{1}{Det}\left[-\left(X + \gamma^b\frac{\partial X}{\partial \gamma^b}\right) + M\ \frac{\partial \sigma}{\partial \gamma^b} + \frac{M}{\gamma^b}\sigma' X\right]\\
&= \frac{1}{Det}\left[-\left(X + \gamma^b\frac{\partial X}{\partial \gamma^b}\right) + \sigma'\frac{M}{(\gamma^b)^2}\left(\hat{p}^s + \alpha^b - c\right)\right] \leq 0,
\end{aligned}$$

where the last equality uses the equilibrium condition $\hat{p}^b + \hat{p}^s - c = \gamma^b X$. Finally,

$$\frac{d\hat{p}^b}{d\gamma^b} + \frac{d\hat{p}^s}{d\gamma^b} = \frac{1}{Det}\left[M\sigma'\frac{X}{\gamma^b} - \sigma\frac{\partial M}{\partial \hat{p}^s}\left(X + \gamma^b\frac{\partial X}{\partial \gamma^b}\right)\right] \geq 0.$$

To prove the second result, we note from the proof of Proposition 5 that $\hat{p}^s - \alpha^s$ is strictly decreasing in α^s. Thus, if α^s is sufficiently small, we have $\hat{p}^s - \alpha^s \geq 0$ so that

$$\frac{\partial M}{\partial \gamma^s} = \frac{1 - G(\frac{\hat{p}^s - \alpha^s}{\gamma^s})}{g(\frac{\hat{p}^s - \alpha^s}{\gamma^s})} + \frac{1 - G(\frac{\hat{p}^s - \alpha^s}{\gamma^s})}{g(\frac{\hat{p}^s - \alpha^s}{\gamma^s})} \left(\frac{\hat{p}^s - \alpha^s}{\gamma^s} \right) \geq 0.$$

Applying Cramer's rule to the following matrix yields the stated results.

$$\begin{bmatrix} 1 - X' & 1 \\ 1 - \frac{M}{\gamma^b}\sigma' & 1 - \sigma \frac{\partial M}{\partial \hat{p}^s} \end{bmatrix} \begin{bmatrix} \frac{d\hat{p}^b}{d\gamma^s} \\ \frac{d\hat{p}^s}{d\gamma^s} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial M}{\partial \gamma^s}\sigma \end{bmatrix}.$$